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AN ALGEBRA OF GENERALIZED ROUMIEU ULTRADISTRIBUTIONS

Abstract. We introduce an algebra of generalized functions containing Roumieu ultradistributions.

1. Introduction

Schwartz distributions [13] have natural extensions Roumieu ultradistributions [12], see [8] for a detailed study and [9] for applications. Gevrey ultradistributions are particular, but important, case of Roumieu ultradistributions.

Colombeau generalized functions [3] and [4], introduced in connection with the problem of multiplication of Schwartz distributions, have been developed and applied in linear and nonlinear problems, see [7].

The problem of multiplication of ultradistributions is still posed. So, it is natural to search for algebras of generalized functions containing spaces of ultradistributions, to study and to apply them. Generalized Gevrey ultradistributions have been introduced and studied in [1] and [2].

The aim of this paper is to introduce an algebra of generalized functions containing Roumieu ultradistributions.

2. Roumieu ultradistributions

Let $(M_p)_{p\in\mathbb{Z}_+}$ be a sequence of positive numbers, recall the following properties:

$(H1)$ Logarithmic convexity :

$$M_p^2 \leq M_{p-1}M_{p+1},\ \forall p \geq 1.$$

$(H2)$ Stability under ultradifferentiation :

$$\exists A > 0, \exists H > 0, M_{p+q} \leq AH^{p+q}M_pM_q, \forall p \geq 0, \forall q \geq 0.$$

$(H2)'$ Stability under differentiation :

$$\exists A > 0, \exists H > 0, M_{p+1} \leq AH^pM_p, \forall p \geq 0.$$

$(H3)'$ Non-quasi-analyticity :

$$\sum_{p=1}^{\infty} \frac{M_{p-1}}{M_p} < +\infty.$$

The associated function of the sequence $(M_p)_{p\in\mathbb{Z}_+}$ is the function defined by

$$M(t) = \sup_p \ln \frac{t^p}{M_p}, t \in \mathbb{R}_+^*$$

EXAMPLE 1. The well-known Gevrey sequence $(M_p)_{p\in\mathbb{Z}_+} = (p!^\sigma)_{p\in\mathbb{Z}_+}, \sigma > 0$, has associated function equivalent to the function $M_\sigma(t) = t^{\frac{1}{\sigma}}$.

An important result on the associated function is given in the following proposition, see [8].

PROPOSITION 1. *Let the sequence $(M_p)_{p\in\mathbb{Z}_+}$ satisfy condition $(H1)$, then it satisfies $(H2)$ if and and only if $\exists A > 0, \exists H > 0, \forall t > 0$,*

$$2M(t) \leq M(Ht) + \ln(AM_0).$$

The class of ultradifferentiable functions of class M, denoted $E^M(\Omega)$, is the space of all $f \in C^\infty(\Omega)$ satisfying for every compact subset K of Ω, $\exists c > 0, \forall \alpha \in \mathbb{Z}_+^n$,

$$\sup_{x\in K} |\partial^\alpha f(x)| \leq c^{|\alpha|+1} M_{|\alpha|} \tag{1}$$

EXAMPLE 2. If $(M_p)_{p\in\mathbb{Z}_+} = (p!^\sigma)_{p\in\mathbb{Z}_+}$ we obtain $E^\sigma(\Omega)$ the Gevrey space of order σ, and $\mathcal{A}(\Omega) := E^1(\Omega)$ is the space of real analytic functions on the open set Ω.

A differential operator of infinite order $P(D) = \sum_{\gamma\in\mathbb{Z}_+^n} a_\gamma D^\gamma$ is called an ultradifferential operator of class $\{M_p\}_{p\in\mathbb{Z}_+}$, if for every $h > 0$ there exist $c > 0$ such that $\forall \gamma \in \mathbb{Z}_+^n$,

$$|a_\gamma| \leq c \frac{h^{|\gamma|}}{M_{|\gamma|}} \tag{2}$$

The basic properties of the space $E^M(\Omega)$ are summarized in the following proposition, for the proof see [10] and [8].

