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Generalized Functions, Linear and Nonlinear Problems, I

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ABSTRACT STOCHASTIC PROBLEMS IN SPACES OF DISTRIBUTIONS

Abstract. The Cauchy problem for the equation $u'(t) = Au(t) + B\mathbb{W}(t)$, $t \geq 0$, with white noise $\mathbb{W}$ and A being the generator of regularized semigroups is studied in different spaces of distributions. Solutions of the problem in spaces of distributions with respect to time variable, random variable and both time and random variables are studied.

1. Introduction

The Cauchy problem for operator-differential equations with white noise as an inhomogeneity often arises as a model of different evolution processes subject to random perturbations. The basic one among them is the Cauchy problem

$$X'(t) = AX(t) + B\mathbb{W}(t), \quad t \in [0, \tau), \ \tau \leq \infty, \quad X(0) = \zeta, \tag{1}$$

where A is the generator of a C_0-semigroup. Because of irregularity of the white noise $\mathbb{W}$ it is usually reduced to an integral equation with the "primitive" of $\mathbb{W}$, i.e. with some Wiener process (e.g. [14, 12]).

Our work is devoted to generalized solutions of the stochastic Cauchy problem (1) with A not necessarily being the generator of a C_0 semigroup, but being the generator of a regularized, namely integrated semigroup $V = \{V(t), \ t \in [0, \tau)\}$ in a Hilbert space H. We suppose $\{\mathbb{W}(t), t \geq 0\}$ to be an $\mathbb{H}$-valued white noise which we define in our work rigorously in different spaces of distributions, $B \in \mathcal{L}(\mathbb{H}, H)$.

The fact that the operator A is generating only an integrated semigroup means that the solution operators $U(t)$, $t \in [0, \tau)$, of the corresponding homogeneous Cauchy problem are not bounded. Therefore one has to introduce some regularized family V instead of $\{U(t)\}$ or consider the solution operators of problem (1) in certain spaces of distributions. At the same time due to irregularity, particularly to discontinuity of the white noise (it is informally defined as a process with independent identically distributed random values with infinite variation) one has to reduce the Cauchy problem to the above-mentioned integral equation with a Wiener process which is defined axiomatically as the infinite dimensional generalization of Brownian motion, or to consider the problem (1) in certain spaces of distributions. The choice of a proper space of distributions depends on the conditions imposed on A and initial value ζ on one hand and on the properties of the noise $\mathbb{W}$ on the other hand.

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In the next section (section 2) we give necessary definitions from the theory of regularized semigroups, Hilbert space-valued (abstract) Wiener processes, abstract Schwartz distributions and stochastic distributions.

In section 3 we consider the problem (1) in spaces of Hilbert space valued distributions with respect to one variable. If it is the time variable t (subsection 3.1) we obtain existence of a unique solution for A generating an n-times integrated semigroup, but $\mathbb{W}$ must be a Q-white noise $\mathbb{W}_Q$, where Q is a nuclear operator in $\mathbb{H}$. If we consider the problem in the space of distributions with respect to the random variable ω (subsection 3.2), we obtain the result for the equation with the singular white noise ($Q = I$), but A must be the generator of a C_0-semigroup. In section 4 we introduce the space of distributions with respect to both t and ω and obtain the result for A generating an n-times integrated semigroup and $\mathbb{W}$ being the singular white noise.

The beginning of research in this direction was made in [14, 11, 10]. In [13, 15] different approaches to defining of distributions in t and ω were studied.

2. Definitions: regularized semigroups, abstract Wiener processes and abstract distributions

2.1. Regularized semigroups

Let A be a closed linear operator and $R(t)$, $t \geq 0$, be bounded linear operators in a Banach space H.

DEFINITION 1. *A strongly continuous family $V = \{V(t),\ t \in [0,\tau)\}$, $\tau \leq \infty$, of bounded operators in H is called an R-regularized semigroup with the generator A if*

$$V(t)A\zeta = AV(t)\zeta, \quad \zeta \in \operatorname{dom} A, \qquad V(t)\zeta = A\int_0^t V(s)\zeta ds + R(t)\zeta, \quad \zeta \in H\,.$$

The semigroup V is called exponentially bounded if $\|V(t)\| \leq Me^{\varpi t}$, $t \geq 0$, for some $M > 0$, $\varpi \in \mathbb{R}$, and local if $\tau < \infty$.

If $R(t) = \left(t^n/n!\right)I$, then V is also an *n-times integrated semigroup*. If $\overline{\operatorname{dom} A} = H$ and $R(t) \equiv R$ is invertible, bounded and densely defined, then V is an *R-semigroup*. If $R = I$, then an R-semigroup is a *C_0-semigroup*.

