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School on Hodge Theory

C. Voisin

LECTURES ON THE HODGE AND GROTHENDIECK–HODGE CONJECTURES*

Abstract. An international school on Hodge theory took place at the CIRM in Trento, Italy, from 31 August to 4 September 2009. This paper is based on the mini-course of five lectures by Claire Voisin, one of which was given each day and accompanied by a lecture by Eduard Looijenga.

Its main topics are a review of the rational Hodge conjecture, including the construction of cycle classes in various cohomology theories, a review of Hodge loci, absolute Hodge classes, and some recent results on fields of definition of Hodge loci.

The article continues with the statement of the generalized Hodge conjecture, involving the coniveau of Hodge substructures. Later sections stress the importance of the generalized Hodge conjecture in the context of algebraic cycles, and discuss its relationship with the generalized Bloch conjecture and the nilpotence conjecture.

Contents and foreword

* Notes by A. Collino, M. Leyenson and S. Salamon
C. Voisin is grateful for their work

The contents are arranged into five sections that correspond closely to the five daily lectures. The first section contains preliminary material and has been expanded somewhat more than the others. Only a couple of dividing lines have been shifted in order to unify the various topics by section, so the generalized Hodge conjecture does not make an appearance before Section 3, and the nilpotence conjecture is the first topic of Section 5.

This article is based on both notes taken by Simon Salamon and a LaTeX file by Maxim Leyenson inspired by the course. The material was edited by Alberto Collino and the lecturer, who also benefitted from the help of Christian Schnell. Relative to the lectures themselves it contains a little additional (and, hopefully, relevant) material, though the scope of this is for the main part limited to extra remarks or examples.

Acknowledgments. Thanks are due to the CIRM for its support of the School and to its director, Fabrizio Catanese, for encouraging a wide participation at both junior and senior level by those working in a broad spectrum of geometry.

1. Introduction to the Hodge conjecture

The Hodge conjecture for a smooth complex projective variety concerns the possible realization of certain cohomology classes by rational combinations of cycles arising from subvarieties. It fails in more general contexts, so one needs to address the question

"what special features do we have in algebraic geometry?"

In order to explain this, we begin with a summary of cohomology theories.

Betti cohomology has an integral structure, whereas algebraic de Rham cohomology has another structure depending on the field of definition. We shall construct the cycle class associated to a subvariety in both theories, and compare the results. The section concludes with a discussion of the Hodge conjecture and related conjectures.

1.1. Cohomology theories

Let X be a smooth complex projective manifold. It comes equipped with both the classical (Euclidean) topology and also the Zariski topology. We shall write X_{cl}, X_{Zar} whenever we need to emphasize one of the respective topologies, whereas X^{an} will indicate X as a complex manifold equipped with the sheaf of holomorphic functions.

Against this background, we shall start with an informal review of several cohomology theories.

Regarding the classical topology of X, there are the so-called Betti cohomology groups

$$H^k(X, \mathbb{Z})_B := H^k(X_{cl}, \mathbb{Z}), \tag{1}$$

defined by means of an acyclic resolution of the constant sheaf with stalk $\mathbb{Z}$ on X_{cl}, such as that provided by singular or Čech cochains in X. These groups can be defined for various coefficients, including $\mathbb{R}$ and $\mathbb{C}$. Over the real field, these groups can also be computed as the de Rham cohomology group

$$H^k_{dR}(X_{cl}, \mathbb{R})$$

of closed k-forms on X_{cl} modulo exact ones. One uses for this the fine Poincaré resolution of the constant sheaf with stalk $\mathbb{R}$ on X_{cl} given by the de Rham complex, which is made of the sheaves of smooth real-valued differential forms linked by exterior differentiation d.

The above spaces do not of course depend on the complex-analytic structure of X, but only on the underlying differentiable manifold. On the other hand, one can define the Dolbeault cohomology groups

$$H^{p,q}(X^{an}) = H^q(X^{an}, \Omega^p_{an})$$

as the Čech cohomology groups of the sheaf Ω^p_{an} of holomorphic p-forms on X. As it is well known, they can be computed via the Dolbeault resolution, which gives the formula

$$H^{p,q}(X^{an}) = \frac{\bar\partial\text{-closed } (p,q)\text{-forms}}{\bar\partial\text{-exact } (p,q)\text{-forms}} \tag{2}$$

formed from the Dolbeault complex of smooth forms of type (p,q) with p fixed.

