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EQUIVALENCE OF SEQUENTIAL DEFINITIONS OF THE CONVOLUTION OF DISTRIBUTIONS

Abstract. The equivalence of various sequential definitions of the convolution of distributions is proved. The list of known equivalent definitions is extended by adding definitions in terms of upper unit-sequences.

1. Introduction

The convolution of distributions (tempered distributions) is closely connected with the space $\mathcal{D}'_{L^1}$ of integrable distributions, the dual of the space $\mathcal{B}_0$. General definitions of the convolution in $\mathcal{D}'$ (in $\mathcal{S}'$), given in different ways in terms of integrability of certain distributions by various authors (C. Chevalley [1], L. Schwartz [10], R. Shiraishi [11]), appeared to be equivalent (see [11]). Later the list of equivalent definitions was gradually extended, for example by adding various sequential definitions (see V.S. Vladimirov [12, pp. 102–105], P. Dierolf–J. Voigt [2], A. Kamiński [4]). Sequential approaches are interesting, because they lead to natural generalizations connected with suitable restrictions of the considered classes of sequences (see [4]; for another type of generalizations see [13]). A similar situation concerns ultradistributions and tempered ultradistributions: various equivalent definitions of the convolution, including sequential ones, are related to integrability of certain ultradistributions (see [9, 5, 6]).

In sequential definitions of convolvable and integrable distributions and ultradistributions, an essential role is played by specific classes of sequences (called *unit-sequences*) of smooth functions of bounded support approximating the constant function 1. In this paper (see also [7]), we study another type (inspired by papers of B. Fisher, see e.g. [3]) of approximation of the function 1 by specific classes of sequences (called *upper unit-sequences*) of smooth functions with supports bounded only from below. In the next section, we give definitions of the classes Π of unit-sequences and Γ of upper unit-sequences as well as the classes $\overline{\Pi}$ and $\overline{\Gamma}$, narrower than Π and Γ.

Using these classes, we give in section 4 several sequential definitions of the convolution in $\mathcal{D}'$ and prove Theorem 3, the main result of the paper, that all of them are equivalent to the classical definitions mentioned above.

In the proof of Theorem 3 we apply the results and methods from [11, 2, 4, 8] (see section 3) as well as Lemma 1 proved in [7]. Note that a counterpart of Theorem 3 for tempered distributions is also true.

2. Preliminaries

The sets of all positive integers, non-negative integers, reals are denoted by $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{R}$ and their Cartesian powers for a fixed $d \in \mathbb{N}$ by $\mathbb{N}^d$, $\mathbb{N}_0^d$, $\mathbb{R}^d$, respectively. Elements of $\mathbb{R}^d$ and $\mathbb{N}_0^d$ are denoted by Latin and their coordinates by the corresponding Greek letters. Our multi-dimensional notation is mostly standard. In particular, for $x=(\xi_1,\dots,\xi_d), y=(\eta_1,\dots,\eta_d) \in \mathbb{R}^d$ and $\alpha \in \mathbb{R}$, the symbols $x \le y$, $x \le \alpha$ and $\alpha \le x$ mean that the respective inequalities $\xi_i \le \eta_i$, $\xi_i \le \alpha$ and $\alpha \le \xi_i$ hold for all $i = 1,\dots,d$. A similar notation concerns strict inequalities. For $a = (\alpha_1,\dots,\alpha_d) \in \mathbb{R}^d$ we set $[a,\infty) := [\alpha_1,\infty)\times\dots\times[\alpha_d,\infty)$. If $a_n = (\alpha_{n,1},\dots,\alpha_{n,d}) \in \mathbb{R}^d$ for $n \in \mathbb{N}$, we write $a_n \to -\infty$ ($a_n \to \infty$) as $n \to \infty$, whenever $\alpha_{n,i} \to -\infty$ ($\alpha_{n,i} \to \infty$) as $n \to \infty$ for every $i = 1,\dots,d$. Moreover, let $\alpha^k := \alpha^{\kappa_1+\dots+\kappa_d}$ for $\alpha \in \mathbb{R}$ and $k = (\kappa_1,\dots,\kappa_d) \in \mathbb{N}_0^d$.

We will consider, beside the usual support, supp φ, also the *unitary support*, $s^1(\varphi) := \{x \in \mathbb{R}^d :\ \varphi(x) = 1\}$, of a function φ on $\mathbb{R}^d$.

