Rend. Sem. Mat. Univ. Politec. Torino
Vol. 69, 3 (2011), 281 – 298

E. Buzano, A. Fino and L. Vezzoni

THE CALABI–YAU EQUATION FOR T^2-BUNDLES OVER T^2: THE NON-LAGRANGIAN CASE

Abstract. In the spirit of [8, 2], we study the Calabi–Yau equation on T^2-bundles over T^2 endowed with an invariant non-Lagrangian almost-Kähler structure showing that for T^2-invariant initial data it reduces to a Monge–Ampère equation having a unique solution. In this way we prove that for every total space M^4 of an orientable T^2-bundle over T^2 endowed with an invariant almost-Kähler structure the Calabi–Yau problem has a solution for every normalized T^2-invariant volume form.

1. Introduction

Let (M^{2n},J,Ω) be a $2n$-dimensional compact Kähler manifold with associated complex structure J and symplectic form Ω. In view of a celebrated Yau's theorem [12] for every volume form σ on M^{2n} satisfying

$$\int_{M^{2n}} \Omega^n = \int_{M^{2n}} \sigma \tag{1}$$

there exists a unique Kähler form $\tilde{\Omega}$ in the same de Rham cohomology class of Ω and such that

$$\tilde{\Omega}^n = \sigma\,. \tag{2}$$

Equation (2) still makes sense in the *almost-Kähler* context when J is merely an almost-complex structure and Ω remains closed. The almost complex structure J is still orthogonal relative to a Riemannian metric g for which $\Omega(X,Y) = g(JX,Y)$, and

$$\tilde{\Omega} = \Omega + d\alpha \tag{3}$$

is again assumed to be a positive-definite $(1,1)$-form relative to J. In this context the equations (1), (2) and (3) constitute the *Calabi–Yau problem*, which in the last years has been intensively studied in four dimensions (see [1, 9, 8, 2] and the references therein).

In [1] Donaldson introduced the Calabi–Yau problem for almost-Kähler manifolds showing that equation (2) has unique solution in dimension four and that it is related to some other central problems in symplectic geometry. In [9] Tosatti, Weinkove and Yau gave a sufficient condition for the existence of solution to the Calabi–Yau equation in terms of the Chern connection. This condition fails in case of the Kodaria–Thurston surface, which is a 4-dimensional nilmanifold, i.e. a compact quotient of the nilpotent Lie group $\mathrm{Nil}^3 \times \mathbb{R}$ by a lattice, where Nil^3 denotes the 3-dimensional real Heisenberg group.

The Kodaira–Thurston surface is a typical example of a compact almost-Kähler 4-dimensional manifold which does not admit any Kähler structure. More precisely, it is the total space of a principal T^2-bundle over a torus $\mathbb{T}^2$ (in our notation from now on, T^2 denotes the 2-torus of the fibre, while $\mathbb{T}^2$ is the base 2-torus) and it has an invariant almost-Kähler structure whose symplectic form vanishes along the fibres of the T^2-fibration, where by invariant structure we mean a structure induced by a left-invariant one on $\mathrm{Nil}^3 \times \mathbb{R}$. The almost-Kähler structures on a total space of a fibration whose symplectic form vanishes along the fibres are usually called *Lagrangian*, since the fibers are Lagrangian submanifolds.

In [8] Tosatti and Weinkove studied the Calabi–Yau equation on the Kodaira–Thurston surface endowed with an invariant Lagrangian almost-Kähler structure, establishing the existence of a solution for every T^2-invariant normalized volume form σ. Then in [2], the previous result obtained by Tosatti and Weinkove was simplified and extended to other T^2-bundles over a $\mathbb{T}^2$ endowed with an invariant Lagrangian almost-Kähler structure.

We recall that in view of [10] every orientable T^2-bundle over a $\mathbb{T}^2$ is a infra-solvmanifold, i.e. a smooth quotient $\Gamma\backslash G$ covered by a solvmanifold $\tilde{\Gamma}\backslash G$, compact quotient by a co-compact discrete subgroup of one of the following four Lie groups

$$\mathbb{R}^4, \quad \mathrm{Nil}^3 \times \mathbb{R}, \quad \mathrm{Nil}^4, \quad \mathrm{Sol}^3 \times \mathbb{R}.$$

These Lie groups are all diffeomorphic to $\mathbb{R}^4$. The Lie groups Nil^3, Nil^4 are nilpotent and Sol^3 is a particular solvable (non nilpotent) Lie group.

In particular, if the total space M^4 of an orientable T^2-bundle over a $\mathbb{T}^2$ is a solvmanifold, then it must be the compact quotient of one of the above Lie groups G. It is well known that all the orientable T^2-bundles over $\mathbb{T}^2$ admit symplectic structures (see [3]). The notion of invariant almost-Kähler structure makes sense for orientable T^2-bundles over $\mathbb{T}^2$, meaning one induced from a left-invariant structure on G which is invariant by the discrete subgroup Γ.

As a main result of [2], it was shown that if $M^4 = \Gamma\backslash G$ is an orientable T^2-bundle over a $\mathbb{T}^2$ with $G = \mathrm{Nil}^3 \times \mathbb{R}$ or Nil^4, and if M^4 admits an invariant Lagrangian almost-Kähler structure (Ω, J), then for every normalized volume form $\sigma = e^F \Omega^2$ with $F \in C^\infty(\mathbb{T}^2)$, the corresponding Calabi–Yau problem has a unique solution.

The Lagrangian condition may or may not apply in the case of $G = \mathrm{Nil}^3 \times \mathbb{R}$, but is automatic when M^4 is modelled on the 3-step nilpotent Lie group Nil^4. In the case of $G = \mathrm{Sol}^3 \times \mathbb{R}$ every invariant almost-Kähler on $\Gamma\backslash G$ is non-Lagrangian.

The aim of this paper is to extend the main result in [2] to the *non-Lagrangian* cases, i.e. to some T^2-fibrations modelled on $\mathrm{Nil}^3 \times \mathbb{R}$ and to all the T^2-fibrations modelled on $\mathrm{Sol}^3 \times \mathbb{R}$.

Our main result is the following

THEOREM 1. *Let $M^4 = \Gamma\backslash G$ be an orientable T^2-bundle over a 2-torus $\mathbb{T}^2$ with $G = \mathrm{Nil}^3 \times \mathbb{R}$ or $\mathrm{Sol}^3 \times \mathbb{R}$, and suppose that M^4 admits an invariant non-Lagrangian almost-Kähler structure (Ω, J). Then for every normalized volume form $\sigma = e^F \Omega^2$ with $F \in C^\infty(\mathbb{T}^2)$, the corresponding Calabi–Yau problem has a unique solution.*

The proof of this theorem consists in showing that the Calabi–Yau problem can be reduced to a single elliptic Monge–Ampère equation which has a solution.

