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THE BESSEL WAVELET CONVOLUTION PRODUCT

Abstract. The convolution product associated with the Bessel wavelet transformation is investigated. Certain norm inequalities for the convolution product are established.

1. Introduction

Cholewinski [1], Haimo [2], Hirschman Jr. [3] and others studied the Hankel convolution for the following form of the Hankel transformation of a function $f \in L^1_\sigma(I)$, where $I = (0,\infty)$ and $L^1_\sigma(I) = \{f : \int_0^\infty |f(x)|\,d\sigma(x) < \infty,\ I = (0,\infty)\}$. Namely,

$$(h_\mu f)(x) = \tilde{f}(x) = \int_0^\infty j_\mu(xt)\,f(t)\,d\sigma(t) \tag{1}$$

where

$$j_\mu(x) = 2^{\mu-\frac{1}{2}}\Gamma(\mu+\tfrac{1}{2})x^{\frac{1}{2}-\mu}J_{\mu-\frac{1}{2}}(x).$$

Here, $J_{\mu-\frac{1}{2}}(x)$ is the Bessel function of order $\mu-\frac{1}{2}$, and

$$d\sigma(t) = \frac{t^{2\mu}}{2^{\mu-\frac{1}{2}}\Gamma(\mu+\frac{1}{2})}dt.$$

We say that $f \in L^p_\sigma(I), 1 \leqslant p < \infty$, if

$$\|f\|_{p,\sigma} = \left(\int_0^\infty |f(x)|^p\,d\sigma(x)\right)^{\frac{1}{p}} < \infty.$$

If $f \in L^1_\sigma(I)$ and $h_\mu f \in L^1_\sigma(I)$ then the inverse Hankel transform is given by

$$f(x) = (h_\mu^{-1}[\tilde{f}])(x) = \int_0^\infty j_\mu(xt)(h_\mu f)(t)\,d\sigma(t). \tag{2}$$

If $f \in L^1_\sigma(I)$, $g \in L^1_\sigma(I)$ then the Hankel convolution is defined by

$$(f\#g)(x) := \int_0^\infty (\tau_x f)(y)g(y)\,d\sigma(y), \tag{3}$$

where the Hankel translation τ_x is given by

$$(\tau_x f)(y) := \tilde{f}(x,y) = \int_0^\infty D(x,y,z)\,f(x)\,d\sigma(z), \tag{4}$$

$$\begin{aligned} D(x,y,z) :=& \int_0^\infty j_\mu(xt)\, j_\mu(yt)\, j_\mu(zt)\, d\sigma(t) \\ =& \ 2^{3\mu-\frac{1}{2}}(\pi)^{3\mu-\frac{5}{2}}[\Gamma(\mu+\tfrac{5}{2})]^2[\Gamma(\mu)]^{-1}(xyz)^{-2\mu+1}[\triangle(x,y,z)]^{-2\mu+1}, \end{aligned}$$

for $\mu > 0$, where $\triangle(x,y,z)$ is the area of a triangle with sides x,y,z if such a triangle exists and zero otherwise.

Here we note that $D(x,y,z)$ is symmetric in x, y, z. Applying (2) to (4) we get the formula

$$\int_0^\infty j_\mu(zt) D(x,y,z)\, d\sigma(z) = j_\mu(xt)\, j_\mu(yt).$$

Setting $t = 0$, we get

$$\int_0^\infty D(x,y,z)\, d\sigma(z) = 1.$$

Therefore in view of (4),

$$\|\tilde{f}(x,y)\|_{1,\sigma} \leqslant \|f\|_{1,\sigma}. \tag{5}$$

Now, using (4) we can write (3) in the following form:

$$(f\#g)(x) = \int_0^\infty \int_0^\infty D(x,y,z) f(z) g(y)\, d\sigma(z)\, d\sigma(y).$$

Some important properties of the Hankel convolution that are relevant are:

