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A COHOMOLOGY VANISHING THEOREM AND LAPLACE HYPERFUNCTIONS WITH HOLOMORPHIC PARAMETERS

Abstract. From 1987 onwards, the theory of Laplace hyperfunctions has been developed by H. Komatsu. Laplace hyperfunctions are represented as a class of holomorphic functions of exponential type. The aim of this paper is to give the vanishing theorem of cohomology groups on a pseudoconvex open subset for holomorphic functions with exponential growth at infinity. As an application of the theorem, we construct the sheaf of Laplace hyperfunctions and that with holomorphic parameters, and we also study several properties of these sheaves. This is a short summary of our paper [1].

1. Introduction

The theory of Laplace hyperfunctions has been developed by H. Komatsu (in [3]–[8]) to give a rigid framework of operational calculus for functions without growth conditions at infinity.

Let us briefly recall the definition of Laplace hyperfunctions with support in $[a,\infty]$ $(a \in \mathbb{R} \sqcup \{+\infty\})$ and that of their Laplace transforms (see [3]–[8]). Let $\mathbb{D}^2$ be the radial compactification $\mathbb{C} \sqcup S^1$ of the complex plane whose topology is defined in the usual way (see the next section). Let $\mathcal{O}_{\mathbb{C}}^{\exp}$ be the sheaf of holomorphic functions of exponential type, that is, if V is an open set in $\mathbb{D}^2$, then $\mathcal{O}_{\mathbb{C}}^{\exp}(V)$ denotes the space of all holomorphic functions $F(z)$ on $V \cap \mathbb{C}$ such that for any compact set K in V there are positive constants H and C for which we have

$$|F(z)| \leqq Ce^{H|z|}, \qquad z \in K \cap \mathbb{C}.$$

Then the space $\mathcal{B}_{[a,\infty]}^{\exp}$ of Laplace hyperfunctions with support in $[a,\infty]$ is defined as the quotient space

$$\mathcal{B}_{[a,\infty]}^{\exp} := \frac{\mathcal{O}_{\mathbb{C}}^{\exp}(\mathbb{D}^2 \setminus [a,\infty])}{\mathcal{O}_{\mathbb{C}}^{\exp}(\mathbb{D}^2)}. \tag{1}$$

Let $f(x)$ be a hyperfunction with support in $[a,\infty]$ with its defining function $F(z) \in \mathcal{O}_{\mathbb{C}}^{\exp}(\mathbb{D}^2 \setminus [a,\infty])$. Then the Laplace transform $\widehat{f}(\lambda)$ of $f(x)$ is defined by the integral

$$\widehat{f}(\lambda) := \int_C e^{-\lambda z} F(z)\, dz,$$

where the path C of integration is composed of a ray from $e^{i\alpha}\infty$ $(-\pi/2 < \alpha < 0)$ to a point $c < a$ and a ray from c to $e^{i\beta}\infty$ $(0 < \beta < \pi/2)$.

As we have seen, the Laplace hyperfunctions are defined by global sections of holomorphic functions of exponential type. Therefore it is an important problem to localize the notion of Laplace hyperfunctions and to construct the sheaf of Laplace hyperfunctions whose global sections with support in $[a,\infty]$ give ones introduced by H. Komatsu. In this paper, we construct the sheaf of Laplace hyperfunctions and that with holomorphic parameters by establishing the vanishing theorem of cohomology groups on a pseudoconvex open subset for holomorphic functions with exponential growth at infinity. The vanishing theorem established here not only plays an important role in the construction of the sheaf of Laplace hyperfunctions but also has independent interest as Example 2 shows. For the details and the proof of the theorems in this paper, see N. Honda and K. Umeta [1]

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2. The vanishing theorem for holomorphic functions of exponential type

We need to introduce several notions before stating our vanishing theorem. Let $n \in \mathbb{N}$, let m be a non-negative integer and let $\mathbb{D}^{2n}$ be the *radial compactification* of $\mathbb{C}^n$, that is, the set $\mathbb{D}^{2n}$ is the disjoint union of $\mathbb{C}^n$ and the real $(2n-1)$-dimensional unit sphere $S^{2n-1} \subset \mathbb{R}^{2n}$.

