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Forty years of Analysis in Turin
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PERIODIC AND HETEROCLINIC TYPE SOLUTIONS FOR SYSTEMS OF ALLEN-CAHN EQUATIONS

Abstract. We consider a class of semilinear elliptic system of the form

$$-\Delta u(x,y)+\nabla W(u(x,y))=0, \quad (x,y)\in\mathbb{R}^2, \tag{1}$$

where $W:\mathbb{R}^2\to\mathbb{R}$ is a double well non negative symmetric potential. We show, via variational methods, that if the set of solutions to the one dimensional system $-\ddot{q}(x)+\nabla W(q(x))=0$, $x\in\mathbb{R}$, which connect the two minima of W as $x\to\pm\infty$ has a discrete structure, then (1) has infinitely many layered solutions with prescribed energy.

1. Introduction

We consider semilinear elliptic system of the form

$$-\Delta u(x,y)+\nabla W(u(x,y))=0, \qquad (x,y)\in\mathbb{R}^2, \tag{2}$$

where $W\in C^2(\mathbb{R}^2,\mathbb{R})$ satisfies

(W_1) there exist $\boldsymbol{a}_\pm\in\mathbb{R}^2$ such that $W(\boldsymbol{a}_\pm)=0$, $W(\xi)>0$ for every $\xi\in\mathbb{R}^2\setminus\{\boldsymbol{a}_\pm\}$ and $D^2W(\boldsymbol{a}_\pm)$ are definite positive;

(W_2) $\liminf_{|\xi|\to+\infty} W'(\xi)\cdot\xi>0$;

(W_3) $W(-x_1,x_2)=W(x_1,x_2)$ for all $(x_1,x_2)\in\mathbb{R}^2$;

The system (2) is the rescaled stationary system associated to the reaction-diffusion system

$$\partial_t u(t,x,y)-\varepsilon^2\Delta u(t,x,y)+\nabla W(u(t,x,y))=0, \quad (x,y)\in\Omega\subset\mathbb{R}^2,\ t>0 \tag{3}$$

which describes two phase physical systems or grain boundaries in alloys. As $\varepsilon\to 0^+$, solutions to (3) tends almost everywhere to global minima of W and sharp phase interfaces appear (see e.g. [10], [18] and [21]). Then, the expansion of such solutions in a point on the interface presents, as first term, the system (2). From this point of view, two layered transition solutions correspond to solutions u of (2) satisfying the asympotic conditions

$$\lim_{x\to\pm\infty} u(x,y)=\boldsymbol{a}_\pm \quad \text{uniformly with respect to } y\in\mathbb{R}. \tag{4}$$

S. Alama, L. Bronsard and C. Gui in [1] studied the existence of solutions to (2) which satisfy the asymptotic condition (4) for $x\to\pm\infty$ while as $y\to\pm\infty$ tends to

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two different one dimensional trajectories, precisely, solutions to the one dimensional associated problem

$$\begin{cases} -\ddot{q}(x) + \nabla W(q(x)) = 0, & x \in \mathbb{R} \\ \lim_{t\to\pm\infty} q(t) = \boldsymbol{a}_\pm. \end{cases} \tag{5}$$

which are furthermore minima of the action

$$V(q) = \int_{\mathbb{R}} \tfrac{1}{2}|\dot{q}|^2 + W(q)\,dx$$

over the class of trajectories connecting $\boldsymbol{a}_\pm$ as $x \to \pm\infty$. Such solutions are found under conditions (W_1) and (W_3), requiring a fast growth at infinity and assuming that there exist a finite number $k \geq 2$ of geometrically distinct one dimensional minimal heteroclinic connections. In [20] M. Schatzman proves the same result, considering a non symmetric potential, assuming that there exists two geometrically distinct one dimensional heteroclinic connections which are supposed to be non degenerate, i.e. the kernel of the corresponding linearized operators are one dimensional.

If u is scalar valued, much is known about the corresponding heteroclinic problem (2)-(4). In this scalar setting, E. De Giorgi in [15] has conjectured that any entire bounded solution of $-\Delta u + u^3 - u = 0$ with $\partial_{x_1} u(x) > 0$ in $\mathbb{R}^n$ for $n \leq 8$ is in fact one-dimensional, i.e., modulo space roto-traslations, it coincides with the unique solution of the one dimensional heteroclinic problem

$$\begin{cases} -\ddot{q}(x) + q(x)^3 - q(x) = 0, & x \in \mathbb{R}, \\ q(0) = 0 \text{ and } \lim_{x\to\pm\infty} q(x) = \pm 1, \end{cases} \tag{6}$$

The conjecture has been proved for $n = 2$ by N. Ghoussoub and C. Gui in [14] and then by L. Ambrosio and X. Cabrè in [7] for $n = 3$ (see also [2]), even for more general double well potentials W. A further step in the proof of the De Giorgi conjecture has been done by O. Savin in [19] where, for $n \leq 8$, the same one dimensional structure is proved for solutions u such that $\partial_{x_1} u(x) > 0$ on $\mathbb{R}^n$ and $\lim_{x_1\to\pm\infty} u(x) = \pm 1$ for all $(x_2, x_3, \ldots, x_n) \in \mathbb{R}^{n-1}$ (see [8], [9] and [13] for related problems). That result is completed in [11] where the existence of entire solutions without any one dimensional symmetry which are increasing and asymptotic to ± 1 with respect to the first variable is proved in dimension $n > 8$.

Here we want to discuss some results obtained in [3] for the problem (2)-(4), where, using a global variational procedure, it is proved that if the minimal set of one dimensional heteroclinic connections satisfies a suitable discreteness assumption then there exist infinitely many solutions to the problem with prescribed *energy*, which can be classified as *homoclinic*, *heteroclinic* or *periodic* solutions.

2. Statement of the main Theorem and outline of the proof

To explain precisely the result and to give an idea of the procedure, let us begin considering the problem already considered in [1] and [20]. So let us define

$$\Gamma = \{q - z_0 \in H^1(\mathbb{R})^2 \mid q(x)_1 = -q(-x)_1,\, q(x)_2 = q(-x)_2\},$$

where $q(x) = (q(x)_1, q(x)_2)$ and $z_0 \in C^\infty(\mathbb{R}, \mathbb{R}^2)$ is fixed in such a way that $z_0(x)_1 = -z_0(-x)_1$, $z_0(x)_2 = z_0(-x)_2$ and $z_0(x) = \boldsymbol{a}_+$ for $x > 1$, be the space of one dimensional symmetric trajectories connecting $\boldsymbol{a}_\pm$ as $x \to \pm\infty$. Setting $m = \inf_\Gamma V(q)$, let

$$\mathcal{M} = \{q \in \Gamma \mid V(q) = m\}$$

be the minimal set of one dimensional symmetric heteroclinic connections. As it is well known, $\mathcal{M}$ is compact, not empty and consists of solutions to (5) in Γ.
Assuming that $\mathcal{M}$ satisfies the discreteness assumption

$$(*) \qquad \mathcal{M} = \mathcal{M}^+ \cup \mathcal{M}^- \text{ with } \mathrm{dist}_{L^2}(\mathcal{M}^+, \mathcal{M}^-) > 0,$$

we will look for bidimensional solution u with prescribed different asymptotes as $y \to \pm\infty$ and precisely

$$\mathrm{dist}_{L^2}(u(\cdot, y), \mathcal{M}^\pm) \to 0 \text{ as } y \to \pm\infty. \tag{7}$$

Note that condition $(*)$ (as the discreteness assumption made in [1] and [20]) does not hold in the scalar case, where the minimal set of one dimensional symmetric solutions $\mathcal{M}$ is in fact constituted by the unique heteroclinic solution of (6).

Under assumption $(*)$, bidimensional solutions satisfying (7) can be obtained using a global variational approach (instead of the approximating procedure used in [1] and [20]), considering a renormalized action functional over a suitable space.

As in [16] and [17] for Hamiltonian ODE systems and in [4] for scalar Allen-Cahn equations, we consider the renormalized action functional

$$\varphi(u) = \int_{\mathbb{R}} \tfrac{1}{2} \|\partial_y u(\cdot, y)\|^2_{L^2(\mathbb{R})^2} + (V(u(\cdot, y)) - m)\, dy$$

which is well defined on the space

$$\mathcal{H} = \{u \in H^1_{loc}(\mathbb{R}^2, \mathbb{R}^2) \mid u(\cdot, y) \in \Gamma \text{ for a.e. } y \in \mathbb{R}\}.$$

Note that φ is weakly lower semicontinuous on $\mathcal{H}$ and for every $u \in \mathcal{H}$, since $V(u(\cdot, y)) \geq m$ for a.e. $y \in \mathbb{R}$, there results $\varphi(u) \geq 0$ while $\varphi(q) = 0$ for all $q \in \mathcal{M}$.
We are interested on solutions which satisfies the right asymptotic conditions as $y \to \pm\infty$. Such solutions can be reached as minima of φ over the class

$$\mathcal{H}_m = \{u \in \mathcal{H} \mid \liminf_{y \to \pm\infty} \mathrm{dist}_{L^2}(u(\cdot, y), \mathcal{M}^\pm) = 0\}.$$

In fact, for all $u \in \mathcal{H}$, for all $y_1 < y_2$ we have

$$\|u(\cdot,y_1) - u(\cdot,y_2)\|_{L^2}^2 \leq (y_2 - y_1) \int_{y_1}^{y_2} \|\partial_y u(\cdot,y)\|_{L^2}^2 \, dy$$

and in particular, if $\varphi(u) < +\infty$ then the map $y \in \mathbb{R} \mapsto u(\cdot,y) \in \Gamma$ is continuous with respect to the $L^2(\mathbb{R})^2$ metric. Moreover, if $u \in \mathcal{H}$ and $y_1 < y_2$, then

$$\varphi(u) \geq \left(\tfrac{2}{y_2 - y_1} \int_{y_1}^{y_2} (V(u(\cdot,y)) - m) \, dy \right)^{1/2} \|u(\cdot,y_1) - u(\cdot,y_2)\|_{L^2}.$$

and in particular, if $V(u(\cdot,y)) \geq m + \nu$ for all $y \in (y_1,y_2)$ and some $\nu > 0$, then

$$\varphi(u) \geq \tfrac{1}{2(y_2-y_1)} \|u(\cdot,y_1) - u(\cdot,y_2)\|_{L^2}^2 + \nu(y_2 - y_1) \geq \sqrt{2\nu} \|u(\cdot,y_1) - u(\cdot,y_2)\|_{L^2}.$$

By the previous estimates, if $u \in \mathcal{H}_m$ we have control on the transition time from $\mathcal{M}^-$ to $\mathcal{M}^+$ and so concentration in the y variable. Indeed it can be proved

LEMMA 1. *There exists $\nu \in (0,m)$ such that if $u \in \mathcal{H}_m$, $\varphi(u) < +\infty$ and for some $y_0 \in \mathbb{R}$, $V(u(\cdot,y_0)) < m + \nu$ then, either*

(i) $\mathrm{dist}_{L^2}(u(\cdot,y), \mathcal{M}^-) \leq d_0$ *for all $y \leq y_0$; or*

(ii) $\mathrm{dist}_{L^2}(u(\cdot,y), \mathcal{M}^+) \leq d_0$ *for all $y \geq y_0$,*

where $d_0 := \frac{1}{5} \mathrm{dist}_{L^2}(\mathcal{M}^+, \mathcal{M}^-)$.

Lemma 1, together with the symmetry in the x variable, allows to get compactness of minimizing sequences in $\mathcal{H}_m$. Indeed, setting $\mu := \inf_{\mathcal{H}} \varphi$ we have

LEMMA 2. *Let $(u_n) \subset \mathcal{H}_m$ be such that $\varphi(u_n) \to \mu$ as $n \to \infty$ and such that* $\mathrm{dist}_{L^2}(u_n(\cdot,0), \mathcal{M}^-) = d_0$ *for all $n \in \mathbb{N}$. Then, there exists $u \in \mathcal{H}$ such that, up to a subsequence,*

(i) *$u_n - u \to 0$ as $n \to \infty$ weakly in $H^1_{loc}(\mathbb{R}^2)^2$,*

(ii) *there exists $L_0 > 0$ such that* $\mathrm{dist}_{L^2}(u(\cdot,y), \mathcal{M}^-) \leq d_0$ *for all $y \leq -L_0$, and* $\mathrm{dist}_{L^2}(u(\cdot,y), \mathcal{M}^+) \leq d_0$ *for all $y \geq L_0$.*

By the invariance with respect to the y-translation of φ and the definition of $\mathcal{H}_m$, we have that there exists a minimizing sequence (u_n) which verifies the condition $\mathrm{dist}_{L^2}(u_n(\cdot,0), \mathcal{M}^-) = d_0$ for all $n \in \mathbb{N}$. Then, by Lemma 2, such sequence weakly converge in $H^1_{loc}(\mathbb{R}^2)^2$ to a function $u \in \mathcal{H}$ such that

$$\lim_{y \to \pm\infty} \mathrm{dist}_{L^2}(u(\cdot,y), \mathcal{M}^\pm) = 0.$$

Then, $u \in \mathcal{H}_m$ and since φ is weakly semicontinuous we can conclude that $\varphi(u) = \mu$, proving the existence of at least one bidimensional solution to (2) in $\mathcal{H}_m$, as already proved in the Theorem by Alama Bronsard and Gui but here in a slightly more general setting

THEOREM 1. *If (W_1)-(W_3) and $(*)$ hold, then there exists $u \in \mathcal{C}^2(\mathbb{R}^2,\mathbb{R}^2)$ solution to (2) such that $u(x,y) \to \boldsymbol{a}_\pm$ as $x \to \pm\infty$ uniformly with respect to. $y \in \mathbb{R}$ and*

$$\lim_{y\to\pm\infty} \mathrm{dist}_{H^1}(u(\cdot,y),\mathcal{M}^\pm) = 0.$$

Now, note that if $u \in \mathcal{H}$ solves the system (2) then

$$\partial_y^2 u(x,y) = \underbrace{-\partial_x^2 u(x,y) + \nabla W(u(x,y))}_{V'(u(\cdot,y))}$$

In other words u defines a trajectory $y \in \mathbb{R} \mapsto u(\cdot,y) \in \Gamma$ solution to the infinite dimensional Lagrangian system

$$\frac{d^2}{dy^2}u(\cdot,y) = V'(u(\cdot,y))$$

which has as equilibria the one dimensional solutions $q \in \mathcal{M}$. From such point of view, bidimensional solutions in $\mathcal{H}_m$ are *heteroclinic type* solutions connecting $\mathcal{M}^\pm$ as $y \to \pm\infty$.

Note that the *energy* is conserved, indeed if $u \in \mathcal{H}$ solves (2) on $\mathbb{R}\times(y_1,y_2)$ then

$$E_u(y) = \tfrac{1}{2}\|\partial_y u(\cdot,y)\|_{L^2}^2 - V(u(\cdot,y))$$

is constant on (y_1,y_2). In particular, for the heteroclinic type solution $u \in \mathcal{H}_m$ given in Theorem 1 we have that $E_u(y) = -m$ for every $y \in \mathbb{R}$ and that it connects in Γ the two component $\mathcal{M}_\pm$ as $y \to \pm\infty$.

Now, note that if we take $c \in (m, m+\lambda)$ with $\lambda > 0$ small enough, by $(*)$ we get

$$(*_c) \qquad \{q \in \Gamma \,|\, V(q) \le c\} = \mathcal{V}_c^- \cup \mathcal{V}_c^+ \text{ with } \mathrm{dist}_{L^2}(\mathcal{V}_c^-,\mathcal{V}_c^+) > 0.$$

A natural problem, which generalizes the above one, is to look for a solution $u \in \mathcal{H}$ to (2) with energy $E_u(y) = -c$ for every $y \in \mathbb{R}$ which connects in Γ the sets $\mathcal{V}_c^\pm$ as $y \to \pm\infty$.

In such a case $V(u(\cdot,y)) = -E_u(y) + \frac{1}{2}\|\partial_y u(\cdot,y)\|_{L^2}^2 \ge c$ for every $y \in \mathbb{R}$ and so solutions with energy $-c$ can be sought as minima of the new renormalized functional

$$\varphi_c(u) = \int_{\mathbb{R}} \tfrac{1}{2}\|\partial_y u(\cdot,y)\|_{L^2}^2 + (V(u(\cdot,y)) - c)\, dy$$

on the space

$$\mathcal{H}_c = \{u \in \mathcal{H} \,|\, \liminf_{y\to\pm\infty} \mathrm{dist}_{L^2}(u(\cdot,y),\mathcal{V}_c^\pm) = 0 \text{ and } V(u(\cdot,y)) \ge c \text{ for a.e. } y \in \mathbb{R}\}.$$

Note that if $u \in \mathcal{H}_c$ then $V(u(\cdot,y)) \ge c$ for a.e. $y \in \mathbb{R}$ and so the functional φ_c is well defined on $\mathcal{H}_c$ with values in $[0,+\infty]$.

The functional φ_c enjoys most of the properties of the functional φ and the above concentration-compacteness results work even in this setting. In particular a suitable y-translated minimizing sequences of φ_c in $\mathcal{H}_c$ weakly converge in $H^1_{loc}(\mathbb{R}^2)^2$ to a function $u_c \in \mathcal{H}$. However, differently to the case considered above, we do not know if the limit point u_c satisfies the constraint $V(u_c(\cdot,y)) \geq c$ for a.e. $y \in \mathbb{R}$ and that hence $u_c \in \mathcal{H}_c$ and $\varphi_c(u_c) = \mu_c := \inf_{\mathcal{H}_c} \varphi_c$. Anyhow we can prove that such condition holds true on the interval (s_c,t_c) where s_c and t_c are defined as

$$s_c = \sup\{y \in \mathbb{R} / \mathrm{dist}_{L^2}(u_c(\cdot,y), \mathcal{V}_c^-) \leq d_0 \text{ and } V(u_c(\cdot,y)) \leq c\}$$

and

$$t_c = \inf\{y > s_c \,/\, V(u_c(\cdot,y)) \leq c\},$$

where we agree that $s_c = -\infty$ whenever $V(u_c(\cdot,y)) > c$ for every $y \in \mathbb{R}$ such that $\mathrm{dist}_{L^2}(u_c(\cdot,y), \mathcal{V}_c^-) \leq d_0$ and that $t_c = +\infty$ whenever $V(u_c(\cdot,y)) > c$ for all $y > s_c$ (note that this is the case that occurs if $c = m$). Indeed we have

LEMMA 3. *For every* $[y_1,y_2] \subset (s_c,t_c)$ *there results* $\inf_{y\in[y_1,y_2]} V(u_c(\cdot,y)) > c$. *Moreover,*

(i) $\lim_{y\to s_c^+} \mathrm{dist}_{L^2} u(\cdot,y), \mathcal{V}_c^-) = \lim_{y\to t_c^-} \mathrm{dist}_{L^2} u(\cdot,y), \mathcal{V}_c^+) = 0$;

(ii) $\liminf_{y\to s_c^+} V(u_c(\cdot,y)) = \liminf_{y\to t_c^-} V(u_c(\cdot,y)) = c$;

(iii) $\varphi_{c,(s_c,t_c)}(u_c) := \int_{s_c}^{t_c} \frac{1}{2}\|\partial_y u(\cdot,y)\|^2_{L^2} + (V(u(\cdot,y)) - c)\, dy = \mu_c$.

In particular, by Lemmas 3 and the definition of s_c and t_c, we derive that if $s_c \in \mathbb{R}$ then $u_c(\cdot,s_c) \in \mathcal{V}_c^-$ and if $t_c \in \mathbb{R}$ then $u_c(\cdot,t_c) \in \mathcal{V}_c^+$. On the other hand, if $s_c = -\infty$ then $\mathrm{dist}_{L^2}(u_c(\cdot,y), \mathcal{V}_c^-) \to 0$ as $y \to -\infty$ while if $t_c = +\infty$ then $\mathrm{dist}_{L^2}(u_c(\cdot,y), \mathcal{V}_c^+) \to 0$ as $y \to +\infty$. Using this properties, by Lemma 3, we get

$$\int_{\mathbb{R}^2} \nabla u_c \nabla h + \nabla W(u_c) h \, dx\, dy = 0 \quad \text{for all } h \in C_0^\infty(\mathbb{R} \times (s_c,t_c))^2$$

and so that that u_c is a *weak* solution to (2) in $\mathbb{R} \times (s_c,t_c)$. Then, it is standard to show that u_c is in fact a *classical* solution to (2) on $\mathbb{R} \times (s_c,t_c)$. Moreover, by Lemma 3, the minimality property of u_c can be used to prove that

$$E_{u_c}(y) = \tfrac{1}{2}\|\partial_y u_c(\cdot,y)\|^2_{L^2} - V(u_c(\cdot,y)) = -c \quad \text{for all } y \in (s_c,t_c)$$

and hence, by Lemma 3-(ii), that u_c verifies the weak Neumann condition

$$\liminf_{y\to s_c^+} \|\partial_y u_c(\cdot,y)\|_{L^2} = \liminf_{y\to t_c^-} \|\partial_y u_c(\cdot,y)\|_{L^2} = 0.$$

In particular, if $s_c, t_c \in \mathbb{R}$, we can recover from u_c, by reflection, a *brake orbit type* entire solution. Precisely, setting $T_c = t_c - s_c$, let

$$v_c(x,y) = \begin{cases} u_c(x, y+s_c) & \text{if } x \in \mathbb{R} \text{ and } y \in [0,T_c) \\ u_c(x, t_c + T_c - y) & \text{if } x \in \mathbb{R} \text{ and } y \in [T_c, 2T_c] \end{cases}$$

and

$$v_c(x,y) = v_c(x, y+2kT_c) \quad \text{for every } (x,y) \in \mathbb{R}^2,\ k \in \mathbb{Z}.$$

Then we have

PROPOSITION 1. *If $s_c, t_c \in \mathbb{R}$, then the function $v_c \in C^2(\mathbb{R}^2, \mathbb{R}^2)$ is a solution of problem (2)-(4). Moreover, $\partial_y v_c(\cdot, 0) \equiv \partial_y v_c(\cdot, T_c) \equiv 0$, $v_c(\cdot, 0) \in \mathcal{V}_c^-$ and $v_c(\cdot, T_c) \in \mathcal{V}_c^+$.*

On the other hand, if $s_c = -\infty$ (resp. $t_c = +\infty$), we can prove that the α-limit (resp. ω-limit) of u_c is constituted by critical points of V at level c.
Hence, if $s_c = -\infty$ and $t_c = +\infty$, then u_c is an entire solution to (2) such that

$$\lim_{y \to \pm\infty} \mathrm{dist}_{H^1}(u_c(\cdot, y), \mathcal{K}_c^\pm) = 0$$

where $\mathcal{K}_c^\pm = \{q \in \mathcal{V}_c^\pm \,|\, V'(q) = 0 \text{ and } V(q) = c\}$. That is, $v_c \equiv u_c$ is an entire solution to (2) of *heteroclinic type*. Note that this is the case if $c = m$.
Finally, if $s_c = -\infty$ and $t_c \in \mathbb{R}$ or $s_c \in \mathbb{R}$ and $t_c = +\infty$, from u_c we can construct, again by reflection, an *homoclinic type* solution. Precisely, if $s_c = -\infty$ and $t_c \in \mathbb{R}$, let us consider the function

$$v_c(x,y) = \begin{cases} u_c(x,y) & \text{if } x \in \mathbb{R} \text{ and } y \le t_c \\ u_c(x, 2t_c - y) & \text{if } x \in \mathbb{R} \text{ and } y > t_c \end{cases}$$

while if $s_c \in \mathbb{R}$ and $t_c = +\infty$, let

$$v_c((x,y) = \begin{cases} u_c(x,y) & \text{if } x \in \mathbb{R} \text{ and } y \ge s_c \\ u_c(x, 2s_c - y) & \text{if } x \in \mathbb{R} \text{ and } y < s_c \end{cases}$$

Then we have

PROPOSITION 2. *If $s_c = -\infty$ and $t_c \in \mathbb{R}$ (or if $s_c \in \mathbb{R}$ and $t_c = +\infty$) then $v_c \in C^2(\mathbb{R}^2, \mathbb{R}^2)$ is a solution of problem (2). Moreover, $v_c(\cdot, t_c) \in \mathcal{V}_c^+$, $\partial_y v_c(\cdot, t_c) \equiv 0$ (resp. $v_c(\cdot, s_c) \in \mathcal{V}_c^-$, $\partial_y v_c(\cdot, s_c) \equiv 0$) and there exists $q_0 \in \mathcal{K}_c^-$ (resp. $q_0 \in \mathcal{K}_c^+$) such that $\liminf_{y \to \pm\infty} \|v_c - q_0\|_{H^1} = 0$.*

Collecting Propositions 1 and 2, we obtain our main result

THEOREM 2. *For every $c \in (m, m+\lambda)$ with $\lambda > 0$ small enough, there exists $v_c \in C^2(\mathbb{R}^2, \mathbb{R}^2)$ solution to (2)-(4) such that $E_{v_c}(y) = -c$ for all $y \in \mathbb{R}$. Moreover*

- (i) *if $t_c = +\infty$ then $\mathrm{dist}_{H^1}(v_c(\cdot, y), \mathcal{K}_c^+) \to 0$ as $y \to +\infty$,*
- (ii) *if $s_c = -\infty$ then $\mathrm{dist}_{H^1}(v_c(\cdot, y), \mathcal{K}_c^-) \to 0$ as $y \to -\infty$,*
- (iii) *if $t_c \in \mathbb{R}$ or $s_c \in \mathbb{R}$ then, respectively, $v_c(\cdot, t_c) \in \mathcal{V}_c^+$ and $\partial_y v_c(\cdot, t_p) \equiv 0$, or $v_c(\cdot, s_c) \in \mathcal{V}_c^-$ and $\partial_y v_c(\cdot, s_c) \equiv 0$.*

In particular if c is a regular value for V then $t_c, s_c \in \mathbb{R}$ *and there exists* $T_c > 0$ *such that* $v_c(x, y+2T_c) = v_c(x,y)$ *for all* $(x,y) \in \mathbb{R}^2$, $\partial_y v_c(\cdot,0) \equiv \partial_y v_c(\cdot,T_c) \equiv 0$, $v_c(\cdot,0) \in \mathcal{V}_c^+$ *and* $v_c(\cdot,T_c) \in \mathcal{V}_c^-$.

Note that the Theorem guarantees the existence of a brake orbit type solution at level c whenever $c \in (m, m+\lambda)$ is a regular value of V. As a consequence of the Sard Smale Theorem and the local compactness properties of V, it can be proved that the set of regular values of V is open and dense in $[m, m+\lambda]$ (see Lemma 2.9 in [6]). Then, Theorem 2 provides in fact the existence of an uncountable set of geometrically distinct two dimensional solutions of (2) of brake orbit type.

The variational procedure that we use was already introduced and used in the framework of scalar non autonomous Allen-Cahn equations in [12] and [5] where the existence of infinitely many bidimensional solutions is given. Energy prescribed brake orbit type solution were introduced and found in [6] for the same kind of non autonomous scalar equations.

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Forty years of Analysis in Turin
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V. Chiadò Piat

HOMOGENIZATION OF SPECTRAL PROBLEMS AND LOCALIZATION EFFECTS

Abstract. We describe classical and new results concerning the limit behaviour of spectral problems in a periodically perforated domain, with special attention to some cases where the eigenfunctions localize.

1. Introduction

My first step into mathematical research was the study of the theory of homogenization, applied to linear e non linear elliptic equations. The subject was suggested to me by professor Angelo Negro, to whom I am deeply grateful for having introduced me to several beautiful topics in mathematical analysis.

The study of spectral problems in periodically perforated domains is developed, since long time, by many authors (see, for example, [15], [13], [11]) and has a number of motivations and applications. One important example is the field of optimal design (see [3], [1], [4]). Starting from the simplest case of the Laplace operator, it is known that the knowledge of its spectrum depends on the boundary conditions and on the geometry of the domain under consideration. An alternative point of view is to say that the knowledge of the spectrum of a boundary value problem gives information about the geometry of the domain. This aspect is particularly important in shape optimization problems, where the shape of the domain is an unknown, and the goal is to choose it in a way to obtain, for example, certain desired modes of vibration. There are several techniques to study the effect of variations of a domain on the corresponding solutions, or eigenvalues and eigenfunctions. Classical methods date back to Hadamard ([10]), and are based on smooth variations of the boundary of a given initial domain, in the normal direction. This approach excludes non smooth boundary, and variations that change the topology, as, for example, create holes. More recent topological optimization methods are able to include topological variations and take into account the knowledge of homogenization of boundary value problems and of spectral problems in perforated domains.

In Section 2 of this paper we present some of the results obtained in collaboration with I. Pankratova and A. Piatnitski. Details and proofs are contained in [6], where we deal with a spectral problem for an elliptic operator in divergence form, complemented by Fourier-type boundary conditions on the surface of the holes. The presence of a non periodic coefficient in the boundary conditions causes a number of interesting effects. First of all, under the assumption that the non periodic coefficient has a unique minimum point, a localization phenomenon holds: namely, for any $k \in \mathbb{N}$ the k-th eigenfunction of the problem is asymptotically localized, in a small neighbourhood of the minimum point, as the periodicity size vanishes. In particular, the principal

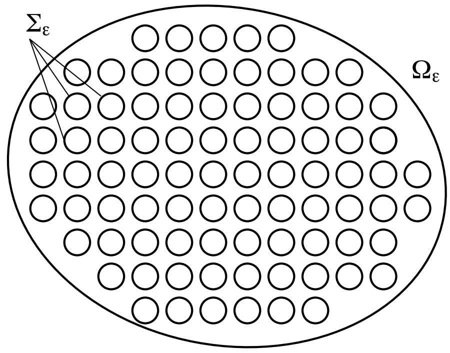

Figure 1: Domain Ω_ε

eigenfunction converges to a δ-function supported at the minimum point. Moreover, the localization process takes place in the scale $\varepsilon^{1/4}$, and it is possible to construct asymptotic expansions which are in integer powers of $\varepsilon^{1/4}$. In this scale the leading term of the asymptotic expansion for the k-th eigenfunction can be proved to be the k-th eigenfunction of an auxiliary harmonic oscillator operator.

Different results for spectral problems of Steklov type are contained in [5], while a study nonlinear variational problems with Fourier boundary conditions can be found in [7].

In Section 3 we address some related papers where other localization phenomena are found out.

2. A problem with Fourier boundary conditions

$$\Omega_\varepsilon = \Omega \setminus \bigcup_{i\in I_\varepsilon} T^i_\varepsilon, \quad I_\varepsilon = \{i \in \mathbb{Z}^d : \ T^i_\varepsilon \subset \Omega\},$$

where $T^i_\varepsilon = \varepsilon(T+i)$, and $T \subset\subset (0,1)^d$ is a compact subset of the unit cube, with non empty interior. We denote by $\omega = (0,1)^d \setminus T$ the open unit cell and by $\Sigma = \partial T$ the boundary of the perforation. In the "periodically perforated" domain $\Omega_\varepsilon \subset \mathbb{R}^d$ we consider the following spectral problem:

$$\text{(1)} \qquad \begin{cases} -\mathrm{div}(a^\varepsilon \nabla u^\varepsilon) = \lambda^\varepsilon u^\varepsilon, \text{ in } \Omega_\varepsilon, \\ a^\varepsilon \nabla u^\varepsilon \cdot n = -q(x) u^\varepsilon, \text{ on } \Sigma_\varepsilon, \\ u^\varepsilon = 0, \qquad\qquad x \in \partial\Omega, \end{cases}$$

where $a^\varepsilon(x) = a\left(\frac{x}{\varepsilon}\right)$. Notice that Ω_ε remains connected, the perforation does not intersect the boundary $\partial\Omega$, and

$$\partial\Omega_\varepsilon = \partial\Omega \bigcup \Sigma_\varepsilon, \quad \Sigma_\varepsilon = \bigcup_{i\in I_\varepsilon} \Sigma^i_\varepsilon, \quad \Sigma^i_\varepsilon = \varepsilon(\Sigma + i).$$

The boundary conditions are known as Fourier, or Robin conditions. We make the following assumptions:

(H0) $\partial\Omega_\varepsilon = \partial\Omega \cap \Sigma_\varepsilon$, where $\Omega \subset \mathbb{R}^d$ is a bounded and regular open set;

(H1) $a(y)$ is a symmetric, uniformly elliptic $d \times d$-matrix in $\mathbb{R}^d$;

(H2) the coefficients $a_{ij}(y) \in L^\infty(\mathbb{R}^d)$ are 1-periodic in all variables;

(H3) the function $q(x) \in C^3(\mathbb{R}^d)$ is non negative;

(H4) the function $q(x)$ attains its global minimum at $x = 0 \in \Omega$, and as $x \to 0$ it satisfies

$$q(x) = q(0) + \frac{1}{2} x^T H(q) x + o(|x|^2),$$

with positive definite Hessian matrix $H(q)$.

One interesting feature of this problem is the presence of the 'slow' variable x in the coefficient q, in the boundary condition. This fact causes a number of interesting effects, among which the localization of eigenfunctions. The eigenvalue problem (1) has the following weak formulation:

find $(\lambda^\varepsilon, u^\varepsilon) \in \mathbb{C} \times H^1(\Omega_\varepsilon)$, $u^\varepsilon = 0$ on $\partial\Omega$ and $u^\varepsilon \neq 0$, such that

$$\int_{\Omega_\varepsilon} a^\varepsilon \nabla u^\varepsilon \cdot \nabla v \, dx + \int_{\Sigma_\varepsilon} q u^\varepsilon v \, d\sigma = \lambda^\varepsilon \int_{\Omega_\varepsilon} u^\varepsilon v \, dx, \quad v \in H_0^1(\Omega). \tag{2}$$

Under the above assumptions **(H0)-(H4)**, it is easy to prove the following result.