The trick of reducing the problem to a Monge–Ampère equation was the core of [2], but the class of equations which appear in the present paper differs from the ones considered in [2].

As a consequence, we show that for every total space M^4 of an orientable T^2-bundle over a $\mathbb{T}^2$ endowed with an invariant almost-Kähler structure (Ω, J) the Calabi–Yau problem has a solution for every normalized T^2-invariant volume form.

The paper is organized as follows. In Section 2 we recall the classification of T^2-bundles over $\mathbb{T}^2$ and we briefly describe the main result of [2]. Sections 3 and 4 contain the proof of Theorem 1 where the case of $G = \mathrm{Nil}^3 \times \mathbb{R}$ and $G = \mathrm{Sol}^3 \times \mathbb{R}$ are treated separately. In each of the two cases we can reduce the problem to a Monge–Ampère equation for which we show the existence of a solution.

Last but not least, we would like to thank the referee for his useful and valuable remarks.

2. The Calabi–Yau equation on T^2-bundles over $\mathbb{T}^2$

Orientable T^2-bundles over a $\mathbb{T}^2$ were classified by Fukuhara and Sakamoto in [7] and it was shown by Ue in [10, 11] that all these manifolds are infra-solvmanifolds. A compact manifold M is called an *infra-solvmanifold* if it admits a finite covering $\pi\colon \tilde{M} \to M$, where $\tilde{M} = \tilde{\Gamma}\backslash G$ is the compact quotient of a solvable Lie group G by a lattice $\tilde{\Gamma}$. Alternatively, M can be written as a quotient $M = \Gamma\backslash G$, where Γ is a discrete group containing a lattice $\tilde{\Gamma}$ of G such that $\tilde{\Gamma}\backslash\Gamma$ is finite. In the case that $\tilde{\Gamma}$ is a lattice, M is simply called a *solvmanifold*.

It turns out that in the classification of T^2-bundles over $\mathbb{T}^2$, the solvable Lie group G must be one of the following four patterns

$$\mathbb{R}^4\,, \quad \mathrm{Nil}^3 \times \mathbb{R}\,, \quad \mathrm{Nil}^4\,, \quad \mathrm{Sol}^3 \times \mathbb{R}\,, \tag{4}$$

while the classification of the possible Γ's determines eight families. For the Lie groups $\mathrm{Nil}^3 \times \mathbb{R}\,, \mathrm{Nil}^4\,, \mathrm{Sol}^3 \times \mathbb{R}$ we have the following description:

(1) Nil^3 is the 3-dimensional Heisenberg group of matrices

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$$

and $\mathrm{Nil}^3 \times \mathbb{R}$ is a 2-step nilpotent Lie group which can be regarded as $\mathbb{R}^4$ with the product

$$(x_0, y_0, z_0, t_0)(x, y, z, t) = (x_0 + x,\ y_0 + y,\ z_0 + z + x_0 y,\ t_0 + t)\,.$$

(2) $\mathrm{Nil}^4 = \mathbb{R} \ltimes \mathbb{R}^3$ is the 3-step nilpotent Lie group of real matrices

$$\begin{pmatrix} 1 & t & \frac{1}{2}t^2 & x \\ 0 & 1 & t & y \\ 0 & 0 & 1 & z \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

(3) $\mathrm{Sol}^3 = \mathbb{R} \ltimes_{\varphi} \mathbb{R}^2$ is a unimodular 2-step solvable Lie group with $\varphi(t) = \begin{pmatrix} \mathrm{e}^t & 0 \\ 0 & \mathrm{e}^{-t} \end{pmatrix}$ and $\mathrm{Sol}^3 \times \mathbb{R}$ can be regarded as $\mathbb{R}^4$ with the product

$$(x_0, y_0, z_0, t_0)(x, y, z, t) = (x_0 + \mathrm{e}^{t_0} x,\ y_0 + \mathrm{e}^{-t_0} y,\ z_0 + z,\ t_0 + t).$$

The diffeomorphism classes of (the total space of) T^2-bundles over $\mathbb{T}^2$ can be summarized in Geiges' eight families [3, Table 1], which can be explicitly described in terms of the generators of the discrete groups Γ, the monodromy matrices along the two curves generating $\pi_1(\mathbb{T}^2)$, as well as the Euler class for the corresponding T^2-bundle.

In the case of $G = \mathrm{Nil}^3 \times \mathbb{R}$ one has two inequivalent fibrations

$$\begin{aligned} \pi_{xy} &: M^4 \to \mathbb{T}^2_{xy}, \\ \pi_{yt} &: M^4 \to \mathbb{T}^2_{yt} \end{aligned}$$

induced from the coordinate mappings

$$\begin{aligned} (x, y, z, t) &\mapsto (x, y), \\ (x, y, z, t) &\mapsto (y, t). \end{aligned}$$

If Γ is not a lattice of G, we have that Γ contains a lattice $\tilde{\Gamma}$ of G such that the quotient $\tilde{\Gamma} \backslash \Gamma$ is a finite group. Therefore there exists a covering map $p \colon \tilde{\Gamma} \backslash G \to \Gamma \backslash G$ which preserves the T^2-bundle structure over $\mathbb{T}^2$.

We recall that an *almost-Kähler structure* on a manifold M is a pair (Ω, J), where Ω is a symplectic form and J is a endomorphism of the tangent bundle to M satisfying $J^2 = -I$ and

$$\Omega(JX, JY) = \Omega(X, Y), \quad \Omega(Z, JZ) > 0$$

for every tangent vector fields X, Y, Z on M with Z nowhere vanishing. Every almost-Kähler structure induces the Riemannian metric $g(X, Y) = \Omega(X, JY)$.

In this paper (as in [2]) we consider on the total space $M^4 = \Gamma \backslash G$ of T^2-bundles over $\mathbb{T}^2$ invariant almost-Kähler structures, that is ones induced from left-invariant structures on G which are invariant by the discrete group Γ and we study for these almost-Kähler manifolds the Calabi–Yau problem. In particular every invariant almost-Kähler structure on $M^4 = \Gamma \backslash G$ induces an invariant almost-Kähler structure on the solvmanifold $\tilde{\Gamma} \backslash G$.

The case $G = \mathbb{R}^4$ (which corresponds to two of the Geiges' families) is not interesting from our point of view, since in this case every invariant almost-Kähler structure is Kähler and Yau's theorem can be applied. For the other cases we have to distinguish the Lagrangian case from the non-Lagrangian one. If M^4 is modelled on $G = \mathrm{Nil}^4$ or on $G = \mathrm{Nil}^3 \times \mathbb{R}$ with bundle structure given by the projection π_{xy}, then every invariant almost-Kähler is Lagrangian. If M^4 is modelled on $\mathrm{Nil}^3 \times \mathbb{R}$ with bundle structure given by the projection π_{yt} then it admits Lagrangian and non-Lagrangian almost-Kähler structures as well. In the case $G = \mathrm{Sol}^3 \times \mathbb{R}$, every invariant almost-Kähler structure is non-Lagrangian.