Let $X := \mathbb{C}^{n+m}$ and $\hat{X}$ be the partial radial compactification $\mathbb{D}^{2n} \times \mathbb{C}^m$ of $\mathbb{C}^{n+m}$. We denote by X_∞ the closed subset $\hat{X} \setminus X$ in $\hat{X}$, and we denote by

$$p_1 : \hat{X} = \mathbb{D}^{2n} \times \mathbb{C}^m \to \mathbb{D}^{2n} \quad (\text{resp. } p_2 : \hat{X} = \mathbb{D}^{2n} \times \mathbb{C}^m \to \mathbb{C}^m)$$

the canonical projection to the first (resp. second) space. A family of fundamental neighborhoods of $(z_0, w_0) \in X \subset \hat{X}$ consists of

$$B_\varepsilon(z_0,w_0) := \{(z,w) \in X;\, |z-z_0| < \varepsilon,\, |w-w_0| < \varepsilon\} \tag{2}$$

for $\varepsilon > 0$, and that of $(z_0, w_0) \in X_\infty$ consists of a product of an open cone and an open ball

$$G_r(\Gamma, w_0) := \left(\left\{z \in \mathbb{C}^n;\, |z| > r,\, \frac{z}{|z|} \in \Gamma\right\} \cup \Gamma\right) \times \left\{w \in \mathbb{C}^m;\, |w-w_0| < \frac{1}{r}\right\}, \tag{3}$$

where $r > 0$ and Γ runs through open neighborhoods of z_0 in S^{2n-1}.

We denote by $\mathcal{O}_X$ the sheaf of holomorphic functions on X.

DEFINITION 1. *Let Ω be an open subset in $\hat{X}$. The set $\mathcal{O}_X^{\exp}(\Omega)$ of holomorphic functions of exponential type on Ω consists of holomorphic functions $f(z,w)$ on $\Omega \cap X$ which satisfy, for any compact set K in Ω,*

$$|f(z,w)| \le C_K e^{H_K|z|}, \qquad ((z,w) \in K \cap X), \tag{4}$$

with some positive constants C_K and H_K. We denote by $\mathcal{O}_X^{\exp}$ the associated sheaf on $\hat{X}$ of the presheaf $\{\mathcal{O}_X^{\exp}(\Omega)\}_\Omega$.

Let A be a subset in $\hat{X}$. We define the set $\mathrm{clos}^1_\infty(A) \subset X_\infty$ as follows. A point $(z, w) \in X_\infty$ belongs to $\mathrm{clos}^1_\infty(A)$ if and only if there exist points $\{(z_k, w_k)\}_{k\in\mathbb{N}}$ in $A \cap X$ that satisfy

$$(z_k, w_k) \to (z, w) \text{ in } \hat{X} \text{ and } \frac{|z_{k+1}|}{|z_k|} \to 1 \quad (k \to \infty).$$

Set

$$N^1_\infty(A) := X_\infty \setminus \mathrm{clos}^1_\infty(X \setminus A).$$

DEFINITION 2. *Let U be an open subset in $\hat{X}$. We say that U is regular at ∞ if $N^1_\infty(U) = U \cap X_\infty$ is satisfied.*

EXAMPLE 1. We give some examples of open subsets which are regular at ∞.

