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PERIODIC AND HETEROCLINIC TYPE SOLUTIONS FOR SYSTEMS OF ALLEN-CAHN EQUATIONS

Abstract. We consider a class of semilinear elliptic system of the form

$$-\Delta u(x,y)+\nabla W(u(x,y))=0, \quad (x,y)\in\mathbb{R}^2, \tag{1}$$

where $W:\mathbb{R}^2\to\mathbb{R}$ is a double well non negative symmetric potential. We show, via variational methods, that if the set of solutions to the one dimensional system $-\ddot{q}(x)+\nabla W(q(x))=0$, $x\in\mathbb{R}$, which connect the two minima of W as $x\to\pm\infty$ has a discrete structure, then (1) has infinitely many layered solutions with prescribed energy.

1. Introduction

We consider semilinear elliptic system of the form

$$-\Delta u(x,y)+\nabla W(u(x,y))=0, \qquad (x,y)\in\mathbb{R}^2, \tag{2}$$

where $W\in C^2(\mathbb{R}^2,\mathbb{R})$ satisfies

(W_1) there exist $\boldsymbol{a}_\pm\in\mathbb{R}^2$ such that $W(\boldsymbol{a}_\pm)=0$, $W(\xi)>0$ for every $\xi\in\mathbb{R}^2\setminus\{\boldsymbol{a}_\pm\}$ and $D^2W(\boldsymbol{a}_\pm)$ are definite positive;

(W_2) $\liminf_{|\xi|\to+\infty} W'(\xi)\cdot\xi>0$;

(W_3) $W(-x_1,x_2)=W(x_1,x_2)$ for all $(x_1,x_2)\in\mathbb{R}^2$;

The system (2) is the rescaled stationary system associated to the reaction-diffusion system

$$\partial_t u(t,x,y)-\varepsilon^2\Delta u(t,x,y)+\nabla W(u(t,x,y))=0, \quad (x,y)\in\Omega\subset\mathbb{R}^2,\ t>0 \tag{3}$$

which describes two phase physical systems or grain boundaries in alloys. As $\varepsilon\to 0^+$, solutions to (3) tends almost everywhere to global minima of W and sharp phase interfaces appear (see e.g. [10], [18] and [21]). Then, the expansion of such solutions in a point on the interface presents, as first term, the system (2). From this point of view, two layered transition solutions correspond to solutions u of (2) satisfying the asympotic conditions

$$\lim_{x\to\pm\infty} u(x,y)=\boldsymbol{a}_\pm \quad \text{uniformly with respect to } y\in\mathbb{R}. \tag{4}$$

S. Alama, L. Bronsard and C. Gui in [1] studied the existence of solutions to (2) which satisfy the asymptotic condition (4) for $x\to\pm\infty$ while as $y\to\pm\infty$ tends to

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two different one dimensional trajectories, precisely, solutions to the one dimensional associated problem

$$(5) \qquad \begin{cases} -\ddot{q}(x) + \nabla W(q(x)) = 0, & x \in \mathbb{R} \\ \lim_{t\to\pm\infty} q(t) = \boldsymbol{a}_{\pm}. \end{cases}$$

which are furthermore minima of the action

$$V(q) = \int_{\mathbb{R}} \tfrac{1}{2}|\dot{q}|^2 + W(q)\,dx$$

over the class of trajectories connecting $\boldsymbol{a}_{\pm}$ as $x \to \pm\infty$. Such solutions are found under conditions (W_1) and (W_3), requiring a fast growth at infinity and assuming that there exist a finite number $k \geq 2$ of geometrically distinct one dimensional minimal heteroclinic connections. In [20] M. Schatzman proves the same result, considering a non symmetric potential, assuming that there exists two geometrically distinct one dimensional heteroclinic connections which are supposed to be non degenerate, i.e. the kernel of the corresponding linearized operators are one dimensional.

