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HOMOGENIZATION OF SPECTRAL PROBLEMS AND LOCALIZATION EFFECTS

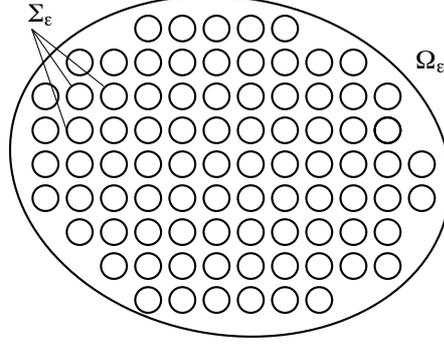
Abstract. We describe classical and new results concerning the limit behaviour of spectral problems in a periodically perforated domain, with special attention to some cases where the eigenfunctions localize.

1. Introduction

My first step into mathematical research was the study of the theory of homogenization, applied to linear e non linear elliptic equations. The subject was suggested to me by professor Angelo Negro, to whom I am deeply grateful for having introduced me to several beautiful topics in mathematical analysis.

The study of spectral problems in periodically perforated domains is developed, since long time, by many authors (see, for example, [15], [13], [11]) and has a number of motivations and applications. One important example is the field of optimal design (see [3], [1], [4]). Starting from the simplest case of the Laplace operator, it is known that the knowledge of its spectrum depends on the boundary conditions and on the geometry of the domain under consideration. An alternative point of view is to say that the knowledge of the spectrum of a boundary value problem gives information about the geometry of the domain. This aspect is particularly important in shape optimization problems, where the shape of the domain is an unknown, and the goal is to choose it in a way to obtain, for example, certain desired modes of vibration. There are several techniques to study the effect of variations of a domain on the corresponding solutions, or eigenvalues and eigenfunctions. Classical methods date back to Hadamard ([10]), and are based on smooth variations of the boundary of a given initial domain, in the normal direction. This approach excludes non smooth boundary, and variations that change the topology, as, for example, create holes. More recent topological optimization methods are able to include topological variations and take into account the knowledge of homogenization of boundary value problems and of spectral problems in perforated domains.

In Section 2 of this paper we present some of the results obtained in collaboration with I. Pankratova and A. Piatnitski. Details and proofs are contained in [6], where we deal with a spectral problem for an elliptic operator in divergence form, complemented by Fourier-type boundary conditions on the surface of the holes. The presence of a non periodic coefficient in the boundary conditions causes a number of interesting effects. First of all, under the assumption that the non periodic coefficient has a unique minimum point, a localization phenomenon holds: namely, for any $k \in \mathbb{N}$ the k -th eigenfunction of the problem is asymptotically localized, in a small neighbourhood of the minimum point, as the periodicity size vanishes. In particular, the principal

Figure 1: Domain Ω_ε

eigenfunction converges to a δ -function supported at the minimum point. Moreover, the localization process takes place in the scale $\varepsilon^{1/4}$, and it is possible to construct asymptotic expansions which are in integer powers of $\varepsilon^{1/4}$. In this scale the leading term of the asymptotic expansion for the k -th eigenfunction can be proved to be the k -th eigenfunction of an auxiliary harmonic oscillator operator.

Different results for spectral problems of Steklov type are contained in [5], while a study nonlinear variational problems with Fourier boundary conditions can be found in [7].

In Section 3 we address some related papers where other localization phenomena are found out.

2. A problem with Fourier boundary conditions

$$\Omega_\varepsilon = \Omega \setminus \bigcup_{i \in I_\varepsilon} T_\varepsilon^i, \quad I_\varepsilon = \{i \in \mathbb{Z}^d : T_\varepsilon^i \subset \Omega\},$$

where $T_\varepsilon^i = \varepsilon(T + i)$, and $T \subset\subset (0, 1)^d$ is a compact subset of the unit cube, with non empty interior. We denote by $\omega = (0, 1)^d \setminus T$ the open unit cell and by $\Sigma = \partial T$ the boundary of the perforation. In the ‘‘periodically perforated’’ domain $\Omega_\varepsilon \subset \mathbb{R}^d$ we consider the following spectral problem:

$$(1) \quad \begin{cases} -\operatorname{div}(a^\varepsilon \nabla u^\varepsilon) = \lambda^\varepsilon u^\varepsilon, & \text{in } \Omega_\varepsilon, \\ a^\varepsilon \nabla u^\varepsilon \cdot n = -q(x) u^\varepsilon, & \text{on } \Sigma_\varepsilon, \\ u^\varepsilon = 0, & x \in \partial\Omega, \end{cases}$$

where $a^\varepsilon(x) = a(\frac{x}{\varepsilon})$. Notice that Ω_ε remains connected, the perforation does not intersect the boundary $\partial\Omega$, and

$$\partial\Omega_\varepsilon = \partial\Omega \cup \Sigma_\varepsilon, \quad \Sigma_\varepsilon = \bigcup_{i \in I_\varepsilon} \Sigma_\varepsilon^i, \quad \Sigma_\varepsilon^i = \varepsilon(\Sigma + i).$$

The boundary conditions are known as Fourier, or Robin conditions. We make the following assumptions:

- (H0) $\partial\Omega_\varepsilon = \partial\Omega \cap \Sigma_\varepsilon$, where $\Omega \subset \mathbb{R}^d$ is a bounded and regular open set;
- (H1) $a(y)$ is a symmetric, uniformly elliptic $d \times d$ -matrix in \mathbb{R}^d ;
- (H2) the coefficients $a_{ij}(y) \in L^\infty(\mathbb{R}^d)$ are 1-periodic in all variables;
- (H3) the function $q(x) \in C^3(\mathbb{R}^d)$ is non negative;
- (H4) the function $q(x)$ attains its global minimum at $x = 0 \in \Omega$, and as $x \rightarrow 0$ it satisfies

$$q(x) = q(0) + \frac{1}{2}x^T H(q)x + o(|x|^2),$$

with positive definite Hessian matrix $H(q)$.

One interesting feature of this problem is the presence of the ‘slow’ variable x in the coefficient q , in the boundary condition. This fact causes a number of interesting effects, among which the localization of eigenfunctions. The eigenvalue problem (1) has the following weak formulation:

find $(\lambda^\varepsilon, u^\varepsilon) \in \mathbb{C} \times H^1(\Omega_\varepsilon)$, $u^\varepsilon = 0$ on $\partial\Omega$ and $u^\varepsilon \neq 0$, such that

$$(2) \quad \int_{\Omega_\varepsilon} a^\varepsilon \nabla u^\varepsilon \cdot \nabla v \, dx + \int_{\Sigma_\varepsilon} q u^\varepsilon v \, d\sigma = \lambda^\varepsilon \int_{\Omega_\varepsilon} u^\varepsilon v \, dx, \quad v \in H_0^1(\Omega).$$

Under the above assumptions (H0)-(H4), it is easy to prove the following result.

LEMMA 1. *The spectrum of problem (2) is real and discrete*

$$0 < \lambda_1^\varepsilon < \lambda_2^\varepsilon \leq \dots \leq \lambda_j^\varepsilon \leq \dots \rightarrow +\infty.$$

Each eigenvalue has finite multiplicity. The corresponding normalized eigenfunctions

$$\int_{\Omega_\varepsilon} u_i^\varepsilon u_j^\varepsilon \, dx = \delta_{ij},$$

form an orthonormal basis of $L^2(\Omega_\varepsilon)$. Moreover, the following variational characterization for λ_1^ε holds true:

$$(3) \quad \lambda_1^\varepsilon = \inf_{\substack{v \in H_0^1(\Omega_\varepsilon, \partial\Omega) \\ \|v\|_{L^2(\Omega_\varepsilon)} = 1}} \int_{\Omega_\varepsilon} a^\varepsilon \nabla v \cdot \nabla v \, dx + \int_{\Sigma_\varepsilon} q(v)^2 \, d\sigma.$$

The main object of our study is the limit behaviour of $(\lambda^\varepsilon, u^\varepsilon)$ as $\varepsilon \rightarrow 0$. Using $q(x) \geq 0$ from below, and $H_0^1(\Omega_\varepsilon) \subset H_0^1(\Omega_\varepsilon, \partial\Omega)$ from above in the variational formula (3) for λ_1^ε , we immediately see that

$$\lambda_{1,N}^\varepsilon \leq \lambda_1^\varepsilon \leq \lambda_{1,D}^\varepsilon.$$