LEMMA 1. *The spectrum of problem* (2) *is real and discrete*

$$0 < \lambda_1^\varepsilon < \lambda_2^\varepsilon \le \cdots \le \lambda_j^\varepsilon \le \cdots \to +\infty.$$

Each eigenvalue has finite multiplicity. The corresponding normalized eigenfunctions

$$\int_{\Omega_\varepsilon} u_i^\varepsilon u_j^\varepsilon \, dx = \delta_{ij},$$

form an orthonormal basis of $L^2(\Omega_\varepsilon)$. Moreover, the following variational characterization for λ_1^ε holds true:

$$\lambda_1^\varepsilon = \inf_{\substack{v \in H_0^1(\Omega_\varepsilon, \partial\Omega) \\ \|v\|_{L^2(\Omega_\varepsilon)} = 1}} \int_{\Omega_\varepsilon} a^\varepsilon \nabla v \cdot \nabla v \, dx + \int_{\Sigma_\varepsilon} q(v)^2 \, d\sigma. \tag{3}$$

The main object of our study is the limit behaviour of $(\lambda^\varepsilon, u^\varepsilon)$ as $\varepsilon \to 0$. Using $q(x) \geq 0$ from below, and $H_0^1(\Omega_\varepsilon) \subset H_0^1(\Omega_\varepsilon, \partial\Omega)$ from above in the variational formula (3) for λ_1^ε, we immediately see that

$$\lambda_{1,N}^\varepsilon \leq \lambda_1^\varepsilon \leq \lambda_{1,D}^\varepsilon.$$

The two constants $\lambda_{1,N}^\varepsilon$, $\lambda_{1,D}^\varepsilon$ are, respectively, the eigenvalues of the homogeneous Neumann problem and of the homogeneous Dirichlet problem, i.e.,

$$\text{(4)} \qquad \begin{cases} -\mathrm{div}(a^\varepsilon \nabla u^\varepsilon) = \lambda_N^\varepsilon u^\varepsilon, & \text{in } \Omega_\varepsilon, \\ a^\varepsilon \nabla u^\varepsilon \cdot n = 0 & \text{on } \Sigma_\varepsilon, \\ u^\varepsilon = 0, & x \in \partial\Omega, \end{cases}$$

and

$$\text{(5)} \qquad \begin{cases} -\mathrm{div}(a^\varepsilon \nabla u^\varepsilon) = \lambda_D^\varepsilon u^\varepsilon, & \text{in } \Omega_\varepsilon, \\ u^\varepsilon = 0 & \text{on } \partial\Omega_\varepsilon. \end{cases}$$

The asymptotic behaviour of both problems has been investigated long ago by Vanninathan in [15], together with the closely related Steklov problem. According to [15], in the Neumann case (4), as $\varepsilon \to 0$ we have

$$\lambda_N^\varepsilon = \lambda_N^0 + \varepsilon \lambda_N^1 + O(\varepsilon^2)$$

and

$$u_\varepsilon = u_0(x) + \varepsilon N\left(\frac{x}{\varepsilon}\right) \cdot Du_0(x) + \dots,$$

with $(\lambda_N^0, u_0(x))$ solutions of the homogenized spectral problem

$$\text{(6)} \qquad \begin{cases} -\mathrm{div}(a^N \nabla u) = \lambda u, & \text{in } \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases}$$

Here a^N is the homogenized matrix of the boundary value problem with Neumann condition on the boundary of the perforation studied by Cioranescu and Saint Jean Paulin in [8]. The vector-valued function N is the first order corrector, defined by the auxiliary boundary-value problem in the perforated periodicity cell

$$\text{(7)} \qquad \begin{cases} -\mathrm{div}(a(y)(\nabla N^j + e_j)) = 0 & \text{in } \omega, \\ a(y)\nabla N^j \cdot n = 0 & \text{on } \Sigma, \\ N^j = N^j(y) \quad \text{periodic}, & \int_\omega N^j(y)dy = 0, \end{cases}$$

and

$$\text{(8)} \qquad a^N = \frac{1}{|\omega|} \int_\omega a(y)(\nabla N^i + e_i)(\nabla N^j + e_j)\, dy.$$

In the Dirichlet case (5), instead,

$$\lambda_D^\varepsilon = \varepsilon^{-2}\lambda_D^0 + O(\varepsilon^2),$$

where λ_D^0 is the first eigenvalue of the Dirichlet spectral problem in the periodicity cell ω

$$(9) \qquad \begin{cases} -\mathrm{div}(a(y)\nabla v) = \lambda v \text{ in } \omega, \\ v = 0 \qquad\qquad \text{on } \Sigma, \\ v = v(y) \qquad \text{periodic .} \end{cases}$$

In this case, the eigenfunctions, upon extension to zero out of Ω_ε, tend strongly to 0 in $H_0^1(\Omega)$.

The Fourier spectral problem with *periodic* cefficients has been studied by Pastukhova in [13], in the case

$$(10) \qquad \begin{cases} -\mathrm{div}(a^\varepsilon\nabla u^\varepsilon) = \lambda^\varepsilon u^\varepsilon, \text{ in } \Omega_\varepsilon, \\ a^\varepsilon\nabla u^\varepsilon\cdot n + b\left(\dfrac{x}{\varepsilon}\right)u^\varepsilon = 0, \text{ on } \Sigma_\varepsilon, \\ u^\varepsilon = 0, \qquad\qquad x\in\partial\Omega. \end{cases}$$

Also the comparison with this problem brings useful information to the solution of our initial spectral problem (1), where the periodically oscillating term $b\left(\frac{x}{\varepsilon}\right)$ is replaced by the function $q(x)$ which depends on the 'slow' variable x. Indeed, for our problem the following lemma holds true.

LEMMA 2. *The first eigenvalue of problem* (1) *satisfies the estimate*

$$\frac{1}{\varepsilon}\frac{|\Sigma|_{d-1}}{|\omega|_d}q(0) + O(1) \le \lambda_1^\varepsilon \le \frac{1}{\varepsilon}\frac{|\Sigma|_{d-1}}{|\omega|_d}q(0) + O(\varepsilon^{-1/2}), \quad \varepsilon\to 0,$$

where $|\omega|_d$ *e* $|\Sigma|_{d-1}$ *indicate, respectively, the* d *and* $(d-1)$ *dimensional measures of the perforated cell* ω *and of the boundary of the perforation* Σ.

To clarify the result, we note that, since $q(x)\ge q(0)$, then

$$\lambda_1^\varepsilon \ge \inf_{\substack{v\in H_0^1(\Omega_\varepsilon,\partial\Omega)\\ \|v\|_{L^2(\Omega_\varepsilon)}=1}} \left\{\int_{\Omega_\varepsilon} a^\varepsilon\nabla v\cdot\nabla v\,dx + q(0)\int_{\Sigma_\varepsilon}(v)^2\,d\sigma\right\} = \nu_1^\varepsilon.$$

But ν_1^ε coincides with the first eigenvalue of del Pastukhova's problem (10), in the case $b(\frac{x}{\varepsilon}) = q(0)$

$$\begin{cases} -\mathrm{div}(a^\varepsilon\nabla w^\varepsilon) = \nu^\varepsilon w^\varepsilon, \quad \text{in } \Omega_\varepsilon, \\ a^\varepsilon\nabla w^\varepsilon\cdot n = -q(0)w^\varepsilon, \quad \text{on } \Sigma_\varepsilon, \\ w^\varepsilon = 0, \qquad\qquad x\in\partial\Omega. \end{cases}$$

In [13] it is proved that

$$\nu_1^\varepsilon = \frac{1}{\varepsilon}\,\frac{|\Sigma|_{d-1}}{|\omega|_d}\,q(0) + O(1), \quad \varepsilon \to 0.$$

Hence, the left-hand side inequality in Lemma 2 follows:

$$\lambda_1^\varepsilon \geq \frac{1}{\varepsilon}\,\frac{|\Sigma|_{d-1}}{|\omega|_d}\,q(0) + O(1), \quad \varepsilon \to 0.$$

Let us now examine the right-hand side inequality in Lemma 2. Choosing any $v \in C_0^\infty(\Omega)$ as test function in the variational characterizazion (3), one gets easily that

$$\lambda_1^\varepsilon \leq C\,\varepsilon^{-1},$$

with a constant C independent of ε. But to get the same constant from above and below requires a different choice of the test function. Choosing $v(x/\varepsilon^\alpha)$ with $v \in C_0^\infty(\Omega)$, $\|v\|_{L^2(\mathbb{R}^d)} = 1$ in the variational characterization (3) one can prove that the optimal estimate is attained when $\alpha = 1/4$ and obtains that

$$\lambda_1^\varepsilon \leq \frac{1}{\varepsilon}\,\frac{|\Sigma|_{d-1}}{|\omega|_d}\,q(0) + O(\varepsilon^{-1/2}), \quad \varepsilon \to 0.$$

One can note that the optimal test functions concentrate at $x = 0$, the minimum point of $q(x)$, as $\varepsilon \to 0$. To be more precise, we can state the following definition and proposition.

DEFINITION 1. *We say that a family* $\{w_\varepsilon(x)\}_{\varepsilon>0}$ *with*

$$0 < c_1 \leq \|w_\varepsilon\|_{L^2(\Omega_\varepsilon)} \leq c_2$$

is concentrated at x_0, *as* $\varepsilon \to 0$, *if for any* $\gamma > 0$ *there is* $\varepsilon_0 > 0$ *such that*

$$\int\limits_{\Omega_\varepsilon \setminus B_\gamma(x_0)} |w_\varepsilon|^2\,dx < \gamma, \qquad \textit{for all } \varepsilon \in (0,\varepsilon_0).$$

Here $B_\gamma(x_0)$ *is a ball of radius* γ *centered at* x_0.

PROPOSITION 1. *The first eigenfunction* u_1^ε *of problem* (1) *is concentrated in the sense of Definition 1 at the minimum point of* $q(x)$, *that is at* $x = 0$.

The asymptotic behaviour of the eigenpairs of problem (1) is described in details by the following theorem.

THEOREM 1. *The following representation holds true*

$$\lambda_j^\varepsilon = \frac{1}{\varepsilon}\,\frac{|\Sigma|_{d-1}}{|\omega|_d}\,q(0) + \frac{\mu_j^\varepsilon}{\sqrt{\varepsilon}}, \quad u_j^\varepsilon(x) = v_j^\varepsilon\Big(\frac{x}{\varepsilon^{1/4}}\Big),$$

where $(\mu_j^\varepsilon, v_j^\varepsilon(z))$ *are such that* $\mu_j^\varepsilon \to \mu_j$*, as* $\varepsilon \to 0$*, and* μ_j *is eigenvalue of the homogenized problem*

$$-\mathrm{div}(a^N \nabla v) + \frac{1}{2}\frac{|\Sigma|_{d-1}}{|\omega|_d}(z^T H(q) z)\, v = \mu v, \quad v \in L^2(\mathbb{R}^d),$$

where $v = v(z)$*, and* a^N *is given by (8). Moreover, if* μ_j *is simple, then*

$$\|v_j^\varepsilon - v_j\|_{L^2(\mathbb{R}^d)} \to 0, \quad \varepsilon \to 0.$$

The proof of the above result is based on the following technique. Subtracting $\frac{1}{\varepsilon}\frac{|\Sigma|}{|\omega|}q(0)$ to both sides of the initial equation, and performing the change of variables $z = \varepsilon^{-1/4}x$, standard manipulations transform the original problem (1) into the following rescaled problem

$$\begin{cases} -\mathrm{div}(a^\varepsilon(z)\nabla v^\varepsilon(z)) - \dfrac{1}{\sqrt{\varepsilon}}\dfrac{|\partial Y|_{d-1}}{|Y|_d}q(0)\, v^\varepsilon = \mu^\varepsilon v^\varepsilon(x), & \text{in } \varepsilon^{-1/4}\Omega_\varepsilon, \\ a^\varepsilon(z)\nabla v^\varepsilon(z)\cdot n = -\varepsilon^{1/4} q(\varepsilon^{1/4}z)\, v^\varepsilon(z), & \text{on } \varepsilon^{-1/4}\Sigma_\varepsilon, \\ v^\varepsilon(z) = 0, & \text{on } \varepsilon^{-1/4}\partial\Omega. \end{cases}$$

where

$$v^\varepsilon(z) = u^\varepsilon\left(\frac{x}{\varepsilon^{1/4}}\right), \quad \mu^\varepsilon = \sqrt{\varepsilon}\Big(\lambda^\varepsilon - \frac{1}{\varepsilon}\frac{|\partial Y|_{d-1}}{|Y|_d}q(0)\Big).$$

The first step in the proof of Theorem 1 is to show an priori estimates for the eigenvalues μ_j^ε

$$\overline{c} \le \mu_1^\varepsilon \le \underline{C}.$$

Then, the proof of the convergence of eigenvalues and eigenfunctions to those of the limit problem in $\mathbb{R}^d$ follows, using various variational and compactness arguments, and scaled trace and Poincaré-type inequalities.

3. Other problems with localization effects

The localization phenomenon in spectral problems should be well-know to physicists, since a long time, and it has been observed in several mathematical works.

In the context of singular perturbation problems, paper [14] deals with the limit behaviour of the first eigenvalue of a singularly perturbed non self-adjoint elliptic operator, with smooth coefficients, defined on a compact Riemannian manifold. Self-adjoint operators on a bounded subset of $\mathbb{R}^d$ are treated as a special case. Here, in particular, the first normalized eigenfunction localises around the mimimum point of the given potential.

In the field of homogenization problems, [2] deals with an operator with a large locally periodic potential has been considered. The localization appears due to the presence of a large factor in the potential and the fact that the operator coefficients depend on slow variable.

In a different context, in [9] the Dirichlet spectral problem for the Laplacian in a thin 2d strip of slowly varying thickness is studied. Here the localization is observed in the vicinity of the point of maximum thickness. The large parameter is the first eigenvalue of 1d Laplacian in the cross-section.

Both in [2] and [9], under natural non-degeneracy conditions, the asymptotics of the eigenpairs are described in terms of the spectrum of an appropriate harmonic oscillator operator. However, the localization scale is of order $\sqrt{\varepsilon}$ with ε being the microscopic length scale.

Localization effect for the negative part of the spectrum are also found in [12] where a spectral problem for locally periodic elliptic operators with sign-changing density function is considered.

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M. Codegone

COMPENSATED COMPACTNESS IN HOMOGENIZATION THEORY

Abstract. Homogenization theory studies the macroscopic behavior of a non-homogeneous medium with a fine periodic structure. If the size of the period goes to zero the solution of the problem, with some suitable hypotheses, converges to the solution of a so-called homogenized problem. The limit problem is achieved through a limit process of a product of weakly convergent function sequences. In order to identify the limit product one can sometimes make use of appropriate test functions, but in general the right tool is compensated compactness. This tool makes it possible to pass to the limit in a product of weakly convergent functions under hypotheses where the curl of one factor is bounded and the divergence of the other is compact. In this way homogenization theory becomes an interesting example where compensated compactness has an important role.

1. Introduction

In the paper [7] A. Negro presents a time periodic boundary value problem for quasi-stationary Maxwell equations in a non-homogeneous multiply connected domain Ω (see figure 1). In this formulation some links are linear, but the characteristic relating the magnetic field H to the magnetic induction B is non-linear. The non-homogeneity of the medium has a natural approximation with a periodic fine structure of period εY and discloses homogenization theory as a worthwhile tool. In the papers [4] and [3] conductivity $\sigma(y)$ is linear, strictly positive and bounded in the simply connected region $Y_1 \subseteq Y$ and is zero in $Y_2 \subseteq Y$, where $Y = Y_1 \bigcup Y_2$ is the reference period, see, in figure 2, the two-dimensional section of the reference period $\mathbf{Y}$ and of the periodical reproduction of $\varepsilon\,\boldsymbol{Y}$. The non-linear magnetic characteristic $\chi^\varepsilon(x,H) = \chi(\frac{x}{\varepsilon},H)$ is supposed be measurable, Lipschitz continuous and strictly monotone in H.

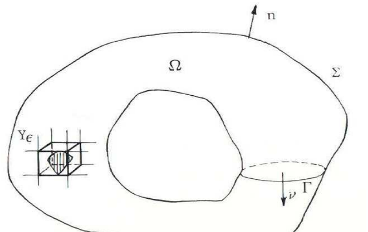

Figure 1: The domain Ω

The problem formulation then presents a linear part related to conductivity σ^ε and a non-linear part related to the magnetic characteristic χ^ε. The parameter εY is the

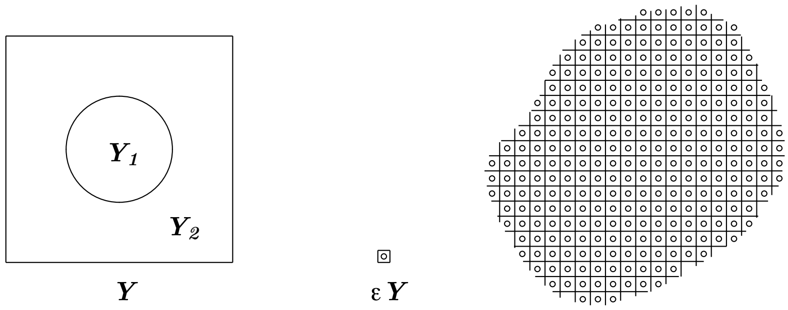

Figure 2: Two-dimensional section of the reference period **Y** and of the periodical reproduction of $\varepsilon\, \boldsymbol{Y}$.

period of the non-homogeneity of the medium and then the limit, when ε goes to zero, is taken in the linear part σ^{ε} and in the non-linear part χ^{ε}. In order to emphasize the use of compensated compactness, we simplify the problem taking into account only the non linear characteristic in a more simple contest.

2. Variational formulation

We denote by

$$\chi : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$$

a function such that, for every $H \in \mathbb{R}^3, \chi(\cdot, H)$ is Lebesgue measurable and εY-periodic and there exist two constants l and L with $0 < l \leq L < +\infty$ such that

$$(\chi(x,H) - \chi(x,H'), H - H') \geq l|H - H'|^2$$
$$|\chi(x,H) - \chi(x,H')| \leq L|H - H'|$$

for a.e. $x \in \mathbb{R}^n$ and for every $H, H' \in \mathbb{R}^n$ and $\chi(x,0) = 0$ for a.e. $x \in \mathbb{R}^n$.

Let us consider the following Dirichlet boundary value problem on the bounded open subset Ω of $\mathbb{R}^n$

$$(1) \qquad \begin{cases} -\operatorname{div}\left(\chi(\frac{x}{\varepsilon}, \nabla u^{\varepsilon})\right) = f_{\varepsilon} \quad \text{on} \quad \Omega \\ u^{\varepsilon} \in H_0^1(\Omega) \end{cases}$$

with f_{ε} strongly converging in $H^{-1}(\Omega)$ to f. We can see that

$$(2) \qquad \begin{cases} \forall \varphi \in C_0^{\infty}(\Omega) \qquad \text{e} \qquad \varphi \geq 0 \\ \displaystyle\int_{\Omega} \left((\chi(x/\varepsilon, H) - \chi(x/\varepsilon, H')) \cdot (H - H')\right) \varphi(x)\, \mathrm{d}x \geq 0 \end{cases}$$

Now we recall some results in homogenization theory.

3. Homogenization results

In order to obtain the limit process (see [8], [9] and [1]) of equation (1) we have to solve a problem in the reference period Y. Find a function $w^\lambda(y)$ such that

$$\text{(3)} \qquad \begin{cases} -\mathrm{div}\left(\chi(y,\lambda+\nabla w^\lambda(y))\right) = 0 \quad \text{in} \quad D'(\mathbb{R}^n) \\ w^\lambda(y) \in H^1(Y), Y-\text{periodic and with mean value zero.} \end{cases}$$

In homogenization theory it is proved that the problem (3) admits a unique solution. We note

$$\text{(4)} \qquad w_\varepsilon^\lambda(x) = \lambda \cdot x + \varepsilon w^\lambda\left(\frac{x}{\varepsilon}\right)$$

then, replacing H and H' by ∇u^ε and $\nabla w_\varepsilon^\lambda$, in equation (2), we get

$$\text{(5)} \qquad \int_\Omega \Big((\chi(x/\varepsilon,\nabla u^\varepsilon) - \chi(x/\varepsilon,\nabla w_\varepsilon^\lambda)) \cdot (\nabla u^\varepsilon - \nabla w_\varepsilon^\lambda)\Big)\,\varphi(x)\,\mathrm{d}x \geq 0.$$

We remark that any term in the equation 5 weakly converges:

$$\text{(6)} \qquad \begin{cases} \chi(x/\varepsilon,\nabla u^\varepsilon) & \rightharpoonup & \chi^0 \\ \chi(x/\varepsilon,\nabla w_\varepsilon^\lambda) & \rightharpoonup & \frac{1}{|Y|}\int_Y \chi(y,\lambda+\nabla w^\lambda(y))\,\mathrm{d}y \\ \nabla u^\varepsilon & \rightharpoonup & \nabla u^0 \\ \nabla w_\varepsilon^\lambda & \rightharpoonup & \lambda \end{cases}$$

and then any factor weakly converges:

$$\text{(7)} \qquad \begin{cases} \left(\chi(x/\varepsilon,\nabla u^\varepsilon) - \chi(x/\varepsilon,\nabla w_\varepsilon^\lambda)\right) & \rightharpoonup & \left(\chi^0 - \frac{1}{|Y|}\int_Y \chi(y,\lambda+\nabla w^\lambda(y))\,\mathrm{d}y\right) \\ \left(\nabla u^\varepsilon - \nabla w_\varepsilon^\lambda\right) & \rightharpoonup & \left(\nabla u^0 - \lambda\right). \end{cases}$$

In order to pass to the limit in (5) we need the result of compensated compactness.

4. Compensated compactness

The product of the two brackets in (5) is only weakly convergent and it is known that the product of two weakly convergent sequences does not converge, in general, to the product of the limits, and this is the main difficulty to characterize χ^0 in terms of u^0. Compensated compactness (see [6], [10]) shows that under some additional assumptions, the product of the two weak convergent sequences converges in the sense of distribution to the product of the limits. We present a general lemma and a particular version of the result which interests the problem of homogenization.

LEMMA 1. *Let V^ε and U^ε be two sequences converging weakly in $L^2(\Omega)$ to V^0 and U^0 respectively. Moreover suppose that:*

$$\begin{cases} \mathrm{div}\, V^\varepsilon & \text{converges strongly in } H^{-1}(\Omega) \\ \mathrm{curl}\, U^\varepsilon & \text{is bounded in } L^2(\Omega) \end{cases}$$

then one has

$$\int_{\Omega}(V^{\varepsilon}\,U^{\varepsilon})\,\varphi\,\mathrm{d}x \quad\longrightarrow\quad \int_{\Omega}\left(V^{0}\,U^{0}\right)\,\varphi\,\mathrm{d}x$$

for every $\varphi\in C_0^{\infty}(\Omega)$.

It is possible to see a proof of the Lemma 1 in [5] and [2].

PROPOSITION 1. *In the hypothesis of equations (1) and (3) we obtain that expression (5) converges:*

$$\begin{aligned}(8)\qquad &\int_{\Omega}\Big((\chi(x/\varepsilon,\nabla u^{\varepsilon})-\chi(x/\varepsilon,\nabla w_{\varepsilon}^{\lambda}))\cdot(\nabla u^{\varepsilon}-\nabla w_{\varepsilon}^{\lambda})\Big)\,\varphi(x)\,\mathrm{d}x\longrightarrow\\ &\longrightarrow\int_{\Omega}\left(\left(\chi^{0}-\frac{1}{|Y|}\int_{Y}\chi(y,\lambda+\nabla w^{\lambda}(y))\,\mathrm{d}y\right)\cdot\left(\nabla u^{0}-\lambda\right)\right)\varphi(x)\,\mathrm{d}x.\end{aligned}$$

Proof. Taking into account equations (1) and (3), we have

$$\begin{cases}-\mathrm{div}(\chi(x/\varepsilon,\nabla u^{\varepsilon})-\chi(x/\varepsilon,\nabla w_{\varepsilon}^{\lambda}))=f_{\varepsilon}\\ \mathrm{curl}(\nabla u^{\varepsilon}-\nabla w_{\varepsilon}^{\lambda})=0\end{cases}$$

and, in order to apply the Lemma 1, we put

$$\begin{cases}V^{\varepsilon}=(\chi(x/\varepsilon,\nabla u^{\varepsilon})-\chi(x/\varepsilon,\nabla w_{\varepsilon}^{\lambda}))\\ U^{\varepsilon}=(\nabla u^{\varepsilon}-\nabla w_{\varepsilon}^{\lambda})\end{cases}$$

and the result is achieved. We can add another proof of this Proposition. We observe that the limit (8) has a direct development by integration by parts

$$\begin{aligned}(9)\qquad &\int_{\Omega}\Big((\chi(x/\varepsilon,\nabla u^{\varepsilon})-\chi(x/\varepsilon,\nabla w_{\varepsilon}^{\lambda}))\cdot(\nabla u^{\varepsilon}-\nabla w_{\varepsilon}^{\lambda})\Big)\,\varphi(x)\,\mathrm{d}x=\\ &={}_{H^{-1}}\left\langle-\mathrm{div}(\chi(x/\varepsilon,\nabla u^{\varepsilon})-\chi(x/\varepsilon,w_{\varepsilon}^{\lambda})),(u^{\varepsilon}-w_{\varepsilon}^{\lambda})\varphi\right\rangle_{H_0^1}+\\ &-\int_{\Omega}\Big((\chi(x/\varepsilon,\nabla u^{\varepsilon})-\chi(x/\varepsilon,\nabla w_{\varepsilon}^{\lambda}))\cdot\nabla\varphi(x)\Big)\;(u^{\varepsilon}-w_{\varepsilon}^{\lambda})\,\mathrm{d}x\end{aligned}$$

for every $\varphi\in C_0^{\infty}$. At the left-hand side of equation (9) there is product of two sequences which converge only in the weak topology, but at the right-hand side the convergence of $u^{\varepsilon}-w_{\varepsilon}^{\lambda}$ is weakly in H^1 and strongly in L^2:

$$(10)\qquad u^{\varepsilon}-w_{\varepsilon}^{\lambda}\quad\longrightarrow\quad u^{0}-(\lambda\cdot x).$$

The strong convergence (10) permits to pass to the limit and the statement is achieved. □

COROLLARY 1. *Let* u^{ε}, u^{0}, $w_{\varepsilon}^{\lambda}$, w^{λ}, χ *and* χ^{0} *be defined as in section 2, 3 and let* $B(\lambda)$ *be defined by*

$$B(\lambda)=\frac{1}{|Y|}\int_{Y}\chi(y,\lambda+\nabla w^{\lambda}(y))$$

for every $\lambda \in \mathbb{R}^3$. *Then*

$$\chi^0 = B(\nabla u^0) \tag{11}$$

and

$$\begin{cases} -\text{div}\left(B(\nabla u^0)\right) = f \quad \text{on} \quad \Omega \\ u^0 \in H_0^1(\Omega). \end{cases} \tag{12}$$

Proof. From (5) and (8) we get

$$\int_\Omega \left(\left(\chi^0 - B(\lambda)\right) \cdot \left(\nabla u^0 - \lambda\right)\right) \varphi(x)\, \mathrm{d}x \geq 0. \tag{13}$$

By taking into account that B is maximal monotone the last inequality (13) ensures identity (11) and then, taking the limit, in the problem (1) we get the homogenized problem (12). □

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CONTINUOUS DEPENDENCE FOR A NONSTANDARD CAHN-HILLIARD SYSTEM WITH NONLINEAR ATOM MOBILITY

Abstract. This note is concerned with a nonlinear diffusion problem of phase-field type, consisting of a parabolic system of two partial differential equations, complemented by Neumann homogeneous boundary conditions and initial conditions. The system arises from a model of two-species phase segregation on an atomic lattice [22]; it consists of the balance equations of microforces and microenergy; the two unknowns are the order parameter ρ and the chemical potential μ. Some recent results obtained for this class of problems are reviewed and, in the case of a nonconstant and nonlinear atom mobility, uniqueness and continuous dependence on the initial data are shown with the help of a new line of argumentation developed in [13].

1. About the model and the mathematical problem

This paper deals with a phase field system that is addressed and investigated in a rather general framework. A special situation has been studied in [9, 12] from the viewpoint of well-posedness and long time behavior. The two papers [10] and [14] are devoted to the optimal control problems for distributed and boundary controls, respectively. The recent contributions [13] and [11] are related to what we are going to discuss and review in this note. As to modeling issues, two directly relevant antecedents have been the papers by Fried & Gurtin [17] and Gurtin [19], while [22], the paper that inspired our research cooperation, led us to begin by studying a system of Allen-Cahn type for phase segregation processes without diffusion [7, 8].

1.1. The nonstandard phase-field system in a simplified form

The initial and boundary value problem introduced in [9] consists in looking for two *fields*, the *chemical potential* μ and the *order parameter* ρ, that solve

(1) $$\varepsilon\,\partial_t\mu + 2\rho\,\partial_t\mu + \mu\,\partial_t\rho - \Delta\mu = 0 \qquad \text{in } \Omega\times(0,T),$$

(2) $$\delta\,\partial_t\rho - \Delta\rho + f'(\rho) = \mu \qquad \text{in } \Omega\times(0,T),$$

(3) $$\partial_n\mu = \partial_n\rho = 0 \qquad \text{on } \Gamma\times(0,T),$$

(4) $$\mu(\cdot\,,0) = \mu_0 \quad \text{and} \quad \rho(\cdot\,,0) = \rho_0 \qquad \text{in } \Omega,$$

where Ω denotes a bounded domain in $\mathbb{R}^3$ with sufficiently smooth boundary Γ and $T > 0$; ε and δ stand for two positive parameters. Moreover, the nonlinearity f is a double-well potential defined in $(0,1)$, whose derivative f' is singular at the endpoints $\rho = 0$ and $\rho = 1$: a relevant example is

(5) $$f(\rho) = \alpha\,\{\rho\,\ln(\rho) + (1-\rho)\,\ln(1-\rho)\} + \beta\,\rho\,(1-\rho),$$

with some positive constants α and β; according to whether or not $\alpha \geq \beta/2$, it turns out that f is convex in the whole of $[0,1]$ or exhibits two wells with a local maximum at $\rho = 1/2$.

The nonstandard phase field model (1)–(4) can be regarded as a variant of the classic Cahn-Hilliard system for diffusion-driven phase segregation by atom rearrangement:

$$\partial_t \rho - \kappa \Delta \mu = 0 \,, \qquad \mu = -\Delta \rho + f'(\rho). \tag{6}$$

As to differences between (1) and $(6)_1$, we point out that the former equation, in which the *mobility* coefficient $\kappa > 0$ has been taken equal to 1, contains a group of terms involving time derivatives, with two nasty nonlinearities. Moreover, (2) differs from $(6)_2$ due to the presence of the viscous contribution $\delta\, \partial_t \rho$; as we shall see, positivity of the coefficient δ is crucial to our analysis. Equations (1)–(2) have the structure of a *phase field system* [5, 21], in which the chemical potential μ takes the place of the more usual temperature variable. Note that, in general, those equations cannot be combined into one higher-order equation, as is instead customarily done with the equations in (6) so as to obtain the well-known *Cahn-Hilliard equation*

$$\partial_t \rho = \kappa \Delta(-\Delta \rho + f'(\rho)). \tag{7}$$

1.2. Generalization of Cahn-Hilliard equation according to Fried and Gurtin

In [17, 19] a broad generalization of (7) was devised, along three directions:

(i) to regard the second of (6) as a *balance of microforces:*

$$\operatorname{div} \boldsymbol{\xi} + \pi + \gamma = 0, \tag{8}$$

where the distance microforce per unit volume is split into an internal part π and an external part γ, and the contact microforce per unit area of a surface oriented by its normal $\boldsymbol{n}$ is measured by $\boldsymbol{\xi} \cdot \boldsymbol{n}$ in terms of the *microstress* vector $\boldsymbol{\xi}$;*

(ii) to regard the first equation in (6) as a *balance law for the order parameter*:

$$\partial_t \rho = -\operatorname{div} \boldsymbol{h} + \sigma, \tag{9}$$

where the pair $(\boldsymbol{h}, \sigma)$ is the *inflow* of ρ;

(iii) to demand that the constitutive choices for $\pi, \boldsymbol{\xi}, \boldsymbol{h}$, and the *free energy density* ψ, be consistent in the sense of Coleman and Noll [6] with an *ad hoc* version of the Second Law of Continuum Thermodynamics:

$$\partial_t \psi + (\pi - \mu)\partial_t \rho - \boldsymbol{\xi} \cdot \nabla(\partial_t \rho) + \boldsymbol{h} \cdot \nabla \mu \leq 0, \tag{10}$$

that is, a postulated "dissipation inequality that accommodates diffusion" (cf. equation (3.6) in [19]).

*In [16] the balance of microforces is stated in the form of a principle of virtual powers for microscopic motions.