Hodge structures

On any complex manifold, the *Frölicher spectral sequence* establishes a link between the Dolbeault groups

$$E_1^{p,q} = H^{p,q}(X^{an})$$

and ordinary de Rham cohomology. This spectral sequence is obtained using the fact that the holomorphic de Rham complex $(\Omega^\bullet_{X^{an}}, d)$ is a resolution of constant sheaf $\mathbb{C}$ on X_{cl}. (This is the so-called holomorphic Poincaré lemma.) We have thus an isomorphism between $H^k(X,\mathbb{C})_B$ and the hypercohomology $\mathbb{H}^k(X_{cl}, \Omega^\bullet_{X^{an}})$. The Frölicher spectral sequence is associated to the naïve filtration

$$F^l(\Omega^\bullet_{X^{an}}) = \Omega^{\bullet \geqslant l}_{X^{an}}$$

on the complex $(\Omega^\bullet_{X^{an}}, d)$.

This filtration induces a decreasing filtration $F^l H^k(X,\mathbb{C})_B$ on the cohomology of any complex manifold X. It is called the Hodge filtration only in the case where X is projective, because it has particular properties in this case.

The classical Hodge theory of harmonic forms shows that on a projective manifold X this "Hodge to de Rham" spectral sequence degenerates at the E_1 stage. To explain this, one first represents each class in the space (2) by a $\overline{\partial}$-harmonic (p,q)-form ω such that

$$0 = \Delta_{\overline{\partial}}\omega = (\overline{\partial}\,\overline{\partial}^* + \overline{\partial}^*\overline{\partial})\omega.$$

But X is a Kähler manifold, and the $\overline{\partial}$-Laplace operator is known to be proportional to d-Laplace operator (see, for example, [15, chapter 0], [36, 6.1.2]). Thus

$$\Delta_d\omega = (dd^* + d^*d)\omega = 0,$$

which implies that $d\omega = 0 = d^*\omega$, and in particular ω is d-closed.

It now follows that $d_1 : E_1^{p,q} \to E_1^{p+1,q}$ vanishes, as do all higher differentials. We also deduce from the comparison of the Laplacians the Hodge decomposition

$$H^k(X,\mathbb{C})_B = \bigoplus_{p+q=k} H^{p,q}(X^{an}), \tag{3}$$

expressing the ordinary Betti groups as a direct sum of the Dolbeault groups (2). Note that, conversely, (3) implies the degeneracy at E_1 because we have

$$\sum_{p,q} \dim E_\infty^{p,q} = \dim H^{p+q}(X,\mathbb{C}) = \sum_{p,q} \dim E_1^{p,q},$$

whence $E_\infty^{p,q} = E_1^{p,q}$. Note also that, under the above decomposition, we have the following formula for the Hodge filtration:

$$F^l H^k(X,\mathbb{C}) = H^{k,0}(X^{an}) \oplus \cdots \oplus H^{l,k-l}(X^{an}) = \bigoplus_{p \geqslant l} H^{p,k-l}(X^{an}). \tag{4}$$

The degeneracy of the Frölicher spectral sequence at E_1 implies that the Hodge filtration has

$$Gr_F^p H^{p+q}(X,\mathbb{C}) \cong H^{p,q}(X^{an})$$

as associated graded pieces, but it is not sufficient to provide the Hodge decomposition (3), if we define the space $H^{p,q}(X^{an}) \subset H^{p+q}(X,\mathbb{C})$ as the space of those de Rham cohomology classes that are representable by a closed form of type (p,q). The case of Hopf surfaces provides a counterexample.

With the definition above of the space $H^{p,q}(X^{an}) \subset H^{p+q}(X,\mathbb{C})_B$, it is obvious that the *Hodge symmetry* property

$$\overline{H^{p,q}(X^{an})} = H^{q,p}(X^{an}) \tag{5}$$

holds, because the complex conjugate of a closed form of type (p,q) is a closed form of type (q,p).

There is a natural pairing on rational cohomology

$$H^k(X,\mathbb{Q})_B \otimes H^{2n-k}(X,\mathbb{Q})_B \to \mathbb{Q} \tag{6}$$

defined by cup product, where $n = \dim_{\mathbb{C}} X$. This gives the Poincaré duality

$$H^k(X,\mathbb{Q}) \cong H^{2n-k}(X,\mathbb{Q})^*,$$

which is compatible the respective Hodge structures, in the sense that $H^{p,q}(X^{an})$ pairs in a non trivial way only with $H^{n-p,n-q}(X^{an})$. Indeed, after tensoring with $\mathbb{C}$, the pairing (6) can be computed in de Rham cohomology by integration:

$$[\omega]\otimes[\eta] \mapsto \int_X \omega\wedge\eta;$$

here ω,η are closed differential forms, and the wedge product of (p,q)-form and a (p',q')-form with $p+p'+q+q' = 2\dim X$ is nonzero only if $p' = n-p$, $q' = n-q$.