1. If $f, g \in L^1_\sigma(I)$ then from [2],

$$\|f\#g\|_{1,\sigma} \leqslant \|f\|_{1,\sigma} \|g\|_{1,\sigma}. \tag{6}$$

2. With the same assumptions,

$$h_\mu(f\#g)(x) = (h_\mu f)(x)(h_\mu g)(x). \tag{7}$$

3. Let $f \in L^1_\sigma(I)$ and $g \in L^p_\sigma(I)$, $p \geqslant 1$. Then $(f\#g)$ exists, is continuous and from [7], we get the inequality

$$\|f\#g\|_{p,\sigma} \leqslant \|f\|_{1,\sigma} \|g\|_{p,\sigma}. \tag{8}$$

4. Let $f \in L^p_\sigma(I)$, $g \in L^q_\sigma(I)$, $1/p + 1/q = 1$. Then $f\#g$ exists, is continuous and from [7] we have

$$\|f\#g\|_{\infty,\sigma} \leqslant \|f\|_{p,\sigma} \|g\|_{q,\sigma}. \tag{9}$$

5. Let $f \in L^p_\sigma(I)$ and $g \in L^q_\sigma(I)$, $1/r = 1/p + 1/q - 1$. Then $(f\#g)$ exists, is continuous and from [7], we get the inequality:

$$\|f\#g\|_{r,\sigma} \leqslant \|f\|_p \|g\|_q. \tag{10}$$

6. Let $f \in L^p_\sigma(I)$, $g \in L^q_\sigma(I)$ and $h \in L^r_\sigma(I)$. Then the weighted norm inequality

$$\left| \int_0^\infty f(x)(g\#h)(x)\, d\sigma(x) \right| \leqslant \|f\|_{p,\sigma} \|g\|_{q,\sigma} \|h\|_{r,\sigma}$$

holds for $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 2$.

As indicated above, the proof of properties 1 to 5 are well known. Hence, we next give the proof of 6.

Using Holder's inequality, we get

$$\left| \int_0^\infty f(x)(g\#h)(x)\, d\sigma(x) \right| \leqslant \|f\|_{p,\sigma} \|g\#h\|_{s,\sigma}, \quad \frac{1}{p} + \frac{1}{s} = 1.$$

Therefore using (9) we have

$$\left| \int_0^\infty f(x)(g\#h)(x)\, d\sigma(x) \right| \leq \|f\|_{p,\sigma} \|g\|_{q,\sigma} \|h\|_{s,\sigma}, \quad \frac{1}{s} = \frac{1}{q} + \frac{1}{r} - 1.$$

From [4], h_μ is isometric on $L^2_\sigma(I)$, $(h_\mu^{-1} h_\mu f) = f$ then Parseval's formula of the Hankel transformation for $f, g \in L^2_\sigma(I)$ is given by

$$\int_0^\infty f(x) g(x)\, d\sigma(x) = \int_0^\infty (h_\mu f)(y)(h_\mu g)(y)\, d\sigma(y). \tag{11}$$

Furthermore, this relation also holds for $f, g \in L^1_\sigma(I)$, see [8].

For $\psi \in L^1_\sigma(I)$, using translation τ given in (4) and dilation $D_a f(x,y) = f(ax, ay)$, the Bessel wavelet [6] is defined by

$$\tilde{\psi}\Big(\frac{t}{a}, \frac{b}{a}\Big) = D_{1/a} \tau_b \psi(t) = \int_0^\infty \psi(z) D\Big(\frac{t}{a}, \frac{b}{a}, z\Big)\, d\sigma(z). \tag{12}$$

The continuous Bessel wavelet transform [6] of a function $f \in L^1_\sigma(I)$ with respect to wavelet $\psi \in L^1_\sigma(I)$ is defined by

$$(B_\psi f)(b,a) = a^{-2\mu-1} \int_0^\infty \tilde{\psi}\Big(\frac{t}{a}, \frac{b}{a}\Big) f(t)\, d\sigma(t), \quad a > 0. \tag{13}$$

By simple modification of (13), we can get

$$(B_\psi f)(b,a) = (f\#\psi)\Big(\frac{b}{a}\Big), \quad a > 0.$$

From (3) and (4) the continuous Bessel wavelet transform of a function $f \in L^1_\sigma(I)$ can be written in the form:

$$(B_\psi f)(b,a) = \int_0^\infty j_\mu(bw)(h_\mu f)(w)(h_\mu \psi)(aw)\, d\sigma(w). \tag{14}$$

Now, we state the Parseval formula of the Bessel wavelet transform from [6, p. 245].

$$\int_0^\infty \int_0^\infty (B_\psi f)(b,a)(B_\psi g)(b,a) \frac{d\sigma(b)\, d\sigma(a)}{a^{2\mu+1}} = C_\psi \langle f, g \rangle, \tag{15}$$

for $f \in L^2_\sigma(I)$ and $g \in L^2_\sigma(I)$.