PROPOSITION 2. *Let the sequence $(M_p)_{p\in\mathbb{Z}_+}$ satisfy condition $(H1)$, then the space $E^M(\Omega)$ is an algebra moreover, if $(M_p)_{p\in\mathbb{Z}_+}$ satisfies $(H2)'$, then $E^M(\Omega)$ is stable by differential operators of finite order with coefficients in $E^M(\Omega)$, and if $(M_p)_{p\in\mathbb{Z}_+}$ satisfies $(H2)$ then any ultradifferential operator of class M operates also as a sheaf homomorphism.*

The space $\mathcal{D}^M(\Omega) = E^M(\Omega) \cap \mathcal{D}(\Omega)$ is not trivial if and only if the sequence $(M_p)_{p\in\mathbb{Z}_+}$ satisfies $(H3)'$.

REMARK 1. The sequence $(p!^\sigma)_{p\in\mathbb{Z}_+}$ satisfies $(H3)'$ if and only if $\sigma > 1$.

DEFINITION 1. *The strong dual of $\mathcal{D}^M(\Omega)$, denoted $\mathcal{D}'_M(\Omega)$, is called the space of Roumieu ultraditributions. The space of Gevrey ultradistributions, denoted $\mathcal{D}'_\sigma(\Omega)$, corresponds to the sequence $M = (p!^\sigma)_{p\in\mathbb{Z}_+}$.*

3. Generalized Roumieu ultradistributions

To define the algebra of generalized Roumieu ultradistributions, we first introduce the algebra of moderate elements and its ideal of null elements.

Let Ω be a non void open set of $\mathbb{R}^n$ and $I =]0,1]$.

REMARK 2. In the sequel, we will always suppose that the sequence $(M_p)_{p\in\mathbb{Z}_+}$ satisfies the conditions $(H1), (H2), (H3)'$ and $M_0 = 1$.

DEFINITION 2. *The space of moderate elements, denoted $\mathcal{E}_m^{\{M\}}(\Omega)$, is the space of $(f_\varepsilon)_\varepsilon \in C^\infty(\Omega)^I$ satisfying for every compact K of Ω, $\forall\alpha \in \mathbb{Z}_+^m, \exists k > 0, \exists c > 0, \exists \varepsilon_0 \in I, \forall \varepsilon \leq \varepsilon_0$,*

$$\sup_{x\in K} |\partial^\alpha f_\varepsilon(x)| \leq c \exp\left(M\left(\frac{k}{\varepsilon}\right)\right). \tag{3}$$

The space of null elements, denoted $\mathcal{N}^{\{M\}}(\Omega)$, is the space of $(f_\varepsilon)_\varepsilon \in C^\infty(\Omega)^I$ satisfying for every compact K of Ω,$\forall\alpha \in \mathbb{Z}_+^m, \forall k > 0, \exists c > 0, \exists \varepsilon_0 \in I, \forall \varepsilon \leq \varepsilon_0$,

$$\sup_{x\in K} |\partial^\alpha f_\varepsilon(x)| \leq c \exp\left(-M\left(\frac{k}{\varepsilon}\right)\right). \tag{4}$$

The main properties of the spaces $\mathcal{E}_m^{\{M\}}(\Omega)$ and $\mathcal{N}^{\{M\}}(\Omega)$ are given in the following proposition.

PROPOSITION 3. *1) The space of moderate elements $\mathcal{E}_m^{\{M\}}(\Omega)$ is an algebra stable by derivation.*

2) The space $\mathcal{N}^{\{M\}}(\Omega)$ is an ideal of $\mathcal{E}_m^{\{M\}}(\Omega)$.

Proof. 1) Let $(f_\varepsilon)_\varepsilon, (g_\varepsilon)_\varepsilon \in \mathcal{E}_m^{\{M\}}(\Omega)$ and K be a compact of Ω, then $\forall\beta \in \mathbb{Z}_+^m$, $\exists k_1 = k_1(\beta) > 0, \exists c_1 = c_1(\beta) > 0, \exists \varepsilon_{1\beta} \in I, \forall \varepsilon \leq \varepsilon_{1\beta}$,

$$\sup_{x\in K} \left|\partial^\beta f_\varepsilon(x)\right| \leq c_1 \exp M\left(\frac{k_1}{\varepsilon}\right), \tag{5}$$

$\forall\beta \in \mathbb{Z}_+^m$, $\exists k_2 = k_2(\beta) > 0, \exists c_2 = c_2(\beta) > 0, \exists \varepsilon_{2\beta} \in I, \forall \varepsilon \leq \varepsilon_{2\beta}$,

$$\sup_{x\in K} \left|\partial^\beta g_\varepsilon(x)\right| \leq c_2 \exp M\left(\frac{k_2}{\varepsilon}\right). \tag{6}$$