If u is scalar valued, much is known about the corresponding heteroclinic problem (2)-(4). In this scalar setting, E. De Giorgi in [15] has conjectured that any entire bounded solution of $-\Delta u + u^3 - u = 0$ with $\partial_{x_1} u(x) > 0$ in $\mathbb{R}^n$ for $n \leq 8$ is in fact one-dimensional, i.e., modulo space roto-traslations, it coincides with the unique solution of the one dimensional heteroclinic problem

$$(6) \qquad \begin{cases} -\ddot{q}(x) + q(x)^3 - q(x) = 0, & x \in \mathbb{R}, \\ q(0) = 0 \text{ and } \lim_{x\to\pm\infty} q(x) = \pm 1, \end{cases}$$

The conjecture has been proved for $n = 2$ by N. Ghoussoub and C. Gui in [14] and then by L. Ambrosio and X. Cabrè in [7] for $n = 3$ (see also [2]), even for more general double well potentials W. A further step in the proof of the De Giorgi conjecture has been done by O. Savin in [19] where, for $n \leq 8$, the same one dimensional structure is proved for solutions u such that $\partial_{x_1} u(x) > 0$ on $\mathbb{R}^n$ and $\lim_{x_1\to\pm\infty} u(x) = \pm 1$ for all $(x_2, x_3, \ldots, x_n) \in \mathbb{R}^{n-1}$ (see [8], [9] and [13] for related problems). That result is completed in [11] where the existence of entire solutions without any one dimensional symmetry which are increasing and asymptotic to ± 1 with respect to the first variable is proved in dimension $n > 8$.

Here we want to discuss some results obtained in [3] for the problem (2)-(4), where, using a global variational procedure, it is proved that if the minimal set of one dimensional heteroclinic connections satisfies a suitable discreteness assumption then there exist infinitely many solutions to the problem with prescribed *energy*, which can be classified as *homoclinic*, *heteroclinic* or *periodic* solutions.

2. Statement of the main Theorem and outline of the proof

To explain precisely the result and to give an idea of the procedure, let us begin considering the problem already considered in [1] and [20]. So let us define

$$\Gamma = \{q - z_0 \in H^1(\mathbb{R})^2 \mid q(x)_1 = -q(-x)_1,\, q(x)_2 = q(-x)_2\},$$

where $q(x) = (q(x)_1, q(x)_2)$ and $z_0 \in C^\infty(\mathbb{R}, \mathbb{R}^2)$ is fixed in such a way that $z_0(x)_1 = -z_0(-x)_1$, $z_0(x)_2 = z_0(-x)_2$ and $z_0(x) = \boldsymbol{a}_+$ for $x > 1$, be the space of one dimensional symmetric trajectories connecting $\boldsymbol{a}_\pm$ as $x \to \pm\infty$. Setting $m = \inf_\Gamma V(q)$, let

$$\mathcal{M} = \{q \in \Gamma \mid V(q) = m\}$$

be the minimal set of one dimensional symmetric heteroclinic connections. As it is well known, $\mathcal{M}$ is compact, not empty and consists of solutions to (5) in Γ.
Assuming that $\mathcal{M}$ satisfies the discreteness assumption

$$(*) \qquad \mathcal{M} = \mathcal{M}^+ \cup \mathcal{M}^- \text{ with } \mathrm{dist}_{L^2}(\mathcal{M}^+, \mathcal{M}^-) > 0,$$

we will look for bidimensional solution u with prescribed different asymptotes as $y \to \pm\infty$ and precisely

$$\text{(7)} \qquad \mathrm{dist}_{L^2}(u(\cdot, y), \mathcal{M}^\pm) \to 0 \text{ as } y \to \pm\infty.$$

Note that condition $(*)$ (as the discreteness assumption made in [1] and [20]) does not hold in the scalar case, where the minimal set of one dimensional symmetric solutions $\mathcal{M}$ is in fact constituted by the unique heteroclinic solution of (6).

Under assumption $(*)$, bidimensional solutions satisfying (7) can be obtained using a global variational approach (instead of the approximating procedure used in [1] and [20]), considering a renormalized action functional over a suitable space.

