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CLASSES OF MULTIVALENT FUNCTIONS INVOLVING THE LIU–OWA OPERATOR

Abstract. Using the Liu–Owa operator we introduce a new class of multivalent analytic functions defined in the unit disc E. We investigate an inclusion relationship and radius problem for the class of functions of p-valent bounded radius rotations which are defined here by means of a certain integral operator $Q^{\alpha}_{\beta,p}$.

1. Introduction

Let $\mathscr{A}(p)$ denote the class of functions $f(z)$ normalized by

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{k+p} z^{k+p}, \quad p \in \mathbb{N} = \{1, 2, \dots\}, \tag{1}$$

which are analytic and p-valent in the open unit disc $E = \{z : |z| < 1\}$. A function $f \in \mathscr{A}(p)$ is said to belong to the class $S(p,\rho)$ of p-valently starlike functions of order ρ (with $0 \leqslant \rho < p$) if it satisfies, for $z \in E$, the conditions

$$\operatorname{Re} \frac{zf'(z)}{f(z)} > \rho \quad \text{and} \quad \int_0^{2\pi} \operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} d\theta = 2p\pi. \tag{2}$$

The class $S(p,\rho)$ was introduced by Goodman [2] and studied in [9] and other papers. For functions $f_j(z) \in \mathscr{A}(p)$, given by

$$f_j(z) = z^p + \sum_{k=1}^{\infty} a_{k+p,j} z^{k+p}, \quad j = 1, 2, \tag{3}$$

we define the Hadamard product (or convolution) of $f_1(z)$ and $f_2(z)$ by

$$(f_1 \star f_2)(z) = z^p + \sum_{k=1}^{\infty} a_{k+p,1} a_{k+p,2} z^{k+p} = (f_2 \star f_1)(z), \quad z \in E. \tag{4}$$

Let $P_k(\rho)$ be the class of functions $p(z)$ analytic in E with $p(0) = 1$ and

$$\int_0^{2\pi} \left| \frac{\operatorname{Re} p(z) - \rho}{1 - \rho} \right| d\theta \leqslant k\pi, \quad z = re^{i\theta}, \tag{5}$$

where $k \geqslant 2$ and $0 \leqslant \rho < 1$. This class was introduced by Padmanabhan et al. in [8]. We note that $P_k(0) = P_k$, see Pinchuk [10], $P_2(\ \rho) = P(\rho)$, the class of analytic functions with positive real part greater than ρ and $P_2(0) = P$, the class of functions with positive real part. From (5) we can easily deduce that $p(z) \in P_k(\rho)$ if and only if there exist $p_1(z),\ p_2(z) \in P(\rho)$ such that for $z \in E$,

$$p(z) = \left(\frac{k}{4} + \frac{1}{2}\right) p_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right) p_2(z). \tag{6}$$

Motivated by Jung et al. [3], Liu and Owa [5] considered the linear operator $Q^{\alpha}_{\beta,p}$: $\mathscr{A}(p) \to \mathscr{A}(p)$ defined by

$$(Q^{\alpha}_{\beta,p} f)(z) = \binom{p+\alpha+\beta-1}{p+\beta-1} \frac{\alpha}{z^{\beta}} \int_0^z \left(1 - \frac{t}{z}\right)^{\alpha-1} t^{\beta-1} f(t)\, dt \tag{7}$$

for $\alpha > 0$ and $\beta > -1$, and

$$(Q^{0}_{\beta,p} f)(z) = f(z) \tag{8}$$

for $\alpha = 0$ and $\beta > -1$.

We note that if $f \in \mathscr{A}(p)$, then from (7) and (8) it follows that

$$(Q^{\alpha}_{\beta,p} f)(z) = z^p + \frac{\Gamma(p+\alpha+\beta)}{\Gamma(p+\beta)} \sum_{k=p+1}^{\infty} \frac{\Gamma(k+\beta)}{\Gamma(k+\alpha+\beta)} a^k z^k,$$

whenever $\alpha \geqslant 0$ and $\beta > -1$. Using the above relation, it is easy to verify that

$$z(Q^{\alpha}_{\beta,p} f)'(z) = (p+\alpha+\beta-1)(Q^{\alpha-1}_{\beta,p} f)(z) - (\alpha+\beta-1)(Q^{\alpha}_{\beta,p} f)(z). \tag{9}$$

For interested readers we refer to the work done by the authors of [7, 1, 4].