Now, we also state from [3, Theorem 2c, p. 312] and [3, corollary 2c, p. 313] which is useful for our approximation results:

THEOREM 1. *Suppose that*

1. $k_n(x) \geq 0, \quad 0 < x < \infty,$
2. $\int_0^\infty k_n(x)\, d\sigma(x) = 1, \quad n = 0,1,2,3,\ldots,$
3. $\lim_{n\to\infty} \int_\delta^\infty k_n(x)\, d\sigma(x) = 0$ *for each* $\delta > 0,$
4. $\phi(x) \in L^\infty_\sigma(I),$
5. ϕ *is continuous at* x_0, $x_0 \in [x-\delta, x+\delta]$ *and* $\delta > 0.$

Then

$$\lim_{n\to\infty} (\phi \# k_n)(x_0) = \phi(x_0).$$

COROLLARY 1. *With the same assumptions on* $k_n(x)$, *if* $f(x) \in L^1_\sigma(I)$ *then*

$$\lim_{n\to\infty} \|f \# k_n - f\|_1 = 0.$$

In this paper, motivated from [5, pp. 129–136] we define convolution product for Bessel wavelet transform and study some of its properties.

2. The Bessel wavelet convolution product

In this section, using properties (5), (11) and (12), we formally define the convolution product for the Bessel wavelet transformation by the relation

$$B_\psi(f \otimes g)(b,a) = (B_\psi f)(b,a)(B_\psi g)(b,a), \tag{16}$$

and investigate its boundedness and approximation properties. This in turn implies that the product of two Bessel wavelet transforms could be wavelet transform under certain conditions.

THEOREM 2. *Let* $f, g, \psi \in L^1_\sigma(I)$ *and* $h_\mu(\psi)(w) \neq 0$. *Then the Bessel wavelet convolution can be written in the form*

$$(f \otimes g)(z) = \int_0^\infty (\tau_{z,a} f)(y) g(y)\, d\sigma(y),$$

where $(\tau_{z,a} f)(y) = \int_0^\infty f(x) D_a(x,y,z)\, d\sigma(x)$,

$$D_a(x,y,z) = \int_0^\infty \int_0^\infty (h_\mu\psi)(at)(h_\mu\psi)(a\xi)\, j_\mu(xt)\, j_\mu(y\xi) L_a(t,\xi,z)\, d\sigma(t)\, d\sigma(\xi), \tag{17}$$

and

$$L_a(t,\xi,z) = \int_0^\infty j_\mu(y\xi)\, j_\mu(yt) Q_a(y,z)\, d\sigma(y), \tag{18}$$

$$Q_a(y,z) = \int_0^\infty \frac{j_\mu(wz)\, j_\mu(wy)}{(h_\mu\psi)(aw)}\, d\sigma(w). \tag{19}$$

Proof. From (14) we have

$$h_\mu[(B_\psi f)(b,a)](w) = (h_\mu\psi)(aw)(h_\mu f)(w). \tag{20}$$

Using (16) and (20) we get

$$\begin{aligned} & h_\mu[(B_\psi(f \otimes g))(b,a)](w) \\ &= h_\mu[(B_\psi f)(b,a)(B_\psi g)(b,a)](w) \\ &= h_\mu[h_\mu^{-1}((h_\mu\psi)(a\cdot)(h_\mu f)(\cdot))\, h_\mu^{-1}((h_\mu\psi)(a\cdot)(h_\mu g)(\cdot))](w). \end{aligned}$$

By property (7) of the Hankel convolution, we have

$$h_\mu[(B_\psi(f \otimes g))(b,a)](w) = [(h_\mu\psi)(a\cdot)(h_\mu f)(\cdot)\#(h_\mu\psi)(a\cdot)(h_\mu g)(\cdot)](w).$$

Therefore by (20), we get

$$(h_\mu\psi)(aw) h_\mu[(f \otimes g)](w) = [(h_\mu\psi)(a\cdot)(h_\mu f)(\cdot)\#(h_\mu\psi)(a\cdot)(h_\mu g)(\cdot)](w). \tag{21}$$

This gives a relation between the Bessel wavelet transform-convolution and the Hankel transform-convolution.