Note that an R-semigroup in [3] is defined as a strongly continuous family of bounded operators satisfying the R-semigroup property

$$V(t+s)R = V(t)V(s), \quad s,t,s+t \in [0,\tau),\ V(0) = R$$

with the infinitesimal generator

$$\mathcal{G} f := (\lambda - L_\lambda^{-1}) f,\ \lambda > \varpi,\ \operatorname{dom} \mathcal{G} = \left\{ f \in H:\ Rf \in \operatorname{ran} L_\lambda \right\},\ L_\lambda f := \int_0^\infty e^{\lambda t} V(t) f dt\,.$$

It is called there a C-semigroup. We prefer the term "R-semigroup" that reflects its regularizing property and makes it differ from C_0-semigroups, where C comes from "continuity".

As to integrated semigroups, they are also defined via corresponding "semigroup property" in [2] with the infinitesimal generator, but we will use the equivalent general Definition 1. We refer to [9, 8] for examples of integrated, convolution, R-semigroups and their generators, including important differential operators.

2.2. Wiener processes

Let $(\Omega, \mathcal{F}, P)$ be a probability space, $\mathbb{H}$ be a Hilbert space and Q be a linear symmetric positive trace class operator with a system of eigenvectors $\{e_i\}$, forming a basis of $\mathbb{H}$, such that $Qe_i = \sigma_i^2 e_i$, $\sum_{i=1}^{\infty} \sigma_i^2 < \infty$.

DEFINITION 2. *A stochastic process* $W_Q = \{W_Q(t),\ t \geq 0\}$ *with values in* $\mathbb{H}$ *is called a Q-Wiener process, if*

$(W1)$ $W_Q(0) = 0$ *a.s.;*

$(W2)$ W_Q *has independent increments;*

$(W3)$ *the increments* $W_Q(t) - W_Q(s)$ *are normally distributed with mean zero and covariance operator equal to* $(t-s)Q$*;*

$(W4)$ *the trajectories of* W_Q *are continuous a.s.*

Thus defined Q-Wiener process is a generalization of Brownian motion. It is well known that Brownian motion $\{\beta(t), t \geq 0\}$, where $\beta(t) = \beta(\omega, t)$, $\omega \in \Omega$, is defined via conditions (W1)–(W4) in the case $\mathbb{H} = \mathbb{R}$ and $Q = I$. A finite-dimensional Brownian motion has form $\sum_{i=1}^{n} \beta_i(t) e_i$, where $\{e_i\}$ is an orthonormal basis in $\mathbb{R}^n$ and β_i are independent Brownian motions. When passing to infinite dimensions, to avoid divergency in $\mathbb{H}$, one has to consider a regularized sum

$$W_Q(t) := \sum_{i=1}^{\infty} \sigma_i \beta_i(t) e_i,\ t \geq 0, \qquad W_Q(t) \in L_2(\Omega; \mathbb{H}),$$

which happens to be an $\mathbb{H}$-valued Q-Wiener process.

The formal series $\sum_{i=1}^{\infty} \beta_i(t) e_i =: W(t)$ is called a *cylindrical Wiener process*.

2.3. Spaces of abstract distributions. White noise in spaces of abstract distributions

For any Banach space $\mathcal{X}$ by $\mathcal{D}'(\mathcal{X})$ we denote the space of all $\mathcal{X}$-valued distributions over the space of test function $\mathcal{D}$. In contrary to the $\mathbb{R}$-valued Schwartz distributions they are called *abstract distributions*. By $\mathcal{D}'_0(\mathcal{X})$ we denote the subspace of distributions having supports in $[0, \infty)$.

Let $\mathbb{H}$ now be a Hilbert space and W_Q be an $\mathbb{H}$-valued Q-Wiener process. Since W_Q has continuous in $t \geq 0$ trajectories for almost all $\omega \in \Omega$, define *Q-white noise* $\mathbb{W}_Q$ (with trajectories) in $\mathcal{D}_0'(\mathbb{H})$ as generalized derivative of W_Q set to be zero at $t < 0$, i.e. by the following equality:

$$\langle \mathbb{W}_Q, \theta \rangle := -\int_0^\infty W_Q(t)\theta'(t)\,dt = \int_0^\infty \theta(t)\,dW_Q(t), \quad \theta \in \mathcal{D}. \tag{2}$$

The first integral in (2) is understood as Bochner integral of an $L_2(\Omega;\mathbb{H})$-valued function, the second one — as an abstract Ito integral with respect to the Wiener process. The equality of the integrals follows from the Ito formula.

We will further use convolution of distributions defined as follows (see. [4]).