In [19], the following list of constitutive prescriptions was shown to be consistent with (iii):

$$\psi = \widehat{\psi}(\rho, \nabla\rho), \;\; \widehat{\pi}(\rho, \nabla\rho, \mu) = \mu - \partial_\rho \widehat{\psi}(\rho, \nabla\rho), \;\; \widehat{\boldsymbol{\xi}}(\rho, \nabla\rho) = \partial_{\nabla\rho} \widehat{\psi}(\rho, \nabla\rho). \tag{11}$$

Within this framework, let also

$$\boldsymbol{h} = -\boldsymbol{M}\nabla\mu, \quad \text{with } \boldsymbol{M} = \widehat{\boldsymbol{M}}(\rho, \nabla\rho, \mu, \nabla\mu), \tag{12}$$

where the tensor-valued *mobility mapping* $\widehat{\boldsymbol{M}}$ satisfies the *residual dissipation inequality*

$$\nabla\mu \cdot \widehat{\boldsymbol{M}}(\rho, \nabla\rho, \mu, \nabla\mu)\nabla\mu \geq 0.$$

With the help of (8), (9), (11), and on taking into account the first of (12), one recovers a general equation for diffusive phase segregation processes:

$$\partial_t \rho = \operatorname{div}\left(\boldsymbol{M}\nabla\left(\partial_\rho \widehat{\psi}(\rho, \nabla\rho) - \operatorname{div}\left(\partial_{\nabla\rho}\widehat{\psi}(\rho, \nabla\rho)\right) - \gamma\right)\right) + \sigma.$$

Then, the Cahn-Hilliard equation (7) is obtained by taking

$$\widehat{\psi}(\rho, \nabla\rho) = f(\rho) + \frac{1}{2}|\nabla\rho|^2, \qquad \boldsymbol{M} = \kappa\boldsymbol{1}, \tag{13}$$

and letting the external distance microforce γ and the order-parameter source term σ be identically null.

1.3. An alternative generalization of Cahn-Hilliard equation

In [22], a modification of the Fried-Gurtin approach to phase-segregation modeling was proposed. While the crucial step (i) was retained, both the order parameter balance (9) and the dissipation inequality (10) were dropped and replaced, respectively, by the *microenergy balance*

$$\partial_t \varepsilon = e + w, \quad e := -\operatorname{div}\overline{\boldsymbol{h}} + \overline{\sigma}, \quad w := -\pi\,\partial_t\rho + \boldsymbol{\xi}\cdot\nabla(\partial_t\rho), \tag{14}$$

and the *microentropy imbalance*

$$\partial_t \eta \geq -\operatorname{div}\boldsymbol{h} + \sigma, \quad \boldsymbol{h} := \mu\overline{\boldsymbol{h}}, \quad \sigma := \mu\overline{\sigma}. \tag{15}$$

As a new feature in this approach, the *microentropy inflow* $(\boldsymbol{h}, \sigma)$ was deemed proportional to the *microenergy inflow* $(\overline{\boldsymbol{h}}, \overline{\sigma})$ through the *chemical potential* μ, a positive field; consistently, the free energy was defined to be

$$\psi := \varepsilon - \mu^{-1}\eta, \tag{16}$$

with the chemical potential playing the same role as *coldness* in the deduction of the heat equation.†

† As much as absolute temperature is a macroscopic measure of microscopic *agitation*, its inverse - the coldness - measures microscopic *quiet*; likewise, as argued in [22], the chemical potential can be seen as a macroscopic measure of microscopic *organization*.

Combining (14)-(16) yields

$$\partial_t \psi \leq -\eta \,\partial_t(\mu^{-1}) + \mu^{-1}\,\overline{\boldsymbol{h}} \cdot \nabla\mu - \pi\,\partial_t \rho + \boldsymbol{\xi} \cdot \nabla(\partial_t \rho), \tag{17}$$

an inequality that replaces (10) in restricting *à la* Coleman-Noll the possible constitutive choices. On taking all of the constitutive mappings delivering $\pi, \boldsymbol{\xi}, \eta$, and $\overline{\boldsymbol{h}}$, dependent in principle on $\rho, \nabla\rho, \mu, \nabla\mu$, and on choosing

$$\psi = \widehat{\psi}(\rho, \nabla\rho, \mu) = -\mu\,\rho + f(\rho) + \frac{1}{2}|\nabla\rho|^2, \tag{18}$$

compatibility with (17) implies that we must have:

$$\left\{ \begin{array}{c} \widehat{\pi}(\rho, \nabla\rho, \mu) = -\partial_\rho \widehat{\psi}(\rho, \nabla\rho, \mu) = \mu - f'(\rho), \\ \widehat{\boldsymbol{\xi}}(\rho, \nabla\rho, \mu) = \partial_{\nabla\rho} \widehat{\psi}(\rho, \nabla\rho, \mu) = \nabla\rho, \\ \widehat{\eta}(\rho, \nabla\rho, \mu) = \mu^2 \partial_\mu \widehat{\psi}(\rho, \nabla\rho, \mu) = -\mu^2 \rho \end{array} \right\} \tag{19}$$

together with

$$\widehat{\overline{\boldsymbol{h}}}(\rho, \nabla\rho, \mu, \nabla\mu) = \widehat{\boldsymbol{H}}(\rho, \nabla\rho, \mu, \nabla\mu)\nabla\mu, \quad \nabla\mu \cdot \widehat{\boldsymbol{H}}(\rho, \nabla\rho, \mu, \nabla\mu)\nabla\mu \geq 0.$$

We now choose for $\widehat{\boldsymbol{H}}$ the simplest expression $\boldsymbol{H} = \kappa \boldsymbol{1}$, implying a constant and isotropic mobility, and once again we assume that the external distance microforce γ and the source $\overline{\sigma}$ are null. Then, with the use of (19) and (16), the microforce balance (8) and the energy balance (14) become, respectively,

$$\Delta\rho + \mu - f'(\rho) = 0 \tag{20}$$

and

$$2\rho\,\partial_t \mu + \mu\,\partial_t \rho - \operatorname{div}(\kappa \nabla\mu) = 0, \tag{21}$$

a nonlinear system for the unknowns ρ and μ.

1.4. Insertion of the two parameters ε and δ

Compare now the systems (20)–(21) and (6): needless to say, (20) is the same ‘static’ relation between μ and ρ as $(6)_2$. However, (21) is rather different from $(6)_1$, for more than one reason:

(R1) (21) is a nonlinear equation, while $\partial_t \rho - \kappa\Delta\mu = 0$ is linear;

(R2) the time derivatives of both ρ and μ are present in (21);

(R3) in front of both $\partial_t \mu$ and $\partial_t \rho$ there are *nonconstant factors* that *should remain nonnegative* during the evolution.

Thus, the system (20)–(21) deserves a careful analysis. We must confess that at the beginning we boldly attacked this problem as it was, prompted to optimism by the previous successful outcome of the joint cooperation for the papers [7, 8], where we had tackled the system of Allen-Cahn type derived via the approach in [22] for no-diffusion phase segregation processes. However, the evolution problem ruled by (20)–(21) turned out to be too difficult for us. Therefore, we decided to study its regularized version (1)–(4) (note that κ has been taken equal to unity in (1)): in fact, this initial-boundary value problem is arrived at by introducing the extra terms $\varepsilon\,\partial_t\mu$ in (21) and $\delta\,\partial_t\rho$ in (20), and by supplementing the so-obtained equations (1) and (2) with homogeneous Neumann conditions (3) at the body's boundary (where ∂_n denotes the outward normal derivative), and with the initial conditions (4).

Of course, the positive coefficients ε and δ are intended to be small. The introduction of the $\varepsilon-$term is motivated by the desire to have a strictly positive coefficient as a factor of $\partial_t\mu$ in (21), in order to guarantee the parabolic structure of equation (1). As to the $\delta-$term, we can say that it transforms (20) into an Allen-Cahn equation with source μ; in fact, it is a sort of regularization already employed in various procedures involving the so-called *viscous Cahn-Hilliard equation* (examples can be found in [2, 3, 18, 20, 23] and references therein).

On the one hand, the presence of the term $\delta\,\partial_t\rho$ with a positive δ is very important for our analysis; on the other hand, nonuniqueness may occur if $\delta = 0$. For instance, take $\rho_0 = 1/2, \mu_0$ constant, and look for a space-independent solution (which is in agreement with homogeneous Neumann boundary conditions (3)). Then, we have that

$$\frac{d}{dt}\Big((\varepsilon+2\rho)^{1/2}\mu\Big) = 0 \quad \text{and} \quad f'(\rho) = \mu.$$

Hence, the solution has the form

$$\mu = z_0\,(\varepsilon+2\rho)^{-1/2} \quad \text{and} \quad f'(\rho) = z_0\,(\varepsilon+2\rho)^{-1/2},$$

for some given constant z_0. Now, choose the potential f such that

$$f'(r) = z_0\,(\varepsilon+2r)^{-1/2} \quad \text{for } r \in [1/3, 2/3],$$

and pick any smooth/irregular $\rho : [0,T] \to [1/3, 2/3]$ with $\rho(0) = 1/2$. We then get infinitely many smooth/irregular solutions! This of course means that uniqueness is out of question; and that, moreover, there is no control on the regularity of solutions in time.

We point out that such a modified system, with positive ε and δ, turns out to be a phase field model with a nonstandard equation (1) for the chemical potential μ, while quite often phase field systems use temperature (in place of chemical potential) and order parameter as variables.

Concerning a physical interpretation of the regularizing perturbations we introduced, to motivate the presence of $\delta\,\partial_t\rho$ is relatively easy. All we need to do in order to let this term appear in the microforce balance is to add $\partial_t\rho$ to the list of state variables we considered to analyze the constitutive consequences of (17). This measure brings in the typical dissipation mechanism of Allen-Cahn nondiffusional segregation processes, where dissipation depends essentially on $(\partial_t\rho)^2$, in addition to Cahn-Hilliard's $|\nabla\mu|^2-$dissipation (cf. [22]), thus opening the way to *split the distance microforce additively* into an equilibrium and a nonequilibrium part, with $\pi^{eq}=-\partial_\rho\widehat{\psi}(\rho,\nabla\rho,\mu)=\mu-f'(\rho)$ the equilibrium part, just as in $(19)_1$, and with $\pi^{neq}=-\delta\,\partial_t\rho$ the nonequilibrium part.

As far as the the introduction of $\varepsilon\,\partial_t\mu$ is concerned, we can say that (formally) the desired term can be made to appear in (1) by modifying the choice of the free energy in (18) as follows:

$$\psi=-\mu\Big(\rho+\frac{\varepsilon}{2}\Big)+f(\rho)+\frac{1}{2}|\nabla\rho|^2. \tag{22}$$

By the way, in [9] we could prove existence and uniqueness of the solution to the initial boundary value problem (1)–(4) with $\varepsilon>0$ and in [12] we discussed the asymptotic behavior of such solutions as $\varepsilon\searrow 0$ by showing a suitable convergence to a (weaker) solution of the limiting problem with $\varepsilon=0$. Thus, in some respect, we can avoid the use of the parameter ε, an issue we expand and make precise in the following subsection.

1.5. Various generalizations

In the first place, we are interested in generalizing the free energy (22). We do this in two ways.

We extend $f(\rho)$ by allowing f to be the sum of a convex and lower semicontinuous function, with proper domain $D(f_1)\subset\mathbb{R}$, and of a smooth function f_2 with no convexity properties (to allow for a double or multi-well potential f). We point out that in this case f_1 need not be differentiable in its domain and, in place of f_1', one should take the subdifferential $\beta:=\partial f_1$ in the order parameter equation. In general, $\beta:=\partial f_1$ is only a graph, not necessarily a function, and may include vertical (and horizontal) lines as in the example $\beta=\partial I_{[0,1]}$, i.e.,

$$\eta\in\partial I_{[0,1]}(u)\quad\text{if and only if}\quad \eta\begin{cases}\le 0 & \text{if } u=0\\ =0 & \text{if } 0<u<1\\ \ge 0 & \text{if } u=1\end{cases}, \tag{23}$$

which corresponds to the potential

$$f_1(u)=I_{[0,1]}(u)=\begin{cases}0 & \text{if } 0\le u\le 1\\ +\infty & \text{elsewhere}\end{cases}. \tag{24}$$

Therefore, f_1 is not required to be smooth so that its subdifferential β might be multivalued.

The other important modification we make in the free energy (22) is that of allowing in the first coupling term a general smooth function, say $h(\rho)$, as factor of $-\mu$ in (22), with the only restriction that $h(\rho)$ be bounded from below by a positive constant. Then, it could be

$$h(\rho)\ge\frac{\varepsilon}{2} \tag{25}$$

to maintain the same notation, and this lower bound should hold at least for the significant values of ρ belonging to the domain of f_1; actually, this was the case for $h(\rho) = \rho + \frac{\varepsilon}{2}$ in the interval $[0,1]$, which is the effective domain of the potential f in (5) (the same domain as in (24)). When one of us was lecturing on our results, an interesting remark by Alexander Mielke was that the behavior of

$$h(\rho) = \rho + \text{small parameter}$$

in a right neighbourhood of 0 ($h(\rho) \approx 0$) differs from that in a left neighbourhood of 1 ($h(\rho) \approx 1$). Instead, assuming only a boundedness from below for h allows many other instances like, e.g., a specular behavior around the extremal points of the domain of f. On the other hand, we stress the fact that f_1 is just supposed to be proper, convex and lower semicontinuous; hence, any form of double-well or multi-well potential, possibly defined on the whole of $\mathbb{R}$, may result from the free energy

$$\psi = \widehat{\psi}(\rho, \nabla\rho, \mu) = -\mu h(\rho) + f_1(\rho) + f_2(\rho) + \frac{1}{2}|\nabla\rho|^2. \tag{26}$$

In this respect, we also cover the case of a free energy ψ which is convex or not with respect to ρ according to whether or not the chemical potential μ is greater or less than a critical value μ_c; e.g., this is the case with f_1 given as in (24) and

$$h(\rho) = \rho(1-\rho), \quad f_2(\rho) = +\mu_c\, \rho(1-\rho).$$

There is also a third novelty in our approach. Indeed, the mobility factor κ appearing in (21) (cf. also the choice for $\widehat{\boldsymbol{H}}$ prior to (21)) is no longer assumed to be constant, but rather to be a nonnegative, continuous and bounded, nonlinear function of μ. In particular, to prove existence of solutions we may let $\kappa(\mu)$ degenerate at $\mu = 0$: indeed, in our model the chemical potential μ is required to take nonnegative values, so that 0 remains critical for μ. The details of such an existence proof are developed in [13], a paper to which we refer frequently in the present note. Let us also mention that in the recent paper [11] an existence theory is presented for a variation of the problem (27)–(30) below, where the conductivity κ in (27) may depend on both variables μ and ρ.

1.6. Aim of this contribution

In this paper, we recall the existence result of [13] and sketch the basic steps of the proof; moreover, in the case when the function $\mu \mapsto \kappa(\mu)$ is Lipschitz continuous and bounded from below by a positive constant, we prove uniqueness and continuous dependence on initial data. This result is new and follows the line of argumentation devised in [13] for the case of κ constant.

We set (cf. (25))

$$g(u) := h(u) - \frac{\varepsilon}{2} \geq 0 \quad \text{for all } u \in D(f_1),$$

and take $\varepsilon = \delta = 1$ for the sake of simplicity. The problem we deal with is:

$$\big(1+2g(\rho)\big)\,\partial_t\mu + \mu\, g'(\rho)\,\partial_t\rho - \operatorname{div}\big(\kappa(\mu)\nabla\mu\big) = 0 \qquad \text{in } \Omega\times(0,T), \tag{27}$$

$$\partial_t\rho - \Delta\rho + \xi + f_2'(\rho) = \mu g'(\rho)\,, \text{ with } \xi\in\beta(\rho), \qquad \text{in } \Omega\times(0,T), \tag{28}$$

$$(\kappa(\mu)\nabla\mu)\cdot \boldsymbol{n}|_\Gamma = \partial_{\boldsymbol{n}}\rho|_\Gamma = 0 \qquad \text{on } \Gamma\times(0,T), \tag{29}$$

$$\mu(\,\cdot\,,0) = \mu_0 \quad \text{and} \quad \rho(\,\cdot\,,0) = \rho_0 \qquad \text{in } \Omega. \tag{30}$$

Clearly, how to select ξ in $\beta(\rho)$ is part of the problem. For (27)–(30) we can prove a well-posedness result. In particular, we think that our continuous dependence proof is a nice piece of work, since it can handle the presence of a multivalued graph β (with vertical segments as, e.g., in (23)) and only exploits the monotonicity property of β. This was not the case for the uniqueness technique used in [9], since there the difference of two equations (1) was tested by the time derivative of the difference of the two ρ components, a procedure that strongly conflicts with nonsmooth potentials.

The longtime behavior of the system (1)–(4) and the structure of the ω-limit set have been analyzed in [9] and in [12]; the latter paper also deals with the $\varepsilon = 0$ problem, as already mentioned. The two papers [10] and [14] are concerned with the study of two optimal control problems for systems similar to (1)–(4); precisely, in [10] a distributed control problem is investigated, while [14] focuses on a boundary control problem.

In this paper, we concentrate on existence and uniqueness. In the next section, we state our assumptions and our results. The existence of a solution to problem (27)–(30) is proved in the Section 3. In Section 4, we show some regularity properties of the solutions. The last section is devoted to proving continuous dependence of the solution on the initial data.

2. Main results

Let Ω be a bounded connected open set in $\mathbb{R}^3$ with smooth boundary Γ (lower-dimensional cases can be treated with minor changes). We introduce a final time $T\in(0,+\infty)$ and set $Q := \Omega\times(0,T)$. Moreover, we set

$$V := H^1(\Omega), \quad H := L^2(\Omega), \quad W := \{v\in H^2(\Omega):\ \partial_{\boldsymbol{n}} v = 0 \text{ on } \Gamma\}, \tag{31}$$

and endow these spaces with their standard norms, for which we use a self-explanatory notation like $\|\cdot\|_V$. For $p\in[1,+\infty]$, we write $\|\cdot\|_p$ both for the usual norm in $L^p(\Omega)$ and for the norm in $L^p(Q)$, since no confusion can arise. Moreover, any of the above symbols for norms is used even for any power of these spaces. We remark that the embeddings $W\subset V\subset H$ are compact, since Ω is bounded and smooth. As V is dense in H, we can identify H with a subspace of V^* in the usual way.

We now introduce the structural assumptions on our system. Firstly, since the chemical potential is expected to be at least nonnegative, we assume that the function

κ is defined just for nonnegative arguments; moreover, we require that

(32) $\kappa : [0,+\infty) \to \mathbb{R}$ is locally Lipschitz continuous,

(33) $\kappa_*, \kappa^* \in (0,+\infty)$ and $\mu_* \in [0,+\infty)$,

(34) $\kappa(r) \le \kappa^*$ for every $r \ge 0$ and $\kappa(r) \ge \kappa_*$ for every $r \ge \mu_*$,

(35) $K(r) := \int_0^r \kappa(s)\,ds$ for $r \ge 0$; K is strictly increasing.

As to the other data, we assume that $f = f_1 + f_2$ and that

(36) $f_1 : \mathbb{R} \to [0,+\infty]$ is convex, proper, lower semicontinuous,

(37) $f_2 : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to [0,+\infty)$ are C^2 functions,

(38) f_2', g, and g' are Lipschitz continuous,

(39) $\beta := \partial f_1$ and $\pi := f_2'$,

(40) $\mu_0 \in V$, $\rho_0 \in W$, $\mu_0 \ge 0$ and $\rho_0 \in D(\beta)$ a.e. in Ω,

(41) there exists some $\xi_0 \in H$ such that $\xi_0 \in \beta(\rho_0)$ a.e. in Ω,

where $D(f_1)$ and $D(\beta)$ $(\subseteq D(f_1))$ denote the effective domains of f_1 and β, respectively. It is known that any proper, convex and lower semicontinuous function is bounded from below by an affine function (see, e.g., [1, Prop. 2.1, p. 51]). Hence, assuming $f_1 \ge 0$ looks reasonable, because one can suitably modify the smooth perturbation f_2 by adding a straight line to it. Another positivity condition, $g \ge 0$, is needed on the set $D(\beta)$, while g can take negative values outside of $D(\beta)$. Finally, since f_1 obeys (36) and f_2 is smooth, assumptions (40)–(41) imply that $f(\rho_0) \in L^1(\Omega)$.

Let us discuss the a priori regularity we ask for any solution (μ,ρ,ξ) to our problem. As (28) reduces for any given μ to a rather standard phase-field equation, it is natural to look for pairs (ρ,ξ) that satisfy

(42) $\rho \in W^{1,\infty}(0,T;H) \cap H^1(0,T;V) \cap L^\infty(0,T;W)$,

(43) $\xi \in L^\infty(0,T;H)$,

and solve the strong form of the relative subproblem, namely,

(44) $\partial_t\rho - \Delta\rho + \xi + \pi(\rho) = \mu g'(\rho)$ and $\xi \in \beta(\rho)$ a.e. in Q,

(45) $\rho(0) = \rho_0$ a.e. in Ω.

We note that (42) also incorporates the Neumann boundary condition for ρ (see (31) for the definition of W).

The situation is different for the component μ: in case of uniform parabolicity, i.e., if $\mu_* = 0$, the coefficient $\kappa(\mu)$ is bounded away from zero, and we require that

(46) $\mu \in H^1(0,T;H) \cap L^\infty(0,T;V)$, $\mu \ge 0$ a.e. in Q,

(47) $\operatorname{div}\bigl(\kappa(\mu)\nabla\mu\bigr) \in L^2(0,T;H)$,

so that μ satisfies

$$\int_\Omega \big(1+2g(\rho(t))\big)\,\partial_t\mu(t)\,v + \int_\Omega \mu(t)\,g'(\rho(t))\,\partial_t\rho(t)\,v + \int_\Omega \kappa(\mu(t))\nabla\mu(t)\cdot\nabla v = 0 \qquad \text{for every } v\in V \text{ and for a.a. } t\in(0,T), \tag{48}$$

$$\mu(0)=\mu_0 \quad \text{a.e. in } \Omega. \tag{49}$$

Thus, equation (27) holds in a strong sense:

$$\big(1+2g(\rho)\big)\,\partial_t\mu + \mu\,g'(\rho)\,\partial_t\rho - \operatorname{div}\big(\kappa(\mu)\nabla\mu\big) = 0 \quad \text{a.e. in } Q, \tag{50}$$

whereas the related Neumann boundary condition in (29) continues to be understood in the usual weak sense. Furthermore, we observe that (46)–(48) imply further regularity for μ whenever κ is smoother, thanks to the regularity theory of quasilinear elliptic equations.

Such a formulation is too strong when μ_* is allowed to be positive, because sufficient information cannot be obtained on the gradient $\nabla\mu$ and the time derivative $\partial_t\mu$. In this case, we rewrite equation (50) as

$$\partial_t\big(1+2g(\rho)\mu\big) - \mu g'(\mu)\partial_t\rho - \Delta K(\mu) = 0, \tag{51}$$

and require lower regularity:

$$\mu\in L^\infty(0,T;H), \quad \mu\ge 0 \quad \text{a.e. in } Q, \quad K(\mu)\in H^1(0,T;H)\cap L^\infty(0,T;V), \tag{52}$$

$$(1+2g(\rho))\mu \in H^1(0,T;V^*). \tag{53}$$

On accounting for the initial and Neumann boundary conditions, we replace (48)–(49) by

$$\langle\partial_t\big((1+2g(\rho))\mu\big)(t),v\rangle - \int_\Omega \big(\mu g'(\rho)\partial_t\rho\big)(t)\,v + \int_\Omega \nabla K(\mu(t))\cdot\nabla v = 0 \qquad \text{for every } v\in V \text{ and for a.a. } t\in(0,T), \tag{54}$$

$$\big((1+2g(\rho))\mu\big)(0) = \big(1+2g(\rho_0)\big)\,\mu_0. \tag{55}$$

Note that the middle term of (54) is meaningful: let us explain why. First, we have that $g'(\rho)\in C^0(\overline{Q})$, because the continuity of ρ,

$$\rho\in C^0([0,T];C^0(\overline{\Omega})) = C^0(\overline{Q}), \tag{56}$$

follows directly from (42) and the compact embedding $W\subset C^0(\overline{\Omega})$ (see, e.g., [24, Sect. 8, Cor. 4]). Next, (52) and the embedding $V\subset L^4(\Omega)$ imply that

$$K(\mu)\in L^\infty(0,T;L^4(\Omega));$$

consequently, $\mu\in L^\infty(0,T;L^4(\Omega))$, as $K(r)$ behaves like r for large $|r|$ (see (34)). Finally, (42) ensures that $\partial_t\rho\in L^\infty(0,T;H)$, whence $\mu g'(\rho)\partial_t\rho\in L^\infty(0,T;L^{4/3}(\Omega))$, and

v is in $L^4(\Omega)$ whenever $v \in V$. We remark that in this framework (51) is only satisfied in a distributional sense.

Here are our results. The first establishes the existence of a weak solution in the general case and the equivalence of strong and weak formulations in the case $\mu_* = 0$; the proof will be outlined in Section 3.

THEOREM 1. *Assume* (32)–(39) *and* (40)–(41). *Then, there exists at least one triplet* (μ, ρ, ξ) *such that*

$$(\mu, \rho, \xi) \text{ satisfies (42)–(43), (52)–(53) and solves problem (44)–(45), (54)–(55).} \tag{57}$$

Moreover, if $\mu_* = 0$ *then any triplet*(μ, ρ, ξ) *as in* (57) *fulfills also* (46)–(49).

Notice that, due to (56), no further assumption is needed to ensure the boundedness of ρ. As to the first component, we have the following boundedness result.

THEOREM 2. *Assume* (32)–(39), (40)–(41)*, and let*

$$\mu_0 \in L^\infty(\Omega). \tag{58}$$

Then, the component μ *of any triplet* (μ, ρ, ξ) *complying with* (57) *is essentially bounded.*

The next result holds if we assume that $\mu_* = 0$.

THEOREM 3. *Assume* (32)–(39), (40)–(41)*,* $\mu_* = 0$*, and*

$$K(\mu_0) \in W. \tag{59}$$

Then,

$$K(\mu) \in W^{1,p}(0,T;H) \cap L^p(0,T;W) \quad \text{for every } p \in [1, +\infty), \tag{60}$$

where μ *is the first component of any triplet* (μ, ρ, ξ) *being as in* (57).

Observe that (59) implies (58), due to the three-dimensional embedding $W \subset L^\infty(\Omega)$ and the strict monotonicity of K^{-1} (see (35)). Uniqueness is a consequence of the following continuous dependence result.

THEOREM 4. *Assume* (32)–(39) *and* $\mu_* = 0$. *Let* $(\mu_{0,i}, \rho_{0,i})$, $i = 1, 2$, *be two sets of initial data satisfying* (40)–(41) *and* (59), *and let* (μ_i, ρ_i, ξ_i), $i = 1, 2$, *be two corresponding triplets fulfilling* (57) *(with the obvious modifications of initial conditions). Then, there exists a constant C, depending on the data through the structural assumptions, such that*

$$\begin{aligned} &\|\mu_1 - \mu_2\|_{L^\infty(0,T;H) \cap L^2(0,T;V)} + \|\rho_1 - \rho_2\|_{L^\infty(0,T;H) \cap L^2(0,T;V)} \\ &\qquad \le C \left\{ \|\mu_{0,1} - \mu_{0,2}\|_H + \|\rho_{0,1} - \rho_{0,2}\|_H \right\}. \end{aligned} \tag{61}$$

Henceforth, we make repeated use of the notation

$$Q_t := \Omega \times (0,t) \quad \text{for } t \in [0,T]. \tag{62}$$

Moreover, we account for the well-known embedding $V \subset L^p(\Omega)$ for $1 \le p \le 6$ and the related Sobolev inequality:

$$\|v\|_p \le C_\Omega \|v\|_V \quad \text{for every } v \in V \text{ and } 1 \le p \le 6, \tag{63}$$

where C depends on Ω only; Hölder inequality and the elementary Young inequality

$$ab \le \varepsilon a^2 + \frac{1}{4\varepsilon} b^2 \quad \text{for every } a,b \ge 0 \text{ and } \varepsilon > 0 \tag{64}$$

are also frequently employed. Finally, throughout the paper we use a small-case italic c for different constants that may only depend on Ω, the final time T, the shape of the nonlinearities f and g, and the properties of the data involved in the statements; the symbol c_ε denotes a constant that depends also on the parameter ε. The meaning of c and c_ε might change from line to line and even in the same chain of inequalities, whereas those constants that we need to refer to are always denoted by capital letters, just like C_Ω in (63).

3. Existence

In this section we sketch the proof of Theorem 1, referring to [13] for details.

Approximation. The approximating problem is based on a time delay in the right-hand side of equation (44). A translation operator $\mathcal{T}_\tau : L^1(0,T;H) \to L^1(0,T;H)$ is considered, depending on a time step $\tau > 0$: for $v \in L^1(0,T;H)$ and for a.a. $t \in (0,T)$, we set:

$$(\mathcal{T}_\tau v)(t) := v(t-\tau) \quad \text{if } t > \tau \quad \text{and} \quad (\mathcal{T}_\tau v)(t) := \mu_0 \quad \text{if } t < \tau; \tag{65}$$

and we replace μ by $\mathcal{T}_\tau \mu$ in (44). At the same time, we modify the equation for μ. Precisely, we force uniform parabolicity and allow the solution to take negative values. Accordingly, we define $\kappa_\tau : \mathbb{R} \to \mathbb{R}$ and the related function K_τ to be

$$\kappa_\tau(r) := \kappa(|r|) + \tau \quad \text{and} \quad K_\tau(r) := \int_0^r \kappa_\tau(s)\, ds \quad \text{for } r \in \mathbb{R}. \tag{66}$$

Then, the approximating problem involves the following equations:

$$\big(1 + 2g(\rho_\tau)\big)\, \partial_t \mu_\tau + \mu_\tau\, g'(\rho_\tau)\, \partial_t \rho_\tau - \operatorname{div}\big(\kappa_\tau(\mu_\tau) \nabla \mu_\tau\big) = 0 \qquad \text{a.e. in } Q, \tag{67}$$

$$\partial_t \rho_\tau - \Delta \rho_\tau + \xi_\tau + \pi(\rho_\tau) = (\mathcal{T}_\tau \mu_\tau)\, g'(\rho_\tau) \quad \text{and} \quad \xi_\tau \in \beta(\rho_\tau) \qquad \text{a.e. in } Q, \tag{68}$$

supplemented by homogeneous Neumann boundary conditions for both μ_τ and ρ_τ, and by the initial conditions $\mu_\tau(0) = \mu_0$ and $\rho_\tau(0) = \rho_0$. It can be easily shown (cf. [13, Lemma 3.1]) that such an initial and boundary value problem has a unique solution $(\mu_\tau, \rho_\tau, \xi_\tau)$, which satisfies the analogues of (42)–(43) and (46)–(47).

Our aim is now to let τ tend to zero in order to obtain a limit triplet (μ, ρ, ξ) complying with (57). Our proof uses compactness arguments and relies on a number of uniform-in-τ a priori estimates. In performing the estimates, τ can be taken as small as desired; it will be convenient to assume $\tau \le \kappa^*$. In order to make the formulas more readable, we omit the index τ in the calculations, and write μ_τ and ρ_τ only when each estimate is established.

First a priori estimate. We test (67) by μ and observe that

$$\big[\big(1+2g(\rho)\big)\,\partial_t\mu + \mu\, g'(\rho)\,\partial_t\rho\big]\mu = \frac{1}{2}\,\partial_t\big[(1+2g(\rho))\mu^2\big].$$

Thus, by integrating over $(0,t)$, where $t \in [0,T]$ is arbitrary, we obtain:

$$\int_\Omega \big(1+2g(\rho(t))\big)|\mu(t)|^2 + 2\int_{Q_t} \kappa_\tau(\mu)|\nabla\mu|^2 = \int_\Omega (1+2g(\rho_0))\mu_0^2\,.$$

Hence, on recalling that $g \ge 0$ and that, in view of (34), $\kappa_\tau^2(r) \le 2\kappa^*\kappa_\tau(r)$ for every $r \in \mathbb{R}$, we are led to

$$\|\mu_\tau\|_{L^\infty(0,T;H)} + \|K_\tau(\mu_\tau)\|_{L^2(0,T;V)} \le c. \tag{69}$$

An analogous test by $-\mu^- = \min\{\mu, 0\}$, and the nonnegativity of μ_0, allow us to deduce that $\mu^- = 0$, whence

$$\mu_\tau \ge 0 \quad \text{a.e. in } Q.$$

Moreover, as K has a linear growth and thanks to (65) and (40), it follows from (69) that

$$\|K_\tau(\mu_\tau)\|_{L^\infty(0,T;H)} + \|\mathcal{T}_\tau\mu_\tau\|_{L^\infty(0,T;H)} + \|\mathcal{T}_\tau K_\tau(\mu_\tau)\|_{L^\infty(0,T;H)\cap L^2(0,T;V)} \le c. \tag{70}$$

The Sobolev inequality (63) and estimate (69) entail $\|K_\tau(\mu_\tau)\|_{L^2(0,T;L^6(\Omega))} \le c$; consequently,

$$\|\mu_\tau\|_{L^2(0,T;L^6(\Omega))} \le c, \tag{71}$$

for (34) implies that $K_\tau(r) \ge \kappa_* r - c$ for every $r \ge 0$.

