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**SOME DIFFERENTIAL SUBORDINATION  
AND SUPERORDINATION PROPERTIES  
OF SYMMETRIC FUNCTIONS**

**Abstract.** In this paper, we study the interesting properties of differential subordination and superordination for the classes of symmetric functions analytic in the unit disc. We derive sandwich results on the basis of this theory.

**1. Introduction**

Let  $H(E)$  denote the class of analytic functions in the open unit disc

$$E = \{z \in \mathbb{C} : |z| < 1\},$$

and let  $H[a, 1]$  denote the subclass of the functions  $f \in H(E)$  of the form

$$f(z) = a + a_1z + a_2z^2 + \cdots, \quad a \in \mathbb{C}.$$

Also, let  $\mathcal{A}$  be the subclass of functions  $f \in H(E)$  of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k.$$

If  $f$  and  $g$  are analytic in  $E$ , we say that  $f$  is subordinate to  $g$ , and we write  $f \prec g$  or  $f(z) \prec g(z)$ , if there exists a Schwarz function  $w$  in  $E$  such that  $f(z) = g(w(z))$ .

Suppose that  $h$  and  $k$  are two analytic functions in  $E$ , and let

$$\varphi(r, s, t; z) : \mathbb{C}^3 \times E \longrightarrow \mathbb{C}.$$

If  $h$  and  $\varphi(h(z), zh'(z), z^2h''(z); z)$  are univalent functions in  $E$  and if  $h$  satisfies the second order superordination

$$(1) \quad k(z) \prec \varphi(h(z), zh'(z), z^2h''(z); z),$$

then  $k$  is said to be a solution of the differential superordination (1). A function  $q$  in  $H(E)$  is called a *subordinant* to (1) if  $q(z) \prec h(z)$  for all the functions  $h$  satisfying condition (1).

A univalent subordinant  $\tilde{q}$  that satisfies  $q(z) \prec \tilde{q}(z)$ , for all subordinants  $q$  of (1), is said to be the *best subordinant*.

Recently, Miller and Mocanu [7] obtained sufficient conditions on functions  $k$ ,  $q$  and  $\varphi$  for which the following implication holds:

$$k(z) \prec \varphi(h(z), zh'(z), z^2h''(z); z) \Rightarrow q(z) \prec h(z).$$

Using these results, the authors of [2] consider certain classes of first-order differential subordinations (see also [4]), as well as subordination-preserving integral operators [3]. Aouf et al. [2, 1] obtained sufficient conditions for certain normalized analytic functions  $f$  to satisfy

$$q_1(z) \prec \frac{zf'(z)}{f(z)} \prec q_2(z),$$

where  $q_1, q_2$  are given univalent normalized functions in  $E$ . Very recently, Shanmugam et al. [13, 14] obtained the so-called sandwich results for certain classes of analytic functions. We refer interested readers to the work done by the authors of [2, 8, 9, 12].

In [11], Sakaguchi defined the class of starlike functions with respect to symmetrical points as follows. Let  $f \in \mathcal{A}$ . Then  $f$  is said to be *starlike with respect to symmetrical points in  $E$*  if, and only if,

$$\operatorname{Re} \frac{zf'(z)}{f(z) - f(-z)} > 0, \quad z \in E.$$

Obviously, this forms a subclass of close-to-convex functions, and its members are therefore univalent. Moreover, this class includes the class of convex functions and odd starlike functions with respect to the origin, see [11].

## 2. Preliminary results

DEFINITION 1 ([7]). Let  $\mathcal{Q}$  be the set of all functions  $f$  that are analytic and injective on  $\bar{E} \setminus U(f)$ , where

$$U(f) = \left\{ \zeta \in \partial E : \lim_{z \rightarrow \zeta} f(z) = \infty \right\},$$

and are such that  $f'(\zeta) \neq 0$  for  $\zeta \in \partial E \setminus U(f)$ .