Let us set

$$\begin{aligned} F_a &= (h_\mu\psi)(a\cdot)(h_\mu f)(\cdot), \\ G_a &= (h_\mu\psi)(a\cdot)(h_\mu g)(\cdot). \end{aligned}$$

Then, by (3) and (4) we get

$$
\begin{aligned}
&(h_\mu\psi)(aw)h_\mu[(f\otimes g)](w)\\
&= \int_0^\infty (\tau_w G_a)(\eta)F_a(\eta)\,d\sigma(\eta)\\
&= \int_0^\infty F_a(\eta)\Big(\int_0^\infty D(w,\eta,\xi)G_a(\xi)\,d\sigma(\xi)\Big)\,d\sigma(\eta)\\
&= \int_0^\infty\int_0^\infty F_a(\eta)G_a(\xi)D(w,\eta,\xi)\,d\sigma(\xi)\,d\sigma(\eta)\\
&= \int_0^\infty\int_0^\infty F_a(\eta)G_a(\xi)\Big(\int_0^\infty j_\mu(wy)\,j_\mu(\eta y)\,j_\mu(\xi y)\,d\sigma(y)\Big)\,d\sigma(\xi)\,d\sigma(\eta)\\
&= \int_0^\infty \Big(\int_0^\infty F_a(\eta)\,j_\mu(\eta y)\,d\sigma(\eta)\Big)\Big(\int_0^\infty G_a(\xi)\,j_\mu(\xi y)\,d\sigma(\xi)\Big)\,j_\mu(wy)\,d\sigma(y)\\
&= \int_0^\infty (h_\mu F_a)(y)(h_\mu G_a)(y)\,j_\mu(wy)\,d\sigma(y).
\end{aligned}
$$

Therefore by the inversion formula of the Hankel transformation (2), we have

$$
\begin{aligned}
&(f\otimes g)(z)\\
&= \int_0^\infty \frac{j_\mu(wz)}{(h_\mu\psi)(aw)}\Big(\int_0^\infty (h_\mu F_a)(y)(h_\mu G_a)(y)\,j_\mu(wy)\,d\sigma(y)\Big)\,d\sigma(w)\\
&= \int_0^\infty (h_\mu F_a)(y)(h_\mu G_a)(y)\Big(\int_0^\infty \frac{j_\mu(wz)\,j_\mu(wy)}{(h_\mu\psi)(aw)}\,d\sigma(w)\Big)\,d\sigma(y).\\
&= \int_0^\infty (h_\mu F_a)(y)(h_\mu G_a)(y)Q_a(y,z)\,d\sigma(y),
\end{aligned}
$$

where $Q_a(y,z)$ is given by (19).

Then by the definition of the Hankel transformation (1),

$$
\begin{aligned}
&(f\otimes g)(z)\\
&= \int_0^\infty\int_0^\infty j_\mu(yt)(h_\mu\psi)(at)(h_\mu f)(t)\,d\sigma(t)\\
&\qquad \Big(\int_0^\infty j_\mu(y\xi)(h_\mu\psi)(a\xi)(h_\mu g)(\xi)\,d\sigma(\xi)\Big)Q_a(y,z)\,d\sigma(y)\\
&= \int_0^\infty\int_0^\infty (h_\mu\psi)(at)(h_\mu\psi)(a\xi)(h_\mu f)(t)(h_\mu g)(\xi)\\
&\qquad \Big(\int_0^\infty j_\mu(y\xi)\,j_\mu(yt)Q_a(y,z)\,d\sigma(y)\Big)\,d\sigma(t)\,d\sigma(\xi)\\
&= \int_0^\infty\int_0^\infty (h_\mu\psi)(at)(h_\mu\psi)(a\xi)(h_\mu f)(t)(h_\mu g)(\xi)L_a(t,\xi,z)\,d\sigma(t)\,d\sigma(\xi),
\end{aligned}
$$