As in [16] and [17] for Hamiltonian ODE systems and in [4] for scalar Allen-Cahn equations, we consider the renormalized action functional

$$\varphi(u) = \int_{\mathbb{R}} \tfrac{1}{2} \|\partial_y u(\cdot, y)\|^2_{L^2(\mathbb{R})^2} + (V(u(\cdot, y)) - m)\, dy$$

which is well defined on the space

$$\mathcal{H} = \{u \in H^1_{loc}(\mathbb{R}^2, \mathbb{R}^2) \mid u(\cdot, y) \in \Gamma \text{ for a.e. } y \in \mathbb{R}\}.$$

Note that φ is weakly lower semicontinuous on $\mathcal{H}$ and for every $u \in \mathcal{H}$, since $V(u(\cdot, y)) \geq m$ for a.e. $y \in \mathbb{R}$, there results $\varphi(u) \geq 0$ while $\varphi(q) = 0$ for all $q \in \mathcal{M}$.
We are interested on solutions which satisfies the right asymptotic conditions as $y \to \pm\infty$. Such solutions can be reached as minima of φ over the class

$$\mathcal{H}_m = \{u \in \mathcal{H} \mid \liminf_{y \to \pm\infty} \mathrm{dist}_{L^2}(u(\cdot, y), \mathcal{M}^\pm) = 0\}.$$

In fact, for all $u \in \mathcal{H}$, for all $y_1 < y_2$ we have

$$\|u(\cdot,y_1) - u(\cdot,y_2)\|_{L^2}^2 \leq (y_2 - y_1) \int_{y_1}^{y_2} \|\partial_y u(\cdot,y)\|_{L^2}^2 \, dy$$

and in particular, if $\varphi(u) < +\infty$ then the map $y \in \mathbb{R} \mapsto u(\cdot,y) \in \Gamma$ is continuous with respect to the $L^2(\mathbb{R})^2$ metric. Moreover, if $u \in \mathcal{H}$ and $y_1 < y_2$, then

$$\varphi(u) \geq \left(\tfrac{2}{y_2 - y_1} \int_{y_1}^{y_2} (V(u(\cdot,y)) - m) \, dy \right)^{1/2} \|u(\cdot,y_1) - u(\cdot,y_2)\|_{L^2}.$$

and in particular, if $V(u(\cdot,y)) \geq m + \nu$ for all $y \in (y_1,y_2)$ and some $\nu > 0$, then

$$\varphi(u) \geq \tfrac{1}{2(y_2-y_1)} \|u(\cdot,y_1) - u(\cdot,y_2)\|_{L^2}^2 + \nu(y_2 - y_1) \geq \sqrt{2\nu} \|u(\cdot,y_1) - u(\cdot,y_2)\|_{L^2}.$$

By the previous estimates, if $u \in \mathcal{H}_m$ we have control on the transition time from $\mathcal{M}^-$ to $\mathcal{M}^+$ and so concentration in the y variable. Indeed it can be proved

LEMMA 1. *There exists $\nu \in (0,m)$ such that if $u \in \mathcal{H}_m$, $\varphi(u) < +\infty$ and for some $y_0 \in \mathbb{R}$, $V(u(\cdot,y_0)) < m + \nu$ then, either*

(i) $\text{dist}_{L^2}(u(\cdot,y), \mathcal{M}^-) \leq d_0$ *for all* $y \leq y_0$*; or*

(ii) $\text{dist}_{L^2}(u(\cdot,y), \mathcal{M}^+) \leq d_0$ *for all* $y \geq y_0$,

where $d_0 := \frac{1}{5}\text{dist}_{L^2}(\mathcal{M}^+, \mathcal{M}^-)$.