To mark that a set $K \subset \mathbb{R}^d$ is compact we will write $K \sqsubset \mathbb{R}^d$. We use the standard notation: L^∞, $\mathcal{C}^\infty$, $\mathcal{E}$, $\mathcal{B}_0$, $\mathcal{B}$, $\mathcal{D}_K$ ($K \sqsubset \mathbb{R}^d$), $\mathcal{D}$, $\mathcal{D}'$, $\mathcal{D}'_{L^1}$ for known spaces of functions and distributions on $\mathbb{R}^d$ and $\langle f,\varphi\rangle$ for the value of $f \in \mathcal{D}'$ on $\varphi \in \mathcal{D}$, or we use the more precise notation: $L^\infty(\mathbb{R}^d),\dots,\mathcal{D}(\mathbb{R}^d)$, $\mathcal{D}'(\mathbb{R}^d)$, $\mathcal{D}'_{L^1}(\mathbb{R}^d)$ and $\langle f,\varphi\rangle_d$ to indicate the dimension d. For $k \in \mathbb{N}_0$, $K \sqsubset \mathbb{R}^d$ and a smooth (i.e. $\mathcal{C}^\infty$) function φ on $\mathbb{R}^d$, we define

$$q_{k,K}(\varphi) := \max_{0\le i\le k}\max_{x\in K}|\varphi^{(i)}(x)|, \quad q_k(\varphi) := \max_{0\le i\le k}\|\varphi^{(i)}\|_\infty,$$

where $\|\cdot\|_\infty$ denotes the supremum norm; evidently, $q_{k,K}(\varphi) \le q_k(\varphi)$.

Recall that the sets $\mathcal{B}_0$; $\mathcal{B}$; and $\mathcal{D}_K$ ($K \sqsubset \mathbb{R}^d$) consist of all smooth functions φ such that $|\varphi^{(i)}(x)| \to 0$ as $|x| \to \infty$ for $i \in \mathbb{N}_0^d$; $q_k(\varphi) < \infty$ for $k \in \mathbb{N}_0$; and supp $\varphi \subset K$, respectively. Moreover, we have $\mathcal{E} = \mathcal{C}^\infty$ and $\mathcal{D} = \cup_{K\sqsubset\mathbb{R}^d}\mathcal{D}_K$ in the sense of equalities of sets. The sets under consideration are endowed with the topologies defined by the respective families of seminorms: $\mathcal{B}_0$ and $\mathcal{B}$ by the family $\{q_k :\ k \in \mathbb{N}_0\}$; $\mathcal{E}$ by the family $\{q_{k,K} :\ k \in \mathbb{N}_0, K \sqsubset \mathbb{R}^d\}$; and $\mathcal{D}_K$ by the family $\{q_{k,K} :\ k \in \mathbb{N}_0\}$ (for $K \sqsubset \mathbb{R}^d$). The space $\mathcal{D}$ is endowed with the inductive limit topology of the spaces $\mathcal{D}_K$. Clearly,

$$q_k(\varphi\psi) \le 2^k q_k(\varphi) q_k(\psi), \qquad \varphi,\psi \in \mathcal{B},\ k \in \mathbb{N}_0. \tag{1}$$

DEFINITION 1. *By unit-sequence we mean a sequence of functions* $\pi_n \in \mathcal{D}$, *convergent to* 1 *in* $\mathcal{E}$, *such that* $\sup_{n\in\mathbb{N}} \|\pi_n^{(k)}\|_\infty < \infty$ *for* $k \in \mathbb{N}_0^d$, *i.e.*

$$\sup_{n\in\mathbb{N}} q_k(\pi_n) =: M_k < \infty, \qquad k \in \mathbb{N}_0^d. \tag{2}$$

By special unit-sequence we mean such a unit-sequence $\{\pi_n\}$ *that for every bounded* $K \subset \mathbb{R}^d$ *there is an* $n_0 \in \mathbb{N}$ *such that* $\pi_n(x) = 1$ *for* $x \in K$, $n \ge n_0$.

DEFINITION 2. *A set $E \subset \mathbb{R}^d$ is called bounded from below if $E \subset [a,\infty)$ for some $a \in \mathbb{R}^d$. By upper unit-sequence we mean a sequence $\{\Gamma_n\}$ of smooth functions, with supports bounded from below, convergent to 1 in $\mathcal{E}$ (i.e., there are $a_n \in \mathbb{R}^d$ with $a_n \to -\infty$ so that $\operatorname{supp}\Gamma_n \subset [a_n,\infty)$ for $n \in \mathbb{N}$) such that $\sup_{n\in\mathbb{N}} \| \Gamma_n^{(k)} \|_\infty < \infty$ for every $k \in \mathbb{N}_0^d$, i.e.*

$$\sup_{n\in\mathbb{N}} q_k(\Gamma_n) =: N_k < \infty, \qquad k \in \mathbb{N}_0^d. \tag{3}$$

By special upper unit-sequence we mean an upper unit-sequence $\{\Gamma_n\}$ such that if $E \subset \mathbb{R}^d$ is bounded from below, then there is an $n_0 \in \mathbb{N}$ so that $s^1(\Gamma_n) \supset E$ for $n \geq n_0$ (i.e. there are $a_n, b_n \in \mathbb{R}^d$, $a_n < b_n$ $(n \in \mathbb{N})$, with $a_n \to -\infty$, and an index n_1 so that $[a_n,\infty) \supset \operatorname{supp}\Gamma_n \supset s^1(\Gamma_n) \supset [b_n,\infty)$ for $n > n_1$).

The classes of all *unit-sequences*, *special unit-sequences*, *upper unit-sequences*, and *special upper unit-sequences* of functions defined on $\mathbb{R}^d$ will be denoted, respectively, by Π, $\overline{\Pi}$, Γ and $\overline{\Gamma}$ or by Π_d, $\overline{\Pi}_d$, Γ_d and $\overline{\Gamma}_d$ to mark the dimension of $\mathbb{R}^d$.