Let $\alpha \in \mathbb{Z}_+^m$, then

$$\left|\partial^\alpha (f_\varepsilon g_\varepsilon)(x)\right| \leq \sum_{\beta=0}^{\alpha} \binom{\alpha}{\beta} \left|\partial^{\alpha-\beta} f_\varepsilon (x)\right| \left|\partial^\beta g_\varepsilon (x)\right|$$

from proposition 1, we have $\exists A>0, \exists H>0, \forall t>0,$

$$2M(t) \leq M(Ht) + \ln (AM_0). \tag{7}$$

For $k = H(\max\{k_1(\beta), k_2(\beta) : \beta \leq \alpha\}), \varepsilon \leq \min\{\varepsilon_{1\beta}, \varepsilon_{2\beta}; |\beta| \leq |\alpha|\}$ and $x \in K,$ we have for $t = \dfrac{k}{\varepsilon}$

$$\begin{aligned}
\exp\left(-M\left(\frac{k}{\varepsilon}\right)\right)\left|\partial^\alpha (f_\varepsilon g_\varepsilon)(x)\right| &\leq \exp(\ln(AM_0)) \sum_{\beta=0}^{\alpha} \binom{\alpha}{\beta} \exp\left(-M\left(\frac{k_1}{\varepsilon}\right)\right) \\
&\quad \times \left|\partial^{\alpha-\beta} f_\varepsilon (x)\right| \exp\left(-M\left(\frac{k_2}{\varepsilon}\right)\right)\left|\partial^\beta g_\varepsilon (x)\right| \\
&\leq A \sum_{\beta=0}^{\alpha} \binom{\alpha}{\beta} c_1(\alpha-\beta) c_2(\beta) = c(\alpha),
\end{aligned}$$

i.e. $(f_\varepsilon g_\varepsilon)_\varepsilon \in \mathcal{E}_m^{\{M\}}(\Omega)$. It is clear, from (5) that for every compact K of Ω, $\forall \beta \in \mathbb{Z}_+^m$, $\exists k_1 = k_1(\beta+1) > 0, \exists c_1 = c_1(\beta+1) > 0, \exists \varepsilon_{1\beta} \in I$ such that $\forall x \in K, \forall \varepsilon \leq \varepsilon_{1\beta}$,

$$\left|\partial^\beta (\partial f_\varepsilon)(x)\right| \leq c_1 \exp\left(M\left(\frac{k_1}{\varepsilon}\right)\right),$$

i.e. $(\partial f_\varepsilon)_\varepsilon \in \mathcal{E}_m^{\{M\}}(\Omega)$.

2) If $(g_\varepsilon)_\varepsilon \in \mathcal{N}^{\{M\}}(\Omega)$, for every K compact of Ω, $\forall \beta \in \mathbb{Z}_+^m, \forall k_2 > 0, \exists c_2 = c_2(\beta, k_2) > 0, \exists \varepsilon_{2\beta} \in I,$

$$\left|\partial^\alpha g_\varepsilon (x)\right| \leq c_2 \exp\left(-M\left(\frac{k_2}{\varepsilon}\right)\right), \forall x \in K, \forall \varepsilon \leq \varepsilon_{2\beta}$$

Let $\alpha \in \mathbb{Z}_+^m$ and $k>0$, then

$$\begin{aligned}
\exp\left(M\left(\frac{k}{\varepsilon}\right)\right)\left|\partial^\alpha (f_\varepsilon g_\varepsilon)(x)\right| &\leq \exp\left(M\left(\frac{k}{\varepsilon}\right)\right) \sum_{\beta=0}^{\alpha} \binom{\alpha}{\beta} \left|\partial^{\alpha-\beta} f_\varepsilon (x)\right| \times \\
&\quad \times \left|\partial^\beta g_\varepsilon (x)\right|.
\end{aligned}$$