Let now $M^4 = \Gamma \setminus G$ be an orientable T^2-bundle over $\mathbb{T}^2$ and denote by $\mathfrak{g}$ the Lie algebra of G. Then every basis (e^i) of the dual space $\mathfrak{g}^*$ induces a global frame of 1-forms on M^4. Furthermore we fix an invariant almost-Kähler structure (Ω, J) on M^4. Let $\sigma = \mathrm{e}^F\, \Omega^2$ be a volume form and let F be a smooth map on the base $\mathbb{T}^2$ of M^4 satisfying

$$\int_{\mathbb{T}^2} (\mathrm{e}^F - 1) = 0.$$

Then in this case the Calabi–Yau problem reads as

$$\begin{cases} (\Omega + da)^2 = \mathrm{e}^F\, \Omega^2\,, \\ J(da) = da\,, \end{cases} \tag{5}$$

on M^4 whose components with respect to the coframe (e^i) are defined on the torus $\mathbb{T}^2$. Thus the Calabi–Yau problem reduces to a system of partial differential equations on the base $\mathbb{T}^2$.

Although the system (5) depends on the choice of G, (Ω, J) and the structure of T^2-fibration, for all the cases we can proceed in the following way: first we parametrize (Ω, J) using a suitable invariant coframe on M^4 in order to simplify the formulation of (5) as far as possible, and then we perform a suitable change of variables transforming the system (5) in a Monge-Ampère equation on the base $\mathbb{T}^2$.

The Lagrangian cases have been considered in [2], where it has been proved that there is a unique solution. In the next two sections we will consider the non-Lagrangian cases for the manifolds modelled on $\mathrm{Nil}^3 \times \mathbb{R}$ and $\mathrm{Sol}^3 \times \mathbb{R}$.

3. Manifolds modelled on $\mathrm{Nil}^3 \times \mathbb{R}$: the non-Lagrangian case

In this section we study the Calabi–Yau problem for T^2-bundles over $\mathbb{T}^2$ modelled on $\mathrm{Nil}^3 \times \mathbb{R}$ and equipped with an invariant non-Lagrangian almost-Kähler structure.

The structure of T^2-bundle over a $\mathbb{T}^2$ is then induced by the projection π_{yt} onto the torus $\mathbb{T}^2_{yt}$. The total space M^4 of the T^2-fibration has the global invariant coframe

$$e^1 = dy\,, \quad e^2 = dx\,, \quad e^3 = dt\,, \quad e^4 = dz - xdy$$

which satisfies the structure equations

$$de^1 = de^2 = de^3 = 0\,, \qquad de^4 = e^{12}\,. \tag{6}$$

LEMMA 1. *Let M^4 be the total space of an oriented T^2-bundle over a 2-torus $\mathbb{T}^2$ modelled on $\mathrm{Nil}^3 \times \mathbb{R}$ and induced by the projection π_{yt} onto the torus $\mathbb{T}^2_{yt}$.*
Let (Ω, J) be an invariant almost-Kähler structure on M^4 with induced Riemannian metric g. Then there exists an orthonormal invariant coframe (f^i) for which

$$\Omega = f^{14} + f^{23}, \tag{7}$$

and

$$f^1 \in \langle e^1 \rangle\,, \quad g(e^3, f^2) = 0\,, \quad g(e^3, f^3) g(e^3, f^4) \geqslant 0. \tag{8}$$

Proof. We can certainly find an orthonormal invariant coframe (f^i) for which (7) is valid and $f^1 \in \langle e^1 \rangle$. Since $f^4 = J(f^1)$, we still have freedom to rotate f^{23} in the plane orthogonal to $\langle f^1, f^4 \rangle$. After a suitable rotation we obtain $g(e^3, f^2) = 0$. If necessary, we may invert the direction of f^2 and f^3 to meet the condition $g(e^3, f^3) g(e^3, f^4) \geqslant 0$, without reversing f^{23}. □

3.1. The Calabi–Yau equation on M^4

Consider on M^4 an invariant non-Lagrangian almost-Kähler structure (Ω, J) with induced Riemannian metric g. Let $\sigma = \mathrm{e}^F\, \Omega^2$ be a volume form where $F = F(y,t)$ is a smooth map on the base satisfying

$$\int_{\mathbb{T}^2} (\mathrm{e}^F - 1) = 0\,. \tag{9}$$

Consider the Calabi–Yau equation

$$\begin{cases} (\Omega + da)^2 = \sigma\,, \\ J(da) = da, \end{cases}$$

where a is a 1-form on M^4 whose components with respect to the basis (e^i) depend on (y,t) only. Let (f^i) be a coframe as in Lemma 1 and set

$$G^i_j = g(e^i, f^j);$$

then we have

$$e^i = G^i_j f^j.$$

In particular we have

$$e^1 = G^1_1 f^1 \tag{10}$$

and

$$G^1_1 \neq 0, \qquad G^1_2 = G^1_3 = G^1_4 = 0. \tag{11}$$

Let $H = G^{-1}$ be the inverse matrix of $G = (G^i_j)$. Then

$$f^i = H^i_j e^j.$$

From (10) and (11), we have

$$H^1_1 = (G^1_1)^{-1} \neq 0,$$

and

$$H^1_2 = H^1_3 = H^1_4 = 0. \tag{12}$$

Thanks to the structure equations (6), we have

$$d f^i = H^i_j de^j = H^i_4 de^4 = H^i_4 e^{12} = H^i_4 G^1_1 (G^2_2 f^{12} + G^2_3 f^{13} + G^2_4 f^{14}).$$

The condition that (Ω, J) be non-Lagrangian implies that

$$G^3_4 \neq 0.$$

Moreover, since $G^3_2 = g(e^3, f^2) = 0$, we have

$$e^3 = G^3_1 f^1 + G^3_3 f^3 + G^3_4 f^4, \tag{13}$$

where

$$G^3_3 G^3_4 \geqslant 0, \tag{14}$$

thanks to (8).