Using the operator $Q^{\alpha}_{\beta,p}$, we now define subclasses of $\mathscr{A}(p)$ as follows:

DEFINITION 1. *Assume that* $\alpha \geqslant 0$, $\beta > -1$ *and* $p \in \mathbb{N}$. *We say that a function* $f(z) \in \mathscr{A}(p)$ *is in the class* $R_k(\alpha, \beta, p, \rho)$ *if and if* $\dfrac{zf'(z)}{f(z)} \in P_k(\rho)$ *for* $z \in E$, $0 \leqslant \rho < p$. *For* $k = 2$ *we obtain* $R_2(p, \rho) = S(p, \rho)$ *as defined by* (2).

DEFINITION 2. *Let* $f \in \mathscr{A}(p)$. *Then* $f \in R_k(\alpha, \beta, p, \rho)$, *if and only if* $Q^{\alpha}_{\beta,p} f$ *belongs to* $R_k(p, \rho)$ *for* $z \in E$ *and* $p \in \mathbb{N}$.

In the present paper, we investigate an inclusion relationship and radius problem for the class of functions of p-valent multivalent functions by means of a certain integral operator $Q^{\alpha}_{\beta,p} f(z)$.

2. Preliminary lemmas

In this section we recall some known results.

LEMMA 1 ([6]). *Let $u = u_1 + iu_2$, $v = v_1 + iv_2$ and $\Psi(u,v)$ be a complex-valued function satisfying the conditions:*

(i) $\Psi(u,v)$ *is continuous in a domain* $D \subset \mathbb{C}^2$,

(ii) $(1,0) \in D$ *and* $\operatorname{Re}\Psi(1,0) > 0$,

(iii) $\operatorname{Re}\Psi(iu_2, v_1) \leqslant 0$, *whenever* $(iu_2, v_1) \in D$ *and* $v_1 \leqslant -\frac{1}{2}(1+u_2^2)$.

If $h(z) = 1 + c_1 z + \cdots$ *is a function analytic in* E *such that* $(h(z), zh'(z)) \in D$ *and* $\operatorname{Re}\Psi(h(z), zh'(z)) > 0$ *for* $z \in E$, *then* $\operatorname{Re} h(z) > 0$ *in* E.

LEMMA 2 ([12]). *Let p be analytic function in E with $p(0) = 1$ and $\operatorname{Re} p(z) > 0$ with $z \in E$. Then, for $s > 0$ and $\mu \neq -1$ (complex number),*

$$\operatorname{Re}\left\{p(z) + \frac{szp'(z)}{p(z)+\mu}\right\} > 0, \quad \textit{for} \quad |z| < r_0,$$

where r_0 is given by

$$r_0 = \frac{|\mu+1|}{\sqrt{A + (A^2 - |\mu^2 - 1|)^{\frac{1}{2}}}} \tag{10}$$

$$A = 2(s+1)^2 + |\mu|^2 - 1, \tag{11}$$

and this result is the best possible.

3. Main results

THEOREM 1. *Assume that $\alpha \geqslant 0$, $\beta > -1$ and $p \in \mathbb{N}$. Let $f \in R_k(\alpha-1, \beta, p, 0)$. Then $f \in R_k(\alpha, \beta, p, \rho)$, in E where*

$$\rho = \frac{2p}{(2b+1) + \sqrt{(2b+1)^2 + 8p}}, \quad \textit{with} \;\; b = \alpha + \beta - 1. \tag{12}$$

Proof. Set

$$\frac{z(Q^{\alpha}_{\beta,p} f)'(z)}{(Q^{\alpha}_{\beta,p} f)(z)} = H(z) = (p-\rho)h(z) + \rho.$$