To establish our main results we need the following lemmas:

LEMMA 1 (Miller and Mocanu [6]). Let  $q$  be univalent in the unit disc  $E$ , and let  $\theta$  and  $\phi$  be analytic in a domain  $D$  containing  $q(E)$ , with  $\phi(w) \neq 0$  when  $w \in q(E)$ . Set  $Q(z) = zq'(z)\phi(q(z))$ ,  $h(z) = \theta(q(z)) + Q(z)$  and suppose that

(i)  $Q$  is a starlike function in  $E$ ,

(ii)  $\operatorname{Re} \frac{zh'(z)}{Q(z)} > 0$  for  $z \in E$ .

If  $p$  is analytic in  $E$  with  $p(0) = q(0)$ ,  $p(E) \subseteq D$  and

$$(2) \quad \theta(p(z)) + zp'(z)\phi(p(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z)),$$

then  $p(z) \prec q(z)$ , and  $q$  is the best dominant of (2).

LEMMA 2 (Shanmugam et al. [14]). Let  $\mu, \gamma \in \mathbb{C}$  with  $\gamma \neq 0$ , and let  $q$  be a convex function in  $E$  with

$$\operatorname{Re} \left( 1 + \frac{zq''(z)}{q'(z)} \right) > \max \left\{ 0, -\operatorname{Re} \frac{\mu}{\gamma} \right\}, \quad z \in E.$$

If  $p$  is analytic in  $E$  and

$$(3) \quad \mu p(z) + \gamma zp'(z) \prec \mu q(z) + \gamma zq'(z),$$

then  $p(z) \prec q(z)$ , and  $q$  is the best dominant of (3).

LEMMA 3 (Bulboacă [5]). Let  $q$  be a univalent function in the unit disc  $E$ , and let  $\theta$  and  $\phi$  be analytic in a domain  $D$  containing  $q(E)$ . Suppose that

$$(i) \operatorname{Re} \frac{\theta'(q(z))}{\phi(q(z))} > 0 \text{ for } z \in E,$$

$$(ii) h(z) = zq'(z)\phi(q(z)) \text{ is starlike in } E.$$

If  $p \in H[q(0), 1] \cap \mathcal{Q}$  with  $p(E) \subseteq D$ ,  $\theta(p(z) + zp'(z))\phi(p(z))$  is univalent in  $E$ , and

$$(4) \quad \theta(q(z)) + zq'(z)\phi(q(z)) \prec \theta(p(z)) + zp'(z)\phi(p(z)),$$

then  $q(z) \prec p(z)$ , and  $q$  is the best subdominant of (4).

LEMMA 4 (Miller and Mocanu [7]). Let  $q$  be convex in  $E$  and let  $\gamma \in \mathbb{C}$ , with  $\operatorname{Re} \gamma > 0$ . If  $p \in H[q(0), 1] \cap \mathcal{Q}$  and  $p(z) + \gamma zp'(z)$  is univalent in  $E$ , then

$$(5) \quad q(z) + \gamma zq'(z) \prec p(z) + \gamma zp'(z)$$

implies  $q(z) \prec p(z)$ , and  $q$  is the best subdominant of (5).

LEMMA 5 (Royster [10]). The function  $q(z) = \frac{1}{(1-z)^{2ab}}$  is univalent in  $E$  if and only if  $|2ab - 1| \leq 1$  or  $|2ab + 1| \leq 1$ .

### 3. Main results

THEOREM 1. Let  $q$  be univalent in  $E$ , with  $q(0) = 1$ , and suppose that

$$(6) \quad \operatorname{Re} \left( 1 + \frac{zq''(z)}{q'(z)} \right) > \max \left\{ 0, -\operatorname{Re} \frac{1}{\lambda} \right\}, \quad z \in E,$$

where  $\lambda \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$ . If  $f \in \mathcal{A}$  satisfies the subordination

$$(7) \quad (1 - \lambda) \left( \frac{f(z) - f(-z)}{2z} \right) + \lambda \left( \frac{f'(z) + f'(-z)}{2} \right) \prec q(z) + \lambda zq'(z),$$

then

$$\frac{f(z) - f(-z)}{2z} \prec q(z),$$

and  $q$  is the best dominant of (7).