For arbitrary $\{\pi_n\} \in \Pi$, $\{\Gamma_n\} \in \Gamma$, $\psi \in \mathcal{B}$ and $k \in \mathbb{N}_0^d$, we have

$$\sup_{n\in\mathbb{N}} q_k(\pi_n\psi) \leq 2^k M_k q_k(\psi); \qquad \sup_{n\in\mathbb{N}} q_k(\Gamma_n\psi) \leq 2^k N_k q_k(\psi). \tag{4}$$

Given a class $\mathcal{Y}$ of sequences of functions consider the property:

$(*)$ *Class $\mathcal{Y}$ satisfies the implication: $\{\rho_n\},\{\sigma_n\} \in \mathcal{Y} \Rightarrow \{\tau_n\} \in \mathcal{Y}$, where the sequence is defined by $\tau_{2n-1} := \rho_n$ and $\tau_{2n} := \sigma_n$ for $n \in \mathbb{N}$.*

Clearly, the above defined classes Π, $\overline{\Pi}$, Γ and $\overline{\Gamma}$ satisfy condition $(*)$.

DEFINITION 3. *A distribution f is called extendible for a function $\psi \in \mathcal{B}$ if $\{\langle f, \pi_n\psi\rangle\}$ is a Cauchy sequence for every $\{\pi_n\} \in \Pi$. The mapping $f_\psi\colon\ \mathcal{D} \cup (\psi) \to \mathbb{C}$ (where (ψ) denotes the singleton set), uniquely defined for such a distribution by*

$$\langle f_\psi, \omega\rangle := \lim_{j\to\infty} \langle f, \pi_j\omega\rangle, \qquad \omega \in \mathcal{D} \cup (\psi), \tag{5}$$

for an arbitrary $\{\pi_n\} \in \Pi$, is called the extension of f for the function ψ.

If f is extendible for a $\psi \in \mathcal{B}$, then the limit in (5) does not depend on the choice of the sequence $\{\pi_n\}$ from Π, because the class Π satisfies (I). Consequently, the left side of (5) is well defined for $\omega = \psi$. Moreover, $\langle f_\psi, \varphi\rangle = \lim_{j\to\infty}\langle f, \pi_j\varphi\rangle = \langle f, \varphi\rangle$ for all $\varphi \in \mathcal{D}$ and $\{\pi_n\} \in \Pi$, due to the continuity of f on $\mathcal{D}$, i.e. $f_\psi|_{\mathcal{D}} = f$.

DEFINITION 4. *A distribution f is called extendible for a sequence $\{\Gamma_n\} \in \Gamma$ if is extendible for all Γ_n. The mapping $f^\diamond\colon\ \mathcal{D}^\diamond \to \mathbb{C}$, where $\mathcal{D}^\diamond := \mathcal{D} \cup \{\Gamma_n : n \in \mathbb{N}\}$, uniquely defined for such a distribution by formula (5) for $\psi \in \mathcal{D}^\diamond$, i.e. $\langle f^\diamond, \Gamma_n\rangle := \lim_{j\to\infty}\langle f, \pi_j\Gamma_n\rangle$ for $n \in \mathbb{N}$, is called the extension of f for the sequence $\{\Gamma_n\}$.*

DEFINITION 5. *A distribution f is called extendible to the space $\mathcal{B}$ if it is extendible for every $\psi \in \mathcal{B}$. The mapping $\tilde{f}\colon\ \mathcal{B} \to \mathbb{C}$, uniquely defined for such a distribution by means of formula (5) for every $\psi \in \mathcal{B}$ and $\{\pi_n\} \in \Pi$, i.e. by*

$$\langle \tilde{f}, \psi \rangle := \lim_{j\to\infty} \langle f, \pi_j \psi \rangle, \qquad \psi \in \mathcal{B}, \tag{6}$$

is called the extension of f to the space $\mathcal{B}$.

3. Integrable distributions

Integrable distributions, elements of the topological dual $\mathcal{B}_0'$ of $\mathcal{B}_0$, were described by P. Dierolf and J. Voigt in [2] by several equivalent conditions. To formulate below the extension of their result, proved in [7] and used in the proof of Theorem 3 in section 4, we apply for $\mathcal{A} \subseteq \mathcal{B}$ and $K \sqsubset \mathbb{R}^d$ the notation: $\mathcal{A}^K := \{\varphi \in \mathcal{A}\colon\ \operatorname{supp}\varphi \cap K = \emptyset\}$.