Lemma 1, together with the symmetry in the x variable, allows to get compactness of minimizing sequences in $\mathcal{H}_m$. Indeed, setting $\mu := \inf_{\mathcal{H}} \varphi$ we have

LEMMA 2. *Let $(u_n) \subset \mathcal{H}_m$ be such that $\varphi(u_n) \to \mu$ as $n \to \infty$ and such that* $\text{dist}_{L^2}(u_n(\cdot,0), \mathcal{M}^-) = d_0$ *for all $n \in \mathbb{N}$. Then, there exists $u \in \mathcal{H}$ such that, up to a subsequence,*

(i) $u_n - u \to 0$ *as $n \to \infty$ weakly in* $H^1_{loc}(\mathbb{R}^2)^2$,

(ii) *there exists $L_0 > 0$ such that* $\text{dist}_{L^2}(u(\cdot,y), \mathcal{M}^-) \leq d_0$ *for all $y \leq -L_0$, and* $\text{dist}_{L^2}(u(\cdot,y), \mathcal{M}^+) \leq d_0$ *for all $y \geq L_0$.*

By the invariance with respect to the y-translation of φ and the definition of $\mathcal{H}_m$, we have that there exists a minimizing sequence (u_n) which verifies the condition $\text{dist}_{L^2}(u_n(\cdot,0), \mathcal{M}^-) = d_0$ for all $n \in \mathbb{N}$. Then, by Lemma 2, such sequence weakly converge in $H^1_{loc}(\mathbb{R}^2)^2$ to a function $u \in \mathcal{H}$ such that

$$\lim_{y \to \pm\infty} \text{dist}_{L^2}(u(\cdot,y), \mathcal{M}^\pm) = 0.$$

Then, $u \in \mathcal{H}_m$ and since φ is weakly semicontinuous we can conclude that $\varphi(u) = \mu$, proving the existence of at least one bidimensional solution to (2) in $\mathcal{H}_m$, as already proved in the Theorem by Alama Bronsard and Gui but here in a slightly more general setting

THEOREM 1. *If (W_1)-(W_3) and $(*)$ hold, then there exists $u \in \mathcal{C}^2(\mathbb{R}^2, \mathbb{R}^2)$ solution to (2) such that $u(x,y) \to \boldsymbol{a}_\pm$ as $x \to \pm\infty$ uniformly with respect to.* $y \in \mathbb{R}$ *and*

$$\lim_{y \to \pm\infty} \mathrm{dist}_{H^1}(u(\cdot, y), \mathcal{M}^\pm) = 0.$$

Now, note that if $u \in \mathcal{H}$ solves the system (2) then

$$\partial_y^2 u(x,y) = \underbrace{-\partial_x^2 u(x,y) + \nabla W(u(x,y))}_{V'(u(\cdot,y))}$$

In other words u defines a trajectory $y \in \mathbb{R} \mapsto u(\cdot, y) \in \Gamma$ solution to the infinite dimensional Lagrangian system

$$\frac{d^2}{dy^2} u(\cdot, y) = V'(u(\cdot, y))$$

which has as equilibria the one dimensional solutions $q \in \mathcal{M}$. From such point of view, bidimensional solutions in $\mathcal{H}_m$ are *heteroclinic type* solutions connecting $\mathcal{M}^\pm$ as $y \to \pm\infty$.

Note that the *energy* is conserved, indeed if $u \in \mathcal{H}$ solves (2) on $\mathbb{R} \times (y_1, y_2)$ then

$$E_u(y) = \tfrac{1}{2} \|\partial_y u(\cdot, y)\|_{L^2}^2 - V(u(\cdot, y))$$

is constant on (y_1, y_2). In particular, for the heteroclinic type solution $u \in \mathcal{H}_m$ given in Theorem 1 we have that $E_u(y) = -m$ for every $y \in \mathbb{R}$ and that it connects in Γ the two component $\mathcal{M}_\pm$ as $y \to \pm\infty$.

Now, note that if we take $c \in (m, m+\lambda)$ with $\lambda > 0$ small enough, by $(*)$ we get

$$(*_c) \qquad \{q \in \Gamma \,|\, V(q) \le c\} = \mathcal{V}_c^- \cup \mathcal{V}_c^+ \text{ with } \mathrm{dist}_{L^2}(\mathcal{V}_c^-, \mathcal{V}_c^+) > 0.$$

A natural problem, which generalizes the above one, is to look for a solution $u \in \mathcal{H}$ to (2) with energy $E_u(y) = -c$ for every $y \in \mathbb{R}$ which connects in Γ the sets $\mathcal{V}_c^\pm$ as $y \to \pm\infty$.