On the other hand, the Poincaré pairing (6) is perfect, and it follows from the above that it induces a perfect pairing between $H^{p,q}(X^{an}) \cong H^q(X,\Omega^p)$ and $H^{n-p,n-q}(X^{an}) \cong H^{n-q}(X,\Omega^{n-p})$: This pairing is in fact Serre's pairing.

We conclude this section by introducing the crucial notion of Hodge class: Starting from the rational vector space $H^k(X,\mathbb{Q})$, one can consider $H^k(X,C) = H^k(X,\mathbb{Q})\otimes\mathbb{C}$. The decomposition (3), together with the symmetry (5), then defines what is called a *Hodge structure of weight* k.

DEFINITION 1. *A Hodge class of degree* $2k$ *on* X *is an element* α *in the space*

$$\mathrm{Hdg}^{2k}(X) = H^{2k}(X,\mathbb{Q})_B \cap H^{k,k}(X).$$

We can give another formula for Hodge classes that is better suited for variational study, namely using the Hodge filtration (4). In fact, we have

$$\alpha \in \mathrm{Hdg}^{2k}(X) \iff \alpha \in H^{2k}(X,\mathbb{Q}) \cap F^k H^{2k}(X,\mathbb{C}). \tag{7}$$

Indeed, a class α in $F^kH^{2k}(X,\mathbb{C})$ can be decomposed as $\alpha = \alpha^{2k,0} + \cdots + \alpha^{k,k}$. If furthermore α is rational, hence real, the vanishing of $\alpha^{p,q}$ for $p < k$ implies by Hodge symmetry the vanishing of $\alpha^{p,q}$ for $q < k$, so that $\alpha = \alpha^{k,k}$.

Algebraic de Rham cohomology

On the other hand, if X is an algebraic variety that is defined over a field K of characteristic 0, one has *algebraic de Rham cohomology*, whose groups

$$H^k_{dR}(X/K) = \mathbb{H}^k(X,\Omega^\bullet_{X/K})$$

are defined as the hypercohomology of the complex of sheaves of algebraic forms on X, in the Zariski topology. Recall that, to do this, one defines the $\mathcal{O}_X$-module of Kähler

differentials Ω^1_{alg} as generated by elements $f\,dg$, for $f, g \in \mathcal{O}_X$, with relations $d\alpha = 0$ for $\alpha \in K$ and
$$d(fg) = f\,dg + g\,df.$$
There is then a K-linear derivation $d\colon \mathcal{O}_X \to \Omega^1_{\mathrm{alg}}$, $f \mapsto df$.

Setting $\Omega^i_{\mathrm{alg}} = \bigwedge^i \Omega^1_{\mathrm{alg}}$, one extends d so as to obtain the algebraic de Rham complex

$$0 \to \mathcal{O}_X \to \Omega^1_{\mathrm{alg}} \to \ \cdots \ \to \Omega^n_{\mathrm{alg}} \to 0 \tag{8}$$

where $n = \dim_K X$. We shall often write $\Omega_{X/K}$ in place of Ω_{alg} to emphasize the base field.

DEFINITION 2. *The algebraic de Rham group* $H^k_{dR}(X/K)$ *is the* K *vector space equal to the hypercohomology* $\mathbb{H}^k(X, \Omega^\bullet_{X/K})$ *of the complex* (8).

By a base change from K to $\mathbb{C}$, we obtain the algebraic de Rham cohomology groups $H^k_{dR}(X/\mathbb{C})$ with complex coefficients.

REMARK 1. The complex-valued cohomology groups
$$H^k_{dR}(X^{an}, \mathbb{C}) = \mathbb{H}^k(X^{an}, \Omega^\bullet_{an})$$
provide a bridge between the Betti and algebraic de Rham theories outlined above. They are computed as the hypercohomology of the de Rham complex of holomorphic differential forms on X. This is a subcomplex of the ordinary smooth de Rham complex with complex coefficients, and the embedding is a quasi-isomorphism.