Second a priori estimate. Add ρ to both sides of (68) and test by $\partial_t\rho$, so as to obtain that

$$\begin{aligned}
&\int_{Q_t} |\partial_t\rho|^2 + \frac{1}{2}\,\|\rho(t)\|_V^2 + \int_\Omega f_1(\rho(t)) \\
&= \frac{1}{2}\|\rho_0\|_V^2 + \int_\Omega f(\rho_0) + \frac{1}{2}\int_\Omega \big(\rho^2(t) - 2f_2(\rho(t))\big) + \int_{Q_t} g'(\rho)(\mathcal{T}_\tau\mu)\partial_t\rho \\
&\le c + c\int_\Omega |\rho(t)|^2 + \frac{1}{4}\int_{Q_t} |\partial_t\rho|^2 + c\|\mathcal{T}_\tau\mu\|^2_{L^\infty(0,T;H)},
\end{aligned}$$

for every $t \in [0,T]$. In view of the chain rule and Young's inequality (64), we have that

$$c\int_\Omega |\rho(t)|^2 \le c\int_\Omega |\rho_0|^2 + \frac{1}{4}\int_{Q_t} |\partial_t\rho|^2 + c\int_0^t \|\rho(s)\|_H^2\,ds.$$

Hence, as f_1 is nonnegative, from (70) and the Gronwall lemma we infer that

$$\|\rho_\tau\|_{H^1(0,T;H)\cap L^\infty(0,T;V)} + \|f_1(\rho_\tau)\|_{L^\infty(0,T;L^1(\Omega))} \le c. \tag{72}$$

Third a priori estimate. Rewrite (68) as

$$-\Delta\rho + \beta(\rho) \ni -\partial_t\rho - \pi(\rho) + (\mathcal{T}_\tau\mu)g'(\rho)$$

and note that the right-hand side is bounded in $L^2(0,T;H)$, thanks to (38)–(39) and to the previous estimates. By a standard argument, that consists in testing formally by either $-\Delta\rho$ or $\beta(\rho)$ and using the regularity theory for elliptic equations, we first recover that

$$\|\Delta\rho(s)\|_H^2 + \|\xi(s)\|_H^2 \le 2\|-\partial_t\rho(s) - \pi(\rho(s)) + ((\mathcal{T}_\tau\mu)g'(\rho))(s)\|_H^2 \tag{73}$$

for a.a. $s \in (0,T)$; finally, we conclude that

$$\|\rho_\tau\|_{L^2(0,T;W)} \le c \quad \text{and} \quad \|\xi_\tau\|_{L^2(0,T;H)} \le c. \tag{74}$$

Fourth a priori estimate. As this estimate is rather long and technical, let us just describe how it can be obtained, referring to [13, Section 4] for details. The aim is improving estimates (72) and (74). By proceeding formally, in particular, by writing $\beta(\rho)$ in place of ξ and treating β like a smooth function, one can differentiate (68) with respect to time and test the resulting equation by $\partial_t\rho$:

$$\begin{aligned}
&\frac{1}{2}\int_\Omega |\partial_t\rho(t)|^2 + \int_{Q_t} |\nabla\partial_t\rho|^2 + \int_{Q_t} \beta'(\rho)|\partial_t\rho|^2 \\
&= \frac{1}{2}\int_\Omega |(\partial_t\rho)(0)|^2 - \int_{Q_t} \big(\pi'(\rho) - g''(\rho)(\mathcal{T}_\tau\mu)\big)\,|\partial_t\rho|^2 + \int_{Q_t} g'(\rho)\partial_t(\mathcal{T}_\tau\mu)\,\partial_t\rho \\
&\le \frac{1}{2}\int_\Omega |(\partial_t\rho)(0)|^2 + c\int_{Q_t} (1+\mathcal{T}_\tau\mu)|\partial_t\rho|^2 + \int_{Q_t} g'(\rho)\partial_t(\mathcal{T}_\tau\mu)\,\partial_t\rho.
\end{aligned} \tag{75}$$

Now, the term difficult to control is the last one on the right-hand side. We compute $\partial_t\mu$ from (67), then integrate by parts and repeatedly use the Hölder, Sobolev, and Young inequalities, so as to obtain:

$$\begin{aligned}
&\int_{Q_t} g'(\rho)\partial_t(\mathcal{T}_\tau\mu)\,\partial_t\rho = \int_0^{t-\tau}\int_\Omega \partial_t\mu(s)\,g'(\rho(s+\tau))\partial_t\rho(s+\tau)\,ds \\
&= -\int_0^{t-\tau}\int_\Omega \kappa_\tau(\mu)(s)\nabla\mu(s)\cdot\nabla\frac{\partial_t g(\rho(s+\tau))}{1+2g(\rho(s))}\,ds \\
&\quad - \int_0^{t-\tau}\int_\Omega \frac{g'(\rho(s))\,g'(\rho(s+\tau))}{1+2g(\rho(s))}\,\mu(s)\partial_t\rho(s)\partial_t\rho(s+\tau)\,ds;
\end{aligned} \tag{76}$$

the last two integrals are treated separately, taking the structural assumptions into account. In the subsequent computations, one takes advantage of the compact embedding $V \subset L^4(\Omega)$ and of the regularity theory for linear elliptic equations. In particular, exploiting (73) entails that

$$\|\nabla\rho(s)\|_V^2 \le c\big(\|\rho(s)\|_V^2 + \|\Delta\rho(s)\|_H^2\big) \le c\big(\|\partial_t\rho(s)\|_H^2 + 1\big),$$

an inequality that turns out to be helpful in the control of one of the terms. At the end, we arrive at

$$\int_\Omega |\partial_t\rho(t)|^2 + \int_{Q_t} |\nabla\partial_t\rho|^2 \le c\int_0^t \phi(s)\|\partial_t\rho(s)\|_H^2\,ds + c,$$
$$\text{where}\quad \phi(s) := \|\mu(s)\|_4^2 + \|\nabla K_\tau(\mu)(s)\|_H^2 + \|\nabla(\mathcal{T}_\tau K_\tau(\mu))(s)\|_H^2\,;$$

hence, as $\phi \in L^1(0,T)$ by (69)–(71), we can apply the Gronwall lemma and conclude that

$$\|\partial_t\rho_\tau\|_{L^\infty(0,T;H)\cap L^2(0,T;V)} \le c. \tag{77}$$

As a consequence, note that $-\Delta\rho + \xi = -\partial_t\rho - \pi(\rho) + g'(\rho)\mathcal{T}_\tau\mu$ belongs to $L^\infty(0,T;H)$ due to (69) and (77). Therefore, by (73), both $-\Delta\rho$ and ξ are in $L^\infty(0,T;H)$. Thanks to elliptic regularity, we conclude that

$$\|\rho_\tau\|_{L^\infty(0,T;W)} \le c \quad\text{and}\quad \|\xi_\tau\|_{L^\infty(0,T;H)} \le c. \tag{78}$$

Fifth a priori estimate. Test (67) by $\partial_t K_\tau(\mu) = \kappa_\tau(\mu)\partial_t\mu$ and obtain

$$\begin{aligned}&\int_{Q_t}(1+2g(\rho))\kappa_\tau(\mu)|\partial_t\mu|^2 + \frac{1}{2}\int_\Omega |\nabla K_\tau(\mu(t))|^2 \\ &\quad = \frac{1}{2}\int_\Omega |\nabla K_\tau(\mu_0)|^2 - \int_{Q_t} g'(\rho)\partial_t\rho\,\mu\,\partial_t K_\tau(\mu)\end{aligned} \tag{79}$$

for every $t \in (0,T)$. The first term on the left-hand side can be estimated from below, as follows:

$$\int_{Q_t}(1+2g(\rho))\kappa_\tau(\mu)|\partial_t\mu|^2 \ge \int_{Q_t}\frac{\kappa_\tau^2(\mu)}{2\kappa^*}|\partial_t\mu|^2 = \frac{1}{2\kappa^*}\int_{Q_t}|\partial_t K_\tau(\mu)|^2. \tag{80}$$

On the right-hand side, the first term is trivial due to $(40)_1$; as to the second one, by the Young, Hölder, and Sobolev inequalities we have that

$$\begin{aligned}-\int_{Q_t} g'(\rho)\partial_t\rho\,\mu\,\partial_t K_\tau(\mu) &\le \frac{1}{4\kappa^*}\int_{Q_t}|\partial_t K_\tau(\mu)|^2 + c\int_0^t \|\mu(s)\|_4^2\|\partial_t\rho(s)\|_4^2\,ds \\ &\le \frac{1}{4\kappa^*}\int_{Q_t}|\partial_t K_\tau(\mu)|^2 + c\int_0^t \|\mu(s)\|_4^2\|\partial_t\rho(s)\|_V^2\,ds.\end{aligned} \tag{81}$$

Next, observe that (34) yields $K_\tau(r) \geq \kappa_* r - c_*$ for every $r \geq 0$, where c_* depends only on the structural assumptions. Hence, on recalling (70), we deduce that

$$\|\mu(s)\|_4^2 \leq c\big(\|K_\tau(\mu)(s)\|_4^2 + 1\big) \leq c\big(\|K_\tau(\mu)(s)\|_V^2 + 1\big) \tag{82}$$

$$\leq c\|\nabla K_\tau(\mu)(s)\|_H^2 + c\|K_\tau(\mu)(s)\|_H^2 + c \leq c\|\nabla K_\tau(\mu)(s)\|_H^2 + c$$

for a.a. $s \in (0,T)$. By combining (80)–(82) with (79), we obtain that

$$\frac{1}{4\kappa^*}\int_{Q_t} |\partial_t K_\tau(\mu)|^2 + \frac{1}{2}\int_\Omega |\nabla K_\tau(\mu(t))|^2 \leq c + c\int_0^t \|\partial_t \rho(s)\|_V^2 \big(\|\nabla K_\tau(\mu)(s)\|_H^2 + 1\big)\, ds.$$

In view of (77), we can apply the Gronwall lemma and conclude that

$$\|K_\tau(\mu_\tau)\|_{H^1(0,T;H)\cap L^\infty(0,T;V)} \leq c. \tag{83}$$

Moreover, arguing as for (71) and (70), we derive that

$$\|\mu_\tau\|_{L^\infty(0,T;L^6(\Omega))} + \|\mathcal{T}_\tau \mu_\tau\|_{L^\infty(0,T;L^6(\Omega))} \leq c. \tag{84}$$

Sixth a priori estimate. Writing (67) as $\partial_t\big((1+2g(\rho))\mu\big) = \Delta K_\tau(\mu) + g'(\rho)\mu\,\partial_t\rho$ and then testing by $v \in L^1(0,T;V)$ leads to

$$\left|\int_Q \partial_t\big((1+2g(\rho))\mu\big)\, v\right| = \left|-\int_Q \nabla K_\tau(\mu)\cdot\nabla v + \int_Q g'(\rho)\mu\,\partial_t\rho\, v\right|$$

$$\leq \|K_\tau(\mu)\|_{L^\infty(0,T;V)}\|v\|_{L^1(0,T;V)} + \|\partial_t\rho\|_{L^\infty(0,T;H)}\|\mu\|_{L^\infty(0,T;L^4(\Omega))}\|v\|_{L^1(0,T;L^4(\Omega))}$$

$$\leq \big(\|K_\tau(\mu)\|_{L^\infty(0,T;V)} + c\|\partial_t\rho\|_{L^\infty(0,T;H)}\|\mu\|_{L^\infty(0,T;L^4(\Omega))}\big)\|v\|_{L^1(0,T;V)}.$$

Hence, (77) and (83)–(84) enable us to infer that

$$\|\partial_t\big((1+2g(\rho_\tau))\mu_\tau\big)\|_{L^\infty(0,T;V^*)} \leq c. \tag{85}$$

Passage to the limit. On setting $\zeta_\tau := (1+2g(\rho_\tau))\mu_\tau$ and recalling the a priori estimates, it turns out that there exist a triplet (μ,ρ,ξ), with $\mu \geq 0$ a.e. in Q, and functions k and ζ such that

$$\mu_\tau \to \mu \quad \text{weakly star in } L^\infty(0,T;L^6(\Omega)), \tag{86}$$

$$\rho_\tau \to \rho \quad \text{weakly star in } W^{1,\infty}(0,T;H)\cap H^1(0,T;V)\cap L^\infty(0,T;W), \tag{87}$$

$$\xi_\tau \to \xi \quad \text{weakly star in } L^\infty(0,T;H), \tag{88}$$

$$K_\tau(\mu_\tau) \to k \quad \text{weakly star in } H^1(0,T;H)\cap L^\infty(0,T;V), \tag{89}$$

$$\zeta_\tau \to \zeta \quad \text{weakly star in } W^{1,\infty}(0,T;V^*)\cap L^\infty(0,T;L^6(\Omega)), \tag{90}$$

at least for a subsequence $\tau = \tau_i \searrow 0$. By (87), (89), and the compact embeddings $W \subset C^0(\overline{\Omega})$ and $V \subset H$, we can apply well-known results (see, e.g., [24, Sect. 8, Cor. 4]) and infer that

$$\rho_\tau \to \rho \quad \text{strongly in } C^0(\overline{Q}), \tag{91}$$

$$K_\tau(\mu_\tau) \to k \quad \text{strongly in } C^0([0,T];H) \text{ and a.e. in } Q. \tag{92}$$

Now, convergences (88) and (91) imply that $\xi \in \beta(\rho)$ a.e. in Q, as is well known (see, e.g., [4, Prop. 2.5, p. 27]). By (91), we also recover the Cauchy condition (45) and the fact that $\phi(\rho_\tau) \to \phi(\rho)$ strongly in $C^0(\overline{Q})$ for every continuous function $\phi : \mathbb{R} \to \mathbb{R}$; of course, this property can be applied to g, g', and π (see (38)). From (86) and (84), it is not difficult to check that $\mathcal{T}_\tau \mu_\tau \to \mu$ weakly star in $L^\infty(0,T;L^6(\Omega))$; hence, the product $\mathcal{T}_\tau \mu_\tau g'(\rho_\tau)$ has the weak star limit $\mu g'(\rho)$ in $L^\infty(0,T;L^6(\Omega))$ and (44) can follow from (68).

Next, we check that μ_τ converges to μ a.e. in Q. Note that K_τ^{-1} converges to K^{-1} uniformly on $[0,R]$ for every $R > 0$. Hence, (92) implies $\mu_\tau \to K^{-1}(k)$ a.e. in Q, and a comparison with (86) enables us to deduce that $K^{-1}(k) = \mu$ (whence $k = K(\mu)$) and

$$\text{(93)} \qquad \mu_\tau \to \mu \quad \text{strongly in } L^p(0,T;L^q(\Omega)), \text{ for every } p < +\infty \text{ and } q < 6,$$

and a.e. in Q (the Egorov theorem is used here). Then, we can also infer that ζ_τ converges to $(1+2g(\rho))\mu$ a.e. in Q, whence $\zeta = (1+2g(\rho))\mu$ by comparing with (90). On the other hand, (90) implies that $\zeta_\tau \to \zeta$ strongly in $C^0([0,T];V^*)$, thus, $\zeta_\tau(0) \to \zeta(0)$ strongly in V^*, so that the Cauchy condition (55) is verified.

It remains for us to identify the limit of $\mu_\tau g'(\rho_\tau)\partial_t \rho_\tau$: we show that it weakly converges to $\mu g'(\rho)\partial_t \rho$ in some L^p-type space. By choosing, e.g., $p = 2$, $q = 4$ in (93) and exploiting the weak star convergence of $\partial_t \rho_\tau$ in $L^\infty(0,T;H)$ (see (87)) and the uniform convergence of $g'(\rho_\tau)$, we deduce that $\mu_\tau g'(\rho_\tau)\partial_t \rho_\tau \to \mu g'(\rho)\partial_t \rho$ weakly in $L^2(0,T;L^{4/3}(\Omega))$. At this point, it is straightforward to derive (54) in an integrated form, namely,

$$\text{(94)} \qquad \int_0^T \langle \partial_t \big((1+2g(\rho))\mu\big)(t), v(t) \rangle \, dt - \int_Q \mu g'(\rho)\partial_t \rho \, v + \int_Q \nabla K(\mu) \cdot \nabla v = 0$$

for any $v \in L^2(0,T;V) \subset L^2(0,T;L^4(\Omega))$, whence the time-pointwise version (54).

End of the proof of Theorem 1. Here, we check the last part of the statement of Theorem 1. In the case $\mu_* = 0$, we have $\kappa(r) \geq \kappa_*$ for every $r \geq 0$. This implies that the inverse function $K^{-1} : [0,+\infty) \to [0,+\infty)$ is Lipschitz continuous. Hence, (52) yields

$$\mu = K^{-1}(K(\mu)) \in H^1(0,T;H) \cap L^\infty(0,T;V),$$

i.e., (46) holds. In particular, we can write

$$\nabla K(\mu) = \kappa(\mu)\nabla\mu \quad \text{and} \quad \partial_t \big((1+2g(\rho))\mu\big) = \mu \, \partial_t(1+2g(\rho)) + (1+2g(\rho))\partial_t \mu$$

and thus replace the weak formulation by the strong one. Next, we point out that (48) implies that (50) holds in the sense of distributions, whence (47) follows by comparison. Finally, (49) is a consequence of (55) and the continuity of μ from $[0,T]$ to H.

4. Regularity properties

In this section, we prove Theorems 2 and 3 and make some remarks on the regularity of solutions. To achieve the first result, we adapt the arguments used in [9, 13].

Proof of Theorem 2. Set $\mu_0^* := \max\{1, \|u_0\|_\infty\}$. We would like to test (54) by $(\mu-k)^+$, for some constant k greater than μ_0^*. We have to check that $(\mu-k)^+$ is an admissible test function, which is not obvious since $\nabla\mu$ might not exist in the usual sense.

Now, thanks to (34)–(35), K is a strictly increasing mapping from $[0,+\infty)$ onto itself and K^{-1} is Lipschitz continuous on the interval $[s_*,+\infty)$, where $s_* := K(\mu_*)$. Therefore, we can choose a strictly increasing map $K_* : [0,+\infty) \to [0,+\infty)$ that is globally Lipschitz continuous and coincides with K^{-1} on $[s_*,+\infty)$. Hence, we have $K_*(K(r)) = r$ for every $r \geq \mu_*$ and $K_*(K(r)) < \mu_*$ for $r < \mu_*$. It follows that $(r-k)^+ = (K_*(K(r))-k)^+$ for every $r \geq 0$ if $k \geq \mu_*$. On the other hand, $K_*(K(\mu)) \in H^1(0,T;H) \cap L^2(0,T;V)$ by (52). Hence, $(\mu-k)^+$ enjoys the same regularity and is an admissible test function in (54) provided that $k \geq \mu_*$. Thus, from now on we assume $k \geq \max\{\mu_0^*, \mu_*\}$. We have from (54) that

$$\int_0^t \langle \partial_t \big[(1+2g(\rho))\mu\big](s), (\mu(s)-k)^+ \rangle\, ds + \int_{Q_t} \nabla K(\mu) \cdot \nabla(\mu-k)^+ = \int_{Q_t} \mu\, \partial_t g(\rho)\, (\mu-k)^+$$

for every $t \in [0,T]$. A simple rearrangement yields:

$$\int_0^t \langle \partial_t \big[(1+2g(\rho))(\mu-k)\big](s), (\mu(s)-k)^+ \rangle\, ds + \int_{Q_t} \nabla K(\mu) \cdot \nabla(\mu-k)^+ = \int_{Q_t} \partial_t g(\rho)\, |(\mu-k)^+|^2 - k \int_{Q_t} \partial_t g(\rho)\, (\mu-k)^+. \tag{95}$$

Note that $1/(1+2g(\rho)) \in H^1(0,T;V) \cap L^\infty(0,T;W)$, in view of (42) and our assumptions on g (cf. (37)–(38)). Then, we can apply the 'chain-rule' Lemma 5.1 in [13] to deduce that

$$\begin{aligned}
&\int_0^t \langle \partial_t \big[(1+2g(\rho))(\mu-k)\big](s), (\mu(s)-k)^+ \rangle\, ds = \int_{Q_t} (\mu-k) \partial_t \big[(1+2g(\rho))(\mu-k)^+\big] \\
&= \int_{Q_t} 2\partial_t g(\rho)\, |(\mu-k)^+|^2 + \int_{Q_t} (\mu-k)(1+2g(\rho))\, \partial_t (\mu-k)^+ \\
&= \frac{1}{2} \int_{Q_t} \partial_t \big[(1+2g(\rho))|(\mu-k)^+|^2\big] + \int_{Q_t} \partial_t g(\rho)\, |(\mu-k)^+|^2.
\end{aligned}$$

On the other hand, we have that

$$\nabla(\mu-k)^+ = \nabla\mu = \nabla K^{-1}(K(\mu)) = (K^{-1})'(K(\mu)) \nabla K(\mu) = \frac{1}{\kappa(\mu)} \nabla K(\mu)$$

almost everywhere in the set where $\mu \geq k$. Furthermore, we observe that $(\mu(0)-k)^+ = 0$ a.e. in Ω on account of $k \geq \mu_0^*$. Hence, (95) yields

$$\frac{1}{2} \int_\Omega (1+2g(\rho(t)))|(\mu(t)-k)^+|^2 + \int_{Q_t} \kappa(\mu) |\nabla(\mu-k)^+|^2 = -k \int_{Q_t} \partial_t g(\rho)\, (\mu-k)^+.$$

As g is nonnegative and $\kappa(r) \geq \kappa_*$ for $r \geq k$ (because $k \geq \kappa_*$), it follows that

$$\frac{1}{2}\int_\Omega |(\mu(t)-k)^+|^2 + \kappa_* \int_{Q_t} |\nabla(\mu-k)^+|^2 \leq k \int_{Q_t} |\partial_t g(\rho)|\,(\mu-k)^+.$$

At this point, we can repeat the argument used in [9]: indeed, the analog of (44) is never used there, and the whole proof is based just on the regularity $\partial_t\rho \in L^\infty(0,T;H) \cap L^2(0,T;V)$. In the present case, we have to exploit the same regularity for $\partial_t g(\rho)$, an easy consequence of (42) and (38).

REMARK 1. The property $\mu \in L^\infty(Q)$ may lead to additional regularitity for ρ, of course under suitable assumptions on the initial data. Indeed, note that (44) yields:

$$\partial_t\rho - \Delta\rho + \xi = \mu g'(\rho) - \pi(\rho) \in L^\infty(Q).$$

So, if we let $\inf\rho_0$ and $\sup\rho_0$ belong to the interior of $D(\beta)$ (assuming that it is not empty, the significant case), one can easily derive that $\xi \in L^\infty(Q)$. Indeed, one can formally multiply by $|\xi|^{p-1}\operatorname{sign}\xi$ and estimate $\|\xi\|_p$ uniformly with respect to p, if the assumption on ρ_0 is satisfied. This implies that

$$\rho \in W^{1,p}(0,T;L^p(\Omega)) \cap L^p(0,T;W^{2,p}(\Omega)) \quad \text{for every } p < +\infty,$$

provided that ρ_0 is smooth enough. However, no further regularity can be proved, in general, since (44) cannot be differentiated.[‡]

Proof of Theorem 3. By virtue of (56), it turns out that $g(\rho)$ is continuous and $g'(\rho)$ is bounded. Moreover, thanks to Theorem 2, μ is bounded too, because $\mu_0 = K^{-1}(K(\mu_0))$ fulfills (58) by virtue of (59).

We point out that (50) can be seen as a uniformly parabolic linear equation for $w = K(\mu)$, with continuous coefficients and a right-hand side belonging to $L^\infty(0,T;H)$. Indeed, as $\partial_t\mu = (\kappa(\mu))^{-1}\partial_t(K(\mu))$, we have that

$$\partial_t w - \frac{\kappa(\mu)}{1+2g(\rho)}\Delta w = -\frac{\mu g'(\rho)\,\partial_t\rho}{1+2g(\rho)}.$$

Therefore, recalling that $w_0 := K(\mu_0) \in W$ and applying the optimal L^p-L^q-regularity results (see, e.g., [15, Thm. 2.3]), we infer (60) and Theorem 3 is proved.

Let us remark that (60) holds under an assumption on w_0 that is actually weaker than $w_0 \in W$. The optimal condition involves a proper Besov space and gives a similar result for a fixed p. We are going to exploit (60) just with $p = 4$ in the proof of our continuous dependence result. As $\mu_* = 0$ and

$$\kappa_*|\nabla\mu| \leq \kappa(\mu)|\nabla\mu| = |\nabla K(\mu)| \quad \text{a.e. in } Q,$$

‡Unless β has a special form, for instance (compute the derivative of the convex part of (5)) ,

$$\beta(\rho) = \ln\frac{\rho}{1-\rho},$$

like in [9]. By the way, in this case the condition $\xi \in L^\infty(Q)$ is equivalent to $\inf\rho > 0$ and $\sup\rho < 1$.

(60) implies that

$$|\nabla\mu| \in L^4(0,T;L^6(\Omega)); \tag{96}$$

this regularity is used for $|\nabla\mu_i|$, $i=1,2$ in the proof of Theorem 4 here below.

5. Continuous dependence

In this section, we prove Theorem 4. We point out that, under the assumptions of this theorem, both solutions (μ_1,ρ_1,ξ_1) and (μ_2,ρ_2,ξ_2) satisfy the regularity properties stated in Theorems 2 and 3. In particular, the following estimate holds true (cf. (96) and (42)):

$$\sum_{i=1}^{2}\Big\{\|\mu_i\|_{L^4(0,T;W^{1,6}(\Omega))}+\|\rho_i\|_{L^4(0,T;W^{1,6}(\Omega))}\Big\}\le c, \tag{97}$$

for some constant c depending only on the data of the problems, including the initial values $(\mu_{0,i},\rho_{0,i})$, $i=1,2$.

As a general strategy for both solutions, we rewrite equation (50) in the form

$$\partial_t\big(\mu/\alpha(\rho)\big)-\alpha(\rho)\,\mathrm{div}\big(\kappa(\mu)\nabla\mu\big)=0, \tag{98}$$

where the function $\alpha:\mathbb{R}\to(0,+\infty)$ is defined by

$$\alpha(r):=\big(1+2g(r)\big)^{-1/2}\quad\text{for } r\in\mathbb{R}. \tag{99}$$

More precisely, let us consider the variational formulation of (98) that accounts for the homogeneous Neumann boundary condition and involves a related unknown function, namely,

$$z:=\frac{\mu}{\alpha(\rho)}\,,$$

with

$$\int_\Omega \partial_t z(t)\,v+\int_\Omega \kappa\big(\alpha(\rho(t))\,z(t)\big)\nabla\big(\alpha(\rho(t))\,z(t)\big)\cdot\nabla\big(\alpha(\rho(t))v\big)=0 \tag{100}$$
$$\text{for a.a. } t\in(0,T) \text{ and for every } v\in V.$$

We point out that, for $i=1,2$, the functions $z_i:=\mu_i/\alpha(\rho_i)$ are bounded, since both μ_i and ρ_i are. Indeed, (56) holds and Theorem 2 can be applied (recall (59) and (58)). Moreover, from (97) we can easily deduce that

$$\sum_{i=1}^{2}\|z_i\|_{L^4(0,T;W^{1,6}(\Omega))}\le c \tag{101}$$

as well. For the sake of convenience, for $i=1,2$ we set

$$a_i:=\alpha(\rho_i),\quad \kappa_i:=\kappa(\mu_i)$$

and observe that (z_i, ρ_i) satisfy (100). In order to simplify formulas and make the proof more readable, let us adopt the notation:

$$\mu := \mu_1 - \mu_2, \quad \rho := \rho_1 - \rho_2, \quad \xi := \xi_1 - \xi_2, \quad z := z_1 - z_2, \quad a := a_1 - a_2,$$
$$\mu_0 := \mu_{0,1} - \mu_{0,2}, \quad \rho_0 := \rho_{0,1} - \rho_{0,2}, \quad \text{and} \quad z_0 := \frac{\mu_{0,1}}{\alpha(\rho_{0,1})} - \frac{\mu_{0,2}}{\alpha(\rho_{0,2})}.$$

Note that z_0 is the initial value of the difference $z_1 - z_2$.

It is our intention to prove the preliminary estimate

$$\int_\Omega |z(t)|^2 + \int_{Q_t} |\nabla(a_1 z)|^2 + \int_\Omega |\rho(t)|^2 + \int_{Q_t} |\nabla\rho|^2 \le c \left\{ \int_\Omega |z_0|^2 + \int_\Omega |\rho_0|^2 \right\}, \tag{102}$$

and then to explain how to derive (61) from (102). Here, c depends on the structure and on an upper bound of the norms of the initial data involved in our assumptions. Indeed, in the subsequent estimates, the (varying) value of the constant c depends on some norms of the considered solutions, e.g., through $\|z_i\|_\infty$. However, such norms can be estimated in terms of an upper bound of the quantities that appear in (40)–(41), (58) and (59).

We proceed as follows. Having written (100) for both solutions and chosen $v = z_1 - z_2$ in the difference, we integrate over $(0,t)$, for an arbitrary $t \in (0,T)$. At the same time, we consider (44) for both solutions and test the difference by $\rho_1 - \rho_2$, integrating over Q_t. Finally, we take a suitable linear combination of the resulting equalities and perform a number of estimates that lead us to apply the Gronwall lemma. However, before starting, we recall a list of inequalities that follow from the boundedness and the Lipschitz continuity of α, α', and $1/\alpha$ (cf. (37)–(38)) and from the Lipschitz continuity of κ. Indeed, in spite of (32), here we may assume κ *globally* Lipschitz continuous, as both μ_1 and μ_2 are bounded. We easily infer that

$$|a| = |\alpha(\rho_1) - \alpha(\rho_2)| \le c|\rho|,$$
$$|\nabla a| = |\alpha'(\rho_1)\nabla\rho + \big(\alpha'(\rho_1) - \alpha'(\rho_2)\big)\nabla\rho_2| \le c|\nabla\rho| + c|\nabla\rho_2|\,|\rho|,$$
$$|\nabla a_i| + |\nabla a_i^{-1}| \le c|\nabla\rho_i|,$$
$$|\kappa_1 - \kappa_2| \le c|\mu|,$$
$$|\mu| \le |a|\,|z_1| + a_2|z| \le c|a| + c|z| \le c|\rho| + c|z|,$$
$$|\nabla z| = |\nabla\big(a_1 z/a_1\big)| \le c|\nabla(a_1 z)| + c|\nabla\rho_1|\,|z|.$$

In what follows, we will repeatedly use these inequalities without alerting the reader. Let us state a lemma that we proved in [13, Section 6].

LEMMA 1. *For each $\varphi \in L^4(0,T;L^6(\Omega))$, we have that*

$$\int_{Q_t} \varphi^2(|z|^2 + |\rho|^2) \le \varepsilon \int_{Q_t} \big(|\nabla(a_1 z)|^2 + |\nabla\rho|^2\big)$$
$$+ c_\varepsilon \int_0^t \big(1 + \|\nabla\rho_1(s)\|_6^4 + \|\varphi(s)\|_6^4\big)\big(\|z(s)\|_2^2 + \|\rho(s)\|_2^2\big)\, ds, \tag{103}$$

for every $\varepsilon > 0$ *and every* $t \in [0,T]$.