Let $k_2 = H.\max\{k_1(\beta), k; \beta \leq \alpha\}$ and $\varepsilon \leq \min\{\varepsilon_{1\beta}, \varepsilon_{2\beta}; \beta \leq \alpha\}$, then $\forall x \in K$, we have for $t = \dfrac{k_2}{\varepsilon}$ in (7)

$$\begin{aligned}
\exp\left(M\left(\frac{k}{\varepsilon}\right)\right)\left|\partial^{\alpha}\left(f_\varepsilon g_\varepsilon\right)(x)\right| &\leq A\sum_{\beta=0}^{\alpha}\binom{\alpha}{\beta}\left[\exp\left(-M\left(\frac{k_1}{\varepsilon}\right)\right)\left|\partial^{\alpha-\beta}f_\varepsilon(x)\right|\right.\\
&\quad \left.\times\exp\left(M\left(\frac{k_2}{\varepsilon}\right)\right)\left|\partial^{\beta}g_\varepsilon(x)\right|\right]\\
&\leq A\sum_{\beta=0}^{\alpha}\binom{\alpha}{\beta}c_1(\alpha-\beta)\,c_2(\beta,k_2)=c(\alpha,k),
\end{aligned}$$

which shows that $(f_\varepsilon g_\varepsilon)_\varepsilon \in \mathcal{N}^{\{M\}}(\Omega)$. □

DEFINITION 3. *The algebra of generalized Roumieu ultradistributions of class* $\{M_p\}_{p\in\mathbb{Z}_+}$*, denoted* $\mathcal{G}^{\{M\}}(\Omega)$*, is the quotient algebra*

$$\mathcal{G}^{\{M\}}(\Omega)=\frac{\mathcal{E}_m^{\{M\}}(\Omega)}{\mathcal{N}^{\{M\}}(\Omega)}.$$

EXAMPLE 3. If $(M_p)_{p\in\mathbb{Z}_+}=(p!^{\sigma})_{p\in\mathbb{Z}_+}$ we obtain $\mathcal{G}^{\sigma}(\Omega)$ the algebra of generalized Gevrey ultradistributions of [1].

4. Embedding of Roumieu ultradistributions with compact support

Let $N=(N_p)_{p\in\mathbb{Z}_+}$ be a sequence satisfying the conditions $(H1),(H2),(H3)'$ and $N_0=1$, the space $\mathcal{S}^{(N)}(\mathbb{R}^n)$ is the space of functions $\varphi\in C^\infty(\mathbb{R}^n)$ such that $\forall b>0$, we have

$$\|\varphi\|_{b,N}=\sup_{\alpha,\beta\in\mathbb{Z}_+^m}\int\frac{|x|^{|\beta|}}{b^{|\alpha+\beta|}N_{|\alpha|}N_{|\beta|}}\left|\partial^{\alpha}\varphi(x)\right|dx<\infty. \tag{8}$$

Define $\sum^{(N)}$ as the set of functions $\phi\in\mathcal{S}^{(N)}(\mathbb{R}^n)$ satisfying

$$\int\phi(x)\,dx=1 \text{ and } \int x^{\alpha}\phi(x)\,dx=0,\forall\alpha\in\mathbb{Z}_+^N\backslash\{0\}.$$

LEMMA 1. *There exist functions in* $\sum^{(N)}$.

DEFINITION 4. *The net* $\phi_\varepsilon=\varepsilon^{-m}\phi(./\varepsilon),\varepsilon\in I$*, where* ϕ *satisfies the conditions of lemma 4, is called a* $N-$*mollifier net*.

The space $\mathcal{E}^{\{M\}}(\Omega)$ is embedded into $\mathcal{G}^{\{M\}}(\Omega)$ by the standard canonical injection

$$\begin{array}{rlll} I: & \mathcal{E}^{\{M\}}(\Omega) & \rightarrow & \mathcal{G}^{\{M\}}(\Omega)\\ & f & \mapsto & [f]=cl(f_\varepsilon), \end{array} \tag{9}$$

where $f_\varepsilon = f$, $\forall\varepsilon \in I$.

The following proposition gives the embedding of Roumieu ultradistributions into $\mathcal{G}^{\{M\}}(\Omega)$. Let M and N two sequences satisfying $(H1),(H2)$ and $(H3)'$ with $M_0 = N_0 = 1$ and $\phi \in \Sigma^{(N)}$.

THEOREM 1. *The map*

$$J_0 : \begin{array}{lcl} \mathcal{E}'_{\{MN\}}(\Omega) & \to & \mathcal{G}^{\{M\}}(\Omega) \\ T & \mapsto & [T] = cl\left(\left(T * \phi_\varepsilon\right)_{/\Omega}\right)_\varepsilon \end{array} \tag{10}$$

is an embedding.