*Proof.* Set

$$\frac{f(z) - f(-z)}{2z} = h(z),$$

where  $h(z)$  is analytic in  $E$  with  $h(0) = 1$ . A simple computation shows that

$$(1 - \lambda) \left( \frac{f(z) - f(-z)}{2z} \right) + \lambda \left( \frac{f'(z) + f'(-z)}{2} \right) = h(z) + \lambda z h'(z),$$

hence the subordination (7) is equivalent to

$$h(z) + \lambda z h'(z) \prec q(z) + \lambda z q'(z).$$

Combining this last relation together with Lemma 2 for special case  $\gamma = \lambda$  and  $\mu = 1$ , we obtain our result.  $\square$

Taking  $q(z) = \frac{1 + Az}{1 + Bz}$  in Theorem 1, where  $-1 \leq B < A \leq 1$ , the condition (6) reduces to

$$(8) \quad \operatorname{Re} \frac{1 - Bz}{1 + Bz} > \max \left\{ 0, -\operatorname{Re} \frac{1}{\lambda} \right\}, \quad z \in E.$$

It is easy to verify that the function

$$\varphi(\zeta) = \frac{1 - \zeta}{1 + \zeta}, \quad |\zeta| < B,$$

is convex in  $E$ , and since  $\varphi(\bar{\zeta}) = \overline{\varphi(\zeta)}$  for all  $|\zeta| < |B|$ , it follows that  $\varphi(E)$  is a convex domain symmetric with respect to the real axis, hence

$$(9) \quad \inf \left\{ \operatorname{Re} \frac{1 - Bz}{1 + Bz} : z \in E \right\} = \frac{1 - |B|}{1 + |B|} > 0.$$

Then, the inequality (8) is equivalent to

$$\operatorname{Re} \frac{1}{\lambda} \geq \frac{|B| - 1}{|B| + 1},$$

hence we have the following result.

**COROLLARY 1.** Let  $\lambda \in \mathbb{C}^*$ ,  $-1 \leq B < A \leq 1$  with

$$\max \left\{ 0, -\operatorname{Re} \frac{1}{\lambda} \right\} \leq \frac{1 - |B|}{1 + |B|}.$$

If  $f \in \mathcal{A}$ , and

$$(10) \quad (1 - \lambda) \left( \frac{f(z) - f(-z)}{2z} \right) + \lambda \left( \frac{f'(z) + f'(-z)}{2} \right) \prec \frac{1 + Az}{1 + Bz} + \lambda \frac{(A - B)z}{(1 + Bz)^2},$$

then

$$\frac{f(z) - f(-z)}{2z} \prec \frac{1 + Az}{1 + Bz},$$

and  $\frac{1 + Az}{1 + Bz}$  is the best dominant of (10).

EXAMPLE 1. Let  $\lambda \in \mathbb{C}^*$  with  $\operatorname{Re}(1/\lambda) \geq 0$ . If  $f \in \mathcal{A}$ , and

$$(11) \quad (1 - \lambda) \left( \frac{f(z) - f(-z)}{2z} \right) + \lambda \left( \frac{f'(z) + f'(-z)}{2} \right) \prec \frac{1+z}{1-z} + \lambda \frac{2z}{(1-z)^2},$$

then

$$\frac{f(z) - f(-z)}{2z} \prec \frac{1+z}{1-z},$$

and  $\frac{1+z}{1-z}$  is the best dominant of (11).

THEOREM 2. Let  $q$  be univalent in  $E$ , with  $q(0) = 1$  and  $q(z) \neq 0$  for all  $z \in E$ . Let  $\gamma, \mu \in \mathbb{C}^*$  and  $v, \eta \in \mathbb{C}$ , with  $v + \eta \neq 0$ . Let  $f \in \mathcal{A}$  and suppose that  $f$  and  $q$  satisfy the following conditions:

$$(12) \quad \frac{vz(f'(z) + f'(-z)) + \eta(f(z) - f(-z))}{(v + \eta)z} \neq 0, \quad z \in E,$$

and

$$(13) \quad \operatorname{Re} \left( 1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)} \right) > 0, \quad z \in E.$$

If

$$(14) \quad 1 + \gamma\mu \left[ \frac{vz(z(f'(z) + f'(-z)))' + \eta z(f'(z) + f'(-z))}{vz(f'(z) + f'(-z)) + \eta(f(z) - f(-z))} - 1 \right] \prec 1 + \gamma \frac{zq'(z)}{q(z)},$$

then

$$\left[ \frac{vz(f'(z) + f'(-z)) + \eta(f(z) - f(-z))}{(v + \eta)z} \right]^\mu \prec q(z),$$

and  $q$  is the best dominant of (14). The power is the principal one.