Let us start our program and, in order to make the argument more transparent, let us deal just with the first equation, if only for a while. We have that

$$\frac{1}{2}\int_\Omega |z(t)|^2 + \int_{Q_t} \big(\kappa_1 \nabla(a_1 z_1) \cdot \nabla(a_1 z) - \kappa_2 \nabla(a_2 z_2) \cdot \nabla(a_2 z)\big) = \frac{1}{2}\int_\Omega |z_0|^2.$$

It is convenient to transform the last integrand on the left-hand side as follows:

$$\begin{aligned}
&\kappa_1 \nabla(a_1 z_1) \cdot \nabla(a_1 z) - \kappa_2 \nabla(a_2 z_2) \cdot \nabla(a_2 z) \\
&= \kappa_1 |\nabla(a_1 z)|^2 + \kappa_1 \nabla(a_1 z_2) \cdot \nabla(a_1 z) - \kappa_2 \nabla(a_2 z_2) \cdot \nabla(a_2 z) \\
&= \kappa_1 |\nabla(a_1 z)|^2 + \kappa_1 \nabla(a z_2) \cdot \nabla(a_1 z) + \kappa_1 \nabla\mu_2 \cdot \nabla(a_1 z) - \kappa_2 \nabla\mu_2 \cdot \nabla(a_2 z). \\
&= \kappa_1 |\nabla(a_1 z)|^2 + \kappa_1 \nabla(a z_2) \cdot \nabla(a_1 z) + \kappa_1 \nabla\mu_2 \cdot \nabla(a z) + (\kappa_1 - \kappa_2) \nabla\mu_2 \cdot \nabla(a_2 z).
\end{aligned}$$

Then, thanks to assumption (34) with $\mu_* = 0$, the above equality yields:

$$\begin{aligned}
(104) \qquad & \frac{1}{2}\int_\Omega |z(t)|^2 + \kappa_* \int_{Q_t} |\nabla(a_1 z)|^2 - \frac{1}{2}\int_\Omega |z_0|^2 \\
& \leq -\int_{Q_t} \kappa_1 \nabla(a z_2) \cdot \nabla(a_1 z) - \int_{Q_t} \kappa_1 \nabla\mu_2 \cdot \nabla(a z) \\
& \quad - \int_{Q_t} (\kappa_1 - \kappa_2) \nabla\mu_2 \cdot \nabla(a_2 z),
\end{aligned}$$

where each term on the right-hand side has to be estimated separately. First, it is straightforward to obtain

$$\begin{aligned}
(105) \qquad & -\int_{Q_t} \kappa_1 \nabla(a z_2) \cdot \nabla(a_1 z) \leq \frac{\kappa_*}{4}\int_{Q_t} |\nabla(a_1 z)|^2 + c\int_{Q_t} \big(z_2^2 |\nabla a|^2 + a^2 |\nabla z_2|^2\big) \\
& \leq \frac{\kappa_*}{4}\int_{Q_t} |\nabla(a_1 z)|^2 + c\int_{Q_t} \big(|\nabla\rho|^2 + |\nabla\rho_2|^2 |\rho|^2\big) + c\int_{Q_t} |\nabla z_2|^2 |\rho|^2 \\
& \leq \frac{\kappa_*}{4}\int_{Q_t} |\nabla(a_1 z)|^2 + C_1 \int_{Q_t} |\nabla\rho|^2 + c\int_{Q_t} \big(|\nabla\rho_2|^2 + |\nabla z_2|^2\big) |\rho|^2,
\end{aligned}$$

where we have denoted by C_1 the constant we want to refer to. As to the second term, we deduce that

$$\begin{aligned}
(106) \qquad & -\int_{Q_t} \kappa_1 \nabla\mu_2 \cdot \nabla(a z) \leq \kappa^* \int_{Q_t} |\nabla\mu_2| \big(|a|\, |\nabla z| + |z|\, |\nabla a|\big) \\
& \leq c\int_{Q_t} |\nabla\mu_2| \big(|\nabla(a_1 z)|\, |\rho| + |z|\, |\nabla\rho_1|\, |\rho| + |z|\, |\nabla\rho| + |z|\, |\nabla\rho_2|\, |\rho|\big) \\
& \leq \frac{\kappa_*}{4}\int_{Q_t} |\nabla(a_1 z)|^2 + c\int_{Q_t} |\nabla\mu_2|^2 |\rho|^2 + \int_{Q_t} |\nabla\rho|^2 \\
& \quad + c\int_{Q_t} |\nabla\mu_2|^2 |z|^2 + c\int_{Q_t} \big(|\nabla\rho_1|^2 + |\nabla\rho_2|^2\big) |\rho|^2.
\end{aligned}$$

For the third and last term, we argue as follows:

$$
\begin{aligned}
&-\int_{Q_t}(\kappa_1-\kappa_2)\nabla\mu_2\cdot\nabla(a_2z)\\
&\le c\int_{Q_t}|\mu|\,|\nabla\mu_2|\,|\nabla(a_2z)| \le c\int_{Q_t}\big(|\rho|+|z|\big)\,|\nabla\mu_2|\,\big(|a_2|\,|\nabla z|+|z|\,|\nabla\rho_2|\big)\\
&\le c\int_{Q_t}\big(|\rho|+|z|\big)\,|\nabla\mu_2|\,\big(|\nabla(a_1z)|+|\nabla\rho_1|\,|z|+|z|\,|\nabla\rho_2|\big)\\
&\le \frac{\kappa_*}{4}\int_{Q_t}|\nabla(a_1z)|^2+c\int_{Q_t}|\nabla\mu_2|^2\big(|\rho|^2+|z|^2\big)\\
&\quad +c\int_{Q_t}|\nabla\mu_2|\big(|\nabla\rho_1|+|\nabla\rho_2|\big)|z|\,\big(|\rho|+|z|\big)\\
&\le \frac{\kappa_*}{4}\int_{Q_t}|\nabla(a_1z)|^2+c\int_{Q_t}\big(|\nabla\mu_2|^2+|\nabla\rho_1|^2+|\nabla\rho_2|^2\big)\big(|\rho|^2+|z|^2\big).
\end{aligned}
$$

Next, we deal with the second equation. Testing the difference of (44) by ρ yields:

$$
\begin{aligned}
&\frac12\int_\Omega|\rho(t)|^2+\int_{Q_t}|\nabla\rho|^2+\int_{Q_t}\xi\rho\\
&=\int_{Q_t}\big(\mu_1g'(\rho_1)-\mu_2g'(\rho_2)-\pi(\rho_1)+\pi(\rho_2)\big)\rho+\frac12\int_\Omega|\rho_0|^2.
\end{aligned}
\tag{107}
$$

We note that the product $\xi\rho$ in the left-hand side is nonnegative by monotonicity, while the first integrand on the right-hand side can be estimated as follows:

$$
\begin{aligned}
&\big(\mu_1g'(\rho_1)-\mu_2g'(\rho_2)-\pi(\rho_1)+\pi(\rho_2)\big)\rho\\
&\le\big(|\mu|\,|g'(\rho_1)|+|\mu_2|\,|g'(\rho_1)-g'(\rho_2)|+|\pi(\rho_1)-\pi(\rho_2)|\big)|\rho|\\
&\le|g'(\rho_1)|\,|\mu|\,|\rho|+c|\mu_2|\,|\rho|^2+c|\rho|^2\le c\big(|\mu|^2+|\rho|^2\big)\le c\big(|z|^2+|\rho|^2\big).
\end{aligned}
$$

Now, on inspecting the coefficients of the integral $\int_{Q_t}|\nabla\rho|^2$ in the right-hand sides of (105) and (106), it appears convenient to multiply (107) by C_1+2 and then add it to (104). Having done this, it is straightforward to deduce that

$$
\begin{aligned}
&\int_\Omega|z(t)|^2+\int_{Q_t}|\nabla(a_1z)|^2+\int_\Omega|\rho(t)|^2+\int_{Q_t}|\nabla\rho|^2\\
&\le c\int_{Q_t}\big(|\nabla\mu_2|+|\nabla\rho_1|+|\nabla\rho_2|+|\nabla z_2|+1\big)^2\big(|z|^2+|\rho|^2\big)+c\int_\Omega|z_0|^2+c\int_\Omega|\rho_0|^2.
\end{aligned}
$$

At this point, we observe that (97) and (101) allow us to apply Lemma 1 with $\varphi=|\nabla\mu_2|+|\nabla\rho_1|+|\nabla\rho_2|+|\nabla z_2|+1$. After such an application, we choose $\varepsilon>0$ small enough and use the Gronwall lemma. Thus, we obtain (102). Now, we easily check that

$$
\begin{aligned}
&|\mu|\le c|\rho|+c|z|,\\
&|\nabla\mu|=|\nabla(a_1z+z_2a)|\le c|\nabla(a_1z)|+c|\nabla z_2|\,|\rho|+z_2|\nabla a|\\
&\qquad\le c|\nabla(a_1z)|+c\,(|\nabla z_2|+|\nabla\rho_2|)\,|\rho|+c|\nabla\rho|
\end{aligned}
$$

almost everywhere in Q and

$$|z_0| \leq c(1/a_1)|\mu_0| + c\, z_{0,2}|a| \leq c\,(|\mu_0| + |\rho_0|)$$

in Ω. By combining these inequalities with (102), we obtain the estimate

$$\int_\Omega |\mu(t)|^2 + \int_{Q_t} |\nabla\mu|^2 + \int_\Omega |\rho(t)|^2 + \int_{Q_t} |\nabla\rho|^2$$
$$\leq c \int_{Q_t} \big(|\nabla z_2| + |\nabla\rho_2|\big)^2 |\rho|^2 + c\int_\Omega |\mu_0|^2 + c\int_\Omega |\rho_0|^2.$$

Hence, we can apply once more Lemma 1 with $\varphi = |\nabla z_2| + |\nabla\rho_2|$, choose $\varepsilon > 0$ small enough, and use again (102), to obtain the following bound:

$$\varepsilon \int_{Q_t} |\nabla(a_1 z)|^2 \leq c \left\{ \int_\Omega |z_0|^2 + \int_\Omega |\rho_0|^2 \right\}.$$

Eventually, we take once more advantage of the Gronwall lemma and plainly conclude that (61) holds true.

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REMARKS ON THE STOCHASTIC TRANSPORT EQUATION WITH HÖLDER DRIFT

Abstract. We consider a stochastic linear transport equation with a globally Hölder continuous and bounded vector field. Opposite to what happens in the deterministic case where shocks may appear, we show that the unique solution starting with a C^1-initial condition remains of class C^1 in space. We also improve some results of [10] about well-posedness. Moreover, we prove a stability property for the solution with respect to the initial datum.

1. Introduction

The aim of this paper is twofold. On one side, we review ideas and recent results about the regularization by noise in ODEs and PDEs (Section 1). On the other, we give detailed proof of two new results of regularization by noise, for linear trasport equations, related to those of the paper [10] (Theorem 4 and the results of section 4).

1.1. The ODE case

A well known but still always surprising fact is the regularization produced by noise on ordinary differential equations (ODEs). Consider the ODE in $\mathbb{R}^d$

$$\frac{d}{dt}X(t) = b(t, X(t)), \quad X(0) = x_0 \in \mathbb{R}^d$$

with $b : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$. If b is Lipschitz continuous and has linear growth, uniformly in t, then there exists a unique solution $X \in C([0,T]; \mathbb{R}^d)$. But when b is less regular there are well-known counterexamples, like the case $d = 1$, $b(x) = 2sign(x)\sqrt{|x|}$, $x_0 = 0$ where the Cauchy problem has infinitely many solutions: $X(t) = 0$, $X(t) = t^2$, $X(t) = -t^2$, and others. The function b of this example is Hölder continuous.

Consider now the stochastic differential equation (SDE)

$$dX(t) = b(t, X(t))\,dt + \sigma dW(t), \quad X(0) = x_0 \in \mathbb{R}^d \tag{1}$$

with $\sigma \in \mathbb{R}$ and $\{W(t)\}_{t \geq 0}$ a d-dimensional Brownian motion on a probability space $(\Omega, \mathcal{F}, P)$. We say that a continuous stochastic process $X(t, \omega)$, $t \geq 0$, $\omega \in \Omega$, adapted to the filtration $\{\mathcal{F}_t^W\}_{t \geq 0}$ of the Brownian motion, is a solution if it satisfies the identity

$$X(t, \omega) = x_0 + \int_0^t b(s, X(s, \omega))\,ds + \sigma W(t, \omega), \quad t \geq 0,$$

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for P-a.e. $\omega \in \Omega$. In the Lipschitz case we have again existence and uniqueness of solutions. But now, we have more: if $\sigma \neq 0$ and $b \in L^\infty\left([0,T]\times\mathbb{R}^d;\mathbb{R}^d\right)$ then there is existence and uniqueness of solutions, [19]. The result is true even when $b \in L^q\left(0,T;L^p\left(\mathbb{R}^d;\mathbb{R}^d\right)\right)$ with $\frac{d}{p}+\frac{2}{q}<1$, $p,q\geq 2$ [14] (the assumptions can be properly localized). Recently, we have proved in [10] the following additional result, which will be used below (the function spaces are defined in Section 1.4).

THEOREM 1. *If* $\sigma \neq 0$ *and* $b \in L^\infty\left(0,T;C_b^\alpha\left(\mathbb{R}^d;\mathbb{R}^d\right)\right)$, $\alpha \in (0,1)$*, then there exists a stochastic flow of diffeomorphisms* $\phi_t = \phi(t,\omega)$ *associated to the SDE, with* $D\phi(t,\omega)$ *and* $D\phi^{-1}(t,\omega)$ *of class* $C^{\alpha'}$ *for every* $\alpha' \in (0,\alpha)$.

By stochastic flow of diffeomorphisms we mean a family of maps $\phi(t,\omega)$: $\mathbb{R}^d \to \mathbb{R}^d$ such that:

i) $\phi(t,\omega)(x_0)$ is the unique solution of the SDE for every $x_0 \in \mathbb{R}^d$;

ii) $\phi(t,\omega)$ is a diffeomorphisms of $\mathbb{R}^d$.

For several results on stochastic flows under more regular conditions on b see [15]. Let us give an idea of the proof assuming $\sigma = 1$. Introduce the vector valued non homogeneous backward parabolic equation

$$\begin{gathered}\frac{\partial U}{\partial t}+b\cdot\nabla U+\frac{1}{2}\Delta U=-b+\lambda U \qquad \text{on } [0,T]\\ U(T,x)=0\end{gathered}$$

with $\lambda \geq 0$. By parabolic regularity theory we have the following result (cf. Theorem 2 in [10]):

THEOREM 2. *If* $b \in L^\infty\left(0,T;C_b^\alpha\left(\mathbb{R}^d;\mathbb{R}^d\right)\right)$, $\alpha \in (0,1)$*, then there exists a unique bounded and locally Lipschitz solution U with the property*

$$\frac{\partial U}{\partial t}\in L^\infty\left(0,T;C_b^\alpha(\mathbb{R}^d;\mathbb{R}^d)\right),\quad D^2U\in L^\infty\left(0,T;C_b^\alpha\left(\mathbb{R}^d;\mathbb{R}^d\otimes\mathbb{R}^d\otimes\mathbb{R}^d\right)\right).$$

Moreover, for large λ *one has, for any* $(t,x)\in[0,T]\times\mathbb{R}^d$,

$$|\nabla U(t,x)|\leq\frac{1}{2}.$$

If $X(t)$ is a solution of the SDE, we apply Itô formula to $U(t,X(t))$ and get

$$U(t,X(t))=U(0,x_0)+\int_0^t \mathcal{L}U(s,X(s))\,ds+\int_0^t \nabla U(s,X(s))\,dW(s)$$

where $\mathcal{L}U=\frac{\partial U}{\partial t}+b\cdot\nabla U+\frac{1}{2}\Delta U$. Hence, being $\mathcal{L}U=-b+\lambda U$,

$$U(t,X(t))=U(0,x_0)+\int_0^t(-b+\lambda U)(s,X(s))\,ds+\int_0^t\nabla U(s,X(s))\,dW(s)$$

and thus

$$\int_0^t b(s,X(s))\,ds = U(0,x_0) - U(t,X(t)) + \int_0^t \lambda U(s,X(s))\,ds + \int_0^t \nabla U(s,X(s))\,dW(s).$$

In other words, we may rewrite the SDE as

$$X(t) = x_0 + U(0,x_0) - U(t,X(t)) + \int_0^t \lambda U(s,X(s))\,ds + \int_0^t \nabla U(s,X(s))\,dW(s) + W(t).$$

The advantage is that U is twice more regular than b and ∇U is once more regular. All terms in this equation are at least Lipschitz continuous.

From the new equation satisfied by $X(t)$ it is easy to prove uniqueness, for instance. But, arguing a little bit formally, it is also clear that we have differentiability of $X(t)$ with respect to the initial condition x_0. Indeed, if $D_hX(t)$ denotes the derivative in the direction h, we (formally) have

$$\begin{aligned} D_hX(t) = h &+ D_hU(0,x_0) - \nabla U(t,X(t))\,D_hX(t) \\ &+ \int_0^t \lambda \nabla U(s,X(s))\,D_hX(s)\,ds \\ &+ \int_0^t D^2U(s,X(s))\,D_hX(s)\,dW(s). \end{aligned}$$

All terms are meaningful (for instance the tensor valued coefficient $D^2U(s,X(s))$ is bounded continuous), $\nabla U(t,X(t))$ has norm less than $1/2$ (hence the term $\nabla U(t,X(t))$ $D_hX(t)$ contracts) and one can prove that this equation has a solution $D_hX(s)$. Along these lines one can build a rigorous proof of differentiability. We do not discuss the other properties.

REMARK 1. *A main open problem is the case when b is random:*$b = b(\omega,t,x)$. *In this case, strong uniqueness statements of the previous form are unknown (when b is not regular).*

1.2. The PDE case

We have seen that noise improves the theory of ODEs. Is it the same for PDEs? We have several more possibilities, several dichotomies:

linear

↗

equations:

↘

non linear

uniqueness (weak solutions)
↗
problems:
↘
blow-up (regular solutions)

additive (like for ODEs)
↗
noise:
↘
bilinear multiplicative.

Let us deal with two of the simplest but not trivial combinations: *linear* transport equations, both the problem of *uniqueness* of weak L^∞ solutions and of *no blow-up* of C^1-solutions, the improvements of the deterministic theory produced by a *bilinear multiplicative* noise.

The linear deterministic transport equation is the first order PDE in $\mathbb{R}^d$

$$\frac{\partial u}{\partial t} + b \cdot \nabla u = 0, \qquad u|_{t=0} = u_0$$

where $b : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ is given and we look for a solution $u : [0,T] \times \mathbb{R}^d \to \mathbb{R}$.

DEFINITION 1. *Assume* $b, \operatorname{div} b \in L^1_{loc} = L^1_{loc}([0,T] \times \mathbb{R}^d)$, $u_0 \in L^\infty(\mathbb{R}^d)$. *We say that u is a weak* L^∞*-solution if:*

i) $u \in L^\infty([0,T] \times \mathbb{R}^d)$

ii) for all $\vartheta \in C_0^\infty(\mathbb{R}^d)$ *one has*

$$\int_{\mathbb{R}^d} u(t,x)\,\vartheta(x)\,dx = \int_{\mathbb{R}^d} u_0(x)\,\vartheta(x)\,dx + \int_0^t \int_{\mathbb{R}^d} u(s,x) \operatorname{div}(b(s,x)\,\vartheta(x))\,dxds$$

Existence of weak L^∞-solutions is a general fact, obtained by weak-star compactness methods. When $b \in L^\infty(0,T; Lip_b(\mathbb{R}^d;\mathbb{R}^d))$, uniqueness can be proved, and also existence of smoother solutions when u_0 is smoother. Moreover, one has the transport relation

$$u(t,\phi(t,x)) = u_0(x)$$

where $\phi(t,x)$ is the deterministic flow associated to the equation of characteristics

$$\frac{d}{dt}\phi(t,x) = b(\phi(t,x)), \quad \phi(0,x) = x.$$

When b is less than Lipschitz continuous, there are counterexamples. For instance, for

$$d = 1, \quad b(x) = 2sign(x)\sqrt{|x|}$$

the PDE has infinitely many solutions from any initial condition u_0. These solutions coincide for $|x| > t^2$, where the flow is uniquely defined, but they can be prolonged almost arbitrarily for $|x| < t^2$, for instance setting

$$u(t,x) = C \text{ for } |x| < t^2$$

with arbitrary C. Remarkable is the result of [5] which states that the solution is unique when (we do not stress the generality of the behavior at infinity)

$$\nabla b \in L^1_{loc}\left([0,T]\times\mathbb{R}^d;\mathbb{R}^d\right), \tag{2}$$

$$\operatorname{div} b \in L^1\left(0,T;L^\infty\left(\mathbb{R}^d,\mathbb{R}^d\right)\right). \tag{3}$$

There are generalizations of this result (for instance [1]), but not so far from it. In these cases the flow exists and is unique but only in a proper generalized sense. The assumption (3) is the quantitative one used to prove the estimate (for simplicity we omit the cut-of needed to localize)

$$\int_{\mathbb{R}^d} u^2(t,x)\,dx = \int_{\mathbb{R}^d} u_0^2(x)\,dx + \int_0^t ds \int_{\mathbb{R}^d} u^2(s,x)\operatorname{div} b(s,x)\,dx$$

$$\leq \int_{\mathbb{R}^d} u_0^2(x)\,dx + \int_0^t \|\operatorname{div} b(s,\cdot)\|_\infty\, ds \int_{\mathbb{R}^d} u^2(s,x)\,dx$$

which implies, by Gronwall lemma, $\int_{\mathbb{R}^d} u^2(t,x)\,dx = 0$ when $u_0 = 0$ (this implies uniqueness, since the equation is linear). The assumption (2) apparently has no role but it is essential to perform these computations rigorously. One has to prove that a weak L^∞-solution u satisfies the previous identity. In order to apply differential calculus to u, one can mollify u but then a remainder, a commutator, appears in the equation. The convergence to zero of this commutator (established by the so called *commutator lemma* of [5]) requires assumption (2). We have recalled these facts since they are a main motiv below.

The problem of no blow-up of C^1 or $W^{1,p}$ solution is open for the deterministic equation, under essentially weaker conditions than Lipschitz continuity of b. The equation satisfied by first derivatives $v_k = \frac{\partial u}{\partial x_k}$ involves derivatives of b as a potential term

$$\frac{\partial v_k}{\partial t} + b\cdot\nabla v_k + \sum_i \frac{\partial b}{\partial x_i} v_i = 0, \qquad v_k|_{t=0} = \frac{\partial u_0}{\partial x_k}$$

and L^∞ bounds on $\frac{\partial b}{\partial x_i}$ seem necessary to control v_k. Again there are simple counterexamples: in the case

$$d = 1, \quad b(x) = -2sign(x)\sqrt{|x|},$$

the equation of characteristics has coalescing trajectories (the solutions from $\pm x_0$ meet at $x = 0$ at time $\sqrt{|x_0|}$) and thus, if we start with a smooth initial condition u_0 such that at some point x_0 satisfies $u_0(x_0) \neq u_0(-x_0)$, then at time $t_0 = \sqrt{|x_0|}$ the solution is discontinuous (unless u_0 is special, the discontinuity appears immediately, for $t > 0$).

Consider the following stochastic version of the linear transport equation:

$$\frac{\partial u}{\partial t} + b\cdot\nabla u + \sigma\nabla u \circ \frac{dW}{dt} = 0, \qquad u|_{t=0} = u_0.$$

The noise W is a d-dimensional Brownian motion, $\sigma \in \mathbb{R}$, the operation $\nabla u \circ \frac{dW}{dt}$ has simultaneously two features: it is a scalar product between the vectors ∇u and $\frac{dW}{dt}$, and has to be interpreted in the Stratonovich sense. The noise has a transport structure as the deterministic part of the equation. It is like to add the fast oscillating term $\sigma \frac{dW}{dt}$ to the drift b:

$$b(x) \longrightarrow b(x) + \sigma \frac{dW}{dt}(t).$$

Concerning Stratonovich calculus and its relation with Itô calculus, see [15]. We recall the so called Wong-Zakai principle (proved as a rigorous theorem in several cases): when one takes a differential equations with a smooth approximation of Brownian motion, and then takes the limit towards true Brownian motion, the correct limit equation involves Stratonovich integrals. Thus equations with Stratonovich integrals are more physically based.

DEFINITION 2. *Assume* $b, \operatorname{div} b \in L^1_{loc}$, $u_0 \in L^\infty(\mathbb{R}^d)$. *We say that a stochastic process* u *is a weak* L^∞*-solution of the SPDE if:*

i) $u \in L^\infty(\Omega \times [0,T] \times \mathbb{R}^d)$

ii) for all $\vartheta \in C_0^\infty(\mathbb{R}^d)$, $\int_{\mathbb{R}^d} u(t,x)\vartheta(x)\,dx$ *is a continuous adapted semimartingale*

iii) for all $\vartheta \in C_0^\infty(\mathbb{R}^d)$, *one has*

$$\int_{\mathbb{R}^d} u(t,x)\vartheta(x)\,dx = \int_{\mathbb{R}^d} u_0(x)\vartheta(x)\,dx + \int_0^t \int_{\mathbb{R}^d} u(s,x)\operatorname{div}(b(s,x)\vartheta(x))\,dx\,ds$$
$$+ \sigma \int_0^t \left(\int_{\mathbb{R}^d} u(s,x)\nabla\vartheta(x)\,dx \right) \circ dW(s).$$

The following theorem is due to [10].

THEOREM 3. *If* $\sigma \neq 0$ *and*

$$b \in L^\infty\left(0,T; C_b^\alpha\left(\mathbb{R}^d;\mathbb{R}^d\right)\right), \quad \operatorname{div} b \in L^p([0,T]\times\mathbb{R}^d), \tag{4}$$

for some $\alpha \in (0,1)$ *and* $p > d \wedge 2$, *then there exists a unique weak* L^∞*-solution of the SPDE. If* $\alpha \in (1/2,1)$ *then we have uniqueness only assuming* $\operatorname{div} b \in L^1_{loc}$. *Moreover, it holds*

$$u(t,\phi(t,x)) = u_0(x)$$

where $\phi(t,x)$ *is the stochastic flow of diffeomorphisms associated to the equation*

$$d\phi(t,x) = b(t,\phi(t,x))\,dt + \sigma dW(t), \quad \phi(0,x) = x$$

given by Theorem 2.

Thus we see that a suitable noise improves the theory of linear transport equation from the view-point of uniqueness of weak solutions. One of the aims of this paper is to prove a variant of this theorem, under different assumptions on b. It requires a new form of commutator lemma with respect to those proved in [5] or [10].

Let us come to the blow-up problem. The following result can be deduced from [10, Appendix A] in which we have considered BV_{loc}-solutions for the transport equation. In Section 2 we will give a direct proof of the existence part which is of independent interest.

THEOREM 4. *If* $\sigma \neq 0$,

$$b \in L^{\infty}(0,T;C_b^{\alpha}(\mathbb{R}^d;\mathbb{R}^d)),$$

for some $\alpha \in (0,1)$ *and* $u_0 \in C_b^1\left(\mathbb{R}^d\right)$*, then there exists a unique classical* C^1*-solution for the transport equation with probability one. It is given by*

$$u(t,x) = u_0\left(\phi_t^{-1}(x)\right) \tag{5}$$

where ϕ_t^{-1} *is the inverse of the stochastic flow* $\phi_t = \phi(t,\cdot)$.

The main claim of this theorem is the regularity of the solution for positive times, which is new with respect to the deterministic case. The uniqueness claim is known, as a particular case of a result in BV_{loc}, see Appendix 1 of [8].

Notice that, for solutions with such degree of regularity (BV_{loc} or C^1), no assumption on $\operatorname{div} b$ is required; $\operatorname{div} b$ does not even appear in the definition of solution (see below). On the contrary, to reach uniqueness in the much wider class of weak L^{∞}-solutions, in [8] we had to impose the additional condition (4) on $\operatorname{div} b$, for some $p > d \wedge 2$ ($\operatorname{div} b$ also appears in the definition of weak L^{∞}-solution); this happens also in the deterministic theory.

1.3. Some other works on regularization by noise

The following list does not aim to be exhaustive, see for instance [7] for other results and references:

- the uniqueness for linear transport equations can be extended to other weak assumptions on the drift, [2], [17]; also no blow-up holds for L^p drift see [6] and [18];
- similar results hold for linear continuity equations, [8], [16]:

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(b\rho) = 0, \qquad \rho|_{t=0} = \rho_0 :$$

  a noise of the form $\nabla\rho \circ \frac{dW}{dt}$ prevents mass concentration;
- analog results hold for the vector valued linear equations

$$\frac{\partial M}{\partial t} + \operatorname{curl}(b \times M) = 0$$

similar to the vorticity formulation of 3-dimensional Euler equations or magneto-hydrodynamics, where the singularities in the deterministic case are not shocks but infinite values of M; a noise of the form $\mathrm{curl}\,(e \times M) \circ \frac{dW}{dt}$ prevents blow-up [12];

- improved Strichartz estimates for a special Schrödinger model with noise

$$i\partial_t u + \Delta u \circ \frac{dW}{dt} = 0$$

have been proved, which are stronger than the corresponding ones for $i\partial_t u + \Delta u = 0$ and allow to prevent blow-up in a non-linear case when blow-up is possible without noise, see [3];

- nonlinear transport type equations of two forms have been investigated: 2D Euler equations and 1D Vlasov-Poisson equations; in these cases non-collapse of measure valued solutions concentrated in a finite number of points has been proved, [11], [4].

We conclude the introduction with some notations.

1.4. Notations

Usually we denote by $D_i f$ the derivative in the i-th coordinate direction and with $(e_i)_{i=1,\dots,d}$ the canonical basis of $\mathbb{R}^d$ so that $D_i f = e_i \cdot Df$. For partial derivatives of any order $n \geq 1$ we use the notation $D^n_{i_1,\dots,i_n}$. If $\eta : \mathbb{R}^d \to \mathbb{R}^d$ is a C^1-diffeomorphism we will denote by $J\eta(x) = \det[D\eta(x)]$ its Jacobian determinant. For a given function f depending on $t \in [0,T]$ and $x \in \mathbb{R}^d$, we will also adopt the notation $f_t(x) = f(t,x)$.

Let $T > 0$ be fixed. For $\alpha \in (0,1)$ define the space $L^\infty\left(0,T; C^\alpha_b(\mathbb{R}^d)\right)$ as the set of all bounded Borel functions $f : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ for which

$$[f]_{\alpha,T} = \sup_{t\in[0,T]} \sup_{x\neq y \in \mathbb{R}^d} \frac{|f(t,x) - f(t,y)|}{|x-y|^\alpha} < \infty$$

($|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^d$ for every d, if no confusion may arise). This is a Banach space with respect to the usual norm $\|f\|_{\alpha,T} = \|f\|_0 + [f]_{\alpha,T}$ where $\|f\|_0 = \sup_{(t,x)\in[0,T]\times\mathbb{R}^d} |f(t,x)|$. Similarly, when $\alpha = 1$ we define $L^\infty\left(0,T; Lip_b(\mathbb{R}^d)\right)$.

We write $L^\infty\left(0,T; C^\alpha_b(\mathbb{R}^d;\mathbb{R}^d)\right)$ for the space of all vector fields $f : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ having all components in $L^\infty\left(0,T; C^\alpha_b(\mathbb{R}^d)\right)$.

Moreover, for $n \geq 1$, $f \in L^\infty\left(0,T; C^{n+\alpha}_b(\mathbb{R}^d)\right)$ if all spatial partial derivatives $D^k_{i_1,\dots,i_k} f \in L^\infty\left(0,T; C^\alpha_b(\mathbb{R}^d)\right)$, for all orders $k = 0,1,\dots,n$. Define the corresponding norm as

$$\|f\|_{n+\alpha,T} = \|f\|_0 + \sum_{k=1}^n \|D^k f\|_0 + [D^n f]_{\alpha,T},$$

where we extend the previous notations $\|\cdot\|_0$ and $[\cdot]_{\alpha,T}$ to tensors. The definition of the space $L^\infty\left(0,T; C^{n+\alpha}_b(\mathbb{R}^d;\mathbb{R}^d)\right)$ is similar. The spaces $C^{n+\alpha}_b(\mathbb{R}^d)$ and $C^{n+\alpha}_b(\mathbb{R}^d;\mathbb{R}^d)$

are defined as before but only involve functions $f:\mathbb{R}^d\to\mathbb{R}^d$ which do not depend on time. Moreover, we say that $f:\mathbb{R}^d\to\mathbb{R}^d$ belongs to $C^{n,\alpha}$, $n\in\mathbb{N}$, $\alpha\in(0,1)$, if f is continuous on $\mathbb{R}^d$, n-times differentiable with all continuous derivatives and the derivatives of order n are locally α-Hölder continuous. Finally, $C_0^0(\mathbb{R}^d)$ denotes the space of all real continuous functions defined on $\mathbb{R}^d$, having compact support and by $C_0^\infty(\mathbb{R}^d)$ its subspace consisting of infinitely differentiable functions.

For any $r>0$ we denote by $B(r)$ the Euclidean ball centered in 0 of radius r and by $C_r^\infty(\mathbb{R}^d)$ the space of smooth functions with compact support in $B(r)$; moreover, $\|\cdot\|_{L_r^p}$ and $\|\cdot\|_{W_r^{1,p}}$ stand for, respectively, the L^p-norm and the $W^{1,p}$-norm on $B(r)$, $p\in[1,\infty]$. We let also $[f]_{C_r^\vartheta}=\sup_{x\neq y\in B(r)}|f(x)-f(y)|/|x-y|^\vartheta$.

We will often use the standard mollifiers. Let $\vartheta:\mathbb{R}^d\to\mathbb{R}$ be a smooth test function such that $0\le\vartheta(x)\le 1$, $x\in\mathbb{R}^d$, $\vartheta(x)=\vartheta(-x)$, $\int_{\mathbb{R}^d}\vartheta(x)dx=1$, $\operatorname{supp}(\vartheta)\subset B(2)$, $\vartheta(x)=1$ when $x\in B(1)$. For any $\varepsilon>0$, let $\vartheta_\varepsilon(x)=\varepsilon^{-d}\vartheta(x/\varepsilon)$ and for any distribution $g:\mathbb{R}^d\to\mathbb{R}^n$ we define the mollified approximation g^ε as

$$g^\varepsilon(x)=\vartheta_\varepsilon * g(x)=g(\vartheta_\varepsilon(x-\cdot)),\quad x\in\mathbb{R}^d. \tag{6}$$

If g depends also on time t, we consider $g^\varepsilon(t,x)=(\vartheta_\varepsilon * g(t,\cdot))(x)$, $t\in[0,T]$, $x\in\mathbb{R}^d$.

Recall that, for any smooth bounded domain $\mathcal{D}$ of $\mathbb{R}^d$, we have: $f\in W^{\vartheta,p}(\mathcal{D})$, $\vartheta\in(0,1)$, $p\ge 1$, if and only if $f\in L^p(\mathcal{D})$ and

$$[f]^p_{W^{\vartheta,p}}=\iint_{\mathcal{D}\times\mathcal{D}}\frac{|f(x)-f(y)|^p}{|x-y|^{\vartheta p+d}}dxdy<\infty.$$

We have $W^{1,p}(\mathcal{D})\subset W^{\vartheta,p}(\mathcal{D})$, $\vartheta\in(0,1)$.

In the sequel we will assume a stochastic basis with a d-dimensional Brownian motion $(\Omega,(\mathcal{F}_t),\mathcal{F},P,(W_t))$ to be given. We denote by $\mathcal{F}_{s,t}$ the completed σ-algebra generated by W_u-W_r, $s\le r\le u\le t$, for each $0\le s<t$.

Let us finally recall our basic assumption on the drift vector field.

HYPOTHESIS 1. There exists $\alpha\in(0,1)$ such that $b\in L^\infty\left(0,T;C_b^\alpha(\mathbb{R}^d;\mathbb{R}^d)\right)$.

2. No blow-up in C^1

This section is devoted to prove Theorem 4. Since the solution claimed by this theorem is regular, we do not need to integrate over test functions in the term $b\cdot\nabla u$ and thus we do not need to require $\operatorname{div} b\in L^1_{loc}$. For this reason, we modify the definition of solution.