Therefore,

$$\begin{aligned}
&(f\otimes g)(z)\\
&= \int_0^\infty\int_0^\infty (h_\mu\psi)(at)(h_\mu\psi)(a\xi)\Big(\int_0^\infty j_\mu(xt)f(x)\,d\sigma(x)\Big)\\
&\qquad \Big(\int_0^\infty j_\mu(y\xi)g(y)\,d\sigma(y)\Big)L_a(t,\xi,z)\,d\sigma(t)\,d\sigma(\xi)\\
&= \int_0^\infty\int_0^\infty f(x)g(y)\Big(\int_0^\infty\int_0^\infty j_\mu(xt)\,j_\mu(y\xi)(h_\mu\psi)(at)(h_\mu\psi)(a\xi)\\
&\qquad L_a(t,\xi,z)\,d\sigma(t)\,d\sigma(\xi)\Big)\,d\sigma(x)\,d\sigma(y)\\
&= \int_0^\infty\int_0^\infty f(x)g(y)D_a(x,y,z)\,d\sigma(x)\,d\sigma(y),
\end{aligned}$$

where

$$D_a(x,y,z)=\int_0^\infty\int_0^\infty j_\mu(xt)\,j_\mu(y\xi)(h_\mu\psi)(at)(h_\mu\psi)(a\xi)L_a(t,\xi,z)\,d\sigma(t)\,d\sigma(\xi).$$

If we define the generalized translation by

$$F_a(z,y)=(\tau_{z,a}f)(y)=\int_0^\infty D_a(x,y,z)\,f(x)\,d\sigma(x),$$

then

$$(f\otimes g)(z)=\int_0^\infty(\tau_{z,a}f)(y)g(y)\,d\sigma(y).$$

□

THEOREM 3. *Assume that* $\inf_\omega |(h_\mu\psi)(aw)|=B_\psi(a)>0$. *Then*

$$\|D_a(x,y,z)\|\leqslant\frac{1}{B_\psi(a)}a^{-2\mu-1}\|\psi\|^2_{1,\sigma}.$$

Proof. From (17) we have

$$\begin{aligned}
&D_a(x,y,z)\\
&= \int_0^\infty\int_0^\infty j_\mu(xt)\,j_\mu(y\xi)(h_\mu\psi)(at)(h_\mu\psi)(a\xi)L_a(t,\xi,z)\,d\sigma(t)\,d\sigma(\xi)\\
&= \int_0^\infty\int_0^\infty j_\mu(xt)\,j_\mu(y\xi)(h_\mu\psi)(at)(h_\mu\psi)(a\xi)\\
&\qquad\Big(\int_0^\infty j_\mu(\eta\xi)\,j_\mu(\eta t)Q_a(\eta,z)\,d\sigma(\eta)\Big)\,d\sigma(t)\,d\sigma(\xi)
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^\infty \int_0^\infty j_\mu(xt) j_\mu(y\xi)(h_\mu\psi)(at)(h_\mu\psi)(a\xi) \\
&\qquad \left(\int_0^\infty j_\mu(\eta\xi) j_\mu(\eta t)\left(\int_0^\infty \frac{j_\mu(wz) j_\mu(\eta w)}{(h_\mu\psi)(aw)}\, d\sigma(w)\right) d\sigma(\eta)\right) d\sigma(t)\, d\sigma(\xi) \\
&= \int_0^\infty \left(\int_0^\infty j_\mu(xt) j_\mu(\eta t)(h_\mu\psi)(at)\, d\sigma(t)\right) \\
&\qquad\qquad \left(\int_0^\infty j_\mu(y\xi) j_\mu(\eta\xi)(h_\mu\psi)(a\xi)\, d\sigma(\xi)\right) Q_a(z,\eta)\, d\sigma(\eta) \\
&= \int_0^\infty h_\mu[j_\mu(xt)(h_\mu\psi)(at)](\eta) h_\mu[j_\mu(y\xi)(h_\mu\psi)(a\xi)](\eta) Q_a(z,\eta)\, d\sigma(\eta) \\
&= \int_0^\infty \int_0^\infty h_\mu[j_\mu(xt)(h_\mu\psi)(at) \# j_\mu(y\xi)(h_\mu\psi)(a\xi)](\eta) \\
&\qquad\qquad j_\mu(w\eta) j_\mu(wz)[(h_\mu\psi)(aw)]^{-1}\, d\sigma(w)\, d\sigma(\eta) \\
&= \int_0^\infty [j_\mu(x\cdot)(h_\mu\psi)(a\cdot) \# j_\mu(y\cdot)(h_\mu\psi)(a\cdot)](w) j_\mu(wz)[(h_\mu\psi)(aw)]^{-1}\, d\sigma(w).
\end{aligned}
$$