The two constants $\lambda_{1,N}^\varepsilon, \lambda_{1,D}^\varepsilon$ are, respectively, the eigenvalues of the homogeneous Neumann problem and of the homogeneous Dirichlet problem, i.e.,

$$(4) \quad \begin{cases} -\operatorname{div}(a^\varepsilon \nabla u^\varepsilon) = \lambda_N^\varepsilon u^\varepsilon, & \text{in } \Omega_\varepsilon, \\ a^\varepsilon \nabla u^\varepsilon \cdot n = 0 & \text{on } \Sigma_\varepsilon, \\ u^\varepsilon = 0, & x \in \partial\Omega, \end{cases}$$

and

$$(5) \quad \begin{cases} -\operatorname{div}(a^\varepsilon \nabla u^\varepsilon) = \lambda_D^\varepsilon u^\varepsilon, & \text{in } \Omega_\varepsilon, \\ u^\varepsilon = 0 & \text{on } \partial\Omega_\varepsilon. \end{cases}$$

The asymptotic behaviour of both problems has been investigated long ago by Vanninathan in [15], together with the closely related Steklov problem. According to [15], in the Neumann case (4), as $\varepsilon \rightarrow 0$ we have

$$\lambda_N^\varepsilon = \lambda_N^0 + \varepsilon \lambda_N^1 + O(\varepsilon^2)$$

and

$$u_\varepsilon = u_0(x) + \varepsilon N\left(\frac{x}{\varepsilon}\right) \cdot Du_0(x) + \dots,$$

with $(\lambda_N^0, u_0(x))$ solutions of the homogenized spectral problem

$$(6) \quad \begin{cases} -\operatorname{div}(a^N \nabla u) = \lambda u, & \text{in } \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases}$$

Here a^N is the homogenized matrix of the boundary value problem with Neumann condition on the boundary of the perforation studied by Cioranescu and Saint Jean Paulin in [8]. The vector-valued function N is the first order corrector, defined by the auxiliary boundary-value problem in the perforated periodicity cell

$$(7) \quad \begin{cases} -\operatorname{div}(a(y)(\nabla N^j + e_j)) = 0 & \text{in } \omega, \\ a(y) \nabla N^j \cdot n = 0 & \text{on } \Sigma, \\ N^j = N^j(y) \text{ periodic, } \int_\omega N^j(y) dy = 0, \end{cases}$$

and

$$(8) \quad a^N = \frac{1}{|\omega|} \int_\omega a(y)(\nabla N^i + e_i)(\nabla N^j + e_j) dy.$$

In the Dirichlet case (5), instead,

$$\lambda_D^\varepsilon = \varepsilon^{-2} \lambda_D^0 + O(\varepsilon^2),$$

where λ_D^0 is the first eigenvalue of the Dirichlet spectral problem in the periodicity cell ω

$$(9) \quad \begin{cases} -\operatorname{div}(a(y)\nabla v) = \lambda v & \text{in } \omega, \\ v = 0 & \text{on } \Sigma, \\ v = v(y) & \text{periodic.} \end{cases}$$

In this case, the eigenfunctions, upon extension to zero out of Ω_ε , tend strongly to 0 in $H_0^1(\Omega)$.

The Fourier spectral problem with *periodic* coefficients has been studied by Pastukhova in [13], in the case

$$(10) \quad \begin{cases} -\operatorname{div}(a^\varepsilon \nabla u^\varepsilon) = \lambda^\varepsilon u^\varepsilon, & \text{in } \Omega_\varepsilon, \\ a^\varepsilon \nabla u^\varepsilon \cdot n + b\left(\frac{x}{\varepsilon}\right) u^\varepsilon = 0, & \text{on } \Sigma_\varepsilon, \\ u^\varepsilon = 0, & x \in \partial\Omega. \end{cases}$$

Also the comparison with this problem brings useful information to the solution of our initial spectral problem (1), where the periodically oscillating term $b\left(\frac{x}{\varepsilon}\right)$ is replaced by the function $q(x)$ which depends on the ‘slow’ variable x . Indeed, for our problem the following lemma holds true.