DEFINITION 3. *Let $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$ be Banach spaces, such that there exists a continuous bilinear operation $(u,v) \mapsto uv \in \mathcal{Z}$ defined on $\mathcal{X}\times\mathcal{Y}$. For any $G \in \mathcal{D}_0'(\mathcal{X})$ and $F \in \mathcal{D}_0'(\mathcal{Y})$ the convolution $G * F \in \mathcal{D}_0'(\mathcal{Z})$ is defined by the equality*

$$\langle G*F, \theta\rangle := \left\langle (g*f)^{(n+m)}, \theta \right\rangle = (-1)^{n+m}\int_0^\infty (g*f)(t)\theta^{n+m}(t)\,dt, \quad \theta \in \mathcal{D},$$

where $g : \mathbb{R} \to \mathcal{X}$, $f : \mathbb{R} \to \mathcal{Y}$ are continuous functions such that

$$\langle G, \theta\rangle = (-1)^n \int_0^\infty g(t)\theta^{(n)}(t)\,dt, \ \langle F, \theta\rangle = (-1)^m \int_0^\infty f(t)\theta^{(m)}(t)\,dt,$$

$(g*f)(t) := \int_0^t g(t-s)f(s)ds$.

Note that in the particular case when G is a regular distribution, i.e. $\langle G, \theta\rangle = \int_0^\infty G(t)\theta(t)\,dt$, the equality $\langle G*F, \theta\rangle = \displaystyle\int_0^\infty G(t)\langle F(\cdot), \theta(t+\cdot)\rangle\,dt$ holds.

2.4. Spaces of abstract stochastic distributions. Singular white noise

The theory of stochastic distributions uses the white noise probability space. It is the triple $(\mathcal{S}', \mathcal{B}(\mathcal{S}'), \mu)$, where $\mathcal{B}(\mathcal{S}')$ is the Borel σ-field of $\mathcal{S}'$ (the Schwartz space of tempered distributions), μ is the centered Gaussian or white noise measure on $\mathcal{B}(\mathcal{S}')$ satisfying the equality

$$\int_{\mathcal{S}'} e^{i\langle\omega,\theta\rangle} d\mu(\omega) = e^{-\frac{1}{2}|\theta|_0^2}, \quad \theta \in \mathcal{S},$$

where $|\cdot|_0$ is the norm of $L_2(\mathbb{R})$. Existence of such measure is stated by the Bochner–Minlos theorem (see, e.g. [6]).

The construction of spaces of *abstract stochastic distributions* [6] is analogous to the construction of the Gelfand triple $\mathcal{S} \subset L_2(\mathbb{R}) \subset \mathcal{S}'$. Its central element is the space (L^2) of all functions of $\omega \in \mathcal{S}'$ which are square integrable with respect to the measure μ. Hermite functions $\xi_k(x) = \pi^{-\frac{1}{4}}\big((k-1)!\big)^{-\frac{1}{2}} e^{-\frac{x^2}{2}} h_{k-1}(x)$ (where

$h_k(x) = (-1)^k e^{\frac{x^2}{2}}(d/dx)^k e^{-\frac{x^2}{2}}$, are Hermite polynomials) are the eigenfunctions of the differential operator $\hat{D} = -\frac{d^2}{dt^2} + t^2 + 1$ with $\hat{D}\xi_k = (2k)\xi_k$, $k \in \mathbb{N}$, and form an orthonormal basis of $L_2(\mathbb{R})$. Stochastic Hermite polynomials $\mathbf{h}_\alpha(\omega) := \prod_k h_{\alpha_k}(\langle\omega, \xi_k\rangle)$, $\omega \in \mathcal{S}'$, where $\alpha \in \mathcal{T}$ (the set of all finite multi-indices) form an orthogonal basis of (L^2) with

$$(\mathbf{h}_\alpha, \mathbf{h}_\beta)_{(L^2)} = \alpha!\,\delta_{\alpha,\beta}, \qquad \alpha! := \prod_k \alpha_k!\,.$$

They are the eigenfunctions of the second quantization operator $\Gamma(\hat{D})$. We have

$$\Gamma(\hat{D})\mathbf{h}_\alpha = \prod_k (2k)^{\alpha_k}\mathbf{h}_\alpha =: (2\mathbb{N})^\alpha \mathbf{h}_\alpha\,.$$

The space of test functions $(\mathcal{S})$ is a countably-Hilbert space $(\mathcal{S}) = \bigcap_{p\in\mathbb{N}} (\mathcal{S}_p)$ with the projective limit topology, where

$$(\mathcal{S}_p) = \left\{\varphi = \sum_{\alpha\in\mathcal{T}} \varphi_\alpha \mathbf{h}_\alpha \in (L^2) : \sum_{\alpha\in\mathcal{T}} \alpha!|\varphi_\alpha|^2(2\mathbb{N})^{2p\alpha} < \infty\right\}$$

with the norm $|\cdot|_p$, generated by the scalar product

$$(\varphi,\psi)_p = (\Gamma(\hat{D})^p\varphi, \Gamma(\hat{D})^p\psi)_{(L^2)} = \sum_{\alpha\in\mathcal{T}} \alpha!\varphi_\alpha\overline{\psi}_\alpha(2\mathbb{N})^{2p\alpha}.$$