*Proof.* We begin by setting

$$(15) \quad \left[ \frac{vz(f'(z) + f'(-z)) + \eta(f(z) - f(-z))}{(v + \eta)z} \right]^\mu = h(z), \quad z \in E,$$

where  $h(z)$  is analytic in  $E$  with  $h(0) = 1$ . Differentiating (15) logarithmically with respect to  $z$ , we have

$$\mu \left[ \frac{vz(z(f'(z) + f'(-z)))' + \eta z(f'(z) + f'(-z))}{vz(f'(z) + f'(-z)) + \eta(f(z) - f(-z))} - 1 \right] = \frac{zh'(z)}{h(z)}.$$

To prove our result we use Lemma 2. Consider in this lemma

$$\theta(w) = 1 \quad \text{and} \quad \varphi(w) = \frac{\gamma}{w},$$

then  $\theta$  is analytic in  $\mathbb{C}$  and  $\varphi(w) \neq 0$  is analytic in  $\mathbb{C}^*$ . Also, if we let

$$Q(z) = zq'(z)\varphi(q(z)) = \gamma \frac{zq'(z)}{q(z)},$$

and

$$g(z) = \theta(q(z)) + Q(z) = 1 + \gamma \frac{zq'(z)}{q(z)},$$

then, since  $Q(0) = 0$  and  $Q'(0) \neq 0$ , the assumption (13) would yield that  $Q$  is a starlike function in  $E$ . From (13) we have

$$\operatorname{Re} \frac{zg'(z)}{Q(z)} = \operatorname{Re} \left( 1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)} \right) > 0, \quad z \in E,$$

and by using Lemma 2 we deduce that the subordination (14) implies  $h(z) \prec q(z)$ , and the function  $q$  is the best dominant of (14).  $\square$

In particular, with  $\nu = 0$ ,  $\eta = \gamma = 1$  and  $q(z) = \frac{1+Az}{1+Bz}$  in Theorem 2, it is easy to see that the assumption (13) holds whenever  $-1 \leq A < B \leq 1$ , which leads to the next result:

**COROLLARY 2.** *Let  $-1 \leq A < B \leq 1$  and  $\mu \in \mathbb{C}^*$ . Let  $f \in \mathcal{A}$  and suppose that  $\frac{f(z) - f(-z)}{2z} \neq 0$  for  $z \in E$ . If*

$$(16) \quad 1 + \mu \left[ \frac{z(f'(z) + f'(-z))}{f(z) - f(-z)} - 1 \right] \prec 1 + \frac{(A-B)z}{(1+Az)(1+Bz)},$$

then

$$\left( \frac{f(z) - f(-z)}{2z} \right)^\mu \prec \frac{1+Az}{1+Bz},$$

and  $\frac{1+Az}{1+Bz}$  is the best dominant of (16). The power is the principal one.

Putting  $\nu = 0$ ,  $\eta = 1$ ,  $\gamma = \frac{1}{ab}$  with  $a, b \in \mathbb{C}^*$ ,  $\mu = a$ , and  $q(z) = \frac{1}{(1-z)^{2ab}}$  in Theorem 2, then combining this together with Lemma 5, we have the next result.

**COROLLARY 3.** *Let  $a, b \in \mathbb{C}^*$  such that  $|2ab - 1| \leq 1$  or  $|2ab + 1| \leq 1$ . Let  $f \in \mathcal{A}$  and let  $\frac{f(z) - f(-z)}{2z} \neq 0$  for all  $z \in E$ . If*

$$(17) \quad 1 + \frac{1}{b} \left( \frac{z(f'(z) + f'(-z))}{f(z) - f(-z)} - 1 \right) \prec \frac{1+z}{1-z},$$

then

$$\left( \frac{f(z) - f(-z)}{2z} \right)^a \prec \frac{1}{(1-z)^{2ab}},$$

and  $\frac{1}{(1-z)^{2ab}}$  is the best dominant of (17). The power is the principal one.