Then $h(z)$ is analytic in E with $h(0) = 1$. Using the identity (9), and after some simple computations, we obtain

$$\begin{aligned} \frac{z(Q^{\alpha-1}_{\beta,p} f)'(z)}{(Q^{\alpha-1}_{\beta,p} f)(z)} &= H(z) + \frac{zH'(z)}{H(z)+\alpha+\beta-1} \\ &= H(z) + \frac{zH'(z)}{H(z)+b} \in P_k(0), \quad z \in E, \quad \text{where } b = \alpha+\beta-1. \end{aligned}$$

Let us define the function $\Phi_b(z)$ by

$$\begin{aligned} \Phi_b(z) &= \frac{1}{1+b}\frac{z^p}{1-z} + \frac{b}{1+b}\frac{z^p}{(1-z)^2}, \quad b = \alpha+\beta-1, \\ H(z) &= (\frac{k}{4}+\frac{1}{2})\Big\{(p-\rho)h_1(z)+\rho\Big\} - (\frac{k}{4}-\frac{1}{2})\Big\{(p-\rho)h_2(z)+\rho\Big\}. \end{aligned}$$

We want to show that $H \in P_k(\rho)$, where ρ is given by (12) or equivalently $h_i \in P$, $i = 1, 2$. Now using the convolution technique [11], we have

$$\begin{aligned} H(z) * \frac{\Phi_b(z)}{z^p} &= H(z) + \frac{zH'(z)}{H(z)+b} \\ &= (\frac{k}{4}+\frac{1}{2})\left\{(p-\rho)\left[h_1(z) * \frac{\Phi_b(z)}{z^p}\right]+\rho\right\} \\ &\qquad - (\frac{k}{4}-\frac{1}{2})\left\{(p-\rho)\left[h_2(z) * \frac{\Phi_b(z)}{z^p}\right]+\rho\right\}. \end{aligned}$$

Therefore it follows that for $i = 1, 2$,

$$\text{Re}\left[(p-\rho)h_i(z)+\rho+\frac{(p-\rho)zh_i'(z)}{(p-\rho)h_i(z)+\rho+\alpha+\beta-1}\right] > 0, \quad z \in E. \tag{13}$$

We formulate the functional $\Psi(u, v)$ by choosing $u = u_1 + iu_2 = h_i(z)$ and $v = v_1 + iv_2 = zh_i'(z)$ in (13). Thus

$$\Psi(u,v) = (p-\rho)u+\rho+\frac{(p-\rho)v}{(p-\rho)u+\rho+\alpha+\beta-1}.$$

It can be easily seen that the first two conditions of Lemma 1 are clearly satisfied. We verify only the third condition, as follows.

$$\begin{aligned} \text{Re}\{\Psi(iu_2, v)\} &= \rho + \text{Re}\left[\frac{(p-\rho)v_1}{(p-\rho)iu_2+\rho+\alpha+\beta-1}\right] \\ &= \rho + \frac{(p-\rho)(\rho+\alpha+\beta-1)v_1}{(p-\rho)^2u_2^2+(\rho+\alpha+\beta-1)^2}. \end{aligned}$$

By putting $v_1 \leqslant -\frac{1(1+u_2^2)}{2}$ in the above equation, we obtain

$$\text{Re}\{\Psi(iu_2,v)\} \leqslant \rho - \frac{1}{2}\frac{(p-\rho)(\rho+\alpha+\beta-1)(1+u_2^2)}{(p-\rho)^2u_2^2+(\rho+\alpha+\beta-1)^2},$$

and the right-hand side equals

$$\begin{aligned} &\frac{(\rho+\alpha+\beta-1)[2\rho(\rho+\alpha+\beta-1)-(p-\rho)]+(p-\rho)[2\rho(p-\rho)-(\rho+\alpha+\beta-1)]u_2^2}{2[(p-\rho)^2u_2^2+(\rho+\alpha+\beta-1)^2]} \\ &= \frac{A+Bu_2^2}{2C}, \end{aligned}$$

where

$$\begin{aligned} A &= (\rho+\alpha+\beta-1)[2\rho^2+[2(\alpha+\beta-1)+1]-p \\ B &= (p-\rho)[2\rho(p-\rho)-(\rho+\alpha+\beta-1) \\ C &= [(p-\rho)^2u_2^2+(\rho+\alpha+\beta-1)^2] > 0. \end{aligned}$$