Its adjoint space $(\mathcal{S})'$ is called the space of stochastic (Hida) distributions (random variables). We have $(\mathcal{S})' = \cup_{p\in\mathbb{N}}(\mathcal{S}_{-p})$ with the inductive limit topology, where $(\mathcal{S}_{-p})$ is the adjoint of $(\mathcal{S}_p)$. The space $(\mathcal{S}_{-p})$ can be identified with the space of all formal expansions $\Phi = \sum_{\alpha\in\mathcal{T}} \Phi_\alpha \mathbf{h}_\alpha$, satisfying $\sum_{\alpha\in\mathcal{T}} \alpha!|\Phi_\alpha|^2(2\mathbb{N})^{-2p\alpha} < \infty$, with scalar product

$$(\Phi,\Psi)_{-p} = (\Gamma(\hat{D})^{-p}\varphi, \Gamma(\hat{D})^{-p}\psi)_{(L^2)} = \sum_{\alpha\in\mathcal{T}} \alpha!\Phi_\alpha\overline{\Psi}_\alpha(2\mathbb{N})^{-2p\alpha}.$$

Denote the corresponding norm by $|\cdot|_{-p}$. We have:

$$\langle\Phi,\varphi\rangle = \sum_{\alpha\in\mathcal{T}} \alpha!\Phi_\alpha\varphi_\alpha \text{ for } \Phi = \sum_{\alpha\in\mathcal{T}} \Phi_\alpha\mathbf{h}_\alpha \in (\mathcal{S})', \quad \varphi = \sum_{\alpha\in\mathcal{T}} \varphi_\alpha\mathbf{h}_\alpha \in (\mathcal{S})\,.$$

Thus we have the following Gelfand triple: $(\mathcal{S}) \subset (L^2) \subset (\mathcal{S})'$.

Define $(\mathcal{S})'(\mathbb{H})$, the space of $\mathbb{H}$-valued generalized random variables over $(\mathcal{S})$ as the space of linear continuous operators $\Phi : (\mathcal{S}) \to \mathbb{H}$ with the topology of uniform convergence on bounded subsets of $(\mathcal{S})$. Denote the action of $\Phi \in (\mathcal{S})'(\mathbb{H})$ on $\varphi \in (\mathcal{S})$ by $\Phi[\varphi]$. The structure of $(\mathcal{S})'(\mathbb{H})$ is due to the next proposition (see the proof in [7]).

PROPOSITION 1. *Any $\Phi \in (\mathcal{S})'(\mathbb{H})$ can be extended to a bounded operator from $(\mathcal{S}_p)$ to $\mathbb{H}$ for some $p \in \mathbb{N}$.*

The space $(\mathcal{S})$ is a nuclear countably Hilbert space since for any $p \in \mathbb{N}$ the embedding $I_{p,p+1} : (\mathcal{S}_{p+1}) \hookrightarrow (\mathcal{S}_p)$ is a Hilbert–Schmidt operator. From this fact and proposition 1 one deduces

COROLLARY 1. *Any $\Phi \in (\mathcal{S})'(\mathbb{H})$ is a Hilbert–Schmidt operator from $(\mathcal{S}_p)$ to $\mathbb{H}$ for some $p \in \mathbb{N}$.*

For any $\Phi \in (\mathcal{S})'(\mathbb{H})$ denote by Φ_j the linear functional defined on $\varphi \in (\mathcal{S})$ by $\langle \Phi_j, \varphi \rangle := (\Phi[\varphi], e_j)$. Let p be such that Φ is Hilbert–Schmidt from $(\mathcal{S}_p)$ to $\mathbb{H}$. Then all Φ_j, $j \in \mathbb{N}$, belong to the corresponding space $(\mathcal{S}_{-p})$, thus we have

$$\Phi_j = \sum_{\alpha \in \mathcal{T}} \Phi_{\alpha,j} \mathbf{h}_\alpha, \quad \sum_{\alpha \in \mathcal{T}} \alpha! |\Phi_{\alpha,j}|^2 (2\mathbb{N})^{-2p\alpha} < \infty.$$