THEOREM 1. *Let $f \in \mathcal{D}'$. The following conditions are equivalent:*

(a_1) *there are an $l \in \mathbb{N}_0$ and a $C > 0$ such that*

$$|\langle f, \varphi \rangle| \le C q_l(\varphi), \qquad \varphi \in \mathcal{D}; \tag{7}$$

(A_1) *f is extendible to $\mathcal{B}$ and its extension $\tilde{f}$ given on $\mathcal{B}$ by (6) is an element of $\mathcal{B}'$, i.e. there are an $l \in \mathbb{N}_0$ and a $C > 0$ such that*

$$|\langle \tilde{f}, \psi \rangle| \le C q_l(\psi), \qquad \psi \in \mathcal{B}; \tag{8}$$

(a_2) *there exists such an $l \in \mathbb{N}_0$ that for every $\varepsilon > 0$ there is a $K \sqsubset \mathbb{R}^d$ such that*

$$|\langle f, \varphi \rangle| \le \varepsilon q_l(\varphi), \qquad \varphi \in \mathcal{D}^K; \tag{9}$$

(A_2) *f is extendible to $\mathcal{B}$ and its extension $\tilde{f}$ given on $\mathcal{B}$ by (6) is an element of $\mathcal{B}'$ with the property: there exists such an $l \in \mathbb{N}_0$ that for every $\varepsilon > 0$ there is a $K \sqsubset \mathbb{R}^d$ for which the inequality holds:*

$$|\langle \tilde{f}, \psi \rangle| \le \varepsilon q_l(\psi), \qquad \psi \in \mathcal{B}^K; \tag{10}$$

(a_3) *there are an $l \in \mathbb{N}_0$, a $C > 0$ and a $K \sqsubset \mathbb{R}^d$ so that (7) holds for all $\varphi \in \mathcal{D}^K$;*

(A_3) *f is extendible to $\mathcal{B}$ and its extension $\tilde{f}$ given on $\mathcal{B}$ by (6) is an element of $\mathcal{B}'$ with the property: there are an $l \in \mathbb{N}_0$, a $C > 0$ and a $K \sqsubset \mathbb{R}^d$ so that (8) holds for all $\varphi \in \mathcal{B}^K$;*

(b) *$\{\langle f, \pi_n \rangle\}$ is a Cauchy sequence for every $\{\pi_n\} \in \Pi$;*

(B) *f is extendible for every sequence $\{\Gamma_n\} \in \Gamma$ and $\{\langle f^\diamond, \Gamma_n\rangle\}$ is a Cauchy sequence, where $f^\diamond$ is the extension of f for the sequence $\{\Gamma_n\}$;*

$(\overline{b})$ *$\{\langle f, \pi_n\rangle\}$ is a Cauchy sequence for every $\{\pi_n\} \in \overline{\Pi}$;*

$(\overline{B})$ *f is extendible for every sequence $\{\Gamma_n\} \in \overline{\Gamma}$ and $\{\langle f^\diamond, \Gamma_n\rangle\}$ is a Cauchy sequence, where $f^\diamond$ is the extension of f for the sequence $\{\Gamma_n\}$.*

If any of the above conditions holds, then

$$\langle \tilde{f}, 1\rangle = \lim_{n\to\infty} \langle f, \pi_n\rangle = \lim_{n\to\infty} \langle f^\diamond, \Gamma_n\rangle, \tag{11}$$

for all $\{\pi_n\} \in \Pi$ and $\{\Gamma_n\} \in \Gamma$.

4. Convolution of distributions

Let $f, g \in \mathcal{D}'(\mathbb{R}^d)$. If the following condition introduced by C. Chevalley in [1]:

$$(C) \qquad (f * \varphi)(\check{g} * \psi) \in L^1(\mathbb{R}^d) \qquad \text{for all } \varphi, \psi \in \mathcal{D}(\mathbb{R}^d),$$

is assumed, there is a unique $f \overset{C}{*} g \in \mathcal{D}'$, the Chevalley convolution of f, g such that

$$\langle (f \overset{C}{*} g) * \varphi, \psi\rangle_d := \int_{\mathbb{R}^d} (f * \varphi)(x)(\check{g} * \psi)(x)\, dx, \qquad \varphi, \psi \in \mathcal{D}(\mathbb{R}^d).$$

L. Schwartz considered in [10] the condition:

$$(S) \qquad (f \otimes g)\,\varphi^\triangle \in \mathcal{D}'_{L^1}(\mathbb{R}^{2d}) \qquad \text{for all } \varphi \in \mathcal{D}(\mathbb{R}^d),$$

where $\varphi^\triangle(x, y) := \varphi(x + y)$ for $x, y \in \mathbb{R}^d$, and R. Shiraishi in [11] the conditions:

$$(S_1) \qquad f(\check{g} * \varphi) \in \mathcal{D}'_{L^1}(\mathbb{R}^d) \qquad \text{for all } \varphi \in \mathcal{D}(\mathbb{R}^d);$$

$$(S_2) \qquad (\check{f} * \varphi)g \in \mathcal{D}'_{L^1}(\mathbb{R}^d) \qquad \text{for all } \varphi \in \mathcal{D}(\mathbb{R}^d),$$

where the symbol $\check{h}$ for a given $h \in \mathcal{D}'(\mathbb{R}^d)$ means the distribution on $\mathbb{R}^d$ defined by $\langle \check{h}, \psi\rangle := \langle h, \check{\psi}\rangle$ and $\check{\psi}(x) := \psi(-x)$ for all $\psi \in \mathcal{D}(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$.