We note $\mathrm{Re}\{\Psi(iu_2,v)\} \leqslant 0$ if and only if $A \leqslant 0$ and $B \leqslant 0$. From $A \leqslant 0$, we obtain ρ as given by (12) and $B \leqslant 0$ gives us $0 \leqslant \rho < p$. Now using Lemma 1 we see that $\mathrm{Re}\,h(z) > 0$ for $z \in E$ and hence $\mathrm{Re}\,\dfrac{z(Q^{\alpha}_{\beta,p}f)'(z)}{Q^{\alpha}_{\beta,p}f)(z)} > \rho$, for $z \in E$ with ρ given by (12) and the proof is complete. $\square$

As a special case when $p = 1$, $b = \alpha + \beta - 1 = 0$ and $k = 2$ we have a well known result that every convex univalent function is starlike univalent of order $\frac{1}{2}$.

Now we take the converse case of Theorem 1.

THEOREM 2. *Assume that* $\alpha \geqslant 0$, $\beta > -1$ *and* $p \in \mathbb{N}$. *Let* $f \in R_k(\alpha,\beta,p,\rho)$ *for* $z \in E$. *Then* $f \in R_k(\alpha-1,\beta,p,0)$ *for* $|z| < r_0$, *where* r_0 *is given by* (10) *with* $\mu = \dfrac{\rho+\alpha+\beta-1}{p-\rho}$ *and* $s = \dfrac{1}{p-\rho}$ *and the value of* r_0 *is the best possible.*

Proof. Let

$$\frac{z(Q^{\alpha}_{\beta,p}f)'(z)}{(Q^{\alpha}_{\beta,p}f)(z)} = (p-\rho)H(z)+\rho, \quad z \in E, \quad \mathrm{Re}\,H(z) > 0.$$

Using identity (9) and proceeding as in Theorem 1, we have

$$\frac{1}{p-\rho}\left\{\frac{z(Q^{\alpha-1}_{\beta,p}f)'(z)}{(Q^{\alpha-1}_{\beta,p}f)(z)} - \rho\right\} = \left[H(z) + \frac{\frac{1}{p-\rho}zH'(z)}{H(z)+\frac{\rho+\alpha+\beta-1}{p-\rho}}\right]$$
$$= (\frac{k}{4}+\frac{1}{2})\left[h_1(z) + \frac{\frac{1}{p-\rho}zh_1'(z)}{h_1(z)+\frac{\rho+\alpha+\beta-1}{p-\rho}}\right] - (\frac{k}{4}-\frac{1}{2})\left[h_2(z) + \frac{\frac{1}{p-\rho}zh_2'(z)}{h_2(z)+\frac{\rho+\alpha+\beta-1}{p-\rho}}\right].$$

Using Lemma 2, with $\mu = \dfrac{\rho+\alpha+\beta-1}{p-\rho} (\neq -1)$ and $s = \dfrac{1}{p-\rho} > 0$, we see that $f \in R_k(\alpha-1,\beta,p,0)$ for $|z| < r_0$ where

$$\begin{aligned} r_0 &= \frac{|\mu+1|}{\sqrt{A+(A^2-|\mu^2-1|)^{\frac{1}{2}}}} \\ A &= 2(s+1)^2+|\mu|^2-1, \end{aligned}$$

and this radius is best possible.

As a special case, we note that if $p = 1$, $\rho+\alpha+\beta-1 = 0$, $\rho = 0$ and $k = 2$, we obtain

$$\begin{aligned} A &= 2.4+0-1 = 7 \\ \mu &= 0, \quad s = 1, \end{aligned}$$

and

$$r_0 = 2 - \sqrt{3} = \frac{1}{\sqrt{7+\sqrt{48}}}.$$

This completes the proof of the theorem. □

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