For the Hilbert–Schmidt norm of $\Phi : (\mathcal{S}_p) \to \mathbb{H}$ we obtain:

$$\|\Phi\|^2_{\mathrm{HS},p} = \sum_{\alpha \in \mathcal{T}} \left\| \Phi \left[\frac{\mathbf{h}_\alpha}{(\alpha!)^{\frac{1}{2}} (2\mathbb{N})^{p\alpha}} \right] \right\|^2 = \sum_{\alpha \in \mathcal{T}} \sum_{j=1}^{\infty} \left| \left\langle \Phi_j, \frac{\mathbf{h}_\alpha}{(\alpha!)^{\frac{1}{2}} (2\mathbb{N})^{p\alpha}} \right\rangle \right|^2$$

$$= \sum_{\alpha \in \mathcal{T}, j \in \mathbb{N}} \alpha! |\Phi_{\alpha,j}|^2 (2\mathbb{N})^{-2p\alpha}.$$

Denote by $(\mathcal{S}_{-p})(\mathbb{H})$ the space of all Hilbert–Schmidt operators acting form $(\mathcal{S}_p)$ to $\mathbb{H}$. It is a separable Hilbert space with an orthogonal basis consisting of operators $\mathbf{h}_\alpha \otimes e_j$, $\alpha \in \mathcal{T}$, $j \in \mathbb{N}$, defined by

$$(\mathbf{h}_\alpha \otimes e_j)\varphi := \big(\mathbf{h}_\alpha, \varphi\big)_{(L^2)} e_j, \quad \varphi \in (\mathcal{S}_p).$$

It follows from corollary 1 that $(\mathcal{S})'(\mathbb{H}) = \bigcup_{p \in \mathbb{N}} (\mathcal{S}_{-p})(\mathbb{H})$ and any $\Phi \in (\mathcal{S})'(\mathbb{H})$ has the decomposition

$$\Phi = \sum_{j \in \mathbb{N}} \Phi_j e_j = \sum_{\alpha \in \mathcal{T}, j \in \mathbb{N}} \Phi_{\alpha,j} (\mathbf{h}_\alpha \otimes e_j) = \sum_{\alpha \in \mathcal{T}} \Phi_\alpha \mathbf{h}_\alpha,$$

where $\Phi_j = (\Phi[\cdot], e_j) \in (\mathcal{S}_{-p})$ for some $p \in \mathbb{N}$, $\Phi_\alpha = \sum_{j \in \mathbb{N}} \Phi_{\alpha,j} e_j \in \mathbb{H}$. For the norm $\| \cdot \|^2_{-p} := \| \cdot \|^2_{\mathrm{HS},p}$ we have

$$\|\Phi\|^2_{-p} = \sum_{j \in \mathbb{N}} |\Phi_j|^2_{-p} = \sum_{\alpha \in \mathcal{T}, j \in \mathbb{N}} \alpha! |\Phi_{\alpha,j}|^2 (2\mathbb{N})^{-2p\alpha} = \sum_{\alpha \in \mathcal{T}} \alpha! \|\Phi_\alpha\|^2 (2\mathbb{N})^{-2p\alpha} < \infty.$$

We evidently have

$$(\mathcal{S}_{-p_1})(\mathbb{H}) \subseteq (\mathcal{S}_{-p_2})(\mathbb{H}) \ \text{ for } \ p_1 < p_2,$$

and

$$\|\Phi\|_{-p_1} \geq \|\Phi\|_{-p_2} \quad \text{for all } \Phi \in (\mathcal{S}_{-p_1})(\mathbb{H}).$$

To define singular white noise in these spaces first define a sequence of independent Brownian motions $\{\beta_j(t)\}$. Let $n:\mathbb{N}\times\mathbb{N}\to\mathbb{N}$ be a bijection with the property $n=n(i,j)\geq ij$. As it was done in [11, 5], we use the Fourier coefficients of the decomposition of Brownian motion $\beta(t)$ in $(L_2)(\mathbb{R})$:

$$\beta(t,\omega)=\langle\omega,\mathbf{1}_{[0,t]}\rangle=\left\langle\omega,\sum_{i=1}^{\infty}\int_0^t\xi_i(s)ds\,\xi_i\right\rangle=\sum_{i=1}^{\infty}\int_0^t\xi_i(s)\,ds\,\mathbf{h}_{\varepsilon_i}\,,$$

where $\varepsilon_i:=(0,0,...,\underset{i}{1},0,...)$. Defining $\beta_j(t)=\sum_{i=1}^{\infty}\int_0^t\xi_i(s)\,ds\,\mathbf{h}_{\varepsilon_{n(i,j)}}$, we obtain the next decomposition for the Wiener process $W(t), t\geq 0$:

$$W(t)=\sum_{j=1}^{\infty}\beta_j(t)\,e_j=\sum_{i,j\in\mathbb{N}}\int_0^t\xi_i(s)ds(\mathbf{h}_{\varepsilon_{n(i,j)}}\otimes e_j)=\sum_{n=1}^{\infty}\int_0^t\xi_{i(n)}(s)ds(\mathbf{h}_{\varepsilon_n}\otimes e_{j(n)}).$$