Assuming (S), (S_1), (S_2), they defined the convolutions $f \overset{S}{*} g$, $f \overset{S_1}{*} g$, $f \overset{S_2}{*} g$:

$$\langle f \overset{S}{*} g, \varphi\rangle_d := \langle (f \otimes g)\,\varphi^\triangle, 1_{2d}\rangle_{2d}, \qquad \varphi \in \mathcal{D}(\mathbb{R}^d);$$

$$\langle f \overset{S_1}{*} g, \varphi\rangle_d := \langle f(\check{g} * \varphi), 1_d\rangle_d, \qquad \varphi \in \mathcal{D}(\mathbb{R}^d);$$

$$\langle f \overset{S_2}{*} g, \varphi\rangle_d := \langle (\check{g} * \varphi)g, 1_d\rangle_d, \qquad \varphi \in \mathcal{D}(\mathbb{R}^d),$$

respectively, where 1_d and 1_{2d} are the constant functions equal to 1 on $\mathbb{R}^d$ and $\mathbb{R}^{2d}$.

R. Shiraishi proved in [11] the following theorem:

THEOREM 2. *Let* $f,g \in \mathcal{D}'(\mathbb{R}^d)$. *Conditions* (C), (S), (S_1) *and* (S_2) *are equivalent. If any of the conditions holds, then* $f \overset{C}{*} g = f \overset{S}{*} g = f \overset{S_1}{*} g = f \overset{S_2}{*} g$.

Due to Theorem 2, we may use for $f,g \in \mathcal{D}'(\mathbb{R}^d)$ the common notation

$$f * g := f \overset{C}{*} g = f \overset{S}{*} g = f \overset{S_1}{*} g = f \overset{S_2}{*} g, \tag{12}$$

whenever one of conditions (C), (S), (S_1) and (S_2) is satisfied.

V.S. Vladimirov gave in [12] for $f,g \in \mathcal{D}'(\mathbb{R}^d)$ the following sequential definition of the convolution, denoted here by $f \overset{V}{*} g$:

$$\langle f \overset{V}{*} g\, , \varphi\rangle_d = \lim_{n\to\infty} \langle f \otimes g, \pi_n \varphi^{\triangle}\rangle_{2d}, \qquad \varphi \in \mathcal{D}(\mathbb{R}^d), \tag{13}$$

whenever the limit in (13) exists for all $\{\pi_n\} \in \overline{\Pi}_{2d}$ or, in other words, whenever

$$(\overline{V}) \qquad \{\langle f \otimes g, \pi_n \varphi^{\triangle}\rangle_{2d}\} \in \mathfrak{C} \qquad \text{for all } \{\pi_n\} \in \overline{\Pi}_{2d} \text{ and } \varphi \in \mathcal{D}(\mathbb{R}^d),$$

where $\mathfrak{C}$ denotes the class of all numerical Cauchy sequences. Clearly, condition $(\overline{V})$ implies that the limit in (13) does not depend on $\{\pi_n\} \in \overline{\Pi}_{2d}$.

P. Dierolf and J. Voigt proved in [2] for $f,g \in \mathcal{D}'(\mathbb{R}^d)$ that Vladimirov's condition $(\overline{V})$ and its extension in the following form:

$$(V) \qquad \{\langle f \otimes g, \pi_n \varphi^{\triangle}\rangle_{2d}\} \in \mathfrak{C} \qquad \text{for all } \{\pi_n\} \in \Pi_{2d} \text{ and } \varphi \in \mathcal{D}(\mathbb{R}^d)$$

are equivalent to conditions (C), (S), (S_1), (S_2) and the convolution $f \overset{V}{*} g$ defined for any $\{\pi_n\}$ from the classes $\overline{\Pi}_{2d}$ and Π_{2d} by common formula (13), coincides with the convolution $f * g$ given by (12).

A. Kamiński considered in [4], in connection with J. Mikusiński's irregular operations, the following conditions for $f,g \in \mathcal{D}'(\mathbb{R}^d)$:

$$(K) \qquad \{\langle (\pi_n f) * (\widetilde{\pi}_n g), \varphi\rangle_d\} \in \mathfrak{C} \quad \text{for all } \{\pi_n\}, \{\widetilde{\pi}_n\} \in \Pi_d \text{ and } \varphi \in \mathcal{D}(\mathbb{R}^d);$$

$$(K_1) \qquad \{\langle (\pi_n f) * g, \varphi\rangle_d\} \in \mathfrak{C} \qquad \text{for all } \{\pi_n\} \in \Pi_d \text{ and } \varphi \in \mathcal{D}(\mathbb{R}^d);$$

$$(K_2) \qquad \{\langle f * (\pi_n g), \varphi\rangle_d\} \in \mathfrak{C} \qquad \text{for all } \{\pi_n\} \in \Pi_d \text{ and } \varphi \in \mathcal{D}(\mathbb{R}^d)$$