Now, set $F_a(t) = j_\mu(xt) h_\mu\psi(at)$ and assume that $\inf_\omega |(h_\mu\psi)(aw)| = B_\psi(a) > 0$. Since $|j_\mu(z)| \leqslant 1$, [2, p. 336], we have

$$
|D_a(x,y,z)| \leqslant \frac{1}{B_\psi(a)} \int_0^\infty |(F_a \# F_a)(w)|\, d\sigma(w).
$$

Using (6), we have

$$
\begin{aligned}
|D_a(x,y,z)| &\leqslant \frac{1}{B_\psi(a)} \|F\|_{1,\sigma} \|F\|_{1,\sigma} \\
&\leqslant \frac{1}{B_\psi(a)} \left[\int_0^\infty |j_\mu(x\nu)(h_\mu\psi)(a\nu)|\, d\sigma(\nu)\right]^2 \\
&\leqslant \frac{1}{B_\psi(a)} \left[\int_0^\infty |\psi(a\nu)|\, d\sigma(\nu)\right]^2 \\
&\leqslant \frac{1}{B_\psi(a)} \left[\|\psi_a\|_{1,\sigma}\right]^2 \\
&\leqslant \frac{a^{-2\mu-1}}{B_\psi(a)} \left[\|\psi\|_{1,\sigma}\right]^2.
\end{aligned}
$$

□

In order to obtain Plancheral formula for the Bessel wavelet transform, we define the space

$$W^2(I\times I)=\left\{g(b,a):\|g\|_{W^2}=\left(\int_0^\infty\int_0^\infty |g(b,a)|^2\frac{d\sigma(b)\,d\sigma(a)}{a^{2\mu+1}}\right)^{\frac{1}{2}}<\infty\right\}.$$

THEOREM 4. *Let $f\in L^2_\sigma(I)$, $\psi\in L^2_\sigma(I)$. Then*

$$\left\|(B_\psi f)(b,a)\right\|_{W^2}=\sqrt{C_\psi}\,\|f\|_{2,\sigma}.$$

Proof. Putting $f=g$ in (15), we prove the above theorem. □

THEOREM 5. *Let $f,g\in L^2_\sigma(I)$ and let $\psi\in L^2_\sigma(I)$ be a Bessel wavelet which satisfies $C_\psi:=\int_0^\infty|(h_\mu\psi)(aw)|^2\frac{d\sigma(a)}{a^{2\mu+1}}>0$. Then*

$$\|f\otimes g\|_{2,\sigma}\leqslant\|f\|_{2,\sigma}\|g\|_{2,\sigma}\|\psi\|_{2,\sigma}.$$

Proof. Using Theorem 4 and (16)

$$\begin{aligned}\sqrt{C_\psi}\,\|f\otimes g\|_{2,\sigma}&\\ &=\left\|B_\psi(f\otimes g)\right\|_{W^2}\\ &=\left\|B_\psi f(b,a)B_\psi g(b,a)\right\|_{W^2}\\ &=\left(\int_0^\infty\int_0^\infty|B_\psi f(b,a)B_\psi g(b,a)|^2\frac{d\sigma(a)\,d\sigma(b)}{a^{2\mu+1}}\right)^{\frac{1}{2}}.\end{aligned}\tag{22}$$

From (14) and (9),we have

$$|B_\psi g(b,a)|\leqslant|(g(a\cdot)\#\psi(\cdot))(b/a)|\leqslant\|g\|_{2,\sigma}\|\psi\|_{2,\sigma}.\tag{23}$$

Applying (22) and (23) we get

$$\sqrt{C_\psi}\,\|f\otimes g\|_{2,\sigma}\leqslant\|g\|_{2,\sigma}\,\|\psi\|_{2,\sigma}\left(\int_0^\infty\int_0^\infty|B_\psi f(b,a)|^2\frac{d\sigma(a)\,d\sigma(b)}{a^{2\mu+1}}\right)^{\frac{1}{2}}.$$

From Theorem 4, we obtain

$$\sqrt{C_\psi}\,\|f\otimes g\|_{2,\sigma}\leqslant\|g\|_{2,\sigma}\,\|\psi\|_{2,\sigma}\sqrt{C_\psi}\,\|f\|_{2,\sigma}.$$

Hence,

$$\|f\otimes g\|_{2,\sigma}\leqslant\|g\|_{2,\sigma}\,\|\psi\|_{2,\sigma}\,\|f\|_{2,\sigma}.$$

□

3. Weighted Sobolev space

In this section we study certain properties of the Bessel wavelet convolution on a weighted Sobolev space defined below.