Proof. Let $T \in \mathcal{E}'_{\{MN\}}(\Omega)$ with $suppT \subset K$, then there exists $P(D) = \sum_{\gamma\in\mathbb{Z}_+^m} a_\gamma D^\gamma$ an ultradifferential operator of class $\{M_pN_p\}_{p\in\mathbb{Z}_+}$,$C > 0$, and continuous functions f_γ with $suppf_\gamma \subset K, \forall\gamma \in \mathbb{Z}_+^m$, and $\sup_{\gamma\in\mathbb{Z}_+^m, x\in K} \left|f_\gamma(x)\right| \le C$, such that

$$T = \sum_{\gamma\in\mathbb{Z}_+^m} a_\gamma D^\gamma f_\gamma.$$

We have

$$T * \phi_\varepsilon(x) = \sum_{\gamma\in\mathbb{Z}_+^m} a_\gamma \frac{(-1)^{|\gamma|}}{\varepsilon^{|\gamma|}} \int f_\gamma(x+\varepsilon y) D^\gamma \phi(y)\, dy.$$

Let $\alpha \in \mathbb{Z}_+^m$, then

$$\left|\partial^\alpha (T * \phi_\varepsilon(x))\right| \le \sum_{\gamma\in\mathbb{Z}_+^m} a_\gamma \frac{1}{\varepsilon^{|\gamma+\alpha|}} \int \left|f_\gamma(x+\varepsilon y)\right| \left|D^{\gamma+\alpha}\phi(y)\right| dy.$$

From (2) and condition $(H2)$, we have $\exists A > 0, \exists H > 0, \forall h > 0, \exists c > 0$, such that

$$\begin{aligned} \left|\partial^\alpha (T * \phi_\varepsilon(x))\right| &\le c \sum_{\gamma\in\mathbb{Z}_+^m} \frac{h^{|\gamma|}}{M_{|\gamma|}N_{|\gamma|}} \frac{1}{\varepsilon^{|\gamma+\alpha|}} \int \left|f_\gamma(x+\varepsilon y)\right| \left|D^{\gamma+\alpha}\phi(y)\right| dy \\ &\le c \sum_{\gamma\in\mathbb{Z}_+^m} \frac{h^{|\gamma|} b^{|\gamma+\alpha|} N_{|\gamma+\alpha|}}{M_{|\gamma|}N_{|\gamma|}} \frac{1}{\varepsilon^{|\gamma+\alpha|}} \int \left|f_\gamma(x+\varepsilon y)\right| \frac{\left|D^{\gamma+\alpha}\phi(y)\right|}{b^{|\gamma+\alpha|}N_{|\gamma+\alpha|}} dy \\ &\le cA \sum_{\gamma\in\mathbb{Z}_+^m} c \frac{h^{|\gamma|} b^{|\gamma+\alpha|} H^{|\gamma+\alpha|} M_{|\alpha|} N_{|\alpha|}}{M_{|\gamma+\alpha|}} \frac{1}{\varepsilon^{|\gamma+\alpha|}} C \|\phi\|_{b,N} \end{aligned}$$

then,

$$\begin{aligned} \frac{(2h)^{|\alpha|}}{M_{|\alpha|}N_{|\alpha|}} \left|\partial^\alpha (T * \phi_\varepsilon(x))\right| &\le cCA \|\phi\|_{b,N} \sum_{\gamma\in\mathbb{Z}_+^m} 2^{-|\gamma|} \frac{(2hHb)^{|\gamma+\alpha|}}{M_{|\gamma+\alpha|}} \frac{1}{\varepsilon^{|\gamma+\alpha|}} \\ &\le c \exp\left(M\left(\frac{k}{\varepsilon}\right)\right), \end{aligned}$$

i.e.

$$|\partial^{\alpha}(T*\phi_{\varepsilon}(x))| \le c(\alpha)\exp\left(M\left(\frac{k}{\varepsilon}\right)\right), \tag{11}$$

where $k = 2hHb$.