Putting  $\nu = 0$ ,  $\eta = \gamma = 1$  and  $q(z) = (1+Bz)^{\mu(A-B)/B}$ ,  $-1 \leq B < A \leq 1$ ,  $B \neq 0$  in Theorem 2, and using Lemma 5, we have the next result.

COROLLARY 4. Let  $-1 \leq B < A \leq 1$  with  $B \neq 0$ , and suppose that

$$\left| \frac{\mu(A-B)}{B-1} \right| \leq 1 \quad \text{or} \quad \left| \frac{\mu(A-B)}{B+1} \right| \leq 1.$$

Let  $f \in \mathcal{A}$  such that  $\frac{f(z) - f(-z)}{2z} \neq 0$  for all  $z \in E$ , and let  $\mu \in \mathbb{C}^*$ . If

$$(18) \quad 1 + \mu \left( \frac{z(f'(z) + f'(-z))}{f(z) - f(-z)} - 1 \right) \prec \frac{1 + [B + \mu(A-B)]z}{1 + Bz},$$

then

$$\left( \frac{f(z) - f(-z)}{2z} \right)^\mu \prec (1 + Bz)^{\mu(A-B)/B},$$

and  $(1 + Bz)^{\mu(A-B)/B}$  is the best dominant of (18). Here the power is the principal one.

By taking  $\nu = 0$ ,  $\eta = 1$ ,  $\gamma = \frac{e^{i\lambda}}{ab \cos \lambda}$  with  $a, b \in \mathbb{C}^*$ ,  $|\lambda| < \pi/2$ ,  $\mu = a$  and  $q(z) = \frac{1}{(1-z)^{2abe^{-i\lambda} \cos \lambda}}$  in Theorem 2, we obtain the following result.

COROLLARY 5. Let  $a, b \in \mathbb{C}^*$  and  $|\lambda| < \pi/2$ , and suppose that

$$\left| abe^{-i\lambda} \cos \lambda - 1 \right| \leq 1 \quad \text{or} \quad \left| abe^{-i\lambda} \cos \lambda + 1 \right| \leq 1.$$

Let  $f \in \mathcal{A}$  be such that  $\frac{f(z) - f(-z)}{2z} \neq 0$  for all  $z \in E$ . If

$$(19) \quad 1 + \frac{e^{i\lambda}}{b \cos \lambda} \left( \frac{z(f'(z) + f'(-z))}{f(z) - f(-z)} - 1 \right) \prec \frac{1+z}{1-z},$$

then

$$\left( \frac{f(z) - f(-z)}{2z} \right)^a \prec \frac{1}{(1-z)^{2abe^{-i\lambda} \cos \lambda}},$$

and  $\frac{1}{(1-z)^{2abe^{-i\lambda} \cos \lambda}}$  is the best dominant of (19). The power is the principal one.

THEOREM 3. Let  $q$  be univalent in  $E$  with  $q(0) = 1$ . Let  $\mu, \gamma \in \mathbb{C}^*$ , let  $\mu, \gamma \in \mathbb{C}^*$  and let  $\delta, \Omega, \nu, \eta \in \mathbb{C}$  with  $\nu + \eta \neq 0$ . Let  $f \in \mathcal{A}$  and suppose that  $f$  and  $q$  satisfy the next two conditions:

$$(20) \quad \frac{\nu z(f'(z) + f'(-z)) + \eta(f(z) - f(-z))}{(\nu + \eta)z} \neq 0, \quad z \in E,$$

and

$$(21) \quad \operatorname{Re} \left( 1 + \frac{zq''(z)}{q'(z)} \right) > \max \left\{ 0, -\operatorname{Re} \frac{\delta}{\gamma} \right\}, \quad z \in E.$$