Differentiating (13) we get

$$G^3_3 d f^3 + G^3_4 d f^4 = (G^3_3 H^3_4 + G^3_4 H^4_4) e^{12} = 0\,,$$

that is,

$$G^3_3 H^3_4 + G^3_4 H^4_4 = 0. \tag{15}$$

Furthermore, the symplectic condition $d\Omega = 0$ gives

$$d f^{23} = 0,$$

i.e.,

$$\begin{aligned} 0 &= H^2_4 G^1_1 (G^2_2 f^{12} + G^2_3 f^{13} + G^2_4 f^{14}) \wedge f^3 - H^3_4 G^1_1 f^2 \wedge (G^2_2 f^{12} + G^2_3 f^{13} + G^2_4 f^{14}) \\ &= G^1_1 (H^2_4 G^2_2 + G^1_1 H^3_4 G^2_3)\, f^{123} - G^1_1 H^2_4 G^2_4\, f^{134} + G^1_1 H^3_4 G^2_4\, f^{124}\,. \end{aligned}$$

It follows that

$$H^2_4 G^2_2 + H^3_4 G^2_3 = 0, \qquad H^2_4 G^2_4 = H^3_4 G^2_4 = 0. \tag{16}$$

From (15) and (16) we have

$$G^2_4 G^3_4 H^4_4 = G^2_4 (G^3_3 H^3_4 + G^3_4 H^4_4) = 0, \tag{17}$$

and, since $H_4^1 = 0$, and H_4^2, H_4^3 and H_4^4 cannot vanish all together, from (12), (16), (17), we obtain that

$$G_4^2 = 0.$$

Write

$$a = a_k f^k$$

and compute

$$\begin{aligned} da =& (G_1^1 a_{2,y} + G_1^1 G_2^2 H_4^k a_k + G_1^3 a_{2,t}) f^{12} + (G_1^1 a_{3,y} + G_1^1 G_3^2 H_4^k a_k + G_1^3 a_{3,t} - G_3^3 a_{1,t}) f^{13} \\ &- G_4^3 a_{2,t} f^{24} + (G_1^1 a_{4,y} + G_1^3 a_{4,t} - G_4^3 a_{1,t}) f^{14} - G_3^3 a_{2,t} f^{23} + (G_3^3 a_{4,t} - G_4^3 a_{3,t}) f^{34}. \end{aligned}$$

Hence da is of type $(1,1)$ with respect to J if and only if

$$\begin{cases} G_1^1 a_{2,y} + G_1^1 G_2^2 (H_4^2 a_2 + H_4^3 a_3 + H_4^4 a_4) + G_1^3 a_{2,t} = -G_3^3 a_{4,t} + G_4^3 a_{3,t}, \\ G_1^1 a_{3,y} + G_1^1 G_3^2 (H_4^2 a_2 + H_4^3 a_3 + H_4^4 a_4) + G_1^3 a_{3,t} - G_3^3 a_{1,t} = -G_3^3 a_{2,t} \end{cases}$$

and in this case da reduces to

$$\begin{aligned} da = & (-G_3^3 a_{4,t} + G_4^3 a_{3,t}) f^{12} - G_3^3 a_{2,t} f^{13} - G_4^3 a_{2,t} f^{24} \\ & + (G_1^1 a_{4,y} + G_1^3 a_{4,t} - G_4^3 a_{1,t}) f^{14} - G_3^3 a_{2,t} f^{23} + (G_3^3 a_{4,t} - G_4^3 a_{3,t}) f^{34}. \end{aligned}$$

The Calabi–Yau equation now reads as

$$\mathrm{e}^F = (1 + G_1^1 a_{4,y} + G_1^3 a_{4,t} - G_4^3 a_{1,t})(1 - G_3^3 a_{2,t}) - G_3^3 G_4^3 (a_{2,t})^2 - (-G_3^3 a_{4,t} + G_4^3 a_{3,t})^2,$$

and the Calabi–Yau problem is equivalent to the following system of partial differential equations:

$$(18) \quad \begin{cases} G_1^1 a_{2,y} + G_1^1 G_2^2 (H_4^2 a_2 + H_4^3 a_3 + H_4^4 a_4) + G_1^3 a_{2,t} + G_3^3 a_{4,t} - G_4^3 a_{3,t} = 0, \\ G_1^1 a_{3,y} + G_1^1 G_3^2 (H_4^2 a_2 + H_4^3 a_3 + H_4^4 a_4) + G_1^3 a_{3,t} - G_3^3 a_{1,t} + G_3^3 a_{2,t} = 0, \\ (1 + G_1^1 a_{4,y} + G_1^3 a_{4,t} - G_4^3 a_{1,t})(1 - G_3^3 a_{2,t}) \\ \qquad - G_3^3 G_4^3 (a_{2,t})^2 - (-G_3^3 a_{4,t} + G_4^3 a_{3,t})^2 = \mathrm{e}^F. \end{cases}$$

In the system (18) the parameter G_3^3 has a special role. We will study separately the cases $G_3^3 = 0$ and $G_3^3 \neq 0$.

3.2. The case $G_3^3 = 0$

This case is quite trivial since condition $G_3^3 = 0$ implies $dt \in \langle f^1, f^4 \rangle$ and $f^{14} = dy \wedge dt$. Therefore if $G_3^3 = 0$ the corresponding Calabi–Yau equation has the explicit solution

$$\tilde{\Omega} = (\mathrm{e}^F - 1) f^{14} + f^{23}.$$

3.3. The Case $G^3_3 \neq 0$

Under this assumption we consider the transformation

$$\begin{aligned} a_1 &= -G^3_3 u_t - G^1_1(H^2_4 G^2_3 + H^4_4 G^2_2)u, \\ a_2 &= -G^3_3 u_t - H^4_4 G^1_1 G^2_2 u, \\ a_3 &= -H^4_4 G^1_1 G^2_3 u, \\ a_4 &= G^1_1 u_y + G^3_1 u_t. \end{aligned}$$

A long but straightforward computation shows that the first two equations of system (18) are identically satisfied, while the third one becomes

$$(u_{yy} + B_{11}u_t + C_{11})(u_{tt} + B_{22}u_t + C_{22}) - (u_{yt} + B_{12}u_t + C_{12})^2 = E_1 + E_2 \mathrm{e}^F, \tag{19}$$

where

$$\begin{aligned} B_{11} &= \frac{G^2_2(G^3_1)^2 H^4_4}{G^1_1 G^3_3} - 2\frac{G^2_3 G^3_1 G^3_4 H^4_4}{G^1_1 G^3_3} + \frac{G^2_3 G^3_4 H^2_4}{G^1_1}, \\ B_{12} &= -\frac{G^2_2 G^3_1 H^4_4}{G^3_3} + \frac{G^2_3 G^3_4 H^4_4}{G^3_3}, \qquad B_{22} = \frac{G^1_1 G^2_2 H^4_4}{G^3_3}, \\ C_{11} &= \frac{1}{(G^1_1)^2} + \frac{(G^3_1)^2}{(G^1_1)^2(G^3_3)^2} + \frac{G^3_4}{(G^1_1)^2 G^3_3}, \\ C_{12} &= -\frac{G^3_1}{G^1_1(G^3_3)^2}, \qquad C_{22} = \frac{1}{(G^3_3)^2}, \\ E_1 &= \frac{G^3_3 G^3_4}{(G^1_1)^2(G^3_3)^4}, \qquad E_2 = (G^1_1 G^3_3)^2. \end{aligned}$$

In particular we have

$$B_{11}B_{22} - (B_{12})^2 = 0$$

and

$$C_{11}C_{22} - (C_{12})^2 = E_1 + E_2.$$

4. Manifolds modelled on $\mathrm{Sol}^3 \times \mathbb{R}$

In this section we study the Calabi–Yau equation for the total spaces M^4 of T^2-bundles over torus $\mathbb{T}^2$ modelled on $\mathrm{Sol}^3 \times \mathbb{R}$.