as well as their variants: $(\overline{K})$, $(\overline{K}_1)$, $(\overline{K}_2)$, in which the class Π_d is replaced by $\overline{\Pi}_d$. He defined the convolutions $f \overset{K}{*} g$, $f \overset{K_1}{*} g$ and $f \overset{K_2}{*} g$ by the following formulas:

$$\langle f \overset{K}{*} g, \varphi\rangle_d := \lim_{n\to\infty} \langle (\pi_n f) * (\widetilde{\pi}_n g), \varphi\rangle_d, \qquad \{\pi_n\}, \{\widetilde{\pi}_n\} \in \Pi_d,\ \varphi \in \mathcal{D}(\mathbb{R}^d);$$

$$\langle f \overset{K_1}{*} g, \varphi\rangle_d := \lim_{n\to\infty} \langle (\pi_n f) * g, \varphi\rangle_d, \qquad \{\pi_n\} \in \Pi_d,\ \varphi \in \mathcal{D}(\mathbb{R}^d);$$

$$\langle f \overset{K_2}{*} g\, , \varphi\rangle_d := \lim_{n\to\infty} \langle f * (\pi_n g), \varphi\rangle_d, \qquad \{\pi_n\} \in \Pi_d,\ \varphi \in \mathcal{D}(\mathbb{R}^d)$$

under conditions (K), (K_1), (K_2), respectively, and by the above formulas restricted to the class $\overline{\Pi}_d$ under conditions $(\overline{K})$, $(\overline{K}_1)$, $(\overline{K}_2)$, respectively. It was shown in [4], due to the results from [11] and [2], that conditions (K), (K_1), (K_2), $(\overline{K})$, $(\overline{K}_1)$, $(\overline{K}_2)$ are equivalent to conditions (V) and $(\overline{V})$ (and to those mentioned previously) and the corresponding convolutions coincide.

Consider now for $f,g \in \mathcal{D}'(\mathbb{R}^d)$, in connection with the classes Γ and $\overline{\Gamma}$ of upper unit-sequences and special upper unit-sequences, the following conditions:

$$(F_V) \qquad \{\langle f \otimes g, \gamma_n \varphi^{\triangle}\rangle_{2d}\} \in \mathfrak{C} \qquad \text{for all } \{\gamma_n\} \in \Gamma_{2d} \text{ and } \varphi \in \mathcal{D}(\mathbb{R}^d);$$

$$(F_K) \qquad \{\langle (\gamma_n f) * (\widetilde{\gamma}_n g), \varphi\rangle_d\} \in \mathfrak{C} \qquad \text{for all } \{\gamma_n\}, \{\widetilde{\gamma}_n\} \in \Gamma_d \text{ and } \varphi \in \mathcal{D}(\mathbb{R}^d)$$

together with their variants $(\overline{F}_V)$ and $(\overline{F}_K)$, in which the classes Γ_{2d} and Γ_d are replaced by $\overline{\Gamma}_{2d}$ and $\overline{\Gamma}_d$, respectively. Define the convolutions $f \overset{F_V}{*} g$ and $f \overset{F_K}{*} g$ by the formulas

$$\langle f \overset{F_V}{*} g, \varphi\rangle_{2d} := \lim_{n\to\infty} \langle f \otimes g, \gamma_n \varphi^{\triangle}\rangle_{2d}, \qquad \{\gamma_n\} \in \Gamma_{2d},\ \varphi \in \mathcal{D}(\mathbb{R}^d);$$

$$\langle f \overset{F_K}{*} g, \varphi\rangle_d := \lim_{n\to\infty} \langle (\gamma_n f) * (\widetilde{\gamma}_n g), \varphi\rangle_d, \qquad \{\gamma_n\}, \{\widetilde{\gamma}_n\} \in \Gamma_d,\ \varphi \in \mathcal{D}(\mathbb{R}^d),$$

under conditions (F_V) and (F_K), respectively, and by the same formulas but restricted to the classes $\overline{\Gamma}_{2d}$ and $\overline{\Gamma}_d$, under conditions $(\overline{F}_V)$ and $(\overline{F}_K)$, respectively.

The following theorem is true:

THEOREM 3. *Let $f,g \in \mathcal{D}'(\mathbb{R}^d)$. Each of the conditions (V), $(\overline{V})$, (K), $(\overline{K})$, (K_1), $(\overline{K}_1)$, (K_2), $(\overline{K}_2)$, (F_V), $(\overline{F}_V)$, (F_K) and $(\overline{F}_K)$ is equivalent to any of the conditions listed in Theorem 2. If any of the conditions is satisfied, then*

$$f * g = f \overset{V}{*} g = f \overset{K}{*} g = f \overset{K_1}{*} g = f \overset{K_2}{*} g = f \overset{F_V}{*} g = f \overset{F_K}{*} g. \tag{14}$$

In the proof of Theorem 3, the following lemma plays an important role:

LEMMA 1. *Let $h \in \mathcal{D}'(\mathbb{R}^{2d})$. Assume that*

$$\operatorname{supp} h \subset K^{\triangle} := \{(x,y) \in \mathbb{R}^{2d} :\ x + y \in K\}$$

for some $K \sqsubset \mathbb{R}^d$ and there is a scalar α such that

$$\lim_{n\to\infty} \langle h, \gamma_n^1 \otimes \gamma_n^2\rangle_{2d} = \alpha \tag{15}$$

for all $\{\gamma_n^1\}, \{\gamma_n^2\} \in \overline{\Gamma}_d$. Then for arbitrary special unit-sequences $\{\pi_n^1\}, \{\pi_n^2\} \in \overline{\Pi}_d$ there exists an increasing sequence $\{q_n\}$ of positive integers such that

$$\lim_{n\to\infty} \langle h, \pi_{q_n}^1 \otimes \pi_{q_n}^2\rangle_{2d} = \alpha. \tag{16}$$

The proof of Lemma 1 is not trivial and requires an induction construction. Its full presentation is beyond the scope of this article. We show a complete proof of Lemma 1 with all its nuances in a separate publication (see [7]).

Proof of Theorem 3. We present the proof of all implications necessary to conclude the formulated equivalence.

$(S) \Rightarrow (F_V)$; $(S) \Rightarrow (V)$. Fix $\varphi \in \mathcal{D}(\mathbb{R}^d)$, denote $h := (f \otimes g)\varphi^{\triangle}$ and assume condition (S), which means that $h \in \mathcal{D}'_{L^1}(\mathbb{R}^{2d})$. By Theorem 1, it follows that

$$\lim_{n\to\infty} \langle h, \Gamma_n \rangle_{2d} = \lim_{n\to\infty} \langle h, \pi_n \rangle_{2d} = \langle h, 1_{2d} \rangle_{2d}$$

for $\{\pi_n\} \in \Pi_{2d}$ and $\{\Gamma_n\} \in \Gamma_{2d}$, i.e. (F_V) and (V) hold. Also $f \overset{F_V}{*} g = f \overset{V}{*} g = f \overset{S}{*} g$.

$(F_V) \Rightarrow (\overline{F}_V)$; $(F_K) \Rightarrow (\overline{F}_K)$; $(V) \Rightarrow (\overline{V})$; $(K_i) \Rightarrow (\overline{K}_i)$ $(i = 1,2)$. The implications are obvious, because of the inclusions $\overline{\Gamma} \subset \Gamma$ and $\overline{\Pi} \subset \Pi$ between the considered classes of (upper) unit-sequences.

$(F_V) \Rightarrow (F_K)$; $(\overline{F}_V) \Rightarrow (\overline{F}_K)$; $(V) \Rightarrow (K)$; $(\overline{V}) \Rightarrow (\overline{K})$. Clearly, the equalities

$$\langle (\Gamma_n^1 f) * (\Gamma_n^2 g), \varphi \rangle_d = \langle (\Gamma_n^1 f) \otimes (\Gamma_n^2 g), \varphi^{\triangle} \rangle_{2d} = \langle (f \otimes g)\varphi^{\triangle}, \Gamma_n^1 \otimes \Gamma_n^2 \rangle_{2d} \tag{17}$$

hold for all $n \in \mathbb{N}$ and $\{\Gamma_n^1\}, \{\Gamma_n^2\} \in \Gamma_d$. Similarly, the equalities

$$\langle (\pi_n^1 f) * (\pi_n^2 g), \varphi \rangle_d = \langle (\pi_n^1 f) \otimes (\pi_n^2 g), \varphi^{\triangle} \rangle_{2d} = \langle (f \otimes g)\varphi^{\triangle}, \pi_n^1 \otimes \pi_n^2 \rangle_{2d} \tag{18}$$

hold for all $n \in \mathbb{N}$ and for all $\{\pi_n^1\}, \{\pi_n^2\} \in \Pi_d$.

If $\{\Gamma_n^1\}$ and $\{\Gamma_n^2\}$ are arbitrary sequences in Γ_d (respectively, in $\overline{\Gamma}_d$), then the sequence $\{\Gamma_n^1 \otimes \Gamma_n^2\}$ is in Γ_{2d} (respectively, in $\overline{\Gamma}_{2d}$). Hence, by (17), condition (F_V) (respectively, $(\overline{F}_V)$) implies condition (F_K) (respectively, $(\overline{F}_K)$). Moreover, $f \overset{F_V}{*} g = f \overset{F_K}{*} g$. Similarly, if $\{\pi_n^1\}$ and $\{\pi_n^2\}$ are in Π_d (respectively, in $\overline{\Pi}_d$), then the sequence $\{\pi_n^1 \otimes \pi_n^2\}$ is in Π_{2d} (respectively, in $\overline{\Pi}_{2d}$) and, due to (18), condition (V) (respectively, $(\overline{V})$) implies condition (K) (respectively, $(\overline{K})$). Moreover, $f \overset{V}{*} g = f \overset{K}{*} g$.