Suppose that $(T*\phi_{\varepsilon})_{\varepsilon} \in \mathcal{N}^{\{M\}}(\Omega)$, then for every compact L of Ω, $\exists c > 0$, $\forall k > 0$, $\exists \varepsilon_0 \in I$,

$$|T*\phi_{\varepsilon}(x)| \le c\exp\left(-M\left(\frac{k}{\varepsilon}\right)\right), \forall x \in L, \forall \varepsilon \le \varepsilon_0 \tag{12}$$

Let $\chi \in \mathcal{D}^{\{MN\}}(\Omega)$ and $\chi = 1$ in a neighborhood of K, then $\forall \psi \in \mathcal{E}^{\{M.N\}}(\Omega)$,

$$\langle T,\psi\rangle = \langle T,\chi\psi\rangle = \lim_{\varepsilon\to 0}\int (T*\phi_{\varepsilon})(x)\chi(x)\psi(x)\,dx$$

Consequently, from (12), we obtain

$$\left|\int (T*\phi_{\varepsilon})(x)\chi(x)\psi(x)\,dx\right| \le c\exp\left(-M\left(\frac{k}{\varepsilon}\right)\right), \forall \varepsilon \le \varepsilon_0,$$

which gives $\langle T,\psi\rangle = 0$. □

NOTATION 1. If $M = (M_p)_{p\in\mathbb{Z}_+}$ and $N = (N_p)_{p\in\mathbb{Z}_+}$ are two sequences, then $MN^{-1} := \left(M_p N_p^{-1}\right)_{p\in\mathbb{Z}_+}$.

In order to show the commutativity of the following diagram of embeddings

$$\begin{array}{ccc} \mathcal{D}^{\left\{MN^{-1}p!\right\}}(\Omega) & \rightarrow & \mathcal{G}^{\{M\}}(\Omega) \\ & \searrow & \uparrow \\ & & \mathcal{E}'_{\{MN\}}(\Omega) \end{array},$$

we have to prove the following fundamental result.

PROPOSITION 4. *Let* $f \in \mathcal{D}^{\left\{MN^{-1}p!\right\}}(\Omega)$ *and* $\phi \in \sum^{(N)}$*, then*

$$\left(f - (f*\phi_{\varepsilon})_{/\Omega}\right)_{\varepsilon} \in \mathcal{N}^{\{M\}}(\Omega).$$

Proof. Let $f \in \mathcal{D}^{\left\{MN^{-1}p!\right\}}(\Omega)$, then there exists a constant $c > 0$, such that

$$|\partial^{\alpha} f(x)| \le c^{|\alpha|+1}\frac{M_p}{N_p}p!, \forall \alpha \in \mathbb{Z}_+^n, \forall x \in \Omega.$$

Let $\alpha \in \mathbb{Z}_+^n$, the Taylor's formula and the properties of ϕ_{ε} give

$$\partial^{\alpha}(f*\phi_{\varepsilon} - f)(x) = \sum_{|\beta|=N}\int \frac{(\varepsilon y)^{\beta}}{\beta!}\partial^{\alpha+\beta} f(\xi)\,\phi(y)\,dy,$$

where $x \le \xi \le x + \varepsilon y$. Consequently, for any $b > 0$, we have

$$\begin{aligned}
\left|\partial^{\alpha}\left(f * \phi_{\varepsilon} - f\right)(x)\right| &\le \varepsilon^{N} \sum_{|\beta|=N} \int \frac{|y|^{|\beta|}}{\beta!} \left|\partial^{\alpha+\beta} f(\xi)\right| |\phi(y)|\, dy \\
&\le \varepsilon^{N} \sum_{|\beta|=N} \frac{b^{|\beta|} N_{|\beta|} M_{|\alpha+\beta|} (\alpha+\beta)!}{\beta! N_{|\alpha+\beta|}} \times \\
&\quad \times \int \frac{N_{|\alpha+\beta|}}{M_{|\alpha+\beta|} (\alpha+\beta)!} \left|\partial^{\alpha+\beta} f(\xi)\right| \frac{|y|^{|\beta|}}{b^{|\beta|} N_{|\beta|}} |\phi(y)|\, dy \\
&\le A \|\phi\|_{b,N}\, c \frac{c^{|\alpha|} H^{|\alpha|} M_{|\alpha|} \alpha!}{N_{|\alpha|}} \varepsilon^{N} \sum_{|\beta|=N} b^{|\beta|} H^{|\beta|} M_{|\beta|} c^{|\beta|}.
\end{aligned}$$