Since the Lie group $\mathrm{Sol}^3 \times \mathbb{R}$ can be seen as $\mathbb{R}^4$ equipped with the product

$$(x_0, y_0, z_0, t_0)(x, y, z, t) = (x_0 + x,\ y_0 + y,\ z_0 + \mathrm{e}^x z,\ t_0 + \mathrm{e}^{-x} t),$$

M^4 inherits the global invariant coframe

$$e^1 = dx, \qquad e^2 = dy, \qquad e^3 = \mathrm{e}^x dz, \qquad e^4 = \mathrm{e}^{-x} dt, \tag{20}$$

satisfying the following structure equations

$$de^1 = de^2 = 0, \qquad de^3 = e^{13}, \qquad de^4 = -e^{14}. \tag{21}$$

Moreover, invariant almost-Kähler structure on M^4 should be parametrized as claimed in the following lemma proved in [2]:

LEMMA 2. *Let (Ω, J) be an invariant almost-Kähler structure on the total space M^4 of a T^2-bundle over a torus $\mathbb{T}^2$ modelled on* $\mathrm{Sol}^3 \times \mathbb{R}$. *Let g be the induced Riemannian metric. Then there exists an orthonormal global coframe (f^i) for which*

$$\Omega = f^{12} + f^{34},$$

and

$$f^1 \in \langle e^1 \rangle, \quad f^3 \in \langle e^3 \rangle, \quad f^4 \in \langle e^3, e^4 \rangle, \tag{22}$$

with

$$g(e^1, f^1) > 0. \tag{23}$$

Notice that in this case every invariant almost-Kähler structure is non-Lagrangian.

4.1. The Calabi–Yau equation on M^4

Let (Ω, J) be an invariant almost-Kähler structure on M^4, (f^k) be a coframe as in the previous lemma and $\sigma = \mathrm{e}^F\,\Omega^2$ be a volume form where $F = F(x,y) \in C^\infty(\mathbb{T}^2)$ satisfies the condition

$$\int_{\mathbb{T}^2} (\mathrm{e}^F - 1) = 0\,.$$

Then we consider the Calabi–Yau equation

$$(\Omega + da)^2 = \sigma, \tag{24}$$

where

$$a = \sum_{k=1}^{4} a_k f^k\,,$$

is a 1-form whose components a_k are functions on the base $\mathbb{T}^2$.

Let g be the Riemannian metric induced by (Ω, J) and set

$$G^i_j = g(e^i, f^j).$$

Then

$$e^i = G^i_j f^j \tag{25}$$

and

$$f^i = H^i_j e^j,$$

where $H = G^{-1}$ is the inverse matrix of $G = (G^i_j)$. In particular this implies that

$$G^3_4 = H^2_4 G^2_2 + H^4_4 G^2_4 = H^3_3 G^4_3 + H^4_3 G^4_4 = 0.$$

From (22) we have that

$$f^1 = H^1_1\, e^1\,, \quad f^2 = H^2_2\, e^2 + H^2_3\, e^3 + H^2_4 e^4\,, \quad f^3 = H^3_3\, e^3\,, \quad f^4 = H^4_3\, e^3 + H^4_4\, e^4\,.$$

Then, making use of (21) and (25), we obtain

$$\begin{aligned} d\,f^1 &= 0, & d\,f^2 &= G^1_1(H^2_3G^3_3 - H^2_4G^4_3)\, f^{13} - G^1_1H^2_4G^4_4 f^{14},\\ d\,f^3 &= G^1_1 f^{13}, & d\,f^4 &= G^1_1(H^4_3G^3_3 - H^4_4G^4_3)\, f^{13} - G^1_1\, f^{14}. \end{aligned}$$

Let

$$a = a_1 f^1 + a_2 f^2 + a_3 f^3 + a_4 f^4$$

be a T^2-invariant form on M^4.

We have

$$\begin{aligned} da = {} & (G^1_1a_{2,x} - G^2_2a_{1,y})\, f^{12}\\ & + \Big(G^1_1a_{3,x} - G^2_3a_{1,y} + G^1_1(H^2_3G^3_3 - H^2_4G^4_3)a_2 + G^1_1a_3 + G^1_1(H^4_3G^3_3 - H^4_4G^4_3)a_4\Big)\, f^{13}\\ & + \Big(G^1_1a_{4,x} - G^2_4a_{1,y} - G^1_1H^2_4G^4_4a_2 - G^1_1a_4\Big)\, f^{14}\\ & + (G^2_2a_{3,y} - G^2_3a_{2,y})\, f^{23} + (G^2_2a_{4,y} - G^2_4a_{2,y})\, f^{24} + (G^2_3a_{4,y} - G^2_4a_{3,y})\, f^{34}\,. \end{aligned}$$

Hence da is J-invariant if and only if its components satisfy

$$\begin{cases} G^1_1a_{3,x} - G^2_3a_{1,y} + G^1_1(H^2_3G^3_3 - H^2_4G^4_3)a_2 + G^1_1a_3 + G^1_1(H^4_3G^3_3 - H^4_4G^4_3)a_4\\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad = G^2_2a_{4,y} - G^2_4a_{2,y},\\ G^1_1a_{4,x} - G^2_4a_{1,y} - G^1_1H^2_4G^4_4a_2 - G^1_1a_4 = G^2_3a_{2,y} - G^2_2a_{3,y}, \end{cases}$$

and equation (24) becomes

$$\begin{aligned} &(1 + G^1_1a_{2,x} - G^2_2a_{1,y})(1 + G^2_3a_{4,y} - G^2_4a_{3,y})\\ &\qquad\qquad - (G^2_2a_{4,y} - G^2_4a_{2,y})^2 - (G^2_2a_{3,y} - G^2_3a_{2,y})^2 = \mathrm{e}^F\,. \end{aligned}$$