$(\overline{F}_K) \Rightarrow (\overline{K})$. Fix $\varphi \in \mathcal{D}(\mathbb{R}^d)$ and denote $h := (f \otimes g)\varphi^{\triangle}$. Assume that condition $(\overline{F}_K)$ holds, i.e. there is a number α such that (15) holds for all $\{\Gamma_n^1\}, \{\Gamma_n^2\} \in \overline{\Gamma}_d$. Fix $\{\pi_n^1\}, \{\pi_n^2\} \in \overline{\Pi}_d$ and let $\{\widetilde{\pi}_n^1\}$ and $\{\widetilde{\pi}_n^2\}$ be arbitrary subsequences of $\{\pi_n^1\}$ and $\{\pi_n^2\}$. By Lemma 1, there exist subsequences $\{\widetilde{\pi}_{q_n}^1\}$ and $\{\widetilde{\pi}_{q_n}^2\}$ of $\{\widetilde{\pi}_n^1\}$ and $\{\widetilde{\pi}_n^2\}$, respectively, such that (16) holds. Then

$$\lim_{n\to\infty} \langle h, \pi_n^1 \otimes \pi_n^2 \rangle_{2d} = \alpha \tag{19}$$

for our arbitrarily fixed sequences $\{\pi_n^1\}, \{\pi_n^2\} \in \overline{\Pi}_d$, i.e. condition $(\overline{K})$ is satisfied. Moreover, $f \overset{F_K}{*} g = f \overset{K}{*} g$ (for the classes $\overline{\Gamma}$ and $\overline{\Pi}$).

$(K) \Rightarrow (K_i)$ $(i = 1,2)$; $(\overline{K}) \Rightarrow (\overline{K}_i)$ $(i = 1,2)$. Fix $\varphi \in \mathcal{D}(\mathbb{R}^d)$ and let $h := (f \otimes g)\varphi^{\triangle}$. Assume that condition (K) (respectively, $(\overline{K})$) is satisfied, i.e. equality (19) is true for some α and for all $\{\pi_n^1\}$ and $\{\pi_n^2\}$ in Π_d (respectively, in $\overline{\Pi}_d$). For arbitrarily fixed sequences $\{\pi_n^1\}$ and $\{\pi_n^2\}$ in Π_d (respectively, in $\overline{\Pi}_d$), the assumption implies

$$\lim_{n,m\to\infty} \langle h, \pi_n^1 \otimes \pi_m^2 \rangle_{2d} = \alpha \tag{20}$$

with the double limit on the left side. If not, there would exist an $\varepsilon_0 > 0$ and increasing sequences $\{p_n\}$, $\{q_n\}$ of indices such that $|\langle h, \pi^1_{p_n} \otimes \pi^2_{q_n} \rangle_{2d} - \alpha| > \varepsilon_0$ for $n \in \mathbb{N}$. But the sequences $\{\pi^1_{p_n}\}$ and $\{\pi^2_{q_n}\}$ are also in Π_d (respectively, in $\overline{\Pi}_d$), so the last inequality contradicts our assumption concerning (19) and proves (20).

Since $\varphi^{\triangle}(\pi^1_n \otimes 1_d)$ and $\varphi^{\triangle}(1_d \otimes \pi^2_m)$ are in $\mathcal{D}(\mathbb{R}^{2d})$ for $n, m \in \mathbb{N}$, we have

$$\lim_{m\to\infty} \langle h, \pi^1_n \otimes \pi^2_m \rangle_{2d} = \langle f \otimes g, \varphi^{\triangle}(\pi^1_n \otimes 1_d) \rangle_{2d} = \langle (\pi^1_n f) * g, \varphi \rangle_d, \qquad n \in \mathbb{N}$$

and

$$\lim_{n\to\infty} \langle h, \pi^1_n \otimes \pi^2_m \rangle_{2d} = \langle f \otimes g, \varphi^{\triangle}(1_d \otimes \pi^2_m) \rangle_{2d} = \langle f * (\pi^2_m g), \varphi \rangle_d, \qquad m \in \mathbb{N}.$$

Hence, by (20),

$$\lim_{n\to\infty} \langle (\pi^1_n f) * g, \varphi \rangle_d = \lim_{n\to\infty} \lim_{m\to\infty} \langle h, \pi^1_n \otimes \pi^2_m \rangle_{2d} = \alpha = \lim_{m\to\infty} \langle f * (\pi^2_m g), \varphi \rangle_d.$$

The implications and the equalities $f \overset{K}{*} g = f \overset{K_1}{*} g = f \overset{K_2}{*} g$ are proved.

$(\overline{K}_i) \Rightarrow (S_i)$ $(i = 1, 2)$. For all $n \in \mathbb{N}$, we have

$$\langle (\pi^1_n f) * g, \varphi \rangle_d = \langle f(\check{g} * \varphi), \pi^1_n \rangle_d \quad \text{and} \quad \langle f * (\pi^2_n g, \varphi \rangle_d = \langle (\check{f} * \varphi) g, \pi^2_n \rangle_d,$$

so conditions $(\overline{K}_1)$ and $(\overline{K}_2)$ imply (S_1) and (S_2), respectively, by Theorem 1.

Since, by Theorem 2, conditions (S), (S_1) and (S_2) are equivalent, the proof of the equivalence of all conditions and of all equalities in (14) is thus completed. $\square$

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