DEFINITION 3. *Assume $b\in L^1_{loc}$, $u_0\in C_b^1\left(\mathbb{R}^d\right)$. We say that a stochastic process $u\in L^\infty(\Omega\times[0,T]\times\mathbb{R}^d)$ is a classical C^1-solution of the stochastic transport equation if:*

i) $u(\omega,t,\cdot)\in C^1(\mathbb{R}^d)$ *for a.e.* $(\omega,t)\in\Omega\times[0,T]$;

ii) *for all* $\vartheta\in C_0^\infty\left(\mathbb{R}^d\right)$, $\int_{\mathbb{R}^d} u(t,x)\,\vartheta(x)\,dx$ *is a continuous adapted semimartingale;*

iii) *for all* $\vartheta\in C_0^\infty\left(\mathbb{R}^d\right)$, *one has*

$$\int_{\mathbb{R}^d} u(t,x)\,\vartheta(x)\,dx = \int_{\mathbb{R}^d} u_0(x)\,\vartheta(x)\,dx - \int_0^t\int_{\mathbb{R}^d} b(s,x)\cdot\nabla u(s,x)\,\vartheta(x)\,dxds$$
$$+\sigma\int_0^t\left(\int_{\mathbb{R}^d} u(s,x)\,\nabla\vartheta(x)\,dx\right)\circ dW(s).$$

If u is a classical C^1-solution and $\operatorname{div} b\in L^1_{loc}\left([0,T]\times\mathbb{R}^d\right)$, then u is also a weak L^∞-solution. Conversely, if u is a weak L^∞-solution, $u_0\in C_b^1\left(\mathbb{R}^d\right)$ and (i) is satisfied then u is a classical C^1-solution.

Before giving the proof we mention the following useful result proved in [10, Theorem 5]:

THEOREM 5. *Assume that Hypothesis 1 holds true for some* $\alpha\in(0,1)$. *Then we have the following facts:*

(i) *(pathwise uniqueness) For every* $s\in[0,T]$, $x\in\mathbb{R}^d$, *the stochastic equation (1) has a unique continuous adapted solution* $X^{s,x}=\left(X_t^{s,x}(\omega), t\in[s,T], \omega\in\Omega\right)$.

(ii) *(differentiable flow) There exists a stochastic flow* $\phi_{s,t}$ *of diffeomorphisms for equation (1). The flow is also of class* $C^{1+\alpha'}$ *for any* $\alpha'<\alpha$.

(iii) *(stability) Let* $(b^n)\subset L^\infty\left(0,T;C_b^\alpha(\mathbb{R}^d;\mathbb{R}^d)\right)$ *be a sequence of vector fields and* ϕ^n *be the corresponding stochastic flows. If* $b^n\to b$ *in* $L^\infty(0,T;C_b^{\alpha'}(\mathbb{R}^d;\mathbb{R}^d))$ *for some* $\alpha'>0$, *then, for any* $p\geq 1$,

$$\lim_{n\to\infty}\sup_{x\in\mathbb{R}^d}\sup_{0\le s\le T} E[\sup_{r\in[s,T]}|\phi^n_{s,r}(x)-\phi_{s,r}(x)|^p]=0 \tag{7}$$

$$\sup_{n\in\mathbb{N}}\sup_{x\in\mathbb{R}^d}\sup_{0\le s\le T} E[\sup_{u\in[s,T]}\|D\phi^n_{s,u}(x)\|^p]<\infty, \tag{8}$$

$$\lim_{n\to\infty}\sup_{x\in\mathbb{R}^d}\sup_{0\le s\le T} E[\sup_{r\in[s,T]}\|D\phi^n_{s,r}(x)-D\phi_{s,r}(x)\|^p]=0. \tag{9}$$

REMARK 2. *We point out that the previous assertions* (7), (8) *and* (9) *also holds when* $\phi^n_{s,r}(x)$ *and* $\phi_{s,r}(x)$ *are replaced respectively by* $(\phi^n_{s,r})^{-1}(x)$ *and* $(\phi_{s,r})^{-1}(x)$.

To see this note that for a fixed $t>0$, $Z_s=(\phi_{s,t})^{-1}(x)$, $s\in[0,t]$, *is measurable with respect to* $\mathcal{F}_{s,t}$ *(the completed* σ*-algebra generated by* W_u-W_r, $s\le r\le u\le t$, *for each* $0\le s<t$*) and solves*

$$Z_s = x-\int_s^t b(r,Z_r)dr-\sigma[W_t-W_s]. \tag{10}$$

This is a simple backward stochastic differential equations, of the same form as the original one (only the drift has opposite sign). Note that for regular functions $f \in C_b^2(\mathbb{R}^d)$*, Itô's formula becomes*

$$f(Z_s) = f(x) - \int_s^t \nabla f(Z_r) \cdot b(r, Z_r) dr - \int_s^t \nabla f(Z_r) \cdot dW_r - \frac{\sigma^2}{2} \int_s^t \triangle f(Z_r) dr$$

where $\int_s^t \nabla f(Z_r) \cdot dW_r$ *is the so called backward Itô integral (is a limit in probability of elementary integrals like* $\sum_k \nabla f(Z_{s_k}) \cdot (W_{s_k} - W_{s_{k-1}})$ *in which we consider the partition* $s_0 = 0 < \ldots < s_N = t$*). Since this stochastic integral enjoys usual properties of the classical Itô integral, one can repeat all the arguments needed to prove* (7)*,* (8) *and* (9) *even for solutions* Z *to* (10).

Proof. (**Theorem 4**) Under the assumptions of the theorem, it has been proved in Appendix 1 of [8] that unqueness holds in BV_{loc}. Hence it holds in C^1. For this result, no assumption on $\operatorname{div} b$ is required.

We show now that (5) is a classical C^1-solution. It is easy to check (i) in Definition 3. Moreover, if $\vartheta \in C_0^\infty(\mathbb{R}^d)$, by changing variable we have:

$$\int_{\mathbb{R}^d} u(t,x)\vartheta(x)dx = \int_{\mathbb{R}^d} u_0(y)\vartheta(\phi_t(y))J\phi_t(y)dy,$$

where $J\phi_t(y) = \det[D\phi_t(y)]$, and so also property (ii) follows. To prove property (iii) consider the flow ϕ_t^ε for the regularized vector field b^ε (see (6)) and let $J\phi_t^\varepsilon(y)$ be its Jacobian determinant. Note that $u_0 \circ (\phi_t^\varepsilon)^{-1} \to u_0 \circ \phi_t^{-1}$ weakly in $L^\infty(\mathbb{R}^d)$, uniformly in $t \in [0,T]$ and P-a.s., indeed for $\vartheta \in C_0^\infty(\mathbb{R}^d)$ we have

$$\int_{\mathbb{R}^d} (u_0 \circ (\phi_t^\varepsilon)^{-1})(y)\vartheta(y)dy = \int_{\mathbb{R}^d} u_0(y)\vartheta(\phi_t^\varepsilon(y))J\phi_t^\varepsilon(y)dy$$

$$\to \int_{\mathbb{R}^d} u_0(y)\vartheta(\phi_t(y))J\phi_t(y)dy,$$

as $\varepsilon \to 0$, using the properties of the stochastic flow stated in Theorem 5. By density we can extend this convergence to any $\vartheta \in L^1(\mathbb{R}^d)$. Moreover since b^ε is smooth, it is easy to prove that

$$dJ_t^\varepsilon(y) = \operatorname{div} b_t^\varepsilon(\phi_t^\varepsilon(y))J\phi_t^\varepsilon(y)dt$$

and by the Itô formula we find

$$\begin{aligned}
&\int_{\mathbb{R}^d} u_0(y)\vartheta(\phi_t^\varepsilon(y))J_t^\varepsilon(y)dy = \\
&\qquad \int_{\mathbb{R}^d} u_0(y)\vartheta(y)dy + \int_0^t ds \int_{\mathbb{R}^d} u_0(y)L^{b^\varepsilon}\vartheta(\phi_s^\varepsilon(y))J\phi_s^\varepsilon(y)dy \\
&\qquad + \int_0^t ds \int_{\mathbb{R}^d} u_0(y)\vartheta(\phi_s^\varepsilon(y))\operatorname{div} b_s^\varepsilon(\phi_s^\varepsilon(y))J\phi_s^\varepsilon(y)dy \\
&\qquad + \sigma \int_0^t dW_s \cdot \int_{\mathbb{R}^d} u_0(y)\nabla\vartheta(\phi_s^\varepsilon(y))J\phi_s^\varepsilon(y)dy,
\end{aligned} \tag{11}$$

where

$$L^{b^\varepsilon}\vartheta(y) = \frac{1}{2}\sigma^2\Delta\vartheta(y) + b_s^\varepsilon(y)\cdot\nabla\vartheta(y).$$

Note that, integrating by parts,

$$\int_0^t ds \int_{\mathbb{R}^d} u_0(y)\vartheta(\phi_s^\varepsilon(y))\operatorname{div} b_s^\varepsilon(\phi_s^\varepsilon(y))J\phi_s^\varepsilon(y)dy$$

$$= \int_0^t ds \int_{\mathbb{R}^d} u_0((\phi_s^\varepsilon)^{-1}(x))\vartheta(x)\operatorname{div} b_s^\varepsilon(x)dx$$

$$= -\int_0^t ds \int_{\mathbb{R}^d} u_0((\phi_s^\varepsilon)^{-1}(x))\nabla\vartheta(x)\cdot b_s^\varepsilon(x)dx$$

$$-\int_0^t ds \int_{\mathbb{R}^d} \nabla u_0((\phi_s^\varepsilon)^{-1}(x))D(\phi_s^\varepsilon)^{-1}(x)\cdot b_s^\varepsilon(x)\,\vartheta(x)dx.$$

Therefore

$$\begin{aligned}\int_{\mathbb{R}^d} u_0(y)\vartheta(\phi_t^\varepsilon(y))J_t^\varepsilon(y)dy = \\ \int_{\mathbb{R}^d} u_0(y)\vartheta(y)dy + \frac{1}{2}\sigma^2\int_0^t ds \int_{\mathbb{R}^d} u_0((\phi_s^\varepsilon)^{-1}(x))\triangle\vartheta(x)dx \\ -\int_0^t ds \int_{\mathbb{R}^d} \nabla u_0((\phi_s^\varepsilon)^{-1}(x))D(\phi_s^\varepsilon)^{-1}(x)\cdot b_s^\varepsilon(x)\,\vartheta(x)dx \\ +\sigma\int_0^t dW_s\cdot\int_{\mathbb{R}^d} u_0(y)\nabla\vartheta(\phi_s^\varepsilon(y))J\phi_s^\varepsilon(y)dy.\end{aligned}$$

By changing variable $y = (\phi_s^\varepsilon)^{-1}(x)$ of the second and third integral in the right-hand side, there are no problems to pass to the limit as $\varepsilon \to 0$, $\mathbb{P}$-a.s., using (iii) in Theorem 5 and Remark 2 (precisely, one can pass to the limit along a suitable sequence $(\varepsilon_n) \subset (0,1)$ converging to 0). To this purpose we only note that for the stochastic integral we have

$$\int_0^t dW_s\cdot\int_{\mathbb{R}^d} u_0(y)\nabla\vartheta(\phi_s^\varepsilon(y))J\phi_s^\varepsilon(y)dy \to \int_0^t dW_s\cdot\int_{\mathbb{R}^d} u_0(y)\nabla\vartheta(\phi_s(y))J\phi_s(y)dy$$

uniformly on $[0,T]$ in $L^2(\Omega)$ as $\varepsilon \to 0$. Finally we get

$$\begin{aligned}\int_{\mathbb{R}^d} u_0((\phi_t)^{-1}(x))\vartheta(x)dx = \int_{\mathbb{R}^d} u_0(y)\vartheta(y)dy + \frac{\sigma^2}{2}\int_0^t ds \int_{\mathbb{R}^d} u_0((\phi_s)^{-1}(x))\triangle\vartheta(x)dx \\ -\int_0^t ds \int_{\mathbb{R}^d} \nabla u_0((\phi_s)^{-1}(x))D(\phi_s)^{-1}(x)\cdot b_s(x)\,\vartheta(x)dx \\ +\sigma\int_0^t dW_s\cdot\int_{\mathbb{R}^d} u_0((\phi_s)^{-1}(x))\nabla\vartheta(x)dx.\end{aligned}$$

By passing from Itô to Stratonovich integral this is exactly the formula we wanted to prove. The proof is complete. □

REMARK 3. *One can show that the boundedness assumption on b is not important to prove the previous Theorem 4. Indeed at least when b is independent on t, one can prove the result with b possibly unbounded, only assuming that its component b_i are "locally uniformly α-Hölder continuous", i.e.,*

$$[b_i]_{\alpha,1} := \sup_{x \neq y \in \mathbb{R}^d} \frac{|b_i(x) - b_i(y)|}{(|x-y|^\alpha \vee |x-y|)} < +\infty, \quad i = 1, \dots, d, \tag{12}$$

where $a \vee b = \max(a,b)$, for $a,b \in \mathbb{R}$. Under (12) *one can still construct a stochastic differentiable flow $\phi_t(x)$ (see Theorem 7 in [9]) which satisfies properties* (8) *and* (9) *(see also Remark 2) and this allows to perform the same proof of Theorem 4.*

3. A stability property

The following result shows a *stability property* for the solutions of the SPDE; such property involves the *weak$*$* topology (or the $\sigma(L^\infty(\mathbb{R}^d), L^1(\mathbb{R}^d))$-topology).

PROPOSITION 1. *Assume that Hypothesis 1 holds true for some $\alpha \in (0,1)$. Moreover, denote by $\phi_t = \phi_{0,t}$ the stochastic flow for equation (1). Then, for any sequence $(v^n) \subset L^\infty(\mathbb{R}^d)$, we have:*

$$v_n \to v \in L^\infty(\mathbb{R}^d) \text{ in weak}* \text{ topology} \implies v_n(\phi_t^{-1}(\cdot)) \to v(\phi_t^{-1}(\cdot))$$
$$\text{in weak}* \text{ topology,}$$

uniformly in $t \in [0,T]$, $P-a.s.$

Proof. We prove that, P-a.s., for any $f \in L^1(\mathbb{R}^d)$ we have

$$a_n = \sup_{t \in [0,T]} \Big| \int_{\mathbb{R}^d} [v_n(\phi_t^{-1}(y)) - v(\phi_t^{-1}(y))] f(y) dy \Big| \to 0, \tag{13}$$

as $n \to \infty$.

Recall that there exists a positive constant M such that $\|v_n\|_0 \le M$, $n \ge 1$, and $\|v\|_0 \le M$ and, moreover, by the separability of $L^1(\mathbb{R}^d)$ there exists a countable dense set $D \subset C_0^\infty(\mathbb{R}^d)$.

It is enough to check (13) when $f \in D$ (with the event of probability one, possibly depending on f). Indeed, if $f \in L^1(\mathbb{R}^d)$, we can consider a sequence $(f_N) \subset D$ which converges to f in $L^1(\mathbb{R}^d)$ and find, $P-$a.s.,

$$a_n \le 2M \int_{\mathbb{R}^d} |f(y) - f_N(y)| dy + \sup_{t \in [0,T]} \Big| \int_{\mathbb{R}^d} [v_n(\phi_t^{-1}(y)) - v(\phi_t^{-1}(y))] f_N(y) dy \Big|;$$

by the previous inequality the assertion follows easily.

To prove (13) for a fixed $f \in D$ we first note that, by changing variable ($J\phi_t(x)$ denotes the Jacobian determinant of ϕ_t at x)

$$\int_{\mathbb{R}^d} [v_n(\phi_t^{-1}(y)) - v(\phi_t^{-1}(y))] f(y) dy = \int_K [v(x) - v_n(x)] f(\phi_t(x)) J\phi_t(x) dx, \tag{14}$$

where we have defined the compact set $K = \pi_2(\{(t,x) \in [0,T] \times \mathbb{R}^d : \phi_t^{-1}(x) \in \mathrm{supp}(f)\})$, with $\pi_2(s,x) = x$, $s \in [0,T]$, $x \in \mathbb{R}^d$.

Using that, P-a.s., the map: $(t,x) \mapsto f(\phi_t(x)) J\phi_t(x)$ is continuous on $[0,T] \times \mathbb{R}^d$, we see from (14) that the map: $t \mapsto \int_{\mathbb{R}^d} [v_n(\phi_t^{-1}(y)) - v(\phi_t^{-1}(y))] f(y) dy$ is continuous on $[0,T]$ and so, P-a.s.,

$$a_n = \sup_{t \in [0,T] \cap \mathbb{Q}} \Big| \int_{\mathbb{R}^d} [v_n(\phi_t^{-1}(y)) - v(\phi_t^{-1}(y))] f(y) dy \Big|. \tag{15}$$

By (14) we also deduce that, P-a.s.,

$$\Big| \int_{\mathbb{R}^d} [v_n(\phi_t^{-1}(y)) - v(\phi_t^{-1}(y))] f(y) dy \Big| \to 0, \quad t \in [0,T] \cap \mathbb{Q}. \tag{16}$$

We finish the proof arguing by contradiction. We consider an event Ω_0 with $P(\Omega_0) = 1$ such that (15), (16) holds for any $\omega \in \Omega_0$ and also $(t,x) \mapsto f(\phi(t,\omega)(x)) J\phi(t,\omega)(x)$ is continuous on $[0,T] \times \mathbb{R}^d$ for any $\omega \in \Omega_0$.

If (13) does not hold for some $\omega_0 \in \Omega_0$, then there exists $\varepsilon > 0$ and $(t_n) \subset [0,T] \cap \mathbb{Q}$ such that

$$\Big| \int_{\mathbb{R}^d} [v_n(\phi_{t_n}^{-1}(y)) - v(\phi_{t_n}^{-1}(y))] f(y) dy \Big| > \varepsilon$$

(we do not indicate dependence on ω_0 to simplify notation; in the sequel we always argue at ω_0 fixed). Possibly passing to a subsequence, we may assume that $t_n \to \hat{t} \in [0,T]$.

By changing variable we have, for any $n \geq 1$,

$$\varepsilon < \Big| \int_K [v(x) - v_n(x)] f(\phi_{t_n}(x)) J\phi_{t_n}(x) dx \Big| \leq (1) + (2),$$

$$(1) = \Big| \int_K [v(x) - v_n(x)] \, [f(\phi_{t_n}(x)) J\phi_{t_n}(x) - f(\phi_{\hat{t}}(x)) J\phi_{\hat{t}}(x)] dx \Big|,$$

$$(2) = \Big| \int_K [v(x) - v_n(x)] \, f(\phi_{\hat{t}}(x)) J\phi_{\hat{t}}(x) dx \Big|.$$

Now

$$(1) \leq 2M \int_K |f(\phi_{t_n}(x)) J\phi_{t_n}(x) - f(\phi_{\hat{t}}(x)) J\phi_{\hat{t}}(x)| dx,$$

which tends to 0, as $n \to \infty$, $P-$a.s., by the dominated convergence theorem (indeed at ω_0 fixed, $(t,x) \mapsto f(\phi(t,\omega_0)(x)) J\phi(t,\omega_0)(x)$ is continuous on $[0,T] \times \mathbb{R}^d$).

Let us consider (2). By uniform continuity of $f(\phi_t(x)) J\phi_t(x)$ on $[0,T] \times K$ we may choose $q \in [0,T] \cap \mathbb{Q}$ such that

$$|f(\phi_{\hat{t}}(x)) J\phi_{\hat{t}}(x) - f(\phi_q(x)) J\phi_q(x)| < \frac{\varepsilon}{4M\lambda(K)},$$

for any $x \in K$ (here $\lambda(K)$ is the Lebesgue measure of K). Now, for any $n \geq 1$,

$$(2) \leq \Big| \int_K [v(x) - v_n(x)] \, [f(\phi_{\hat{t}}(x)) J\phi_{\hat{t}}(x) - f(\phi_q(x)) Jf(\phi_q(x))] dx \Big|$$
$$+ \Big| \int_K [v(x) - v_n(x)] \, f(\phi_q(x)) J\phi_q(x) dx \Big|$$
$$\leq \varepsilon/2 + \Big| \int_K [v(x) - v_n(x)] \, f(\phi_q(x)) J\phi_q(x) dx \Big|.$$

Since $x \mapsto f(\phi_q(x)) J\phi_q(x)$ is integrable on $\mathbb{R}^d$, we find that the last term tends tends to 0, as $n \to \infty$.

We have found a contradiction. The proof is complete. □

4. New uniqueness results

The aim of this section is to prove some new uniqueness results for L^∞ weak solutions of the SPDE obtained extending the key estimates in fractional Sobolev spaces.

Unlike Theorem 4 we will assume more conditions on b. On the other hand we will allow $u_0 \in L^\infty(\mathbb{R}^d)$ and prove stronger uniqueness results in the larger class of weak solutions. Recall that the uniqueness statement, in a class of so regular solutions, of Theorem 4 is rather obvious and does not require special effort and assumptions on the drift. On the contrary, the uniqueness claims in a class of weak solutions of Theorems 6 and 7 below are quite delicate and require suitable conditions on the drift.

The first result is the following:

THEOREM 6. *Let $d \geq 2$ and $u_0 \in L^\infty(\mathbb{R}^d)$. Assume Hypothesis 1 and also that*

$$\operatorname{div} b \in L^q(0,T; L^p(\mathbb{R}^d))$$

for some $q > 2 \geq p > \frac{2d}{d+2\alpha}$. Then there exists a unique weak L^∞-solution u of the Cauchy problem for the transport equation and $u(t,x) = u_0(\phi_t^{-1}(x))$.

The main interest of this result is due to the fact that we can consider some p in the critical interval $(1,2]$ not covered by Hypothesis 2 in [10]; recall that this requires that there exists $p \in (2,+\infty)$, such that

$$\operatorname{div} b \in L^p([0,T] \times \mathbb{R}^d), \quad d \geq 2. \tag{17}$$

The next uniqueness result requires an additional hypothesis of Sobolev regularity for b (beside the usual Hölder regularity) but allows to avoid *global* integrability assumptions on $\operatorname{div} b$.

THEOREM 7. *Assume $u_0 \in L^\infty(\mathbb{R}^d)$, $\operatorname{div} b \in L^1_{\mathrm{loc}}([0,T] \times \mathbb{R}^d)$ and*

$$b \in L^1(0,T; W^{\vartheta,1}_{\mathrm{loc}}(\mathbb{R}^d)) \cap L^\infty(0,T; C^\alpha(\mathbb{R}^d)) \tag{18}$$

with $\alpha, \vartheta \in (0,1)$ and $\alpha + \vartheta > 1$. Then there exists a unique weak L^∞-solution u of the Cauchy problem for the transport equations and $u(t,x) = u_0(\phi_t^{-1}(x))$.

REMARK 4. Recall that $b \in L^1(0,T;W^{\vartheta,1}_{\mathrm{loc}}(\mathbb{R}^d))$ if $\int_0^T \|b(s,\cdot)\|_{W^{\vartheta,1}(\mathcal{D})} ds < \infty$, for any smooth bounded domain $\mathcal{D} \subset \mathbb{R}^d$. Since $C^\alpha(\mathcal{D}) \subset W^{\vartheta,1}(\mathcal{D})$, for any $\vartheta < \alpha$, we deduce that Hypothesis 1 implies (18) when $\alpha > 1/2$; in particular Theorem 7 follows from Theorem 3 but only when $\alpha > 1/2$.

The proofs of both theorems follow ideas of [10, Section 5], using the results below on the commutator and on the regularity of the Jacobian of the flow. The following commutator estimates follows from [10, Lemma 22].

COROLLARY 1. *Assume* $v \in L^\infty_{loc}(\mathbb{R}^d,\mathbb{R}^d)$, $\operatorname{div} v \in L^1_{loc}(\mathbb{R}^d)$, $g \in L^\infty_{loc}(\mathbb{R}^d)$ *and* $\rho \in C^\infty_r(\mathbb{R}^d)$.

(i) If there exists $\vartheta \in (0,1)$ *such that* $v \in W^{\vartheta,1}_{\mathrm{loc}}(\mathbb{R}^d,\mathbb{R}^d)$, *then*

$$\left|\int_{\mathbb{R}^d} \mathcal{R}_\varepsilon [g,v](x)\rho(x)dx\right| \leq C_r \|g\|_{L^\infty_{r+1}} \left(\|\rho\|_{L^\infty_r} \|\operatorname{div} v\|_{L^1_{r+1}} + [\rho]_{C^{1-\vartheta}_r} [v]_{W^{\vartheta,1}_{r+1}}\right).$$

(ii) If there exists $\alpha \in (0,1)$ *such that* $v \in C^\alpha_{\mathrm{loc}}(\mathbb{R}^d,\mathbb{R}^d)$, *then*

$$\left|\int_{\mathbb{R}^d} \mathcal{R}_\varepsilon [g,v](x)\rho(x)dx\right| \leq C_r \|g\|_{L^\infty_{r+1}} \left(\|\rho\|_{L^\infty_r} \|\operatorname{div} v\|_{L^1_{r+1}} + [v]_{C^\alpha_{r+1}} [\rho]_{W^{1-\alpha,1}_r}\right).$$

Proof. We have

$$\left|\iint g(x')D_x\vartheta_\varepsilon(x-x')\left(\rho(x)-\rho(x')\right)[v(x)-v(x')]\,dxdx'\right|$$

$$\leq \frac{\varepsilon^{1-\vartheta}}{\varepsilon}[\rho]_{C^{1-\vartheta}_r}\|g\|_{L^\infty_{r+1}}\frac{1}{\varepsilon^d}\iint_{B(r+1)^2}|D_x\vartheta(\frac{x-x'}{\varepsilon})|\,\frac{|v(x)-v(x')|}{|x-x'|^{\vartheta+d}}|x-x'|^{\vartheta+d}dxdx'$$
$$\leq [\rho]_{C^{1-\vartheta}_r}\|g\|_{L^\infty_{r+1}}\|D\vartheta\|_\infty [v]_{W^{\vartheta,1}_{r+1}}$$

The second statement has a similar proof. □

The previous result can be extended to the case in which commutators are composed with a flow.

LEMMA 1. *Let* ϕ *be a* C^1*-diffeomorphism of* $\mathbb{R}^d$ *(*$J\phi$ *denotes its Jacobian). Assume* $v \in L^\infty_{loc}(\mathbb{R}^d,\mathbb{R}^d)$, $\operatorname{div} v \in L^1_{loc}(\mathbb{R}^d)$, $g \in L^\infty_{loc}(\mathbb{R}^d)$.

Then, for any $\rho \in C^\infty_r(\mathbb{R}^d)$ *and any* $R > 0$ *such that* $\operatorname{supp}(\rho \circ \phi^{-1}) \subseteq B(R)$, *we have a uniform bound of* $\int \mathcal{R}_\varepsilon [g,v](\phi(x))\rho(x)dx$ *under one of the following conditions:*

(i) there exists $\vartheta \in (0,1)$ *such that* $v \in W^{\vartheta,1}_{\mathrm{loc}}(\mathbb{R}^d,\mathbb{R}^d)$, $J\phi \in C^{1-\vartheta}_{\mathrm{loc}}(\mathbb{R}^d)$;

(ii) there exists $\alpha \in (0,1)$ *such that* $J\phi \in W^{1-\alpha,1}_{\mathrm{loc}}(\mathbb{R}^d)$, $v \in C^\alpha_{\mathrm{loc}}(\mathbb{R}^d,\mathbb{R}^d)$.

Moreover, under one of the previous conditions, we also have

$$\lim_{\varepsilon\to 0}\int \mathcal{R}_{\varepsilon}\left[g,v\right]\left(\phi\left(x\right)\right)\rho\left(x\right)dx=0.$$

Proof. By a change of variables $\int \mathcal{R}_\varepsilon[g,v](\phi(x))\rho(x)dx = \int \mathcal{R}_\varepsilon[g,v](y)\rho_\phi(y)dx$ where the function $\rho_\phi(y)=\rho(\phi^{-1}(y))J\phi^{-1}(y)$ has the support strictly contained in the ball of radius R. Clearly, $\|\rho_\phi\|_{L^\infty_R}\le\|\rho\|_{L^\infty_r}\|J\phi^{-1}\|_{L^\infty_R}$. To prove the result, we have to check that Corollary 1 can be applied with ρ_ϕ instead of ρ.

(i) To apply Corollary 1 (i), we need to check that $\rho_\phi\in C^{1-\vartheta}_{loc}$. This follows since

$$[\rho_\phi]_{C^{1-\vartheta}_R}\le\|J\phi^{-1}\|_{L^\infty_R}[\rho(\phi^{-1}(\cdot))]_{C^{1-\vartheta}_R}+\|\rho\|_{L^\infty_r}[J\phi^{-1}]_{C^{1-\vartheta}_R}$$

$$\le\|D\phi^{-1}\|_{L^\infty_R}\|D\rho\|_{L^\infty_r}[D\phi^{-1}]_{C^{1-\vartheta}_R}+\|\rho\|_{L^\infty_r}[D\phi^{-1}]_{C^{1-\vartheta}_R}.$$

and the bound follows.

(ii) To apply Corollary 1 (ii), we need to check that $\rho_\phi\in W^{1-\alpha,1}_{loc}$: first

$$[\rho_\phi]_{W^{1-\alpha,1}_R}\le\|J\phi^{-1}\|_{L^\infty_R}[\rho\circ\phi^{-1}]_{W^{1-\alpha,1}_R}+[J\phi^{-1}]_{W^{1-\alpha,1}_R}\|\rho\|_{L^\infty_r}$$

and since

$$[\rho\circ\phi^{-1}]_{W^{1-\alpha,1}_R}\le\|D(\rho\circ\phi^{-1})\|_{L^1_R}\le\|D\rho\|_{L^1_r}\|D\phi^{-1}\|_{L^\infty_R}$$

we find

$$[\rho_\phi]_{W^{1-\alpha,1}_R}\le C_R\|D\rho\|_{L^1_r}\|D\phi^{-1}\|_{L^\infty_R}\|J\phi^{-1}\|_{L^\infty_R}+[J\phi^{-1}]_{W^{1-\alpha,1}_R}\|\rho\|_{L^\infty_r}$$

and the bound follows. □

Finally the next theorem extends the analysis of the Jacobian of the flow presented in Section 2 and links the regularity condition on $J\phi$ required in Lemma 1 (ii) to the assumption on the divergence of b stated in Theorem 6.

THEOREM 8. *Let $d\ge 2$. Assume Hypothesis 1 and the existence of $p\in(\frac{2d}{d+2\alpha},2]$ and $q>2$ such that* $\operatorname{div} b\in L^q(0,T;L^p(\mathbb{R}^d))$. *Then, for any $r>0$, $J\phi\in L^p(0,T;W^{1-\alpha,p}_r)$, P-a.s.*

Proof. In the sequel we assume $\sigma=1$ to simplify notation.

The first part of the proof is similar to the one of [10, Theorem 11]. Indeed Step 1 can be carried on thanks to the chain rule for fractional Sobolev spaces: if $f:\mathbb{R}^d\to\mathbb{R}$ is a continuous function, of class $W^{1-\alpha,p}_{loc}(\mathbb{R}^d)$ and $g:\mathbb{R}\to\mathbb{R}$ is a C^∞ function, then $g\circ f\in W^{1-\alpha,p}_{loc}(\mathbb{R}^d)$ and

$$[(g\circ f)]^p_{W^{1-\alpha,p}_r}\le\left(\sup_{x\in B(r)}\left|g'(f(x))\right|\right)^p[f]^p_{W^{1-\alpha,p}_r},$$

for every $r > 0$. The modification of Step 2 does not pose any problem, so we only consider the last steps of the proof.

Step 3. To prove the assertion it is enough to check that the family $(\psi_\varepsilon)_{\varepsilon>0}$ is bounded in $L^p(\Omega \times (0,T); W_r^{1-\alpha,p})$.

Indeed, once we have proved this fact, we can extract from the previous sequence ψ_{ε_n} a subsequence which converges weakly in $L^p(\Omega \times (0,T); W_r^{1-\alpha,p})$ to some γ. This also implies that such subsequence converges weakly in $L^p(\Omega \times (0,T), L_r^p)$ to γ so we must have that $\gamma = J\phi$.

We introduce the following Cauchy problem, for $\varepsilon \geq 0$,

$$\begin{cases} \dfrac{\partial F^\varepsilon}{\partial t} + \dfrac{1}{2}\Delta F^\varepsilon + DF^\varepsilon \cdot b^\varepsilon = \operatorname{div} b^\varepsilon, & t \in [0,T[\\ F^\varepsilon(T,x) = 0, & x \in \mathbb{R}^d. \end{cases} \tag{19}$$

This problem has a unique solution F^ε in the space $L^q(0,T; W^{2,p}(\mathbb{R}^d)$. Moreover, there exists a positive constant $C = C(p,q,d,T,\|b\|_\infty)$ such that

$$\|F^\varepsilon\|_{L^q(0,T;W^{2,p}(\mathbb{R}^d))} \leq C\|\operatorname{div} b\|_{L^q(0,T;L^p(\mathbb{R}^d))}, \tag{20}$$

for any $\varepsilon \geq 0$. This result can be proved by using [13, Theorem 1.2] and repeating the argument of the proof in [14, Theorem 10.3]. This argument works without difficulties in the present case in which b (and so b^ε) is globally bounded and $\operatorname{div} b \in L^q(0,T; L^p(\mathbb{R}^d))$ with $p,q \in (1,+\infty)$.