DEFINITION 1. *The Zemanian space $H_\mu(I), I=(0,\infty)$, is the set of all infinitely differentiable functions ϕ on $(0,\infty)$ such that*

$$\gamma^{\mu}_{m,k}(\phi)=\sup_{x\in(0,\infty)}\left|x^m\left(x^{-1}\frac{d}{dx}\right)^k x^{-\mu-\frac{1}{2}}\phi(x)\right|<\infty$$

for all $m,k\in\mathbb{N}_0$. Then $f\in H'_\mu(I)$ is defined by the following way:

$$\langle f,\phi\rangle=\int_0^\infty f(x)\phi(x)dx,\ \phi\in H_\mu(I).$$

DEFINITION 2. *Let $k(w)$ be an arbitrary weight function. Then a function $\Phi\in[H_\mu(I)]'$ is said to belong to the weighted Sobolev space $G^p_{\mu,k}(I)$ for $\mu\in\mathbb{R}$, $1\leqslant p<\infty$, if it satisfies*

$$\|\Phi\|_{p,\mu,\sigma,k}=\left(\int_0^\infty|k(w)(h_\mu\Phi)(w)|^p\,d\sigma(w)\right)^{1/p}, \tag{24}$$

where $a>0$ and $\Phi\in L^p_\sigma(I)$.

In what follows we shall assume that $k(w)=|(h_\mu\psi)(aw)|$ for fixed $a>0$.

THEOREM 6. *Let $f\in G^1_{\mu,k}(I)$ and $g\in G^p_{\mu,k}(I)$, $p\geq 1$. Then*

$$\|f\otimes g\|_{p,\mu,\sigma,k}\leqslant\|f\|_{1,\mu,\sigma,k}\|g\|_{p,\mu,\sigma,k}.$$

Proof. In view of (24) we have

$$\|f\otimes g\|_{p,\mu,\sigma,k}=\left(\int_0^\infty|k(w)h_\mu(f\otimes g)(w)|^p\,d\sigma(w)\right)^{1/p}.$$

By (8) and (21) we have

$$\begin{aligned}\|f\otimes g\|_{p,\mu,\sigma,k}&\leq\|F_a(w)\|_{1,\mu,\sigma,k}\|G_a(w)\|_{p,\mu,\sigma,k}\\&\leq\big\|(h_\mu\psi)(aw)(h_\mu f)(w)\big\|_{1,\mu,\sigma,k}\\&\qquad\big\|(h_\mu\psi)(aw)(h_\mu g)(w)\big\|_{p,\mu,\sigma,k}.\end{aligned}\tag{25}$$

From Definition 2, we get

$$\|f\otimes g\|_{p,\mu,\sigma,k}\leqslant\|f\|_{1,\mu,\sigma,k}\|g\|_{p,\mu,\sigma,k}. \tag{26}$$

□

THEOREM 7. *Let $f\in G^p_{\mu,k}(I)$ and $g\in G^q_{\mu,k}(I)$, with $1\leqslant p,\ q<\infty$ and $1/r=1/p+1/q-1$. Then*

$$\|f\otimes g\|_{r,\mu,\sigma,k}\leqslant\|f\|_{p,\mu,\sigma,k}\|g\|_{q,\mu,\sigma,k}. \tag{27}$$

Proof. Using (10) and (24) we get (27). □

Approximation properties of the Bessel wavelet convolution are given next.