Therefore the Calabi–Yau problem reduces to the following system of partial differential equations

$$(26)\quad \begin{cases} G^1_1a_{3,x} - G^2_3a_{1,y} + G^1_1(H^2_3G^3_3 - H^2_4G^4_3)a_2 + G^1_1a_3 + G^1_1(H^4_3G^3_3 - H^4_4G^4_3)a_4\\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad = G^2_2a_{4,y} - G^2_4a_{2,y},\\ G^1_1a_{4,x} - G^2_4a_{1,y} - G^1_1H^2_4G^4_4a_2 - G^1_1a_4 = G^2_3a_{2,y} - G^2_2a_{3,y},\\ (1 + G^1_1a_{2,x} - G^2_2a_{1,y})(1 + G^2_3a_{4,y} - G^2_4a_{3,y})\\ \qquad\qquad - (G^2_2a_{4,y} - G^2_4a_{2,y})^2 - (G^2_2a_{3,y} - G^2_3a_{2,y})^2 = \mathrm{e}^F\,. \end{cases}$$

4.2. Reduction of (26) to a single equation

Consider $u \in C^2(\mathbb{T}^2)$ such that

$$\int_{\mathbb{T}^2} u = 0, \tag{27}$$

and let

$$\begin{cases} a_1(x,y) = -\dfrac{H_1^1 G_2^2 \big(u_y(x,y) - u_y(x,0)\big) + 2H_4^4 G_3^4 \big(u(x,y) - u(x,0)\big)}{(G_3^2)^2 + (G_4^2)^2} \\ \qquad - \dfrac{G_1^1 H_2^2}{(G_3^2)^2 + (G_4^2)^2} \left(\displaystyle\int_0^y \big(u_{xx}(x,t) - u(x,t)\big)\, dt - y \int_0^1 \big(u_{xx}(x,t) - u(x,t)\big)\, dt \right), \\ a_2(x,y) = -\dfrac{1}{(G_3^2)^2 + (G_4^2)^2} \left(\displaystyle\int_0^x \left(\int_0^1 u(s,t)\, dt \right) ds - \int_0^1 \big(u_x(x,t) - u_x(0,t)\big)\, dt \right), \\ a_3(x,y) = -\dfrac{H_2^2 G_3^2 u_x(x,y) + H_1^1 G_4^2 u_y(x,y) - H_2^2 (G_3^2 - 2G_4^2 H_4^4 G_3^4) u(x,y)}{(G_3^2)^2 + (G_4^2)^2} \\ \qquad - \dfrac{H_2^2 G_3^2}{(G_3^2)^2 + (G_4^2)^2} \left(\displaystyle\int_0^x \left(\int_0^1 u(s,t)\, dt \right) ds - \int_0^1 \big(u_x(x,t) - u_x(0,t)\big)\, dt \right), \\ a_4(x,y) = -\dfrac{H_2^2 G_4^2 u_x(x,y) - H_1^1 G_3^2 u_y(x,y) + H_2^2 G_4^2 u(x,y)}{(G_3^2)^2 + (G_4^2)^2} \\ \qquad - \dfrac{H_2^2 G_4^2}{(G_3^2)^2 + (G_4^2)^2} \left(\displaystyle\int_0^x \left(\int_0^1 u(s,t)\, dt \right) ds - \int_0^1 \big(u_x(x,t) - u_x(0,t)\big)\, dt \right). \end{cases}$$

Thanks to condition (27) we have that the functions a_1 to a_4 are periodic. A long computation shows that the first two equations of system (26) are identically satisfied, while the third one becomes

$$(u_{xx} + B_{11} u_y + C_{11} + Du)(u_{yy} + B_{22} u_y + C_{22}) - (u_{xy} + B_{12} u_y)^2 = E_1 + E_2 \mathrm{e}^F, \tag{28}$$

where

$$\begin{aligned} B_{11} &= \frac{2H_1^1 G_2^2 G_3^2 (G_4^2 + G_3^2 H_4^4 G_3^4)}{(G_3^2)^2 + (G_4^2)^2}, \\ B_{12} &= \frac{(G_4^2)^2 - (G_3^2)^2 + 2G_3^2 G_4^2 H_4^4 G_3^4}{(G_3^2)^2 + (G_4^2)^2}, \\ B_{22} &= -\frac{2G_1^1 H_2^2 G_4^2 (G_3^2 - G_4^2 H_4^4 G_3^4)}{(G_3^2)^2 + (G_4^2)^2}, \\ C_{11} &= H_1^1 \Big((G_2^2)^2 + (G_3^2)^2 + (G_4^2)^2 \Big), \qquad C_{22} = G_1^1, \\ D &= -1, \qquad E_1 = (G_2^2)^2, \qquad E_2 = (G_3^2)^2 + (G_4^2)^2. \end{aligned}$$

In particular we have

$$B_{11}B_{22} - (B_{12})^2 = -1$$

and

$$C_{11}C_{22} - (C_{12})^2 = E_1 + E_2.$$

5. The Monge–Ampère equation

Both equations (19) and (28) are generalized Monge-Ampère equations of the following type:

$$A_{11}[u]A_{22}[u] - \big(A_{12}[u]\big)^2 = E_1 + E_2\, \mathrm{e}^F, \tag{29}$$

where

$$\begin{aligned} A_{11}[u] &= u_{xx} + B_{11}u_y + C_{11} + Du, \\ A_{12}[u] &= u_{xy} + B_{12}u_y + C_{12}, \\ A_{22}[u] &= u_{yy} + B_{22}u_y + C_{22}, \end{aligned}$$

with B_{ij}, C_{ij}, D, E_i real numbers such that

$$C_{11} + C_{22} > 0, \qquad D \leqslant 0, \tag{30}$$

$$E_1 > 0, \qquad E_2 > 0, \tag{31}$$

$$B_{11}B_{22} - (B_{12})^2 = D \tag{32}$$

and

$$C_{11}C_{22} - (C_{12})^2 = E_1 + E_2. \tag{33}$$

Moreover

$$F \in C^\infty(\mathbb{T}^2), \tag{34}$$

and it satisfies the condition

$$\int_{\mathbb{T}^2} (\mathrm{e}^F - 1) = 0. \tag{35}$$

In Theorem 4 we shall prove that equation (29) has a solution belonging to $C^\infty(\mathbb{T}^2)$, and satisfying the condition

$$\int_{\mathbb{T}^2} u = 0. \tag{36}$$

For all $n \in \mathbb{N}$, $0 < \varepsilon < 1$, consider the semi-norms

$$|u|_{C^n} = \max_{0 \leqslant j \leqslant n} \sup_{(x,y) \in \mathbb{R}^2} \left| \partial_x^j \partial_y^{n-j} u(x,y) \right|$$

$$|u|_{C^{n,\varepsilon}} = \max_{0 \leqslant j \leqslant n} \sup_{(x,y) \in \mathbb{R}^2} \sup_{(h,k) \in \mathbb{R}^2 \setminus \{(0,0)\}} \frac{\left| \partial_x^j \partial_y^{n-j} u(x+h, y+k) - \partial_x^j \partial_y^{n-j} u(x,y) \right|}{(h^2+k^2)^{\varepsilon/2}},$$

and the norms

$$\|u\|_{C^n} = \max_{0 \leqslant k \leqslant n} |u|_{C^k}, \qquad \|u\|_{C^{n,\varepsilon}} = \max\left\{ \|u\|_{C^n}, |u|_{C^{n,\varepsilon}} \right\}.$$