In such a case $V(u(\cdot, y)) = -E_u(y) + \frac{1}{2}\|\partial_y u(\cdot, y)\|_{L^2}^2 \ge c$ for every $y \in \mathbb{R}$ and so solutions with energy $-c$ can be sought as minima of the new renormalized functional

$$\varphi_c(u) = \int_{\mathbb{R}} \tfrac{1}{2} \|\partial_y u(\cdot, y)\|_{L^2}^2 + (V(u(\cdot, y)) - c)\, dy$$

on the space

$$\mathcal{H}_c = \{u \in \mathcal{H} \,|\, \liminf_{y \to \pm\infty} \mathrm{dist}_{L^2}(u(\cdot, y), \mathcal{V}_c^\pm) = 0 \text{ and } V(u(\cdot, y)) \ge c \text{ for a.e. } y \in \mathbb{R}\}.$$

Note that if $u \in \mathcal{H}_c$ then $V(u(\cdot, y)) \ge c$ for a.e. $y \in \mathbb{R}$ and so the functional φ_c is well defined on $\mathcal{H}_c$ with values in $[0, +\infty]$.

The functional φ_c enjoys most of the properties of the functional φ and the above concentration-compacteness results work even in this setting. In particular a suitable y-translated minimizing sequences of φ_c in $\mathcal{H}_c$ weakly converge in $H^1_{loc}(\mathbb{R}^2)^2$ to a function $u_c \in \mathcal{H}$. However, differently to the case considered above, we do not know if the limit point u_c satisfies the constraint $V(u_c(\cdot,y)) \geq c$ for a.e. $y \in \mathbb{R}$ and that hence $u_c \in \mathcal{H}_c$ and $\varphi_c(u_c) = \mu_c := \inf_{\mathcal{H}_c} \varphi_c$. Anyhow we can prove that such condition holds true on the interval (s_c,t_c) where s_c and t_c are defined as

$$s_c = \sup\{y \in \mathbb{R} / \mathrm{dist}_{L^2}(u_c(\cdot,y), \mathcal{V}_c^-) \leq d_0 \text{ and } V(u_c(\cdot,y)) \leq c\}$$

and

$$t_c = \inf\{y > s_c \,/\, V(u_c(\cdot,y)) \leq c\},$$

where we agree that $s_c = -\infty$ whenever $V(u_c(\cdot,y)) > c$ for every $y \in \mathbb{R}$ such that $\mathrm{dist}_{L^2}(u_c(\cdot,y), \mathcal{V}_c^-) \leq d_0$ and that $t_c = +\infty$ whenever $V(u_c(\cdot,y)) > c$ for all $y > s_c$ (note that this is the case that occurs if $c = m$). Indeed we have

LEMMA 3. *For every* $[y_1,y_2] \subset (s_c,t_c)$ *there results* $\inf_{y\in[y_1,y_2]} V(u_c(\cdot,y)) > c$. *Moreover,*

(i) $\lim\limits_{y\to s_c^+} \mathrm{dist}_{L^2} u(\cdot,y), \mathcal{V}_c^-) = \lim\limits_{y\to t_c^-} \mathrm{dist}_{L^2} u(\cdot,y), \mathcal{V}_c^+) = 0$;

(ii) $\liminf\limits_{y\to s_c^+} V(u_c(\cdot,y)) = \liminf\limits_{y\to t_c^-} V(u_c(\cdot,y)) = c$;

(iii) $\varphi_{c,(s_c,t_c)}(u_c) := \int_{s_c}^{t_c} \frac{1}{2}\|\partial_y u(\cdot,y)\|^2_{L^2} + (V(u(\cdot,y)) - c)\,dy = \mu_c$.