Serre's GAGA gives an equivalence between the categories of analytic and algebraic coherent sheaves for the projective variety X, which induces an isomorphism on H^0, hence on all the cohomology groups H^i [27]. It also results in a comparison theorem, due to Grothendieck (who also proved a similar statement in the non-projective case), for the two hypercohomology groups
$$\mathbb{H}^q(X_{Zar}, \Omega^\bullet_{\mathrm{alg}}) \cong \mathbb{H}^q(X^{an}, \Omega^\bullet_{an}).$$
We have seen that the Frölicher spectral sequence of X^{an} is associated to the naïve filtration of the holomorphic de Rham complex $\Omega^\bullet_{an}$ on X^{an}. There is an analogous spectral sequence for algebraic de Rham cohomology starting from $E_1^{p,q} = H^q(X_{Zar}, \Omega^p_{\mathrm{alg}})$. However, the spectral sequences for the algebraic and analytic de Rham cohomology coincide at the E_1 level by GAGA, hence at any level. It follows that the Frölicher spectral sequence for the algebraic de Rham complex is also degenerate at the E_1 level. Another (algebraic) proof of this result has been given by Deligne and Illusie [11].

1.2. Construction of cycle classes

Given a subvariety Z in a smooth complex algebraic variety X, we want to define the cycle class $[Z]$ in the cohomology groups of X, for various cohomology theories.

We shall define cycle classes in Betti cohomology first. If Z is smooth, there is no difficulty. Otherwise, one method is to pass in some way to a smooth variety and use Poincaré duality. Let Z be a reduced irreducible subvariety of codimension k in X. By Hironaka's theorem, there is a desingularization

$$\tilde{i}\colon\ \tilde{Z} \longrightarrow Z \subset X$$

of Z, and we may consider

$$H^{2n-2k}(X,\mathbb{Q})_B \xrightarrow{\tilde{i}^*} H^{2n-2k}(\tilde{Z},\mathbb{Q})_B \xrightarrow{\cong} \mathbb{Q}.$$

With the help of Poincaré duality, the composition determines a class

$$\beta \in H^{2n-2k}(X,\mathbb{Q})_B^* \cong H^{2k}(X,\mathbb{Q})_B.$$

This β vanishes on $H^{p',q'}$ if $(p',q') \neq (n-k,n-k)$, so

$$\beta \in H^{k,k}(X^{\mathrm{an}}).$$

This is the rational *Betti cycle class* of Z.

An important point is the fact that *Betti cycle classes are Hodge classes*. Indeed, they are rational, and we just proved that they are also of type (k,k).

REMARK 2. An alternative approach (avoiding desingularizations) is to construct the fundamental homology class of Z as a singular homology class, by choosing a triangulation of Z. For example, any closed analytic subset has a triangulation subordinate to the equisingular stratification. Let Sing Z denote the subvariety consisting of singular points of Z, and $U = Z \setminus \mathrm{Sing}\ Z$ the smooth locus of Z. Since Sing Z has real codimension at least 2 in Z and U is a complex manifold of dimension $n-k$, hence a real oriented manifold of dimension $2n-2k$, there is a fundamental class $[Z]_{\mathrm{fund}} \in H_{2n-2k}(Z,\mathbb{Z})$, which provides a homology class $i_*[Z]_{\mathrm{fund}} \in H_{2n-2k}(X,\mathbb{Z})$. The ambient space X is smooth compact oriented, so this time we can apply the Poincaré isomorphism

$$H_{2n-2k}(X,\mathbb{Z}) \xrightarrow{\cong} H^{2k}(X,\mathbb{Z})_B$$

to $i_*[Z]_{\mathrm{fund}}$ to obtain the *integral* cycle class $[Z]_B \in H^{2k}(X,\mathbb{Z})_B$ in cohomology.