THEOREM 8. *Let $\psi_{n,a}(w) = \psi_n(aw)$, $n = 0,1,2,\dots$ be the sequence of basic wavelet functions such that*

1. $\psi_{n,a}(w) \geqslant 0, \quad 0 < w < \infty,$
2. $\int_0^\infty \psi_{n,a}(w)\, d\sigma(w) = 1,$
3. $\lim_{n\to\infty} \int_\varepsilon^\infty \psi_{n,a}(w)\, d\sigma(w) = 0,$ *for each* $\varepsilon > 0,$
4. $(h_\mu \psi_{n,a})(w) \in L^1_\sigma(I),$
5. $h_\mu^{-1}[(h_\mu \psi_{n,a})(w)] = \psi_{n,a}(w).$

Then

$$\lim_{n\to\infty} \| f(b) - (B_{\psi_n} f)(b,a)\|_{1,\sigma} = 0.$$

Proof. See [3, pp. 315–316]. □

THEOREM 9. *Let $k_n(w) = (h_\mu \psi)(aw)(h_\mu g_n)(w)$ for fixed $a > 0, n \in \mathbb{N}$, and $\phi(w) = (h_\mu \psi)(aw)(h_\mu f)(w)$ satisfy:*

1. $k_n(w) \geqslant 0, \quad 0 < w < \infty,$
2. $\int_0^\infty k_n(w)\, d\sigma(w) = 1, \quad w = 0,1,2,3,\dots,$
3. $\lim_{n\to\infty} \int_\delta^\infty k_n(w)\, d\sigma(w) = 0,$ *for each* $\delta > 0,$
4. $\phi(w) \in L^\infty_\sigma(I),$
5. ϕ *is continuous at* w_0*, and* $(h_\mu \psi)(aw_0) \neq 0$ *for* $w_0 \in [w-\delta, w+\delta]$*,* $\delta > 0.$

Then

$$\lim_{n\to\infty} h_\mu(f \otimes g_n)(w_0) = (h_\mu f)(w_0).$$

Proof. In view of relation (21) we have

$$(h_\mu \psi)(aw) h_\mu(f \otimes g_n)(w) = (\phi \# k_n)(w).$$

Now, using Theorem1.1,we have

$$\begin{aligned} \lim_{n\to\infty} (h_\mu \psi)(aw_0) h_\mu(f \otimes g_n)(w_0) &= \lim_{n\to\infty} (\phi \# k_n)(w_0) \\ &= \phi(w_0) \\ &= (h_\mu \psi)(aw_0)(h_\mu f)(w_0). \end{aligned}$$

This implies that

$$\lim_{n\to\infty} h_\mu(f\otimes g_n)(w_0) = (h_\mu f)(w_0).$$

□

THEOREM 10. *Let $f,\psi \in L^1_\sigma(I)$, and $k_n(w)$ be the same as Theorem 9, which satisfies all the four properties of Theorem 8. Then*

$$\lim_{n\to\infty} \left\|(h_\mu\psi)(aw_0)(h_\mu f)(w_0) - (h_\mu\psi)(aw_0)h_\mu(f\otimes g_n)(w_0)\right\|_{1,\sigma} = 0.$$

Proof. Using (21), we have

$$\begin{aligned} &\lim_{n\to\infty} \left\|(h_\mu\psi)(aw_0)(h_\mu f)(w_0) - (h_\mu\psi)(aw_0)h_\mu(f\otimes g_n)(w_0)\right\|_{1,\sigma} \\ &= \lim_{n\to\infty} \Big\|(h_\mu\psi)(aw_0)(h_\mu f)(w_0) \\ &\qquad -[(h_\mu\psi)(a\cdot)(h_\mu f)(\cdot)\#(h_\mu\psi)(a\cdot)(h_\mu g_n)(a\cdot)](w_0)\Big\|_{1,\sigma} \\ &= \lim_{n\to\infty} \left\|\phi(w_0) - (\phi\# k_n)(w_0)\right\|_{1,\sigma}. \end{aligned}$$

Since $f,\psi_a \in L^1_\sigma(I)$, $\phi(w) = (h_\mu f)(h_\mu\psi_a) = h_\mu(f\#\psi_a) \in L^1_\sigma(I)$. Therefore using the tools of [3, Corollary 2c, pp. 313–314], we have

$$\lim_{n\to\infty} \left\|(h_\mu\psi)(aw_0)(h_\mu f)(w_0) - (h_\mu\psi)(aw_0)h_\mu(f\otimes g_n)(w_0)\right\|_{1,\sigma} = 0.$$

□

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