Let $k > 0$ and $T > 0$, then

$$\left|\partial^{\alpha}\left(f * \phi_{\varepsilon} - f\right)(x)\right| \le c(\alpha)\, \varepsilon^{N} M_{N} (kT)^{-N} \sum_{|\beta|=N} (kTbHc)^{|\beta|}$$

where $c(\alpha) = A \|\phi\|_{b,N}\, c \frac{c^{|\alpha|} H^{|\alpha|} M_{|\alpha|} \alpha!}{N_{|\alpha|}}$. Taking $kTbHc \le \frac{1}{2a}$, with $a > 1$, we obtain

$$\begin{aligned}
\left|\partial^{\alpha}\left(f * \phi_{\varepsilon} - f\right)(x)\right| &\le c(\alpha)\, \varepsilon^{N} M_{N} (kT)^{-N} a^{-N} \sum_{|\beta|=N} \left(\frac{1}{2}\right)^{|\beta|} \\
&\le c(\alpha)\, \varepsilon^{N} M_{N} (kT)^{-N} a^{-N}. \qquad (13)
\end{aligned}$$

Let $\varepsilon_0 \in I$ such that $\varepsilon_0 M_1 < 1$ and take $T \ge \frac{M_{p-1} M_{p+1}}{M_p^2}, \forall p \ge 1$.

Then, see [11], there exists $N = N(\varepsilon) \in \mathbb{Z}^{+}$, such that

$$1 \le \frac{\varepsilon}{k} (M_N)^{N} \le T$$

which gives

$$a^{-N} \le \exp\left(-M\left(\frac{k}{\varepsilon}\right)\right) \qquad \text{and} \qquad \varepsilon^{N} M_{N} (kT)^{-N} < 1,$$

if we take $a \ge 2$. Finally, from (13), we have

$$\left|\partial^{\alpha}\left(f * \phi_{\varepsilon} - f\right)(x)\right| \le c \exp\left(-M\left(\frac{k}{\varepsilon}\right)\right), \qquad (14)$$

i.e. $\left(f * \phi_{\varepsilon} - f\right)_{\varepsilon} \in \mathcal{N}^{\{M\}}(\Omega)$. □

REMARK 3. Microlocal analysis suitable for the algebra of generalized Roumieu ultradistributions will be discussed in a separate paper.

References

[1] BENMERIEM, K., AND BOUZAR, C. Generalized gevrey ultradistributions. *New York J. Math.*, 15 (2009), 37–72.

[2] BENMERIEM, K., AND BOUZAR, C. Generalized gevrey ultradistributions and their microlocal analysis. *Operator Theory : Advances and Applications*, 213 (2011), 235–250.

[3] COLOMBEAU, J. F. *New generalized functions and multiplication of distributions*. North Holland, 1984.

[4] COLOMBEAU, J. F. *Elementary introduction to new generalized functions*. North Holland, 1985.

[5] GEL'FAND, I., AND SHILOV, G. *Generalized functions*, vol. 2. Academic Press, 1967.

[6] HÖRMANDER, L. *The analysis of linear partial differential operators, I, Distribution theory and Fourier analysis*. Springer, 1983.

[7] KAMINSKI, A., OBERGUGGENBERGER, M., AND PILIPOVIC, S. Linear and non-linear theory of generalized functions and its applications. *Banach Center Publications*, 88 (2010).

[8] KOMATSU, H. Ultradistributions i. *J. Fac.Sci. Univ. Tokyo, Sect. IA*, 20 (1973), 25–105.

[9] LIONS, J. L., AND MAGENES, E. *Non-homogeneous boundary value problems and applications*, vol. 3. Springer, 1973.

[10] MANDELBROJT, S. *Séries adhérentes. Régularisation des suites. Applications*. Gauthier-Villars, 1952.

[11] PILIPOVIC, S., AND SCARPALÉZOS, D. Colombeau generalized ultradistributions. *Math. Proc. Camb. Phil. Soc.*, 130 (2001), 541–553.

[12] ROUMIEU, C. Sur quelques extentions de la notion de distributions. *Ann. Sc. Ec. Norm.*, 77 (1960), 41–121.

[13] SCHWARTZ, L. *Théorie des distributions*, 2 ed. Herman, 1966.

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