- Let U be the open set $G_r(\Gamma, 0) \cup \tilde{U}$, where $\tilde{U}$ is a bounded open subset in X and the cone $G_r(\Gamma, 0)$ is defined by (3) with $r > 0$ and Γ an open subset in S^{2n-1}. Then U is regular at ∞. In particular, $\mathbb{D}^2$ and $\mathbb{D}^2 \setminus [a, +\infty]$ $(a \in [-\infty, \infty))$ are regular at ∞.
- For the set $U := \mathbb{D}^2 \setminus \{1, 2, 3, 4, \ldots, +\infty\}$ we have $N^1_\infty(U) = S^1 \setminus \{+\infty\}$, and hence U is regular at ∞. However, $U := \mathbb{D}^2 \setminus \{1, 2, 4, 8, 16, \ldots, +\infty\}$ is not regular because of $N^1_\infty(U) = S^1$.

For a subset A in X, we denote by $\mathrm{dist}(p, A)$ the distance between a point p and A, i.e.,

$$\mathrm{dist}(p, A) := \inf_{q\in A} |p - q|.$$

For convenience, set $\mathrm{dist}(p, A) = +\infty$ if A is empty. We also define, for $q = (z, w) \in X$,

$$\mathrm{dist}_{\mathbb{D}^{2n}}(q, A) := \mathrm{dist}\left(q, A \cap p_2^{-1}(p_2(q))\right) = \inf_{(\zeta, w)\in A} |z - \zeta|.$$

Let Ω be an open subset in $\hat{X}$. We set

$$(5)\qquad \begin{aligned} &\psi(p) := \min\left\{\frac{1}{2}, \frac{\mathrm{dist}_{\mathbb{D}^{2n}}(p, X \setminus \Omega)}{1 + |z|}\right\}, \qquad (p = (z, w) \in X), \\ &\Omega_\varepsilon := \left\{p = (z, w) \in \Omega \cap X; \mathrm{dist}(p, X \setminus \Omega) > \varepsilon, |w| < \frac{1}{\varepsilon}\right\}, \quad (\varepsilon > 0). \end{aligned}$$

Now we give the main theorem.

THEOREM 1. *Assume the following two conditions:*

1. *$\Omega \cap X$ is pseudoconvex in X and Ω is regular at ∞.*
2. *At a point in $\Omega \cap X$ sufficiently close to $z = \infty$ the function $\psi(z, w)$ is continuous and uniformly continuous with respect to the variables w, that is, for any $\varepsilon > 0$,*

there exist $\delta_\varepsilon > 0$ *and* $R_\varepsilon > 0$ *for which* $\psi(z,w)$ *is continuous on the open set* $\Omega_{\varepsilon,R_\varepsilon} := \Omega_\varepsilon \cap \{|z| > R_\varepsilon\}$ *and satisfies*

$$\bigl|\psi(z,w) - \psi(z,w')\bigr| < \varepsilon, \qquad \bigl((z,w),(z,w') \in \Omega_{\varepsilon,R_\varepsilon},\ |w-w'| < \delta_\varepsilon\bigr).$$

Then we have

$$H^k(\Omega, \mathcal{O}_X^{\exp}) = 0, \qquad (k \neq 0). \tag{6}$$

As condition 2. in the theorem is automatically satisfied for a product of open sets, we have the following corollary.

COROLLARY 1. *Let* U *(resp.* W*) be an open subset in* $\mathbb{D}^{2n}$ *(resp.* $\mathbb{C}^n$*). If* $U \cap \mathbb{C}^n$ *and* W *are pseudoconvex in* $\mathbb{C}^n$ *and* $\mathbb{C}^m$ *respectively and if* U *is regular at* ∞ *in* $\mathbb{D}^{2n}$*, then* (6) *holds for* $\Omega := U \times W$*.*

Note that, in the later section, we will see that, if $n = 1$, the vanishing theorem still holds for an open subset of product type without the regularity condition at ∞. However, if n is greater than one, one cannot expect the vanishing theorem anymore without the regularity condition at ∞ as the following example shows.