Its derivative with respect to t is called singular $\mathbb{H}$-valued white noise. It has the following decomposition:

$$\mathbb{W}(t)=\sum_{i,j\in\mathbb{N}}\xi_i(t)(\mathbf{h}_{\varepsilon_{n(i,j)}}\otimes e_j)=\sum_{i,j\in\mathbb{N}}\mathbb{W}_{\varepsilon_{n(i,j)}}(t)\mathbf{h}_{\varepsilon_{n(i,j)}}\,,\quad \mathbb{W}_{\varepsilon_{n(i,j)}}(t)=\xi_i(t)e_j\,.$$

By the well known estimates $\left|\int_0^t\xi_i(s)\,ds\right|^2=O(i^{-\frac{3}{2}})$ and $|\xi_i(t)|=O(i^{-1/4})$ of the Hermite functions, we obtain

$$\|W(t)\|_{-1}^2=\sum_{i,j\in\mathbb{N}}\left|\int_0^t\xi_i(s)\,ds\right|^2\big(2n(i,j)\big)^{-2}\leq C\sum_{i,j\in\mathbb{N}}i^{-7/2}\,j^{-2}<\infty,$$

$$\|\mathbb{W}(t)\|_{-1}^2=\sum_{i,j\in\mathbb{N}}|\xi_i(t)|^2\big(2n(i,j)\big)^{-2}\leq C\sum_{i,j\in\mathbb{N}}i^{-5/2}\,j^{-2}<\infty.$$

Thus $W(t)\in(\mathcal{S}_{-1})(\mathbb{H})\subset(\mathcal{S})'(\mathbb{H})$ and $\mathbb{W}(t)\in(\mathcal{S}_{-1})(\mathbb{H})\subset(\mathcal{S})'(\mathbb{H})$ for all $t\geq 0$.

Convergence in the space $(\mathcal{S})'(\mathbb{H})$ is characterized by the next proposition [7].

PROPOSITION 2. *Let* $\Phi_n=\sum_\alpha\Phi_\alpha^{(n)}\mathbf{h}_\alpha\,,\Phi=\sum_\alpha\Phi_\alpha\mathbf{h}_\alpha\in(\mathcal{S})'(\mathbb{H})$. *The following assertions are equivalent:*

(i) $\Phi_n\to\Phi$ *in* $(\mathcal{S})'(\mathbb{H})$;

(ii) all elements of the sequence $\{\Phi_n\}$ *and* Φ *belong to* $(\mathcal{S}_{-p})(\mathbb{H})$ *for some* $p\in\mathbb{N}$ *and* $\lim_{n\to\infty}\|\Phi_n-\Phi\|_{-p}=0$,

It follows from this propositions that differentiation with respect to t of an $(\mathcal{S})'(\mathbb{H})$-valued function $\Phi(t)$ is equivalent to its differentiation as of a function with values in $(\mathcal{S}_{-p})(\mathbb{H})$ for some $p\in\mathbb{N}_0$. It is easy to see that $\frac{d}{dt}W(t)=\mathbb{W}(t)$ for all $t\in\mathbb{R}$. We will call an $(\mathcal{S})'(\mathbb{H})$-valued function $\Phi(t)$ integrable on $[a;b]\subset\mathbb{R}$ if it is Bochner integrable as an $(\mathcal{S}_{-p})(\mathbb{H})$-valued function for some p.

3. Solutions of stochastic Cauchy problem generalized with respect to one of the variables

3.1. Generalized solutions with respect to t

Let A be a closed linear operator acting from H to $[\mathrm{dom} A]$ (the domain of A endowed with the graph-norm), $B \in \mathcal{L}(\mathbb{H}; H)$, $\zeta \in H$ and let $\mathbb{W}_Q$ be an $\mathbb{H}$-valued Q-white noise, defined by (2).

We define the *generalized solution* of the Cauchy problem (1) with $\mathbb{W} = \mathbb{W}_Q$ to be a distribution $X \in \mathcal{D}'_0(L_2(\Omega; [\mathrm{dom} A]))$ satisfying the equation

$$P * X = \delta \otimes \zeta + B\mathbb{W}_Q, \tag{3}$$ [1]

where $P := \delta' \otimes I - \delta \otimes A \in \mathcal{D}'_0\big(\mathcal{L}([\mathrm{dom} A], H)\big)$

A distribution $G \in \mathcal{D}'_0\big(\mathcal{L}(H, [\mathrm{dom} A])\big)$ is called the *convolution inverse* for $P \in \mathcal{D}'_0\big(\mathcal{L}([\mathrm{dom} A], H)\big)$ if $G * P = \delta \otimes I_{[\mathrm{dom} A]}$, $P * G = \delta \otimes I_H$.