LEMMA 3. *Under the hypotheses* (31)*, for all* $u \in C^2(\mathbb{T}^2)$ *satisfying* (29) *we have that*

$$\begin{cases} A_{11}[u] > 0, \\ A_{22}[u] > 0. \end{cases} \tag{37}$$

Proof. Equation (29) implies that $A_{11}[u]A_{22}[u] > E_1$. Then $A_{11}[u]$ and $A_{22}[u]$ never vanish and have the same sign. At a point where u reaches its minimum value, we have $u_y = 0$ and $u_{yy} \leqslant 0$. Then

$$A_{22}[u] = u_{yy} + C_{22} > 0$$

and both $A_{11}[u]$ and $A_{22}[u]$ must be positive everywhere. □

LEMMA 4. *Consider a function* $u \in C^2(\mathbb{T}^2)$ *satisfying equation* (29)*. Under the hypotheses* (30) *and* (31) *we have that*

$$Du(x,y) \geqslant C_{11}, \qquad \forall (x,y) \in \mathbb{R}^2. \tag{38}$$

Proof. Consider a point where Du attains its minimum value. Since $D \leqslant 0$, this corresponds to a point where u reaches its maximum value. Then we have $u_y = 0$ and $u_{xx} \leqslant 0$ and from (37) we have

$$C_{11} + Du \geqslant u_{xx} + C_{11} + Du > 0,$$

which implies

$$Du \geqslant C_{11},$$

at the maximum and therefore everywhere. □

We need Lemma 6.3 of [2]:

LEMMA 5. *Consider* $w \in C^2(\mathbb{T})$ *and two real numbers* α *and* β *such that*

$$w''(t) + \alpha w'(t) \geqslant \beta, \qquad \forall t \in \mathbb{R}. \tag{39}$$

Then we have

$$\left| w'(t) \right| \leqslant 2\,|\beta|\, \mathrm{e}^{2|\alpha|}, \qquad \forall t \in \mathbb{R}. \tag{40}$$

THEOREM 2. *Assume hypotheses* (30) *and* (33) *are satisfied. Then all solutions of* (29) *satisfy the following estimate:*

$$\|u\|_{C^2} \leq 2\big(|B_{11}|+1\big)\,|B_{22}|\,\mathrm{e}^{2C_{22}} + C_{11} + C_{22}.$$

Proof. From (37) we obtain that

$$u_{yy} + B_{22}u_y \geqslant -C_{22},$$

hence from Lemma 5 we obtain that

$$|u_y| \leqslant 2\,|B_{22}|\,\mathrm{e}^{2C_{22}}. \tag{41}$$

From (37), (38) and (41), we obtain

$$u_{xx} \geqslant -2\,|B_{11}B_{22}|\,\mathrm{e}^{2C_{22}} - C_{11} - C_{22},$$

hence from Lemma 5 we obtain

$$|u_x| \leqslant 2\,|B_{11}B_{22}|\,\mathrm{e}^{2C_{22}} + C_{11} + C_{22}. \tag{42}$$

Now consider a point $(x_0, y_0) \in [0,1] \times [0,1]$ where u vanishes. Then we have

$$\begin{aligned} u(x,y) = &\int_0^1 u_x\big((1-t)x + tx_0, (1-t)y + ty_0\big)\,dt\,(x-x_0) \\ &+ \int_0^1 u_y\big((1-t)x + tx_0, (1-t)y + ty_0)\big)\,dt\,(y-y_0), \end{aligned}$$

which, together with periodicity, implies

$$|u|_{C^0} \leqslant 2\,|u|_{C^1}.$$

This estimate, together (41) and (42), implies

$$|u|_{C^0} \leqslant 2\big(|B_{11}|+1\big)\,|B_{22}|\,\mathrm{e}^{2C_{22}} + C_{11} + C_{22}. \qquad \square$$

Let $\tau \in [0,1]$ and set

$$\mathfrak{S}_\tau = \Big\{u \in C^2(\mathbb{T}^2) \mid A_{11}[u]A_{22}[u] - \big(A_{12}[u]\big)^2 = E_1 + (1-\tau)E_2 + \tau E_2\mathrm{e}^F,\ \textstyle\int_{\mathbb{T}^2} u = 0\Big\}. \tag{43}$$

THEOREM 3. *Assume hypotheses* (30) *to* (34) *are satisfied. Then*

$$\mathfrak{S}_\tau \subset C^{2,1/2}(\mathbb{T}^2), \qquad \forall \tau \in [0,1],$$

and

$$\sup_{0\leq\tau\leqslant 1}\ \sup_{u\in\mathfrak{S}_\tau} \|u\|_{C^{2,1/2}} < \infty.$$

Proof. Thanks to Lemma 3 and hypothesis (31) the equation

$$A_{11}[u]A_{22}[u] - \big(A_{12}[u]\big)^2 = E_1 + (1-\tau)E_2 + \tau E_2 \mathrm{e}^F$$

is uniformly elliptic and we can apply Theorem 2 of [5]. ☐

COROLLARY 1. *Under the same hypotheses of Theorem 3 we have that*

$$\mathfrak{S}_\tau \subset C^\infty(\mathbb{T}^2)$$

for all $0 \leqslant \tau \leqslant 1$.

Proof. This follows from Theorems 1 and 3 of [6]. ☐

THEOREM 4. *Under the hypotheses* (30) *to* (35)*, we have that for all* $\tau \in [0,1]$ *the equation*

$$A_{11}[u]A_{22}[u] - \big(A_{12}[u]\big)^2 = E_1 + (1-\tau)E_2 + \tau E_2 \mathrm{e}^F$$

has a solution in $C^\infty(\mathbb{T}^2)$ *satisfying condition* (36)*.*

In particular, for $\tau = 1$ *we obtain that equation* (29) *is solvable.*

Proof. In view of Corollary 1, it is sufficient to prove the existence of a C^2-solution. Hence we have to prove that $\mathfrak{S}_\tau(\mathbb{T}^2) \neq \varnothing$ for all $\tau \in [0,1]$. If $\tau = 0$, thanks to (33) we have that $0 \in \mathfrak{S}_0$. Then we may set

$$\rho = \sup\big\{\sigma \in [0,1] \mid \mathfrak{S}_\tau \neq \varnothing, \forall \tau \in [0,\sigma]\big\}$$

We must show that $\mathfrak{S}_\rho \neq \varnothing$ and that $\rho = 1$.