If

$$(22) \quad \Psi(z) \equiv \left[ \frac{vz(f'(z) + f'(-z)) + \eta(f(z) - f(-z))}{(v + \eta)z} \right]^\mu \times \left[ \delta + \gamma\mu \left( \frac{vz(z(f'(z) + f'(-z)))' + \eta z(f'(z) + f'(-z))}{vz(f'(z) + f'(-z)) + \eta(f(z) - f(-z))} - 1 \right) \right] + \Omega,$$

and

$$(23) \quad \Psi(z) \prec \delta q(z) + \gamma z q'(z) + \Omega,$$

then

$$\left[ \frac{vz(f'(z) + f'(-z)) + \eta(f(z) - f(-z))}{(v + \eta)z} \right]^\mu \prec q(z),$$

and  $q$  is the best dominant of (23). All the powers are the principal ones.

*Proof.* We begin by setting

$$(24) \quad \left[ \frac{vz(f'(z) + f'(-z)) + \eta(f(z) - f(-z))}{(v + \eta)z} \right]^\mu = h(z).$$

Then  $h(z)$  is analytic in  $E$  with  $h(0) = 1$ . Logarithmic differentiation of (24) yields

$$\mu \left( \frac{vz(z(f'(z) + f'(-z)))' + \eta z(f'(z) + f'(-z))}{vz(f'(z) + f'(-z)) + \eta(f(z) - f(-z))} - 1 \right) = \frac{zh'(z)}{h(z)},$$

and hence

$$\mu h(z) \left( \frac{vz(z(f'(z) + f'(-z)))' + \eta z(f'(z) + f'(-z))}{vz(f'(z) + f'(-z)) + \eta(f(z) - f(-z))} - 1 \right) = zh'(z).$$

Let us consider the functions:

$$\begin{aligned} \theta(w) &= \delta w + \Omega, & \varphi(w) &= \gamma, & w &\in \mathbb{C}, \\ Q(z) &= zq'(z)\varphi(q(z)) = \gamma zq'(z), & z &\in E, \end{aligned}$$

and

$$g(z) = \theta(q(z)) + Q(z) = \delta q(z) + \gamma zq'(z) + \Omega, \quad z \in E.$$

From the assumption (21), we see that  $Q$  is starlike in  $E$  and, that

$$\operatorname{Re} \frac{zg'(z)}{Q(z)} = \operatorname{Re} \left( \frac{\delta}{\gamma} + 1 + \frac{zq''(z)}{q'(z)} \right) > 0, \quad z \in E.$$

Thus, applying Lemma 2, this completes the proof.  $\square$



Taking  $q(z) = \frac{1+Az}{1+Bz}$  in Theorem 3, where  $-1 \leq B < A \leq 1$  and according to (9) the condition (21) becomes

$$\max \left\{ 0, -\operatorname{Re} \frac{\delta}{\gamma} \right\} \leq \frac{1-|B|}{1+|B|}.$$

Hence, for the special case  $\nu = 1 = \gamma, \eta = 0$ , we have the following result:

**COROLLARY 6.** *Let  $-1 \leq B < A \leq 1$  and let  $\delta \in \mathbb{C}$  with*

$$\max \left\{ 0, -\operatorname{Re} \delta \right\} \leq \frac{1-|B|}{1+|B|}.$$

*Let  $f \in \mathcal{A}$  and suppose that*

$$\frac{f'(z) + f'(-z)}{2} \neq 0, \quad z \in E$$

*and let  $\mu \in \mathbb{C}^*$ . If*

$$(25) \quad \left[ \frac{f'(z) + f'(-z)}{2} \right]^\mu \left[ \delta + \mu \left( \frac{z(f'(z) + f'(-z))}{f(z) - f(-z)} - 1 \right) \right] + \Omega < \delta \frac{1+Az}{1+Bz} + \Omega + \frac{z(A-B)}{(1+Bz)^2},$$

*then*

$$\left( \frac{f'(z) + f'(-z)}{2} \right)^\mu < \frac{1+Az}{1+Bz},$$

*and  $\frac{1+Az}{1+Bz}$  is the best dominant of (25). All the powers are the principal ones.*

Taking  $\gamma = \eta = 1, \nu = 0$  and  $q(z) = \frac{1+z}{1-z}$  in Theorem 3, we obtain:

**COROLLARY 7.** *Let  $f \in \mathcal{A}$  be such that  $\frac{f(z) - f(-z)}{2z} \neq 0$  for all  $z \in E$ , and let  $\mu \in \mathbb{C}^*$ . If*

$$(26) \quad \left[ \frac{f(z) - f(-z)}{2z} \right]^\mu \left[ \delta + \mu \left( \frac{z(f'(z) + f'(-z))}{f(z) - f(-z)} - 1 \right) \right] + \Omega < \delta \frac{1+z}{1-z} + \Omega + \frac{2z}{(1-z)^2},$$

*then*

$$\left[ \frac{f(z) - f(-z)}{2z} \right]^\mu < \frac{1+z}{1-z},$$

*and  $\frac{1+z}{1-z}$  is the best dominant of (26). All the powers are the principal ones.*

#### 4. Superordination and sandwich results

**THEOREM 4.** Let  $q$  be convex in  $E$  with  $q(0) = 1$ , let  $\lambda \in \mathbb{C}^*$  with  $\operatorname{Re} \lambda > 0$ . Let  $f \in \mathcal{A}$  and suppose that  $\frac{f(z) - f(-z)}{2z} \in H[q(0), 1] \cap \mathcal{Q}$ . If the function

$$(1 - \lambda) \left( \frac{f(z) - f(-z)}{2z} \right) + \lambda \left( \frac{f'(z) + f'(-z)}{2} \right)$$

is univalent in the unit disc  $E$ , and

$$(27) \quad q(z) + \lambda z q'(z) \prec (1 - \lambda) \left( \frac{f(z) - f(-z)}{2z} \right) + \lambda \left( \frac{f'(z) + f'(-z)}{2} \right),$$

then

$$q(z) \prec \frac{f(z) - f(-z)}{2z},$$

and  $q$  is the best subordinator of (27).

*Proof.* Set

$$\frac{f(z) - f(-z)}{2z} = h(z), \quad z \in E.$$

Then  $h(z)$  is analytic in  $E$  with  $h(0) = 1$ . Taking logarithmic differentiation with respect  $z$ , we have

$$(28) \quad z \left( \frac{f'(z) + f'(-z)}{f(z) - f(-z)} \right) - 1 = \frac{zh'(z)}{h(z)}.$$

A simple computations show that

$$h(z) + \lambda zh'(z) = (1 - \lambda) \left( \frac{f(z) - f(-z)}{2z} \right) + \lambda \left( \frac{f'(z) + f'(-z)}{2} \right),$$

and now, by using Lemma 5, we obtain the desired result.  $\square$

Taking  $q(z) = \frac{1 + Az}{1 + Bz}$  in Theorem 4, where  $-1 \leq B < A \leq 1$ , we obtain the next result.

**COROLLARY 8.** Let  $q$  be convex in  $E$  with  $q(0) = 1$ , let  $\lambda \in \mathbb{C}^*$  with  $\operatorname{Re} \lambda > 0$ . Let  $f \in \mathcal{A}$  and suppose that  $\frac{f(z) - f(-z)}{2z} \in H[q(0), 1] \cap \mathcal{Q}$ . If the function

$$(1 - \lambda) \left( \frac{f(z) - f(-z)}{2z} \right) + \lambda \left( \frac{f'(z) + f'(-z)}{2} \right)$$

is univalent in the unit disc  $E$ , and

$$(29) \quad \frac{1 + Az}{1 + Bz} + \frac{\lambda(A - B)z}{(1 + Bz)^2} \prec (1 - \lambda) \left( \frac{f(z) - f(-z)}{2z} \right) + \lambda \left( \frac{f'(z) + f'(-z)}{2} \right),$$

then

$$\frac{1 + Az}{1 + Bz} \prec \frac{f(z) - f(-z)}{2z},$$

and  $\frac{1 + Az}{1 + Bz}$  is the best subordinant of (29), where  $-1 \leq B < A \leq 1$ .

Using the same techniques as we used in Theorem 3, and then applying Lemma 3, we have the following theorem.