In particular, by Lemmas 3 and the definition of s_c and t_c, we derive that if $s_c \in \mathbb{R}$ then $u_c(\cdot,s_c) \in \mathcal{V}_c^-$ and if $t_c \in \mathbb{R}$ then $u_c(\cdot,t_c) \in \mathcal{V}_c^+$. On the other hand, if $s_c = -\infty$ then $\mathrm{dist}_{L^2}(u_c(\cdot,y), \mathcal{V}_c^-) \to 0$ as $y \to -\infty$ while if $t_c = +\infty$ then $\mathrm{dist}_{L^2}(u_c(\cdot,y), \mathcal{V}_c^+) \to 0$ as $y \to +\infty$. Using this properties, by Lemma 3, we get

$$\int_{\mathbb{R}^2} \nabla u_c \nabla h + \nabla W(u_c) h\, dx\, dy = 0 \quad \text{for all } h \in C_0^\infty(\mathbb{R} \times (s_c,t_c))^2$$

and so that that u_c is a *weak* solution to (2) in $\mathbb{R} \times (s_c,t_c)$. Then, it is standard to show that u_c is in fact a *classical* solution to (2) on $\mathbb{R} \times (s_c,t_c)$. Moreover, by Lemma 3, the minimality property of u_c can be used to prove that

$$E_{u_c}(y) = \tfrac{1}{2}\|\partial_y u_c(\cdot,y)\|^2_{L^2} - V(u_c(\cdot,y)) = -c \quad \text{for all } y \in (s_c,t_c)$$

and hence, by Lemma 3-(ii), that u_c verifies the weak Neumann condition

$$\liminf_{y\to s_c^+} \|\partial_y u_c(\cdot,y)\|_{L^2} = \liminf_{y\to t_c^-} \|\partial_y u_c(\cdot,y)\|_{L^2} = 0.$$

In particular, if $s_c,t_c \in \mathbb{R}$, we can recover from u_c, by reflection, a *brake orbit type* entire solution. Precisely, setting $T_c = t_c - s_c$, let

$$v_c(x,y) = \begin{cases} u_c(x,y+s_c) & \text{if } x \in \mathbb{R} \text{ and } y \in [0,T_c) \\ u_c(x,t_c+T_c-y) & \text{if } x \in \mathbb{R} \text{ and } y \in [T_c,2T_c] \end{cases}$$

and

$$v_c(x,y) = v_c(x, y+2kT_c) \quad \text{for every } (x,y) \in \mathbb{R}^2,\ k \in \mathbb{Z}.$$

Then we have

PROPOSITION 1. *If $s_c, t_c \in \mathbb{R}$, then the function $v_c \in C^2(\mathbb{R}^2, \mathbb{R}^2)$ is a solution of problem (2)-(4). Moreover, $\partial_y v_c(\cdot, 0) \equiv \partial_y v_c(\cdot, T_c) \equiv 0$, $v_c(\cdot, 0) \in \mathcal{V}_c^-$ and $v_c(\cdot, T_c) \in \mathcal{V}_c^+$.*

On the other hand, if $s_c = -\infty$ (resp. $t_c = +\infty$), we can prove that the α-limit (resp. ω-limit) of u_c is constituted by critical points of V at level c.
Hence, if $s_c = -\infty$ and $t_c = +\infty$, then u_c is an entire solution to (2) such that

$$\lim_{y \to \pm\infty} \mathrm{dist}_{H^1}(u_c(\cdot, y), \mathcal{K}_c^{\pm}) = 0$$

where $\mathcal{K}_c^{\pm} = \{q \in \mathcal{V}_c^{\pm} \,|\, V'(q) = 0 \text{ and } V(q) = c\}$. That is, $v_c \equiv u_c$ is an entire solution to (2) of *heteroclinic type*. Note that this is the case if $c = m$.
Finally, if $s_c = -\infty$ and $t_c \in \mathbb{R}$ or $s_c \in \mathbb{R}$ and $t_c = +\infty$, from u_c we can construct, again by reflection, an *homoclinic type* solution. Precisely, if $s_c = -\infty$ and $t_c \in \mathbb{R}$, let us consider the function