By the properties of the convolution inverse it is proved in [1] that the generalized random process X defined by

$$\langle X, \theta \rangle := \langle G\zeta, \theta \rangle + \langle G * B\mathbb{W}_Q, \theta \rangle, \quad \theta \in \mathcal{D}, \tag{4}$$

is the unique solution of (3) in the space $\mathcal{D}'_0(L_2(\Omega, [\mathrm{dom} A]))$. As a consequence we obtain the next result.

THEOREM 1. *Suppose that A is the generator of an n-times integrated semigroup $V = \{V(t),\ t \geq 0\}$. The Cauchy problem (1) with $\mathbb{W} = \mathbb{W}_Q$ has a unique solution $X \in \mathcal{D}'_0(L_2(\Omega; [\mathrm{dom} A]))$ given by the formula*

$$\langle X, \theta \rangle = (-1)^n \left[\int_0^\infty \theta^{(n)}(t) V(t) \zeta \, dt - \int_0^\infty \theta^{(n+1)}(t) \, dt \int_0^t V(t-s) B W_Q(s) \, ds \right].$$

This follows from the fact that the convolution inverse G in this case is the generalized derivative of V of order n; therefore, by (4) and (2) we obtain the result.

3.2. Generalized solutions with respect to ω

To consider the Cauchy problem (1) in the space $(\mathcal{S})'(H)$ we define the action of a linear closed operator $A : H \to H$ and linear bounded operator $B : \mathbb{H} \to H$ in the next way.

For any $\Phi = \sum_\alpha \Phi_\alpha \mathbf{h}_\alpha \in (\mathcal{S})'(\mathbb{H})$ define $B\Phi := \sum_\alpha B\Phi_\alpha \mathbf{h}_\alpha$. Thus B evidently becomes a linear continuous mapping of $(\mathcal{S})'(\mathbb{H})$ into $(\mathcal{S})'(H)$.

[1] For $u \in \mathcal{D}'$, $h \in H$ by $u \otimes h$ we denote the distribution from $\mathcal{D}'(H)$ defined by the equality $\langle u \otimes h, \theta \rangle := \langle u, \theta \rangle h$.

Define $(\mathrm{dom}A)$ to be the set of all $\Phi = \sum_\alpha \Phi_\alpha \mathbf{h}_\alpha \in (\mathcal{S})'(H)$ such that $\Phi_\alpha \in \mathrm{dom}A$ for all $\alpha \in \mathcal{T}$ and $\sum_\alpha \|A\Phi_\alpha\|_H^2 (2\mathbb{N})^{-2p\alpha} < \infty$ for some p. For any $\Phi = \sum_\alpha \Phi_\alpha \mathbf{h}_\alpha \in (\mathrm{dom}A)$ define $A\Phi := \sum_\alpha A\Phi_\alpha \mathbf{h}_\alpha$.

The following theorem is proved in [1].

THEOREM 2. *Let A be the generator of a C_0-semigroup $\{U(t),\ t \geq 0\}$ in a Hilbert space H. Then for any $\zeta \in (\mathrm{dom}A) \subset (\mathcal{S})'(H)$ the Cauchy problem (1) with the singular white noise $\mathbb{W}$ has the unique solution*

$$X(t) = U(t)\zeta + \int_0^t U(t-s)B\mathbb{W}(s)ds \in (\mathcal{S})'(H), t \geq 0.$$

The solution is constructed as the series $X(t) = \sum_\alpha X_\alpha(t)\mathbf{h}_\alpha, t \geq 0$, where

$$X_\alpha(t) = \begin{cases} U(t)\zeta_{\varepsilon_n} + \int_0^t U(t-s)B\mathbb{W}_{\varepsilon_n}(s)\,ds, & \alpha = \varepsilon_n, \\ U(t)\zeta_\alpha, & \alpha \neq \varepsilon_n \end{cases}$$

are the solutions of the well-posed Cauchy problems

$$X'_{\varepsilon_n}(t) = AX_{\varepsilon_n}(t) + B\mathbb{W}_{\varepsilon_n}(t), \quad X_{\varepsilon_n}(0) = \zeta_{\varepsilon_n},$$
$$X'_\alpha(t) = AX_\alpha(t), \quad X_\alpha(0) = \zeta_\alpha \text{ for } \alpha \neq \varepsilon_n.$$

4. Generalized solutions with respect to t and ω

We see from the results of the previous section that in order to solve the Cauchy problem (1) with weaker conditions imposed on A, namely with A generating an n-times integrated semigroup, one has to consider it in the space $\mathcal{D}_0'(L_2(\Omega;H))$ of distributions in variable t. At the same time this forces one to take a Q-white noise $\mathbb{W}_Q$ with a nuclear operator Q as $\mathbb{W}$. In order to introduce the white noise with $Q = I$ into the equation one has to state the problem (1) in the space $(\mathcal{S})'(H)$ of distributions with respect to the random variable ω, but under this approach one has to impose more restrictive conditions on A, namely it must be the generator of a C_0-semigroup. This suggests the idea of combining the two approaches and considering the problem (1) in a suitable space of distributions in both t and ω.