From the previous result we can also deduce, since we are assuming $q > 2$, that $F^\varepsilon \in C([0,T]; W^{1,p}(\mathbb{R}^d))$, for any $\varepsilon \geq 0$, and moreover there exists a positive constant $C = C(p,q\, d,T,\|b\|_\infty)$ such that

$$\sup_{t\in[0,T]} \|F^\varepsilon(t,\cdot)\|_{W^{1,p}(\mathbb{R}^d)} \leq C\|\operatorname{div} b\|_{L^q(0,T;L^p(\mathbb{R}^d))}. \tag{21}$$

We only give a sketch of proof of (21). Define $u^\varepsilon(t,x) = F^\varepsilon(T-t,x)$; we have the explicit formula

$$u^\varepsilon(t,x) = \int_0^t P_{t-s} g^\varepsilon(s,\cdot)(x)ds,$$

where (P_t) is the heat semigroup and $g^\varepsilon(t,x) = Du^\varepsilon(t,x) \cdot b^\varepsilon(T-t,x) - \operatorname{div} b^\varepsilon(T-t,x)$. We get, since $q > 2$ and $q' = \frac{q}{q-1} < 2$,

$$\begin{aligned} \|D_x u^\varepsilon(t,\cdot)\|_{L^p} &\leq c\int_0^t \frac{1}{(t-s)^{1/2}} \|g^\varepsilon(s,\cdot)\|_{L^p} ds \\ &\leq C\Big(\int_0^T \frac{1}{s^{q'/2}} ds\Big)^{1/q'} \Big(\int_0^T \|\operatorname{div} b(s,\cdot)\|_{L^p}^q ds\Big)^{1/q} \end{aligned}$$

and so (21) holds. Using Itô formula we find (remark that $F^\varepsilon(t,\cdot) \in C_b^2(\mathbb{R}^d)$)

(22)

$$F^\varepsilon(t,\phi_t^\varepsilon(x)) - F^\varepsilon(0,x) - \int_0^t DF^\varepsilon(s,\phi_s^\varepsilon(x)) \cdot dW_s = \int_0^t \operatorname{div} b^\varepsilon(s,\phi_s^\varepsilon(x))\, ds = \psi_\varepsilon(t,x).$$

Since we already know that $(\psi_\varepsilon)_{\varepsilon>0}$ is bounded in $L^p(\Omega\times(0,T),L^p_r)$ and since $p\le 2$, to verify that $(\psi_\varepsilon)_{\varepsilon>0}$ is bounded in $L^p(\Omega\times(0,T);W_r^{1-\alpha,p})$, it is enough to prove that $E\int_0^T[\psi_\varepsilon(t,\cdot)]^2_{W_r^{1-\alpha,2}}dt\le C$, for any $\varepsilon>0$. We give details only for the most difficult term $\int_0^t DF^\varepsilon(s,\phi_s^\varepsilon(x))dW_s$ in (22). The $F(0,x)$ term can be controlled using (21) and the others are of easier estimation. We show that there exists a constant $C>0$ (independent on ε) such that

$$E\int_0^T dt\left[\int_0^t DF^\varepsilon(s,\phi_s^\varepsilon(\cdot))\,dW_s\right]^2_{W_r^{1-\alpha,2}}\le C \tag{23}$$

We have

$$\begin{aligned}
&E\left[\int_0^T dt\int_{B(r)}\int_{B(r)}\frac{|\int_0^t(DF^\varepsilon(s,\phi_s^\varepsilon(x))-DF^\varepsilon(s,\phi_s^\varepsilon(x')))dW_s|^2}{|x-x'|^{(1-\alpha)2+d}}dx\,dx'\right]\\
&=\int_0^T\int_{B(r)}\int_{B(r)}E\int_0^t\frac{|DF^\varepsilon(s,\phi_s^\varepsilon(x))-DF^\varepsilon(s,\phi_s^\varepsilon(x'))|^2}{|x-x'|^{(1-\alpha)2+d}}ds\,dx\,dx',\\
&=E\int_0^T dt\int_0^t ds\int_{B(r)}\int_{B(r)}\frac{|DF^\varepsilon(s,\phi_s^\varepsilon(x))-DF^\varepsilon(s,\phi_s^\varepsilon(x'))|^2}{|x-x'|^{(1-\alpha)2+d}}dx\,dx'\\
&\le TE\left[\int_0^T ds\int_{B(r)}\int_{B(r)}\frac{|DF^\varepsilon(s,\phi_s^\varepsilon(x))-DF^\varepsilon(s,\phi_s^\varepsilon(x'))|^2}{|x-x'|^{(1-\alpha)2+d}}dx\,dx'\right],\\
&\le TE\int_0^T[DF^\varepsilon(s,\phi_s^\varepsilon(\cdot))]^2_{W_r^{1-\alpha,2}}ds
\end{aligned}$$

By the Sobolev embedding the $W_r^{1-\alpha,2}$-seminorm can be controlled by the norm in $W_r^{1,p}$ if

$$1-\frac{d}{p}\ge(1-\alpha)-\frac{d}{2}.$$

This holds if $p\ge\frac{2d}{d+2\alpha}$. Then we consider p_1 such that $p>p_1>\frac{2d}{d+2\alpha}$ and show that

$$E\int_0^T\|DF^\varepsilon(s,\phi_s^\varepsilon(\cdot))\|^2_{W_r^{1,p_1}}ds\le C<\infty, \tag{24}$$

where C is independent on ε.

Step 4. To obtain (24) we estimate

$$E\int_0^T ds\Big(\int_{B(r)}|D^2F^\varepsilon(s,\phi_s^\varepsilon(x))D\phi_s^\varepsilon(x)|^{p_1}dx\Big)^{\frac{2}{p_1}}$$

A similar term has been already estimated in the proof of Theorem 11 in [10]. Since

$$\int_{B(r)}\left(\int_0^T E\left[|D\phi_s^\varepsilon(x)|^r\right]ds\right)^\gamma dx<\infty,$$

for every $r,\gamma \geq 1$ (see (8)), by the Hölder inequality, it is sufficient to prove that

$$\int_0^T E\left[\left(\int_{B(r)} \left|D^2 F^\varepsilon(s,\phi_s^\varepsilon(x))\right|^p dx\right)^{\frac{2}{p}}\right] dt \leq C < \infty.$$

We have

$$\begin{aligned}
&\int_0^T E\left[\left(\int_{B(r)} \left|D^2 F^\varepsilon(s,\phi_s^\varepsilon(x))\right|^p dx\right)^{\frac{2}{p}}\right] dt \\
&= E\left[\int_0^T ds\left(\int_{\phi_s^\varepsilon(B(r))} \left|D^2 F^\varepsilon(s,y)\right|^p J(\phi_s^\varepsilon)^{-1}(y)dy\right)^{\frac{2}{p}}\right] \\
&\leq \sup_{s\in[0,T],y\in\mathbb{R}^d} E[J(\phi_s^\varepsilon)^{-1}(y)]^{2/p} \int_0^T \left(\int_{\mathbb{R}^d} \left|D^2 F^\varepsilon(s,y)\right|^p dy\right)^{\frac{2}{p}} \leq C < \infty,
\end{aligned}$$

where, using the results of [10, Section 3] and the bound (20), C is independent on $\varepsilon > 0$. The proof is complete. □

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A QUALITATIVE MODEL FOR AGGREGATION-FRAGMENTATION AND DIFFUSION OF β-AMYLOID IN ALZHEIMER'S DISEASE

Abstract. In this paper we present a mathematical model for the aggregation, fragmentation and diffusion of Aβ amyloid in the brain affected by Alzheimer's disease. The model is based on a classical discrete Smoluchowski aggregation-fragmentation equation modified to take diffusion into account.

Ad Angelo con stima e affetto.

Alzheimer's disease (AD) is nowadays one of the most common late life dementia: current estimates of AD incidence are above 24 million of affected persons worldwide, a number that is expected to double every 20 years. Due to the consistent economic costs this will imply for the whole society, not to mention the disease caused to the affected patients and their families, it is clear that considerable efforts are made at all possible levels of research (medical, biological, pharmacological and even mathematical) to make any kind of feasible progress in the study of the disease.

From the mathematical point of view, even if in recent years several models have been developed for the description and the study of pathologies such as tumors, the modeling for the study of AD is far less developed. Besides the classical approaches *in vivo* and *in vitro*, there has been an increasing interest toward the approach *in silico*, i.e. toward mathematical modeling and computer simulations. We refer for instance to [15], [3], and, first of all, to the remarkably exhaustive and deep paper [5].

It is important to stress that, despite the large number of experimental data that can be extracted from biomedical literature and incorporated in mathematical models as in [5], mathematical models do not currently have a "predictive" value; rather, they are what physicists call "toy models", i.e. simplified formal models that can be used in order to test preliminary new theories, quickly identifying, for instance, the most relevant hypotheses or rejecting those less likely to lead to new insights. In this sense, qualitative models take a place beside more specific fully quantitative models, and can be used for reducing experimental costs or for overcoming structural difficulties.

In this spirit, in our recent paper [1] we have provide an elementary mathematical model of the diffusion and agglomeration of the β-amyloid (Aβ hereafter) in the brain affected by Alzheimer's disease (AD). For a detailed review of the current knowledge on the role of Aβ in AD (the so-called *amyloid cascade hypothesis*), we refer to [6]. Roughly speaking, Aβ is produced normally by the intramembranous proteolysis

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of APP (amyloid precursor protein) throughout life, but a change in the metabolism may increase the total production of Aβ, and, in particular, the production – among other isoforms – of the monomeric $A\beta_{40}$ and $A\beta_{42}$ that are highly toxic. From now on, for sake of simplicity, we shall simply write Aβ.

Successively, Aβ oligomers are subject to three different phenomena:

- diffusion through the microscopic tortuosity of the brain tissue;
- agglomeration, leading eventually to the formation of long, insoluble amyloid fibrils, which accumulate in spherical microscopic deposits known as senile plaques;
- fragmentation, leading to the formation of small oligomers through the breakage of longer amyloid fibrils.

In our paper [1] we have considered only agglomeration and diffusion of Aβ oligomers, completely neglecting fragmentation phenomena. This choice was motivated by the fact that we were primarily interested in considering the early stage of AD, when small amyloid fibrils are free to move and to coalesce in the brain. Therefore we discarded deliberately fibril fragmentation, which can be considered as a secondary process in the mechanism of amyloid self-assembly ([7], [17]), especially when oligomers of small size are involved. In the present paper, however, we include also fragmentation and we generalize the results obtained in [1].

A natural way to describe the agglomeration (and also fragmentation) phenomena is by means of the so-called Smoluchowski equations (classical references are [14] and [4]).

Since the fibrils at the stage of the disease under consideration are relatively small, diffusion also plays a key role in the description of the behaviours of the fibrils, as recently discussed e.g. in [10], [9].

In [1] we fixed as spatial scale of reference a size comparable to a multiple of the size of a neuron, and we avoided the description of intracellular phenomena, as well as of the clinical manifestations of the disease at a macroscopic scale.

With this choice of scale, we coherently assumed a uniform diffusion, and therefore we modeled it by the usual Fourier linear diffusion equation. Indeed, if one considers a large (i.e. macroscopic) portion of the brain tissue, it has been recently proved that the diffusion of the amyloid is affected by the metabolic activity and therefore may change from one region to another according to the neuronal activity ([2]). On the contrary, since we focus on a small portion of the celebral tissue (typically the affected areas are the hippocampus or of the cerebral cortex), linear diffusion appears to be the most appropriate (see, for instance, [11]).

Moreover, we assumed that "large" assemblies do not aggregate with each other. This assumption was related to technical aspects of the model (basically, it is meant to prevent blow-up phenomena for solutions at a finite time), but was also coherent with experimental data.

We now briefly recall all the notations introduced in [1]: The portion of cerebral tissue we consider is represented by a bounded smooth region $\Omega_0 \subset \mathbb{R}^3$ (since only qualitative information is desired we shall take $\Omega_0 \subset \mathbb{R}^2$ in our simulations for keeping the computing effort small enough), whereas the neurons are represented by a family of regular regions Ω_j such that

1. $\overline{\Omega}_j \subset \Omega_0$ if $j = 1, \dots, M$;
2. $\overline{\Omega}_i \cap \overline{\Omega}_j = \emptyset$ if $i \neq j$.

We set

$$\Omega := \Omega_0 \setminus \bigcup_{j=1}^{M} \overline{\Omega}_i.$$

From the mathematical point of view, we will consider a vector-valued function $u = (u_1, \dots, u_N)$, where $N \in \mathbb{N}$ and $u_j = u_j(t,x)$, $t \in \mathbb{R}$, $t \geq 0$ (the time), and $x \in \Omega$:

- if $1 \leq j < N-1$, then $u_j(t,x)$ is the (molar) concentration at the time t at the point x of an Aβ assembly of j monomers;
- u_N takes into account aggregations of more than $N-1$ monomers. Although u_N has a different meaning from the other u_m's, we keep the same letter u in order to avoid cumbersome notations.

With these notations we were lead to the following Cauchy-Neumann problem:

$$(1) \qquad \begin{cases} \frac{\partial}{\partial t} u_1 &= d_1 \Delta_x u_1 - u_1 \sum_{j=1}^{N} a_{1,j} u_j, \\ \frac{\partial}{\partial t} u_m &= d_m \Delta_x u_m - u_m \sum_{j=1}^{N} a_{m,j} u_j + \frac{1}{2} \sum_{j=1}^{m-1} a_{j,m-j} u_j u_{m-j} \\ & \qquad \text{(for } 1 < m < N\text{)}, \\ \frac{\partial}{\partial t} u_N &= d_N \Delta_x u_N + \frac{1}{2} \sum_{j+k \geq N, k<N, j<N} a_{j,k} u_j u_k, \end{cases}$$

with Neumann boundary conditions:

$$(2) \qquad \begin{cases} \dfrac{\partial u_m}{\partial \nu} = 0 & \text{on } \partial\Omega_0, \text{ for } m = 1, \dots, N \\ \dfrac{\partial u_1}{\partial \nu} = \psi_j & \text{on } \partial\Omega_j, j = 1, \dots, M \\ \dfrac{\partial u_m}{\partial \nu} = 0 & \text{on } \partial\Omega_j, j = 1, \dots, M \quad \text{for } m = 2, \dots, N, \end{cases}$$

where $0 \leq \psi_j \leq 1$ is a smooth function for $j = 1, \dots, M$, and, eventually, with Cauchy data:

$$
(3) \qquad \begin{cases} u_1(0,\cdot) & = U_1 \geq 0 \\ \\ u_m(0,\cdot) & = 0 \quad \text{for } m = 2,\dots,N. \end{cases}
$$

The coefficient $a_{i,j} \geq 0$ take into account the rate of coagulation of oligomers of length i and j, respectively, whereas $0 \leq \psi_j \leq 1$ is a smooth function for $j = 1,\dots,M$, describing the production of the amyloid near the membrane of the neuron. We only took into account neurons affected by the disease, i.e. we assume $\psi_j \not\equiv 0$ for $j = 1,\dots,M$. Moreover, to avoid technicalities, we assumed that U_1 is smooth, more precisely $U_1 \in \boldsymbol{C}^{2+\alpha}(\Omega)$ for some $\alpha \in (0,1)$, and that $\frac{\partial U_1}{\partial \nu} = \psi_j$ on $\partial\Omega_j$, $j = 0,\dots,M$.

For the system above we were able to prove in [1] the existence of a unique positive classical solution, existing for all positive times, as well as some estimates on its asymptotic behaviour.

As explained above, our previous model can be improved if we take into account that the Aβ-oligomers are also subject to a secondary process of fragmentation. More precisely, we assume that oligomers of length $i+j$ is subject to a fragmentation phenomenon, yielding oligomers of length i and j, with fragmentation rate $\beta_{i,j} = \beta_{j,i} \geq 0$. Coherently with our model, we make the following assumptions:

- Large agglomerate are stable. Then the equation for $m = N$ does not contain fragmentation terms. Analogously, the equation for $m = N-1$ contains only fragmentation terms corresponding to t loss of mass. For the same reason, we assume $b_{i,j} = 0$ if $i + j \geq N$;

- since in the disease coagulation prevails over fragmentation, we assume there exists $\gamma \in (0,1)$ such that

$$
(4) \qquad b_{i,j-i} \leq \gamma a_{i,j} \quad \text{for } i,j = 1,\dots,N-1, i < j.
$$

Thus, we are lead to the following coagulation-fragmentation system

$$
(5) \qquad \begin{cases}
\frac{\partial}{\partial t}u_1 & = d_1\Delta_x u_1 - u_1\sum_{j=1}^{N} a_{1,j}u_j + \sum_{j=1}^{N-2} b_{1,j}u_{j+1}, \\
\\
\frac{\partial}{\partial t}u_m & = d_m\Delta_x u_m - u_m\sum_{j=1}^{N} a_{m,j}u_j \\
& \quad + \frac{1}{2}\sum_{j=1}^{m-1} a_{j,m-j}u_j u_{m-j} \\
& \quad - \frac{1}{2}\sum_{j=1}^{m-1} b_{m-j,j}u_m + \sum_{j=1}^{N-m-1} b_{m,j}u_{j+m} \\
& \qquad \text{(for } 1 < m < N-1\text{)}, \\
\frac{\partial}{\partial t}u_{N-1} & = d_{N-1}\Delta_x u_{N-1} - u_{N-1}\sum_{j=1}^{N} a_{N-1,j}u_j \\
& \quad + \frac{1}{2}\sum_{j=1}^{N-2} a_{j,N-1-j}u_j u_{N-1-j} \\
& \quad - \frac{1}{2}\sum_{j=1}^{N-2} b_{N-1-j,j}u_{N-1}, \\
\\
\frac{\partial}{\partial t}u_N & = d_N\Delta_x u_N + \frac{1}{2}\sum_{j+k\geq N, k<N, j<N} a_{j,k}u_j u_k,
\end{cases}
$$

with the same Neumann boundary conditions and Cauchy initial data as above.

If we set $b_{i,0} = b_{0,i} = 0$, then equations (5) can be written in in the simpler way

$$(6) \quad \begin{cases} \frac{\partial}{\partial t} u_1 &= d_1 \Delta_x u_1 - u_1 \sum_{j=1}^N a_{1,j} u_j + \sum_{j=1}^{N-2} b_{1,j} u_{j+1}, \\ \frac{\partial}{\partial t} u_m &= d_m \Delta_x u_m - u_m \sum_{j=1}^N a_{m,j} u_j + \frac{1}{2} \sum_{j=1}^{m-1} a_{j,m-j} u_j u_{m-j} \\ & \quad - \frac{1}{2} \sum_{j=0}^{m-1} b_{m-j,j} u_m + \sum_{j=0}^{N-m-1} b_{m,j} u_{j+m} \\ & \qquad (\text{for } 1 < m \le N-1), \\ \frac{\partial}{\partial t} u_N &= d_N \Delta_x u_N + \frac{1}{2} \sum_{j+k \ge N, k<N, j<N} a_{j,k} u_j u_k, \end{cases}$$

In addition, to avoid technicalities, we assume that U_1 is smooth, more precisely $U_1 \in \boldsymbol{C}^{2+\alpha}(\Omega)$ for some $\alpha \in (0,1)$, and that $\frac{\partial U_1}{\partial \nu} = \psi_j$ on $\partial \Omega_j$, $j = 0, \dots, M$.

THEOREM 1. *If* $T > 0$ *then the Neumann-Cauchy problem* (6) *has a unique classical positive solution* $u \in \boldsymbol{C}^{1+\alpha/2, 2+\alpha}([0,T] \times \Omega)$.

Proof. Let $g \in \boldsymbol{C}^{2+\alpha}(\Omega)$ be such that

$$\frac{\partial g}{\partial \nu} = 0 \quad \text{on } \partial \Omega_0$$

and

$$\frac{\partial g}{\partial \nu} = \psi_j \quad \text{on } \partial \Omega_j, \, j = 1, \dots, M.$$

Set now $v_1 := u_1 - g$, $v_m := u_m$ for $m > 1$. Equations (6) become

$$(7) \quad \begin{cases} \frac{\partial}{\partial t} v_1 &= d_1 \Delta_x v_1 - v_1 \big(a_{1,1} v_1 + \sum_{j=2}^N a_{1,j} v_j + 2 a_{1,1} g \big) \\ & \quad + d_1 \Delta_x g - a_{1,1} g^2 - g \sum_{j=2}^N a_{1,j} v_j + \sum_{j=1}^{N-1} b_{1,j} v_{j+1} \\ \frac{\partial}{\partial t} v_m &= d_m \Delta_x v_m - v_m \big(\sum_{j=1}^N a_{m,j} v_j + a_{m,1} g \big) \\ & \quad + \frac{1}{2} \sum_{j=1}^{m-1} a_{j,m-j} v_j v_{m-j} + a_{1,m-1} g v_{m-1} \\ & \quad - \frac{1}{2} \sum_{j=0}^{m-1} b_{m-j,j} u_m + \sum_{j=0}^{N-m-1} b_{m,j} u_{j+m} \\ & \qquad (\text{for } 1 < m < N), \\ \frac{\partial}{\partial t} v_N &= d_N \Delta_x v_N + \frac{1}{2} \sum_{j+k \ge N} a_{j,k} v_j v_k \\ & \quad + g \big(a_{1,N-1} v_{N-1} + a_{1,N} v_N \big), \end{cases}$$

with homogeneous boundary conditions and Cauchy data

$$(8) \quad \begin{cases} v_1(0,\cdot) &= U_1 - g \\ v_m(0,\cdot) &= 0 \quad \text{for } m = 2, \dots, N. \end{cases}$$

By [8] and the parabolic maximum principle (see, e.g., [13], Theorem 9.6), there exists $\tau_{\max} > 0$ such that the above Cauchy-Dirichlet problem has a local *positive* classical maximal solution

$$v \in \boldsymbol{C}^{1+\alpha/2,2+\alpha}([0,\tau]\times\Omega)$$

for every $\tau \in (0,\tau_{\max})$. Therefore, the original Neumann-Dirichlet problem has a local positive classical situation in $[0,\tau_{\max})\times\Omega$.

To achieve the proof of the theorem, we have but to show that u can be continued on all $[0,T]\times\Omega$, and therefore, by [8] (1.3), or [12], Theorem 1 (iii), we have but to show that

$$\sup_{0\le t<\tau_{\max}} \|u(t,\cdot)\|_{(L^\infty(\Omega))^N} < \infty. \tag{9}$$

We can argue as in [16] and [1] by induction on the components of u. Let $g \in \boldsymbol{C}^2(\Omega)$ be such that

$$\frac{\partial g}{\partial\nu} = 1 \quad \text{on } \partial\Omega_0$$

and

$$\frac{\partial g}{\partial\nu} = 1 \quad \text{on } \partial\Omega_j,\, j = 1,\dots,M.$$

Without loss of generality, we may assume $g \ge 0$. We set $C := \max_\Omega d_1|\Delta_x g|$, $u_0 := g + Ct$ and $v_1 := u_1 - u_0$. We have

$$\begin{cases} \frac{\partial}{\partial t} v_1 = d_1\Delta_x v_1 - a_{1,1}v_1^2 \\ \qquad - v_1\left(\sum_{j=2}^N a_{1,j}u_j + 2a_{1,1}u_0\right) + \sum_{j=2}^N b_{1,j-1}u_j + \\ \qquad + d_1\Delta_x u_0 - \frac{\partial u_0}{\partial t} - a_{1,1}u_0^2 - u_0\sum_{j=2}^N a_{1,j}u_j \\ \frac{\partial v_1}{\partial\nu} = -1 \quad \text{on } \partial\Omega_0 \\ \frac{\partial v_1}{\partial\nu} = \psi_j - 1 \le 0 \quad \text{on } \partial\Omega_j,\, j=1,\dots,M \\ v_1(x,0) = U_1(x) - g(x), \quad x \in \Omega. \end{cases} \tag{10}$$

We set $h := -\sum_{j=2}^N \left(a_{1,j}v_1 - b_{1,j-1}\right)u_j$ and

$$\begin{aligned} k : &= -d_1\Delta_x u_0 + C + a_{1,1}u_0^2 + u_0\sum_{j=2}^N a_{1,j}u_j \\ &\ge -d_1\Delta_x u_0 + C \ge 0. \end{aligned}$$

Thus, the equation in (10) becomes

$$\frac{\partial}{\partial t}v_1 - d_1\Delta_x v_1 = -a_{1,1}v_1^2 - 2a_{1,1}u_0v_1 + h - k \le -2a_{1,1}u_0v_1 + h. \tag{11}$$

We take now $k_1 := \max\{\gamma, \|U_1(x) - g(x)\|_{L^\infty(\Omega)}\}$. We multiply equation (11) by $(v_1 - k_1)_+$ and we integrate on $[0,t]\times\Omega$, for $t < \tau_{\max}$. Keeping into account that $u_0 \ge 0$ and hence $u_0 v_1(v_1-k_1)_+ \geq 0$, we get

$$\frac{1}{2}\|(v_1(t,\cdot)-k_1)_+^2\|_{L^2(\Omega)} + \int_0^t \|\nabla(v_1(s,\cdot)-k_1)_+^2\|_{L^2(\Omega)}\,ds$$
$$-\int_0^t \Big(\int_{\partial\Omega} \frac{\partial v_1}{\partial \nu}(v_1(s,\cdot)-k_1)_+\Big)\,ds$$
$$\leq -\sum_{j=2}^N \int_0^t \int_\Omega (v_1(s,x)-k_1)_+\big(a_{1,j}v_1(s,x)-b_{1,j-1}\big)u_j(s,x)\,dx\,ds$$
$$\leq -\sum_{j=2}^N a_{1,j}\int_0^t \int_\Omega (v_1(s,x)-k_1)_+\big(v_1(s,x)-\gamma\big)u_j(s,x)\,dx\,ds,$$

by (4). Suppose now $(v_1(s,x)-k_1)_+ > 0$. Then $v_1(s,x) > k_1 \geq \gamma$, and hence $(v_1(s,\cdot)-k_1)_+\big(v_1(s,\cdot)-\gamma\big)u_j(s,\cdot) \geq 0$ in Ω, so that

$$\frac{1}{2}\|(v_1(t,\cdot)-k_1)_+^2\|_{L^2(\Omega)} + \int_0^t \|\nabla(v_1(s,\cdot)-k_1)_+^2\|_{L^2(\Omega)}\,ds$$
$$-\int_0^t \Big(\int_{\partial\Omega} \frac{\partial v_1}{\partial \nu}(v_1(s,\cdot)-k_1)_+\,d\mathcal{H}^{n-1}\Big)\,ds \leq 0.$$

Since $\frac{\partial v_1}{\partial \nu} \leq 0$ on $\partial\Omega$, we can conclude that $\|(v_1(t,\cdot)-k_1)_+^2\|_{L^2(\Omega)} \leq 0$, and then that $v_1 \leq k_1$, so that

$$0 \leq u_1 \leq k_1 + C\tau_{\max}.$$

Suppose now

$$\|u_j(t,\cdot)\|_{(L^\infty(\Omega))^N} \leq C_j \quad \text{for } t \in (0,\tau_{\max})$$

$j = 1,\dots,m-1$. If we choose $C \geq d_1 \max_\Omega |\Delta_x u_0|$, $u_0 := g + Ct$ (where g is as above) and $v_m := u_m - u_0$, we have

$$\text{(12)}\qquad \begin{cases} \frac{\partial}{\partial t}v_m = d_m\Delta_x v_m - v_m\sum_{j=1}^N a_{m,j}u_j \\ \qquad\qquad + d_m\Delta_x u_0 - \frac{\partial u_0}{\partial t} - u_0\sum_{j=1}^N a_{m,j}u_j \\ \qquad\qquad + \frac{1}{2}\sum_{j=1}^{m-1} a_{j,m-j}u_j(x,t)u_{m-j}(x,t) \\ \qquad\qquad - \frac{1}{2}\sum_{j=0}^{m-1} b_{m-j,j}u_m + \sum_{j=0}^{N-m-1} b_{m,j}u_{j+m}, \\ \frac{\partial v_m}{\partial \nu} = -1 \quad \text{on } \partial\Omega_0 \\ \frac{\partial v_m}{\partial \nu} = -1 \quad \text{on } \partial\Omega_j,\ j=1,\dots,M \\ v_m(x,0) = -g(x), \quad x \in \Omega. \end{cases}$$

Since $d_m\Delta_x u_0 - \frac{\partial u_0}{\partial t} - u_0\sum_{j=1}^N a_{m,j}u_j \geq 0$, the equation in (12) yields

$$\frac{\partial}{\partial t}v_m - d_m\Delta_x v_m$$
$$\leq -v_m\sum_{j=1}^N a_{m,j}u_j + \frac{1}{2}\sum_{j=1}^{m-1} a_{j,m-j}u_j(x,t)u_{m-j}(x,t)$$
$$-\frac{1}{2}\sum_{j=0}^{m-1} b_{m-j,j}u_m + \sum_{j=0}^{N-m-1} b_{m,j}u_{j+m}$$

We can repeat now the arguments of the proof of Lemma 2.2 in [16] to obtain the a priori bound (9). This achieves the proof of the Theorem.

□

If $m = 1,\ldots,N-1$, we multiply by m the equation for u_m in (6) and we sum up for $m = 1,\ldots,N-1$. We obtain

$$\begin{aligned}
&\frac{\partial}{\partial t}\sum_{m=1}^{N-1} m u_m = \Delta_x \sum_{m=1}^{N-1} d_m m u_m \\
&-\sum_{m=1}^{N-1}\sum_{j=1}^{N} m a_{m,j} u_m u_j + \frac{1}{2}\sum_{m=2}^{N-1}\sum_{j=1}^{m-1} m a_{j,m-j} u_j u_{m-j} \\
&-\frac{1}{2}\sum_{m=2}^{N-1} m \sum_{j=0}^{m-1} b_{m-j,j} u_m + \sum_{m=1}^{N-1} m \sum_{j=0}^{N-m-1} b_{m,j} u_{j+m}.
\end{aligned} \tag{13}$$

Now, as in [1],

$$\begin{aligned}
&-\sum_{m=1}^{N-1}\sum_{j=1}^{N} m a_{m,j} u_m u_j + \frac{1}{2}\sum_{m=2}^{N-1}\sum_{j=1}^{m-1} m a_{j,m-j} u_j u_{m-j} \\
&\qquad -\sum_{m=1}^{N-1} m a_{m,N} u_m u_N.
\end{aligned}$$

On the other hand, keeping in mind that $b_{i,j} = 0$ if $i + j \geq N$, we can write

$$\begin{aligned}
&-\frac{1}{2}\sum_{m=1}^{N-1} m \sum_{j=0}^{m-1} b_{m-j,j} u_m + \sum_{m=1}^{N-1} m \sum_{j=0}^{N-m-1} b_{m,j} u_{j+m} \\
&\quad = -\frac{1}{2}\sum_{m=1}^{\infty} m \sum_{j=0}^{m-1} b_{m-j,j} u_m + \sum_{m=1}^{\infty} m \sum_{j=0}^{\infty} b_{m,j} u_{j+m} \\
&\quad = -\frac{1}{2}\sum_{m=1}^{\infty}\sum_{j=0}^{\infty} (m+j) b_{m,j} u_m + \sum_{m=1}^{\infty} m \sum_{j=0}^{\infty} b_{m,j} u_{j+m} \\
&\quad = -\frac{1}{2}\sum_{m=0}^{\infty}\sum_{j=0}^{\infty} j b_{m,j} u_m + \frac{1}{2}\sum_{m=0}^{\infty} m \sum_{j=0}^{\infty} b_{m,j} u_{j+m} = 0.
\end{aligned}$$

In other words, the stability of large agglomerates yields that the global mass of the soluble oligomers is not affected by the fragmentation. Thus, in particular, the following asymptotic estimate holds

PROPOSITION 1 (see [1], Proposition 3.7). *If we set*

$$\Phi(t) := \sum_{m=1}^{N-1} \fint_{\Omega} m u_m(t,x)\, dx$$

(in other words, Φ is the total mass of soluble oligomers), then there exists $a > 0$ such that for $t > 1$ we have

$$\Phi(t) \leq e^{-a(t-1)}\Phi(1) + \frac{d_1 \sum_{j=1}^{M} \int_{\partial\Omega_j} \psi_j \, d\mathcal{H}^{n-1}}{a\lambda_1 |\Omega|}(1 - e^{-a(t-1)}). \tag{14}$$

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Forty years of Analysis in Turin
A conference in honour of Angelo Negro

E. Serra*

ON A CONJECTURE OF DE GIORGI CONCERNING NONLINEAR WAVE EQUATIONS

Abstract. We discuss a conjecture by De Giorgi, which states that global weak solutions to the Cauchy problem associated to certain nonlinear wave equations can be obtained as limits of minimizers of suitable convex functionals. There is no restriction on the growth of the nonlinearity, and the method is easily extended to more general equations.

Dedicated to Angelo Negro on the occasion of his 70*th birthday.*

1. The conjecture

In this talk I will report on a joint work with Paolo Tilli, discussing a conjecture of Ennio De Giorgi related to some classes of nonlinear wave equations.
We consider minimization/evolution problems in space time, $\mathbb{R}\times\mathbb{R}^n$, $n\geq 0$; the accent on *minimization* or *evolution* depends on the point of view, and as we will see this is at the core of the problem.

In a paper published in the Duke Mathematical Journal, [1], De Giorgi stated the following conjecture.