Consider a sequence $\tau_n \in [0,\rho]$ converging to ρ and such that $\mathfrak{S}_{\tau_n} \neq \varnothing$. Let $u_n \in \mathfrak{S}_{\tau_n}$ for all n. By Theorem 3, the sequence (u_n) is bounded in $C^{2,1/2}(\mathbb{T}^2)$, hence, by the Ascoli–Arzelà theorem, it contains a subsequence (v_n) that converges in $C^2(\mathbb{T}^2)$ to a function v, which is a solution belonging to $\mathfrak{S}_\rho$.

Now we show that $\rho = 1$. Assume by contradiction $\rho < 1$ and let $C_*^{k,1/2}(\mathbb{T}^2)$ be the space of functions $u \in C^{k,1/2}(\mathbb{T}^2)$ satisfying $\int_{\mathbb{T}^2} u = 0$. Consider the map

$$T : C_*^{2,1/2}(\mathbb{T}^2) \times [0,1] \to C_*^{0,1/2}(\mathbb{T}^2),$$

defined as

$$T(u,\tau) = A_{11}[u]A_{22}[u] - \big(A_{12}[u]\big)^2 - E_1 - (1-\tau)E_2 - \tau E_2 \mathrm{e}^F.$$

Observe that

$$\int_{\mathbb{T}^2} T(u,\tau) = 0,$$

thanks to (32), (33) and (35).

We know that there exists $v \in \mathfrak{S}_\rho \subset C_*^{2,1/2}(\mathbb{T}^2)$ such that $T(v,\rho) = 0$. We have

$$T'[v,\rho](w,0) = Lw,$$

with

$$L : C_*^{2,1/2}(\mathbb{T}^2) \to C_*^{0,1/2}(\mathbb{T}^2)$$

given by

$$\begin{aligned} Lw = {} & \big(A_{22}[v] + C_{22}\big) w_{xx} - 2\big(A_{12}[v] + C_{12}\big) w_{xy} + \big(A_{11}[v] + C_{11}\big) w_{yy} \\ & + \Big(B_{11}\big(A_{22}[v] + C_{22}\big) - 2B_{12}\big(A_{12}[v] + C_{12}\big) + B_{22}\big(A_{11}[v] + C_{11}\big)\Big) w_y \\ & + D\big(A_{22}[u] + C_{22}\big) w. \end{aligned} \tag{44}$$

Now from Lemma 3 and hypotheses (31) the matrices

$$\begin{bmatrix} A_{11}[v] & A_{12}[v] \\ A_{12}[v] & A_{22}[v] \end{bmatrix}, \qquad \begin{bmatrix} C_{11} & C_{12} \\ C_{12} & C_{22} \end{bmatrix}$$

are positive, so their sum is positive too, and the operator L is uniformly elliptic. Since $D\big(A_{22}[u] + A_{22}[v]\big) \leqslant 0$, we may apply the strong maximum principle ([4], Theorem 3.5) and obtain that $Lw = 0$ implies that w is constant, that is $w = 0$, by condition (36). Ellipticity and classical Schauder estimates ([4, Theorem 6.2]) show that L is onto. Since L is one-to-one, it must be an isomorphism. Then, by the implicit function theorem, there exists $\varepsilon > 0$ such that $\mathfrak{S}_\tau(\mathbb{T}^2) \neq \varnothing$ for $\rho < \tau < \rho + \varepsilon$, contradicting the definition of ρ. □

References

[1] DONALDSON, S. K. Two-forms on four-manifolds and elliptic equations. In *Inspired by S.S. Chern*. World Scientific, Hackensack NJ, 2006, pp. 153–172. Nankai Tracts Math., Vol. 11.

[2] FINO, A., LI, Y. Y., SALAMON, S., AND VEZZONI, L. The Calabi-Yau equation on 4-manifolds over 2-tori. To appear in Trans. Amer. Math. Soc., math.DG/1103.3995.

[3] GEIGES, H. Symplectic structures on T^2-bundles over T^2. *Duke Math. J. 67* (1992), 539–555.

[4] GILBARG, D., AND TRUDINGER, N. S. *Elliptic Partial Differential Equations of Second Order*, vol. 224 of *Grundlehren der Mathematischen Wissenschaften*. Springer-Verlag, Berlin, 1983.

[5] HEINZ, E. Interior estimates for solutions of elliptic Monge-Ampère equations. In *Proc. Sympos. Pure Math., Vol. 4*. American Mathematical Society, Providence RI, 1961, pp. 149–155.

[6] NIRENBERG, L. On nonlinear elliptic partial differential equations and Hölder continuity. *Comm. Pure Appl. Math. 6* (1953), 103–156; addendum 395.

[7] SAKAMOTO, K., AND FUKUHARA, S. Classification of T^2-bundles over T^2. *Tokyo J. Math. 6* (1983), 311–327.

[8] TOSATTI, V., AND WEINKOVE, B. The Calabi-Yau equation on the Kodaira-Thurston manifold. *J. Inst. Math. Jussieu 10* (2011), 437–447.

[9] TOSATTI, V., WEINKOVE, B., AND YAU, S. T. Taming symplectic forms and the Calabi-Yau equation. *Proc. London Math. Soc. 97* (2008), 401–424.

[10] UE, M. On the 4-dimensional Seifert fiberings with Euclidean base orbifolds. In *A Fête of Topology*. Academic Press, Boston, 1988, pp. 471–523.

[11] UE, M. Geometric 4-manifolds in the sense of Thurston and Seifert 4-manifolds, I. *J. Math. Soc. Japan 42* (1990), 511–540.

[12] YAU, S. T. On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation, I. *Comm. Pure Appl. Math. 31* (1978), 339–411.

AMS Subject Classification: 53C55 (32Q25, 32Q60, 53C21), 58J05

Ernesto BUZANO
Dipartimento di Matematica, Università di Torino
Via Carlo Alberto 10, 10123 Torino, ITALIA
e-mail: `ernesto.buzano@unito.it`

Anna FINO
Dipartimento di Matematica, Università di Torino
Via Carlo Alberto 10, 10123 Torino, ITALIA
e-mail: `annamaria.fino@unito.it`

Luigi VEZZONI
Dipartimento di Matematica, Università di Torino
Via Carlo Alberto 10, 10123 Torino, ITALIA
e-mail: `luigi.vezzoni@unito.it`

Lavoro pervenuto in redazione il 09.01.2012 e, in forma definitiva, il 04.03.2012