$$v_c(x,y) = \begin{cases} u_c(x,y) & \text{if } x \in \mathbb{R} \text{ and } y \le t_c \\ u_c(x, 2t_c - y) & \text{if } x \in \mathbb{R} \text{ and } y > t_c \end{cases}$$

while if $s_c \in \mathbb{R}$ and $t_c = +\infty$, let

$$v_c((x,y) = \begin{cases} u_c(x,y) & \text{if } x \in \mathbb{R} \text{ and } y \ge s_c \\ u_c(x, 2s_c - y) & \text{if } x \in \mathbb{R} \text{ and } y < s_c \end{cases}$$

Then we have

PROPOSITION 2. *If $s_c = -\infty$ and $t_c \in \mathbb{R}$ (or if $s_c \in \mathbb{R}$ and $t_c = +\infty$) then $v_c \in C^2(\mathbb{R}^2, \mathbb{R}^2)$ is a solution of problem (2). Moreover, $v_c(\cdot, t_c) \in \mathcal{V}_c^+$, $\partial_y v_c(\cdot, t_c) \equiv 0$ (resp. $v_c(\cdot, s_c) \in \mathcal{V}_c^-$, $\partial_y v_c(\cdot, s_c) \equiv 0$) and there exists $q_0 \in \mathcal{K}_c^-$ (resp. $q_0 \in \mathcal{K}_c^+$) such that $\liminf_{y \to \pm\infty} \|v_c - q_0\|_{H^1} = 0$.*

Collecting Propositions 1 and 2, we obtain our main result

THEOREM 2. *For every $c \in (m, m+\lambda)$ with $\lambda > 0$ small enough, there exists $v_c \in C^2(\mathbb{R}^2, \mathbb{R}^2)$ solution to (2)-(4) such that $E_{v_c}(y) = -c$ for all $y \in \mathbb{R}$. Moreover*

(*i*) *if $t_c = +\infty$ then $\mathrm{dist}_{H^1}(v_c(\cdot, y), \mathcal{K}_c^+) \to 0$ as $y \to +\infty$,*

(*ii*) *if $s_c = -\infty$ then $\mathrm{dist}_{H^1}(v_c(\cdot, y), \mathcal{K}_c^-) \to 0$ as $y \to -\infty$,*

(*iii*) *if $t_c \in \mathbb{R}$ or $s_c \in \mathbb{R}$ then, respectively, $v_c(\cdot, t_c) \in \mathcal{V}_c^+$ and $\partial_y v_c(\cdot, t_p) \equiv 0$, or $v_c(\cdot, s_c) \in \mathcal{V}_c^-$ and $\partial_y v_c(\cdot, s_c) \equiv 0$.*

In particular if c is a regular value for V then t_c, $s_c \in \mathbb{R}$ and there exists $T_c > 0$ such that $v_c(x,y+2T_c) = v_c(x,y)$ for all $(x,y) \in \mathbb{R}^2$, $\partial_y v_c(\cdot,0) \equiv \partial_y v_c(\cdot,T_c) \equiv 0$, $v_c(\cdot,0) \in \mathcal{V}_c^+$ and $v_c(\cdot,T_c) \in \mathcal{V}_c^-$.

Note that the Theorem guarantees the existence of a brake orbit type solution at level c whenever $c \in (m, m+\lambda)$ is a regular value of V. As a consequence of the Sard Smale Theorem and the local compactness properties of V, it can be proved that the set of regular values of V is open and dense in $[m, m+\lambda]$ (see Lemma 2.9 in [6]). Then, Theorem 2 provides in fact the existence of an uncountable set of geometrically distinct two dimensional solutions of (2) of brake orbit type.

The variational procedure that we use was already introduced and used in the framework of scalar non autonomous Allen-Cahn equations in [12] and [5] where the existence of infinitely many bidimensional solutions is given. Energy prescribed brake orbit type solution were introduced and found in [6] for the same kind of non autonomous scalar equations.

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