LEMMA 2. *The first eigenvalue of problem (1) satisfies the estimate*

$$\frac{1}{\varepsilon} \frac{|\Sigma|_{d-1}}{|\omega|_d} q(0) + O(1) \leq \lambda_1^\varepsilon \leq \frac{1}{\varepsilon} \frac{|\Sigma|_{d-1}}{|\omega|_d} q(0) + O(\varepsilon^{-1/2}), \quad \varepsilon \rightarrow 0,$$

where $|\omega|_d$ and $|\Sigma|_{d-1}$ indicate, respectively, the d and $(d-1)$ dimensional measures of the perforated cell ω and of the perforation Σ .

To clarify the result, we note that, since $q(x) \geq q(0)$, then

$$\lambda_1^\varepsilon \geq \inf_{\substack{v \in H_0^1(\Omega_\varepsilon, \partial\Omega) \\ \|v\|_{L^2(\Omega_\varepsilon)} = 1}} \left\{ \int_{\Omega_\varepsilon} a^\varepsilon \nabla v \cdot \nabla v \, dx + q(0) \int_{\Sigma_\varepsilon} (v)^2 \, d\sigma \right\} = v_1^\varepsilon.$$

But v_1^ε coincides with the first eigenvalue of del Pastukhova’s problem (10), in the case $b\left(\frac{x}{\varepsilon}\right) = q(0)$

$$\begin{cases} -\operatorname{div}(a^\varepsilon \nabla w^\varepsilon) = v^\varepsilon w^\varepsilon, & \text{in } \Omega_\varepsilon, \\ a^\varepsilon \nabla w^\varepsilon \cdot n = -q(0) w^\varepsilon, & \text{on } \Sigma_\varepsilon, \\ w^\varepsilon = 0, & x \in \partial\Omega. \end{cases}$$

In [13] it is proved that

$$v_1^\varepsilon = \frac{1}{\varepsilon} \frac{|\Sigma|_{d-1}}{|\omega|_d} q(0) + O(1), \quad \varepsilon \rightarrow 0.$$

Hence, the left-hand side inequality in Lemma 2 follows:

$$\lambda_1^\varepsilon \geq \frac{1}{\varepsilon} \frac{|\Sigma|_{d-1}}{|\omega|_d} q(0) + O(1), \quad \varepsilon \rightarrow 0.$$

Let us now examine the right-hand side inequality in Lemma 2. Choosing any $v \in C_0^\infty(\Omega)$ as test function in the variational characterization (3), one gets easily that

$$\lambda_1^\varepsilon \leq C \varepsilon^{-1},$$

with a constant C independent of ε . But to get the same constant from above and below requires a different choice of the test function. Choosing $v(x/\varepsilon^\alpha)$ with $v \in C_0^\infty(\Omega)$, $\|v\|_{L^2(\mathbb{R}^d)} = 1$ in the variational characterization (3) one can prove that the optimal estimate is attained when $\alpha = 1/4$ and obtains that

$$\lambda_1^\varepsilon \leq \frac{1}{\varepsilon} \frac{|\Sigma|_{d-1}}{|\omega|_d} q(0) + O(\varepsilon^{-1/2}), \quad \varepsilon \rightarrow 0.$$

One can note that the optimal test functions concentrate at $x = 0$, the minimum point of $q(x)$, as $\varepsilon \rightarrow 0$. To be more precise, we can state the following definition and proposition.

DEFINITION 1. *We say that a family $\{w_\varepsilon(x)\}_{\varepsilon>0}$ with*

$$0 < c_1 \leq \|w_\varepsilon\|_{L^2(\Omega_\varepsilon)} \leq c_2$$

is concentrated at x_0 , as $\varepsilon \rightarrow 0$, if for any $\gamma > 0$ there is $\varepsilon_0 > 0$ such that

$$\int_{\Omega_\varepsilon \setminus B_\gamma(x_0)} |w_\varepsilon|^2 dx < \gamma, \quad \text{for all } \varepsilon \in (0, \varepsilon_0).$$

Here $B_\gamma(x_0)$ is a ball of radius γ centered at x_0 .

PROPOSITION 1. *The first eigenfunction u_1^ε of problem (1) is concentrated in the sense of Definition 1 at the minimum point of $q(x)$, that is at $x = 0$.*

The asymptotic behaviour of the eigenpairs of problem (1) is described in details by the following theorem.