CONJECTURE 1. Let $p\in\mathbb{N}$ be an even number. For $\varepsilon>0$, let $v_\varepsilon(t,x)$ denote the minimizer of the convex functional

$$F_\varepsilon(v)=\int_0^\infty\int_{\mathbb{R}^n} e^{-t/\varepsilon}\big\{|v''(t,x)|^2+\frac{1}{\varepsilon^2}|\nabla v(t,x)|^2+\frac{1}{\varepsilon^2}|v(t,x)|^p\big\}\,dxdt$$

subject to the boundary conditions

$$v(0,x)=\alpha(x),\quad v'(0,x)=\beta(x),\quad x\in\mathbb{R}^n,$$

where $\alpha,\beta\in C_0^\infty(\mathbb{R}^n)$ are given functions. Then, for almost every $(t,x)\in\mathbb{R}^+\times\mathbb{R}^n$, the limit

$$w(t,x)=\lim_{\varepsilon\downarrow 0}v_\varepsilon(t,x)$$

exists and the function $w(t,x)$ solves in $\mathbb{R}^+\times\mathbb{R}^n$ the nonlinear wave equation

$$w''-\Delta w+\frac{p}{2}w^{p-1}=0 \tag{1}$$

with initial conditions

$$w(0,x)=\alpha(x),\quad w'(0,x)=\beta(x),\quad x\in\mathbb{R}^n. \tag{2}$$

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REMARK 1. Existence and uniqueness of a minimizer for the functional F_ε are straightforward. Basically one can consider the largest space of $L^1_{\mathrm{loc}}(\mathbb{R}\times\mathbb{R}^n)$ functions where F_ε is finite and minimize among functions that satisfy the boundary condition (in a suitable sense). Coercivity and strict convexity easily provide existence and uniqueness of a minimizer v_ε.

The above conjecture casts a completely new bridge between *hard evolution problems* and more easily tractable *convex minimization problems*. Indeed, if proven true, it provides a method to approximate nonlinear (defocusing) wave equations by convex minimization problems. The variational approach is by genuine minimization, and not by Critical Point Theory, where one would have to use functionals that behave rather badly from the point of view of existence results. Notice also that the nonlinearity exponent p can be arbitrarily large.

We also point out that the approach is new (in spirit) even for the linear wave equation $w'' - \Delta w = 0$ or for the linear Klein–Gordon equation $w'' - \Delta w + w = 0$.

A further point of interest is the possibility to extend the method to other classes of evolution equations.

A proof of this conjecture has to face a series of difficulties. Among others, we list the following ones.

- The functionals involve first order spatial derivatives, but second order time derivatives.
- The weight $e^{-t/\varepsilon}$ in each single functional (ε fixed) decays very rapidly as $t\to\infty$.
- For fixed $t_2 > t_1$, the weight ratio $e^{-t_1/\varepsilon}/e^{-t_2/\varepsilon}$ diverges as $\varepsilon\to 0$.
- The time–scale depends on ε, making it difficult to compare two minimizers v_{ε_1} and v_{ε_2}.
- As $\varepsilon\to 0$, $e^{-t/\varepsilon}$ concentrates close to $t=0$, and rescaled functionals Γ-converge to a constant functional, thereby exhibiting a strong loss of information.

The following is our main result.

THEOREM 1 ([2]). *For every real $p\geq 2$ and for initial data α,β in $H^1\cap L^p$, the conjecture is true, up to subsequences.*

REMARK 2. Passing to subsequences is *not* necessary if the Cauchy problem (1)–(2) has uniqueness. However uniqueness for this problem is not known for large p.

REMARK 3. The solution of the Cauchy problem (1)–(2) obtained in the above theorem is of *energy class*, i.e. the function

$$\mathcal{E}(t) := \int_{\mathbb{R}^n}\left(|w'(t,x)|^2 + |\nabla w(t,x)|^2 + |w(t,x)|^p\right)dx$$

satisfies the energy inequality $\mathcal{E}(t)\leq\mathcal{E}(0)$.

We recall that *conservation* of energy for the Cauchy problem (1)–(2) is not known for large p.

2. The main ideas of the proof

We now sketch some of the main ideas involved in the proof. It is clear that, in order to pass to the limit in the Euler–Lagrange equation associated to the functionals F_ε, some estimates are needed. The type of estimates that we obtain, and that are sufficient to complete the limit procedure, can be summarized in the following list.

- A localized L^2 estimate for ∇v_ε, with values in $L^2(\mathbb{R}^n)$:

$$\int_t^{t+T} \int_{\mathbb{R}^n} |\nabla v_\varepsilon(s,x)|^2 \, dx\, ds \le CT, \qquad t \ge 0, \quad T \ge \varepsilon.$$

- A localized L^p estimate for v_ε, with values in $L^p(\mathbb{R}^n)$:

$$\int_t^{t+T} \int_{\mathbb{R}^n} |v_\varepsilon(s,x)|^p \, dx\, ds \le CT, \qquad t \ge 0, \quad T \ge \varepsilon.$$

- A global L^∞ estimate for v'_ε, with values in $L^2(\mathbb{R}^n)$:

$$\int_{\mathbb{R}^n} |v'_\varepsilon(t,x)|^2 \, dx \le C, \qquad t \ge 0.$$

These estimates provide convergence (up to subsequences) to some $w(t,x)$, with

$$w \in L^\infty(\mathbb{R}^+; L^p), \quad \nabla w \in L^\infty(\mathbb{R}^+; L^2), \quad w' \in L^\infty(\mathbb{R}^+; L^2),$$

for which the energy function

$$\mathcal{E}(t) := \int_{\mathbb{R}^n} \left(|w'|^2 + |\nabla w|^2 + |w|^p \right) dx$$

is finite for a.e. $t > 0$.

Moreover, w solves (in weak sense) the wave equation

$$w'' - \Delta w + \frac{p}{2} |w|^{p-2} w = 0,$$

as one sees by passing to the limit in the Euler–Lagrange equation of v_ε. In this context, it is interesting to note that the weight $e^{-t/\varepsilon}$ can be absorbed inside the test function during the limit process.

Indeed, let $\eta \in C_0^\infty(\mathbb{R}^+ \times \mathbb{R}^n)$ be a test function. Since v_ε is the global minimizer for F_ε, it satisfies the Euler–Lagrange equation that, written in weak form, is

$$\int_0^\infty \int_{\mathbb{R}^n} e^{-t/\varepsilon} \big(\varepsilon^2 v''_\varepsilon \eta'' + \nabla v_\varepsilon \nabla \eta + \frac{p}{2} |v_\varepsilon|^{p-2} v_\varepsilon \eta \big) \, dx\, dt = 0.$$

Integrating once by parts in time yields

$$\int_0^\infty \int_{\mathbb{R}^n} \big(-\varepsilon^2 v_\varepsilon' \big(e^{-t/\varepsilon}\eta''\big)' + e^{-t/\varepsilon}(\nabla v_\varepsilon \nabla \eta + \frac{p}{2}|v_\varepsilon|^{p-2} v_\varepsilon \eta)\big)\, dx\, dt = 0.$$

Now choosing $\eta = e^{t/\varepsilon}\varphi$, with $\varphi \in C_0^\infty(\mathbb{R}^+ \times \mathbb{R}^n)$, the preceding identity reads

$$\int_0^\infty \int_{\mathbb{R}^n} \big(-v_\varepsilon'(\varepsilon^2\varphi'' + 2\varepsilon\varphi' + \varphi)' + \nabla v_\varepsilon \nabla \varphi + \frac{p}{2}|v_\varepsilon|^{p-2} v_\varepsilon \varphi\big) dx\, dt = 0$$

As $\varepsilon \to 0$, from $v_\varepsilon \to w$ (weakly in H^1, strongly in $L^{p-1}, \dots$) we obtain

$$\int_0^\infty \int_{\mathbb{R}^n} \big(-w'\varphi' + \nabla w \nabla \varphi + \frac{p}{2}|w|^{p-2} w\varphi\big) dx\, dt = 0 \qquad \forall \varphi \in C_0^\infty(\mathbb{R}^+ \times \mathbb{R}^n),$$

namely the weak form of the wave equation

$$w'' - \Delta w + \frac{p}{2}|w|^{p-2} w = 0.$$

REMARK 4. Also the two initial conditions

$$v_\varepsilon(0,x) = \alpha(x) \qquad \text{and} \qquad v_\varepsilon'(0,x) = \beta(x)$$

pass to the limit as $\varepsilon \to 0$. For the former, the $L^\infty(\mathbb{R}^+; L^2)$ bound on v_ε' is enough. For the latter, we need estimates on v_ε'', uniform in ε. These are obtained in L^∞, with values in the dual of $H^1 \cap L^p$, by a careful choice of test functions in the Euler–Lagrange equation for v_ε.

We now sketch the main argument to obtain the *a priori* estimates that allowed us to carry out the preceding limit procedure. First of all it is convenient to get rid of the parameter ε in the weight: setting

$$u_\varepsilon(t,x) = v_\varepsilon(\varepsilon t, x),$$

we see that v_ε minimizes F_ε if and only if u_ε minimizes

$$J_\varepsilon(u) = \int_0^\infty \int_{\mathbb{R}^n} e^{-t} \big(|u''|^2 + \varepsilon^2|\nabla u|^2 + \varepsilon^2 |u|^p\big)\, dx\, dt$$

with boundary contidions

$$\begin{cases} u(0,x) = \alpha \\ u'(0,x) = \varepsilon\beta \end{cases}$$

Precisely, $J_\varepsilon(u_\varepsilon) = \varepsilon F_\varepsilon(v_\varepsilon)$.

Now a crucial role is played by the function

$$E(t) = \int_{\mathbb{R}^n} |u_\varepsilon'|^2\, dx - 2\int_{\mathbb{R}^n} u_\varepsilon' u_\varepsilon''\, dx + e^t \int_t^\infty e^{-s} L(s)\, ds$$

where L is

$$L(s)=\int_{\mathbb{R}^n}|u''_\varepsilon(s,x)|^2+\varepsilon^2|\nabla u_\varepsilon(s,x)|^2+\varepsilon^2|u_\varepsilon(s,x)|^p\,dx.$$

The function $E:\mathbb{R}^+\to\mathbb{R}$ is a sort of *energy*, and indeed it is strongly related to the energy $\mathcal{E}$ of the wave equation. Its properties are summarized int he following result.

THEOREM 2 (Energy lemma). *Let u_ε be the minimizer for J_ε and let*

$$E(t)=\int_{\mathbb{R}^n}|u'_\varepsilon|^2\,dx-2\int_{\mathbb{R}^n}u'_\varepsilon u''_\varepsilon\,dx+e^t\int_t^\infty e^{-s}L(s)\,ds.$$

Then E is positive and decreasing; precisely

$$E'=-4\int_{\mathbb{R}^n}|u''_\varepsilon|^2\,dx \qquad \textit{in the sense of distributions}$$

and

$$0\le\frac{1}{\varepsilon^2}E(t)\le\mathcal{E}(0)+O(\varepsilon),$$

where

$$\mathcal{E}(0):=\int_{\mathbb{R}^n}\left(\beta^2+|\nabla\alpha|^2+|\alpha|^p\right)dx.$$

The proof of this result *could* be obtained, formally, by multiplying by u'_ε the Euler–Lagrange equation, but the integral

$$\int_{\mathbb{R}^n}|u_\varepsilon|^{p-2}u_\varepsilon u'_\varepsilon\,dx$$

is (a priori) meaningless for large p.

Instead, we make use of inner variations: we build competitors for u_ε of the form

$$U_\delta(t,x)=u_\varepsilon(t+\delta\eta(t),x),\quad \eta\in C_0^\infty(\mathbb{R}^+),$$

and compute $\dfrac{d}{d\delta}J_\varepsilon(U_\delta)$ at $\delta=0$. This is essentially the procedure that is used to derive the Du Bois–Reymond equation in the Calculus of Variations.

The other tools to complete the argument are the following.

- A level estimate:

$$J_\varepsilon(u_\varepsilon)\le J_\varepsilon(\alpha+\varepsilon t\beta)\le C\varepsilon^2.$$

- An energy estimate:

$$E(0)=\varepsilon^2\mathcal{E}(0)+O(\varepsilon^3)\le C\varepsilon^2.$$

- A consequence of the Energy lemma:

$$\int_{\mathbb{R}^n}|u'_\varepsilon(t)|^2dx+e^t\int_t^\infty\int_s^\infty e^{-\tau}L(\tau)\,d\tau\,ds\le E(t)\le E(0)\le C\varepsilon^2.$$

Setting

$$H(t)=\int_t^\infty e^{-\tau}L(\tau)\,d\tau,$$

the last inequality can be written more concisely

$$\int_{\mathbb{R}^n}|u_\varepsilon'(t)|^2dx+e^t\int_t^\infty H(s)\,ds\le C\varepsilon^2. \tag{3}$$

From this we first derive a pointwise estimate on H. Since H is decreasing by definition,

$$H(t+1)\le\int_t^{t+1}H(s)\,ds\le\int_t^\infty H(s)\,ds$$

Multiplying by e^{t+1} and using (3) yields

$$e^{t+1}H(t+1)\le e\,e^t\int_t^\infty H(s)\,ds\le C\varepsilon^2,$$

that is,

$$e^tH(t)\le C\varepsilon^2 \qquad \forall t\ge 1.$$

But if $t\in[0,1]$,

$$e^tH(t)\le eH(t)\le eH(0)=eJ_\varepsilon(u_\varepsilon)\le C\varepsilon^2,$$

so that

$$e^tH(t)\le C\varepsilon^2 \qquad \forall t\ge 0.$$

We are now in a position to conclude. Due to the preceding discussion we can proceed by estimating

$$\begin{aligned}C\varepsilon^2&\ge e^tH(t)=e^t\int_t^\infty e^{-s}L(s)\,ds\ge e^t\int_t^{t+1}e^{-s}L(s)\,ds\\&\ge e^te^{-t-1}\int_t^{t+1}L(s)\,ds=e^{-1}\int_t^{t+1}\int_{\mathbb{R}^n}|u_\varepsilon''|^2+\varepsilon^2|\nabla u_\varepsilon|^2+\varepsilon^2|u_\varepsilon|^p\,dx\,ds\\&\ge e^{-1}\varepsilon^2\int_t^{t+1}\int_{\mathbb{R}^n}|\nabla u_\varepsilon|^2+|u_\varepsilon|^p\,dx\,ds.\end{aligned}$$

Dividing by ε^2 we obtain

$$\int_t^{t+1}\int_{\mathbb{R}^n}|\nabla u_\varepsilon|^2dx\,ds\le C \qquad \forall t\ge 0,$$

$$\int_t^{t+1}\int_{\mathbb{R}^n}|u_\varepsilon|^pdx\,ds\le C \qquad \forall t\ge 0$$

and, directly from (3),

$$\int_{\mathbb{R}^n}|u_\varepsilon'(t)|^2\,dx\le C\varepsilon^2 \qquad \forall t\ge 0.$$

When we scale back to v_ε by $u_\varepsilon(s,x) = v_\varepsilon(\varepsilon s, x)$ and we change variables these estimates take the form

$$\frac{1}{\varepsilon}\int_{\varepsilon t}^{\varepsilon t+\varepsilon}\int_{\mathbb{R}^n}|\nabla v_\varepsilon|^2 dx\,ds \le C \qquad \forall t \ge 0,$$

$$\frac{1}{\varepsilon}\int_{\varepsilon t}^{\varepsilon t+\varepsilon}\int_{\mathbb{R}^n}|v_\varepsilon|^p dx\,ds \le C \qquad \forall t \ge 0$$

and

$$\varepsilon^2\int_{\mathbb{R}^n}|v_\varepsilon'(t)|^2\,dx \le C\varepsilon^2 \qquad \forall t \ge 0.$$

Since t is arbitrary, we can rename εt by t and obtain

$$\int_{t}^{t+\varepsilon}\int_{\mathbb{R}^n}|\nabla v_\varepsilon|^2 dx\,ds \le C\varepsilon \qquad \forall t \ge 0,$$

$$\int_{t}^{t+\varepsilon}\int_{\mathbb{R}^n}|v_\varepsilon|^p dx\,ds \le C\varepsilon \qquad \forall t \ge 0,$$

$$\int_{\mathbb{R}^n}|v_\varepsilon'(t)|^2\,dx \le C \qquad \forall t \ge 0.$$

The last one is the global L^2 estimate on v_ε'. As for the remaining two, given $T \ge \varepsilon$, the interval $[t, t+T]$ can be covered by $O(T/\varepsilon)$ adjacent subintervals of length ε. On each of these intervals we use the above estimates and we add the results, arriving at

$$\int_{t}^{t+T}\int_{\mathbb{R}^n}|\nabla v_\varepsilon|^2\,dx\,ds \le C\varepsilon O(T/\varepsilon) \le CT, \qquad t \ge 0, \qquad T \ge \varepsilon$$

$$\int_{t}^{t+T}\int_{\mathbb{R}^n}|v_\varepsilon|^p\,dx\,ds \le C\varepsilon O(T/\varepsilon) \le CT, \qquad t \ge 0, \qquad T \ge \varepsilon,$$

which are the localized estimates we were looking for.

3. Some open problems

Here is a very short list of open problems that arise from the preceding discussion.

- Proving the conjecture without passing to subsequences. This is related, as we said, to the presence of uniqueness for the Cauchy problem (1)–(2), when p is large. If there is uniqueness, we know that there is no need for subsequences. If, on the contrary, there is no uniqueness, the situation could be even more interesting. Indeed, if one could prove the conjecture without passing to subsequences, then one would have a way to select a privileged solution to the Cauchy problem that could be referred to, for example, as the "Variational Solution".

- Other equations. Just to make an example, what about

$$w'' - \frac{2}{q}\operatorname{div}\left(|\nabla w|^{q-2}\nabla w\right) + \frac{p}{2}|w|^{p-2}w = 0,$$

the wave equation for the q-Laplacian with defocusing nonlinearity?

This would correspond to the functional

$$\int_0^\infty \int_{\mathbb{R}^n} e^{-t/\varepsilon}\left(\varepsilon^2|v''|^2 + |\nabla v|^q + |v|^p\right) dx\,dt.$$

As far as we know, even the *existence* of global weak solutions (to the Cauchy problem) for large q is unknown. Does the method of De Giorgi work to solve this problem?

- The abstract form of De Giorgi's Conjecture. Consider any convex functional of the Calculus of Variations,

$$F(u) = \int_\Omega f(x, u, \nabla u, \ldots)\, dx$$

Let $v_\varepsilon(t,x)$ be the minimizer of

$$\int_0^\infty e^{-t/\varepsilon}\left(\int_\Omega \varepsilon^2 |v_\varepsilon''(t,x)|^2\, dx + F(v_\varepsilon(t,\cdot))\right) dt$$

with given boundary conditions $v_\varepsilon(0,\cdot)$ and $v_\varepsilon'(0,\cdot)$

As $\varepsilon \to 0$, does v_ε converge to some w, which solves the Cauchy Problem for the equation

$$w'' + \nabla F(w) = 0 \quad ?$$

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WEAK INSTABILITY OF HAMILTONIAN EQUILIBRIA

Abstract. This is an expository paper on Lyapunov stability of equilibria of autonomous Hamiltonian systems. Our aim is to clarify the concept of *weak instability*, namely instability without non-constant motions which have the equilibrium as limit point as time goes to minus infinity. This is done by means of some examples. In particular, we show that a weakly unstable equilibrium point can be stable for the linearized vector field.

1. Introduction

Stability of the equilibrium is a mathematical field more then two centuries old. Indeed, Lagrange stated the celebrated Lagrange-Dirichlet theorem in the eighteenth century, and some so called converses of that statement are still proved nowadays. So many mathematicians have been interested in stability that we refrain from mentioning them with the exception of the most important, Lyapunov, who defended his doctoral thesis "The general problem of the stability of motion" in 1892. The applications are also countless in mechanics and in most sciences. To start with the rich literature on this matter, see Arnold et al. [1], Meyer et al. [3], and Rouche et al. [6].

Important mathematical objects related to the instability of the equilibrium are *asymptotic motions*. Before their formal definition, let us mention that the upper position of a simple pendulum, and zero velocity, constitute an unstable equilibrium and its asymptotic motions are neither rotations (when the pendulum swings around and around) nor librations (when it swings back and forth), and they stay between the two behaviors.

Let us consider a smooth vector field f on an open set $A \subseteq \mathbb{R}^N$ with an equilibrium point $\hat{x} \in A$, so $f(\hat{x}) = 0$. We say that $\phi : (-\infty, b) \to A$ is an *asymptotic motion* in the past to the equilibrium point $\hat{x}$, if $\phi(t)$ is a non-constant solution to the o.d.e. $\dot{x} = f(x)$ such that $\phi(t) \to \hat{x}$ as $t \to -\infty$. In the sequel we briefly write 'asymptotic motion' instead of 'asymptotic motion in the past' since we are only concerned with this kind of asymptotic motions. Of course the existence of an asymptotic motion implies the Lyapunov instability of the equilibrium point. The basic sufficient condition for the existence of an asymptotic motion is the presence of an eigenvalue of $f'(\hat{x})$ with strictly positive real part, see for instance Hartman [2] remark to Corollary 6.1, p. 243.

In this paper we focus on *autonomous Hamiltonian systems* so in the sequel $N = 2n$, $x = (q, p)$, $q, p \in \mathbb{R}^n$, and the vector field is

$$(\partial_p H(q,p), -\partial_q H(q,p)) \tag{1}$$

for some smooth H called the Hamiltonian function. Our aim is to clarify the concept

*This paper is based on a lecture given in Turin on June 1, 2012, for Angelo Negro's 70th birthday.

of *Lyapunov instability without asymptotic motions* that we briefly call *weak instability*. This is done by means of some examples.

Section 2 deals with linear systems. Of course there is a trivial situation where weak instability appears: the free particle. The equilibrium is non-isolated and the eigenvalues vanish, the example can be done in one degree of freedom so in dimension 2. A more subtle instability of the equilibrium for a linear system is obtained when the eigenvalues are purely imaginary and some Jordan blocks have dimension greater than one, of course this can happen only in dimension at least 4. The example we are going to see comes from the planar restricted 3-body problem at one of the relative equilibria, the Lagrange equilateral points, also called the Trojan points, at the critical Routh value of the mass ratio of the primaries.

In Section 3 we move on nonlinear systems. Their equilibria can be unstable even if we have stability for the linearized system as the Cherry Hamiltonian in dimension 4 shows by means of an asymptotic motion. Cherry's system is the third example of this paper, it was published in 1925 and, in the last 20 years, it became important in plasma physics, see Pfirsch [5] and the references therein.

Our fourth example, also in dimension 4, comes from [10] and shows that *we can have weak instability of an Hamiltonian equilibrium which is linearly stable*. Some systems, produced by Barone-Netto and myself [11] and [9], preceded [10], they give non-Hamiltonian examples of weak instability for linearly stable equilibria.

Hopefully, the concept of weak instability will stimulate further researches in stability within mathematical physics, together with other fresh notions like the "weak asymptotic stability" introduced by Ortega, Planas-Bielsa and Ratiu, see [4] and the references therein.

2. Weak instability for linear systems

2.1. Free particle

Our first example is a particle on a straight line under no forces

$$H(q,p) = \frac{p^2}{2}, \qquad q,p \in \mathbb{R}. \tag{2}$$

The Hamiltonian vector field is

$$\big(\partial_p H(q,p), -\partial_q H(q,p)\big) = (p,0). \tag{3}$$

It is a linear field with the double eigenvalue 0. The integral curves are

$$q(t) = q(0) + p(0)t\,, \qquad p(t) = p(0). \tag{4}$$

Each $(q_0,0) \in \mathbb{R}^2$ is an equilibrium point and its instability can be shown by means of the sequence $(q(0),p(0)) = (q_0,1/m) \to (q_0,0)$ as $m \to +\infty$. There are no asymptotic motions.

2.2. Linearization at L_4

Our second example is the quadratic part of the Hamiltonian function of the planar restricted 3-body problem at one of the relative equilibria, the Lagrange libration point L_4 at the critical Routh value of the mass ratio of the primaries. In the sequel $q = (q_1, q_2), p = (p_1, p_2), (q, p) = (q_1, q_2, p_1, p_2)$, and

$$H(q,p) = \frac{1}{\sqrt{2}}\det(p,q) + \frac{1}{2}|q|^2 = \frac{1}{\sqrt{2}}\big(p_1q_2 - p_2q_1\big) + \frac{1}{2}\big(q_1^2 + q_2^2\big). \tag{5}$$

The Hamiltonian vector field is

$$\begin{aligned} &(\partial_{p_1}H(q,p), \partial_{p_2}H(q,p), -\partial_{q_1}H(q,p), -\partial_{q_2}H(q,p)) = \\ &= \Big(q_2/\sqrt{2}, -q_1/\sqrt{2}, -q_1 + p_2/\sqrt{2}, -q_2 - p_1/\sqrt{2}\Big), \end{aligned} \tag{6}$$

see H_0 and the o.d.e. at the end of p. 256, with $\xi = q$, $\eta = p$, $\omega = 1/\sqrt{2}$, $\delta = 1$, and also H_0 at p. 258 in Meyer et al. [3].

It is a linear vector field with the double eigenvalues $\lambda = \pm i/\sqrt{2}$ and Jordan blocks $\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$. The origin is now the unique equilibrium point.

The function $|q|^2$ is a first integral. Suppose the integral curve $(q(t), p(t)) \to 0$ as $t \to -\infty$, then $|q(t)|^2 \equiv 0$ and this fact further implies that $|p(t)|^2 \equiv 0$, indeed for $q(t) \equiv 0$ we have

$$\frac{d}{dt}|p(t)|^2 = 2p(t) \cdot \Big(-q_1(t) + p_2(t)/\sqrt{2}, -q_2(t) - p_1(t)/\sqrt{2}\Big) = 0. \tag{7}$$

So the integral curve is constant and we do not have asymptotic motions to the equilibrium point.

The origin is an unstable equilibrium point as we can see with

$$\begin{aligned} q_1(t) &= \frac{1}{m}\cos\frac{t}{\sqrt{2}}, & q_2(t) &= -\frac{1}{m}\sin\frac{t}{\sqrt{2}}, \\ p_1(t) &= -\frac{t}{m}\cos\frac{t}{\sqrt{2}}, & p_2(t) &= \frac{t}{m}\sin\frac{t}{\sqrt{2}}, \end{aligned} \tag{8}$$

for w $(q_1(0), q_2(0), p_1(0), p_2(0)) = (1/m, 0, 0, 0) \to 0$ as $m \to +\infty$.

Incidentally, in connection with the nonlinear 3-body problem which has the Hamiltonian vector field defined by (6) as linearization at L_4, the book [3] at the end of Sec. 13.6 says that in 1977 two papers claimed to have proved the stability of the equilibrium, however one proof is wrong and the other is unconvincing. The last sentence is: "It would be interesting to give a correct proof of stability in this case, because the linearized system is not simple, and so the linearized equations are unstable".

3. Instability for linearly stable equilibria

3.1. Cherry Hamiltonian

Next, the famous Cherry Hamiltonian system shows that the equilibrium can be unstable even if it is stable for the linearized system, briefly even if it is *linearly stable*. In Cherry [7] p. 199, or in Whittaker [8] p. 412, we can see the Hamiltonian function $H : \mathbb{R}^4 \to \mathbb{R}$

$$H(q,p) = \frac{1}{2}\left(q_1^2 + p_1^2\right) - \left(q_2^2 + p_2^2\right) + \sigma\Big(q_2\left(q_1^2 - p_1^2\right) - 2q_1p_1p_2\Big). \tag{9}$$

The Hamiltonian vector field, written as a column vector, is

$$\begin{pmatrix} p_1 - 2\sigma q_2 p_1 - 2\sigma q_1 p_2 \\ -2p_2 - 2\sigma q_1 p_1 \\ -q_1 - 2\sigma q_2 q_1 + 2\sigma p_1 p_2 \\ 2q_2 + \sigma p_1^2 - \sigma q_1^2 \end{pmatrix}. \tag{10}$$

The linearized vector field $(p_1, -2p_2, -q_1, 2q_2)$ is obtained for $\sigma = 0$. The origin is stable for the linearized systems which consists of two harmonic oscillators: $\ddot{q}_1 = -q_1$, $\ddot{q}_2 = -4q_2$. The eigenvalues are distinct $\pm i$, $\pm 2i$. However, the origin is Lyapunov unstable for the vector field (10) whenever $\sigma \neq 0$ since it has the following asymptotic motion defined for $t < 0$

$$\begin{aligned} q_1(t) &= \frac{\sin t}{\sqrt{2}\,\sigma t}\,, \qquad & q_2(t) &= \frac{\sin(2t)}{2\sigma t}\,, \\ p_1(t) &= \frac{\cos t}{\sqrt{2}\,\sigma t}\,, \qquad & p_2(t) &= -\frac{\cos(2t)}{2\sigma t}\,. \end{aligned} \tag{11}$$

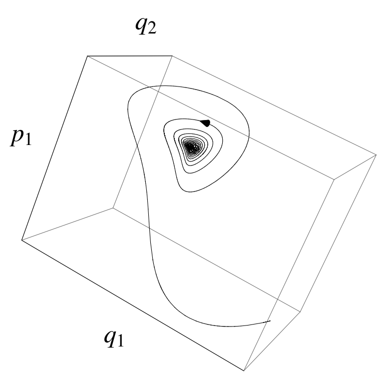

Figure 1: Asymptotic motion for Cherry Hamiltonian

3.2. Variation-like Hamiltonian

Our final example shows that the origin is an unstable equilibrium point which is linearly stable and has no asymptotic motions for the system defined by

$$H(q,p) = p_1p_2 + q_1q_2 + \sigma q_1^2 q_2, \qquad \sigma \neq 0. \tag{12}$$

It is a particular case of the following Hamiltonian function introduced in [10]

$$H(q,p) = p_1p_2 + g(q_1)q_2, \qquad g(0) = 0,\ g'(0) > 0, \tag{13}$$

where $g \in C^1$ on a neighborhood of 0. The Hamiltonian vector field is

$$\begin{pmatrix} p_2 \\ p_1 \\ -g'(q_1)q_2 \\ -g(q_1) \end{pmatrix} = \begin{pmatrix} p_2 \\ p_1 \\ -g'(0)q_2 \\ -g'(0)q_1 \end{pmatrix} + o\big(|(q,p)|\big). \tag{14}$$

The origin is stable for the linearized system which consists of two harmonic oscillators: $\ddot{q}_1 = -g'(0)q_1$, $\ddot{q}_2 = -g'(0)q_2$. In this case the eigenvalues are double $\pm i\sqrt{g'(0)}$ however the Jordan blocks are one-dimensional.

The subsystem of the first and last canonical equations

$$\dot{q}_1 = p_2, \qquad \dot{p}_2 = -g(q_1), \tag{15}$$

separates. If we take a solution $(q_1(t), p_2(t))$ of this subsystem and plug $q_1(t)$ into the second and third canonical equations, we then get the equations of variation of (15) along the solution $(q_1(t), p_2(t))$. This is why the function in formula (13) is called *variation-like Hamiltonian* in the title of this subsection.

There are no asymptotic motions, indeed if the solution

$$\big(q_1(t), q_2(t), p_1(t), p_2(t)\big) \to 0 \qquad \text{as} \quad t \to -\infty \tag{16}$$

then $(q_1(t), p_2(t)) \equiv 0$, since the origin is a local center for (15), and this implies $(q_2(t), p_1(t)) \equiv 0$ too.

In spite of this fact, the origin is unstable for (14) for most functions g as above. Theorem 3.3 in [10] proves that stability is equivalent to the isochrony of the periodic solutions of the subsystem (15) in a neighborhood of $0 \in \mathbb{R}^2$, and this implies the isochronous periodicity of all integral curves of (14) in a neighborhood of $0 \in \mathbb{R}^4$. Moreover, Corollary 2.3 in [10] for a smooth g provides

$$g'''(0) = \frac{5g''(0)^2}{3g'(0)} \tag{17}$$

as the simplest necessary condition for (local isochrony and then) stability. So the choice $g(q_1) = q_1 + \sigma q_1^2$ of the Hamiltonian (12) gives instability for all $\sigma \neq 0$. In Figure 2 we can see the projection on the q_1, q_2-plane of the integral curve of the Hamiltonian vector field given by (12).

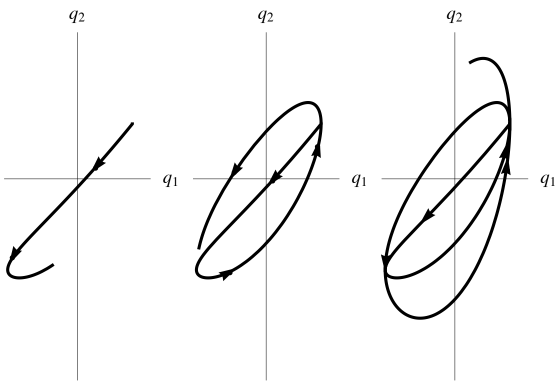

Figure 2: Projection of an unbounded orbit for $H = p_1 p_2 + q_1 q_2 + q_1^2 q_2$

Finally, let us remark that the Hamiltonian (12), composed with the symplectic transformation $(Q,P) \mapsto (Q_1 + Q_2, Q_1 - Q_2, P_1 + P_2, P_1 - P_2)/\sqrt{2}$, becomes

$$\frac{1}{2}\left(Q_1^2 + P_1^2\right) - \frac{1}{2}\left(Q_2^2 + P_2^2\right) + \frac{\sigma}{2\sqrt{2}}(Q_1 + Q_2)\left(Q_1^2 - Q_2^2\right) \tag{18}$$

a function with some features in common with Cherry's Hamiltonian (9).

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