EXAMPLE 2. Assume $n = 2$ and $m = 0$, i.e., $X = \mathbb{C}^2_{(z_1,z_2)}$ and $\hat{X} = \mathbb{D}^4$. Set

$$\begin{aligned} &U := \left\{(z_1,z_2) \in X;\, |\arg(z_1)| < \frac{\pi}{4},\, |z_2| < |z_1|\right\}, \\ &\Omega := \bigl(\overline{U}\bigr)^\circ \setminus \{p_\infty\} \subset \hat{X}, \end{aligned}$$

where p_∞ denotes the point $(1,0,0,0)$ in $S^3 \subset \mathbb{D}^4$. The open subset $\Omega \cap X = U$ is pseudoconvex in X, while Ω is not regular at ∞. Then we have $H^1(\Omega, \mathcal{O}_X^{\exp}) \neq 0$.

3. Laplace hyperfunctions with holomorphic parameters

By Theorem 1, we can construct cohomologically the sheaf $\mathcal{B}_{\mathbb{R}}^{\exp}$ of Laplace hyperfunctions and the sheaf $\mathcal{B}\mathcal{O}_N^{\exp}$ of Laplace hyperfunctions with holomorphic parameters.

Let $N = \mathbb{R} \times \mathbb{C}^m (m \geq 0)$, and let $\overline{N} = \overline{\mathbb{R}} \times \mathbb{C}^m$ be the closure of N inside $\hat{X} = \mathbb{D}^2 \times \mathbb{C}^m$. Then we have the following theorem.

THEOREM 2. *The closed set* $\overline{N}$ *is purely* 1*-codimensional with respect to the sheaf* $\mathcal{O}_X^{\exp}$*, i.e.,*

$$\mathscr{H}^k_{\overline{N}}(\mathcal{O}_X^{\exp}) = 0, \quad (k \neq 1).$$

Here $\mathscr{H}^k_{\overline{N}}(\mathcal{O}_X^{\exp})$ *is the k-th derived sheaf of* $\mathcal{O}_X^{\exp}$ *with support in* $\overline{N}$*.*

As a particular case, we have the following corollary.

COROLLARY 2. *$\overline{\mathbb{R}}$ is purely 1-codimensional with respect to the sheaf $\mathcal{O}_{\mathbb{C}}^{\exp}$, that is,*

$$\mathscr{H}_{\overline{\mathbb{R}}}^{k}(\mathcal{O}_{\mathbb{C}}^{\exp}) = 0 \quad (k \neq 1).$$

DEFINITION 3. *The sheaf $\mathcal{BO}_N^{\exp}$ of Laplace hyperfunctions of one variable with holomorphic parameters is defined by*

$$\mathcal{BO}_N^{\exp} := \mathscr{H}_{\overline{N}}^{1}(\mathcal{O}_X^{\exp}) \underset{\mathbb{Z}_{\overline{N}}}{\otimes} \omega_{\overline{N}},$$

where $\mathbb{Z}_{\overline{N}}$ denotes the constant sheaf on $\overline{N}$ having stalk $\mathbb{Z}$ and $\omega_{\overline{N}}$ denotes the orientation sheaf $\mathscr{H}_{\overline{N}}^{1}(\mathbb{Z}_{\hat{X}})$ on $\overline{N}$.

The global sections of the sheaf $\mathcal{BO}_N^{\exp}$ can be written in terms of cohomology groups by Theorem 2. For an open set $\Omega \subset \overline{\mathbb{R}}$ and a pseudoconvex open subset $T \subset \mathbb{C}^m$, by taking a complex neighborhood V of Ω in $\mathbb{D}^2$, we have

$$\mathcal{BO}_N^{\exp}(\Omega \times T) = H^1_{\Omega \times T}(V \times T, \mathcal{O}_X^{\exp}) = \frac{\mathcal{O}_X^{\exp}((V \setminus \Omega) \times T)}{\mathcal{O}_X^{\exp}(V \times T)}.$$

Note that the above representation does not depend on a choice of the complex neighborhood V.