THEOREM 1. *The following representation holds true*

$$\lambda_j^\varepsilon = \frac{1}{\varepsilon} \frac{|\Sigma|_{d-1}}{|\omega|_d} q(0) + \frac{\mu_j^\varepsilon}{\sqrt{\varepsilon}}, \quad u_j^\varepsilon(x) = v_j^\varepsilon\left(\frac{x}{\varepsilon^{1/4}}\right),$$

where $(\mu_j^\varepsilon, v_j^\varepsilon(z))$ are such that $\mu_j^\varepsilon \rightarrow \mu_j$, as $\varepsilon \rightarrow 0$, and μ_j is eigenvalue of the homogenized problem

$$-\operatorname{div}(a^N \nabla v) + \frac{1}{2} \frac{|\Sigma|_{d-1}}{|\omega|_d} (z^T H(q) z) v = \mu v, \quad v \in L^2(\mathbb{R}^d),$$

where $v = v(z)$, and a^N is given by (8). Moreover, if μ_j is simple, then

$$\|v_j^\varepsilon - v_j\|_{L^2(\mathbb{R}^d)} \rightarrow 0, \quad \varepsilon \rightarrow 0.$$

The proof of the above result is based on the following technique. Subtracting $\frac{1}{\varepsilon} \frac{|\Sigma|}{|\omega|} q(0)$ to both sides of the initial equation, and performing the change of variables $z = \varepsilon^{-1/4} x$, standard manipulations transform the original problem (1) into the following rescaled problem

$$\begin{cases} -\operatorname{div}(a^\varepsilon(z) \nabla v^\varepsilon(z)) - \frac{1}{\sqrt{\varepsilon}} \frac{|\partial Y|_{d-1}}{|Y|_d} q(0) v^\varepsilon = \mu^\varepsilon v^\varepsilon(x), & \text{in } \varepsilon^{-1/4} \Omega_\varepsilon, \\ a^\varepsilon(z) \nabla v^\varepsilon(z) \cdot n = -\varepsilon^{1/4} q(\varepsilon^{1/4} z) v^\varepsilon(z), & \text{on } \varepsilon^{-1/4} \Sigma_\varepsilon, \\ v^\varepsilon(z) = 0, & \text{on } \varepsilon^{-1/4} \partial \Omega. \end{cases}$$

where

$$v^\varepsilon(z) = u^\varepsilon \left(\frac{x}{\varepsilon^{1/4}} \right), \quad \mu^\varepsilon = \sqrt{\varepsilon} \left(\lambda^\varepsilon - \frac{1}{\varepsilon} \frac{|\partial Y|_{d-1}}{|Y|_d} q(0) \right).$$

The first step in the proof of Theorem 1 is to show a priori estimates for the eigenvalues μ_j^ε

$$\bar{c} \leq \mu_1^\varepsilon \leq \underline{C}.$$

Then, the proof of the convergence of eigenvalues and eigenfunctions to those of the limit problem in \mathbb{R}^d follows, using various variational and compactness arguments, and scaled trace and Poincaré-type inequalities.

3. Other problems with localization effects

The localization phenomenon in spectral problems should be well-known to physicists, since a long time, and it has been observed in several mathematical works.

In the context of singular perturbation problems, paper [14] deals with the limit behaviour of the first eigenvalue of a singularly perturbed non self-adjoint elliptic operator, with smooth coefficients, defined on a compact Riemannian manifold. Self-adjoint operators on a bounded subset of \mathbb{R}^d are treated as a special case. Here, in particular, the first normalized eigenfunction localises around the minimum point of the given potential.

In the field of homogenization problems, [2] deals with an operator with a large locally periodic potential has been considered. The localization appears due to the

presence of a large factor in the potential and the fact that the operator coefficients depend on slow variable.

In a different context, in [9] the Dirichlet spectral problem for the Laplacian in a thin 2d strip of slowly varying thickness is studied. Here the localization is observed in the vicinity of the point of maximum thickness. The large parameter is the first eigenvalue of 1d Laplacian in the cross-section.

Both in [2] and [9], under natural non-degeneracy conditions, the asymptotics of the eigenpairs are described in terms of the spectrum of an appropriate harmonic oscillator operator. However, the localization scale is of order $\sqrt{\varepsilon}$ with ε being the microscopic length scale.

Localization effect for the negative part of the spectrum are also found in [12] where a spectral problem for locally periodic elliptic operators with sign-changing density function is considered.

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