Recall that the singular white noise $\mathbb{W}(t)$ belongs to the Hilbert space $(\mathcal{S}_{-1})(\mathbb{H}) \subset (\mathcal{S})'(\mathbb{H})$ of all Hilbert–Schmidt operators acting from $(\mathcal{S}_1)$ to $\mathbb{H}$ for each $t \in \mathbb{R}$. Consider the space $\mathcal{D}_0'((\mathcal{S}_{-1})(\mathbb{H}))$ of abstract $(\mathcal{S}_{-1})(\mathbb{H})$-valued distributions with supports in $[0;\infty)$ over the space $\mathcal{D}$ of test functions. Denote by $\mathbb{W}$ the distribution defined by

$$\langle \mathbb{W}, \theta \rangle = \int_0^\infty \mathbb{W}(t)\theta(t)dt, \quad \theta \in \mathcal{D}.$$

It is easy to see that $\mathbb{W}(t)$ is a continuous $(\mathcal{S}_{-1})(\mathbb{H})$-valued function of t, therefore $\mathbb{W} \in \mathcal{D}_0'((\mathcal{S}_{-1})(\mathbb{H}))$.

In section 3.2 we defined the action of $B \in \mathcal{L}(\mathbb{H}, H)$ as a linear continuous mapping of $(\mathcal{S})'(\mathbb{H})$ into $(\mathcal{S})'(H)$. Denote by the same symbol its restriction to $(\mathcal{S}_{-1})(\mathbb{H})$. It is easy to see that it is a linear bounded operator from $(\mathcal{S}_{-1})(\mathbb{H})$ to $(\mathcal{S}_{-1})(H)$.

Now we define the action of A in $(\mathcal{S}_{-1})(H)$. By $(\mathrm{dom}A)_{-1}$ we denote the set of all $\Phi = \sum_\alpha \Phi_\alpha \mathbf{h}_\alpha \in (\mathcal{S}_{-1})(H)$ such that

$$\Phi_\alpha \in \mathrm{dom}A \ \text{ for all } \ \alpha \in \mathcal{T} \ \text{ and } \ \sum_\alpha \|A\Phi_\alpha\|_H^2 (2\mathbb{N})^{-2\alpha} < \infty.$$

For any $\Phi = \sum_\alpha \Phi_\alpha \mathbf{h}_\alpha \in (\mathrm{dom}A)_{-1}$ define $A\Phi := \sum_\alpha A\Phi_\alpha \mathbf{h}_\alpha$. Since A is linear and closed as an operator in H, this defines A as a closed linear operator in $(\mathcal{S}_{-1})(H)$. Denote by $[(\mathrm{dom}A)_{-1}]$ the space $(\mathrm{dom}A)_{-1}$ with the graph norm. With such defined A we can consider the operator $P = \delta' \otimes I - \delta \otimes A$ as a distribution belonging to the space $\mathcal{D}_0'\Big(\mathcal{L}\big([(\mathrm{dom}A)_{-1}]; (\mathcal{S}_{-1})(H)\big)\Big)$.

We will call $X \in \mathcal{D}_0'\big([(\mathrm{dom}A)_{-1}]\big)$ a solution of problem (1) with $\zeta \in (\mathcal{S}_{-1})(H)$ if it satisfies the equation $P * X = \delta \otimes \zeta + B\mathbb{W}$.

As a straightforward generalization of Theorem 1 to the case of $(\mathcal{S}_{-1})(H)$-valued functions we obtain the following result.

THEOREM 3. *Let A be the generator of an n-times integrated semigroup $V = \{V(t), t \geq 0\}$. Then for any $\zeta \in (\mathcal{S}_{-1})(H)$ the Cauchy problem (1) has a unique solution $X \in \mathcal{D}_0'\big([(\mathrm{dom}A)_{-1}]\big)$ given by the formula*

$$\langle X, \theta \rangle = (-1)^n \left[\int_0^\infty \theta^{(n)}(t) V(t) \zeta dt + \int_0^\infty \theta^{(n)}(t) dt \int_0^t V(t-s) B\mathbb{W}(s) ds \right].$$

where the integrals are understood as the Bochner integrals of $(\mathcal{S}_{-1})(H)$-valued functions.

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