DEFINITION 4. *We define the sheaf $\mathcal{B}_{\mathbb{R}}^{\exp}$ of Laplace hyperfunctions of one variable on $\overline{\mathbb{R}}$ by*

$$\mathcal{B}_{\mathbb{R}}^{\exp} := \mathscr{H}_{\overline{\mathbb{R}}}^{1}(\mathcal{O}_{\mathbb{C}}^{\exp}) \underset{\mathbb{Z}_{\overline{\mathbb{R}}}}{\otimes} \omega_{\overline{\mathbb{R}}},$$

where $\mathbb{Z}_{\overline{\mathbb{R}}}$ denotes the constant sheaf on $\overline{\mathbb{R}}$ having stalk $\mathbb{Z}$ and $\omega_{\overline{\mathbb{R}}}$ denotes the orientation sheaf $\mathscr{H}_{\overline{\mathbb{R}}}^{1}(\mathbb{Z}_{\hat{X}})$ on $\overline{\mathbb{R}}$.

The restriction of $\mathcal{B}_{\mathbb{R}}^{\exp}$ to $\mathbb{R}$ is isomorphic to the sheaf $\mathcal{B}_{\mathbb{R}}$ of ordinary hyperfunctions because of $\mathcal{O}_{\mathbb{C}}^{\exp}|_{\mathbb{C}} = \mathcal{O}_{\mathbb{C}}$. By Corollary 2 we have

$$\Gamma_{[a,\infty]}(\overline{\mathbb{R}}, \mathcal{B}_{\mathbb{R}}^{\exp}) = \frac{\mathcal{O}_{\mathbb{C}}^{\exp}(\mathbb{D}^2 \setminus [a,\infty])}{\mathcal{O}_{\mathbb{C}}^{\exp}(\mathbb{D}^2)}.$$

Hence the set $\mathcal{B}_{[a,\infty]}^{\exp}$ defined by H. Komatsu coincides with $\Gamma_{[a,\infty]}(\overline{\mathbb{R}}, \mathcal{B}_{\mathbb{R}}^{\exp})$ in our framework.

4. Several properties of $\mathcal{BO}_N^{\exp}$

We can also show the vanishing theorem on an open subset which is not necessarily regular at ∞ if $n = 1$. This fact is deeply related to the flabbiness of $\mathcal{BO}_N^{\exp}$.

THEOREM 3. *Let U be an open subset in $\mathbb{D}^2$, and W a pseudoconvex open subset in $\mathbb{C}^m$. Then we have*

$$H^k(U \times W, \mathcal{O}_X^{\exp}) = 0, \qquad (k \neq 0).$$

The sets N, $\overline{N}$ and $\hat{X}$ are the same as those in the previous section. Now we state the theorems for the flabbiness and the unique continuation property of $\mathcal{BO}_N^{\exp}$.

THEOREM 4. *Let Ω_1 and Ω_2 be open subsets in $\overline{\mathbb{R}}$ with $\Omega_1 \subset \Omega_2$, and let W be a pseudoconvex open subset in $\mathbb{C}^m$. Then the restriction*

$$\mathcal{BO}_N^{\exp}(\Omega_2 \times W) \to \mathcal{BO}_N^{\exp}(\Omega_1 \times W)$$

is surjective.

COROLLARY 3 ([3]). *The sheaf $\mathcal{B}_{\mathbb{R}}^{\exp}$ of Laplace hyperfunctions is flabby.*

The following theorem shows that the sheaf $\mathcal{BO}_N^{\exp}$ has a unique continuation property with respect to holomorphic parameters.

THEOREM 5. *Let W_1 and W_2 be non-empty connected open subsets in $\mathbb{C}^m$ with $W_1 \subset W_2$ and Ω an open subset in $\overline{\mathbb{R}}$. Then the restriction*

$$\mathcal{BO}_N^{\exp}(\Omega \times W_2) \longrightarrow \mathcal{BO}_N^{\exp}(\Omega \times W_1)$$

is injective.

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