**THEOREM 5.** Let  $q$  be convex in  $E$  with  $q(0) = 1$ , suppose that  $\mu, \gamma \in \mathbb{C}^*$ , and let  $\delta, \Omega, \nu, \eta \in \mathbb{C}$  with  $\nu + \eta \neq 0$  and  $\operatorname{Re}(\delta/\gamma) > 0$ .

Let  $f \in \mathcal{A}$  and suppose that  $f$  satisfies the following conditions:

$$\frac{\nu z(f'(z) + f'(-z)) + \eta(f(z) - f(-z))}{(\nu + \eta)z} \neq 0, \quad z \in E,$$

and

$$\left[ \frac{\nu z(f'(z) + f'(-z)) + \eta(f(z) - f(-z))}{(\nu + \eta)z} \right]^\mu \in H[q(0), 1] \cap \mathcal{Q}.$$

If the function  $\psi$  given by equation (22) is univalent in  $E$ , and

$$(30) \quad \delta q(z) + \gamma z q'(z) + \Omega \prec \psi(z),$$

then

$$q(z) \prec \left[ \frac{\nu z(f'(z) + f'(-z)) + \eta(f(z) - f(-z))}{(\nu + \eta)z} \right]^\mu,$$

and  $q$  is the best subordinate of (30) (all powers are the principal ones).

Observe that by combining Theorem 1 with Theorem 4 and Theorem 3 with Theorem 5, respectively, we have the next two sandwich theorems:

**THEOREM 6.** Let  $q_1, q_2$  be two convex functions in  $E$  with  $q_1(0) = q_2(0) = 1$ , let  $\lambda \in \mathbb{C}^*$  with  $\operatorname{Re} \lambda > 0$ . Let  $f \in \mathcal{A}$  and suppose that  $\frac{f(z) - f(-z)}{2z} \in H[q(0), 1] \cap \mathcal{Q}$ .

If the function

$$(1 - \lambda) \left( \frac{f(z) - f(-z)}{2z} \right) + \lambda \left( \frac{f'(z) + f'(-z)}{2} \right)$$

is univalent in the unit disc  $E$ , and

$$(31) \quad q_1(z) + \lambda z q_1'(z) \prec (1 - \lambda) \left( \frac{f(z) - f(-z)}{2z} \right) + \lambda \left( \frac{f'(z) + f'(-z)}{2} \right) \prec q_2(z) + \lambda z q_2'(z),$$

then

$$q_1(z) \prec \frac{f(z) - f(-z)}{2z} \prec q_2(z),$$

and  $q_1$  and  $q_2$  are, respectively, the best subordinate and the best dominant of (31).

**THEOREM 7.** Let  $q_1, q_2$  be two convex functions in  $E$  with  $q_1(0) = q_2(0) = 1$ , let  $\mu, \gamma \in \mathbb{C}^*$ , and let  $\delta, \Omega, \nu, \eta \in \mathbb{C}$  with  $\nu + \eta \neq 0$  and  $\operatorname{Re}(\delta/\gamma) > 0$ . Let  $f \in \mathcal{A}$  satisfy the following conditions:

$$\left[ \frac{\nu z(f'(z) + f'(-z)) + \eta(f(z) - f(-z))}{(\nu + \eta)z} \right] \neq 0, \quad z \in E,$$

and

$$\left[ \frac{\nu z(f'(z) + f'(-z)) + \eta(f(z) - f(-z))}{(\nu + \eta)z} \right]^\mu \in H[q(0), 1] \cap \mathcal{Q}.$$

If the function  $\psi$  given by (22) is univalent in  $E$ , and

$$(32) \quad \delta q_1(z) + \gamma z q_1'(z) + \Omega \prec \psi(z) \prec \delta q_2(z) + \gamma z q_2'(z) + \Omega,$$

then

$$q_1(z) \prec \left[ \frac{\nu z(f'(z) + f'(-z)) + \eta(f(z) - f(-z))}{(\nu + \eta)z} \right]^\mu \prec q_2(z),$$

and  $q_1$  and  $q_2$  are, respectively, the best subordinate and the best dominant of (32) (all the powers are the principal ones).

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