To construct the cohomology class $[Z]_{dR}$ in algebraic de Rham cohomology, we start with the subsheaf $\Omega^{k,c}_{\mathrm{alg}}$ of closed forms of degree k over a field K. This maps naturally to the truncated complex

$$0 \to \Omega^k_{\mathrm{alg}} \to \cdots \to \Omega^n_{\mathrm{alg}} \to 0,$$

itself a subcomplex of the full de Rham complex. In this way, we get

$$\begin{array}{ccc} H^l(X,\Omega^{k,c}_{\mathrm{alg}}) & \longrightarrow & \mathbb{H}^{l+k}(X,\Omega^{\bullet\geqslant k}_{\mathrm{alg}}) = F^k\mathbb{H}^{l+k}(X,\Omega^\bullet_{\mathrm{alg}}) \\ & & \big\downarrow \\ & & H^{k+l}_{\mathrm{dR}}(X/K) = \mathbb{H}^{k+l}(X,\Omega^\bullet_{\mathrm{alg}}). \end{array}$$

We now give Bloch's construction, which works for an arbitrary locally complete intersection Z in X [4, Paragraph 5]. Let Z be a subvariety of codimension k in X, both defined over a field K, and consider the inclusions

$$Z \overset{i}{\hookrightarrow} X \overset{j}{\hookleftarrow} Z\setminus X. \tag{9}$$

Choose an open set $U \subset X$ such that the subset $U \cap Z$ of U is defined by k equations $f_1, \ldots, f_k$. Then $W = U \setminus (U \cap Z)$ is covered by $U_1, \ldots, U_k$ where U_i is the subset for which $f_i \neq 0$. Consider the closed differential form

$$\omega_U = \frac{df_1}{f_1} \wedge \cdots \wedge \frac{df_k}{f_k} \in \Omega^{k,c}_{\text{alg}}(U_1 \cap \cdots \cap U_k).$$

This determines a Čech class of degree $k-1$ for the sheaf $\Omega^{k,c}_{X/K}$ restricted to W, and so a class in

$$H^{k-1}(W, \Omega^{k,c}_{X/K}) \to \mathbb{H}^{2k-1}(W, \Omega^{\bullet \geqslant k}_{X/K}) \to \mathbb{H}^{2k}_{Z\cap U}(U, \Omega^{\bullet \geqslant k}_{X/K}),$$

and the result on the right is in fact independent of the choice of f_i's.

We can glue these locally defined classes to get a global section

$$\gamma \in H^0(X, \mathcal{H}^{2k}_Z(\Omega^{\bullet \geqslant k}_{X/K})) \overset{\cong}{\longleftarrow} \mathbb{H}^{2k}_Z(X, \Omega^{\bullet \geqslant k}_{X/K}).$$

On the left, $\mathcal{H}^{2k}_Z(\cdot)$ is the sheaf of local cohomology groups with support on Z [17], and the isomorphism follows from the local to global spectral sequence, and the fact that $\mathcal{H}^i_Z(\Omega^{\bullet \geqslant k}_{X/K})$ vanishes for $i \leqslant 2k-1$. Using the natural maps

$$\mathbb{H}^{2k}_Z(X, \Omega^{\bullet \geqslant k}_{X/K}) \to \mathbb{H}^{2k}(X, \Omega^{\bullet \geqslant k}_{X/K}) \to \mathbb{H}^{2k}(X, \Omega^{\bullet}_{X/K}),$$

we therefore end up with a well-defined class in $H^{2k}_{\text{dR}}(X/K)$, by definition of this K-vector space of finite dimension. This is the *algebraic de Rham cycle class* of Z, denoted by $[Z]_{\text{dR}}$.

REMARK 3. Given a smooth algebraic variety X and a smooth subvariety Z in X of codimension k, Grothendieck constructed the cycle class of Z in the cohomology $H^k(X_{\text{Zar}}, \Omega^k_{X/K})$. Bloch's construction gives a class in

$$F^k H^{2k}_{dR}(X/K) = \mathbb{H}^{2k}(X, \Omega^{\bullet \geqslant k}_{X/K}),$$

while the Grothendieck class is its image in the last quotient of this space for the Hodge filtration.

For the sequel, we shall stick to our algebraic construction of the cycle class $[Z]_{\text{dR}}$ because we want to exploit the fact that if X and Z are defined over a subfield K of $\mathbb{C}$, so is $[Z]_{\text{dR}}$. This is crucial for the prediction concerning the fields of definition of the locus of Hodge classes [33, 39]. In the next subsection, we shall compare $[Z]_{\text{dR}}$ with the topological cycle class, and discuss further the first Chern class.

1.3. The comparison theorem

Assume that K is a subfield of $\mathbb{C}$, so X/K extends to the complex variety $X/\mathbb{C}$, and $\Omega_{X/\mathbb{C}} = \Omega_{X/K} \otimes_K \mathbb{C}$. It follows that each complex algebraic de Rham cohomology group

$$H^l_{dR}(X/\mathbb{C}) = H^l_{dR}(X/K) \otimes_K \mathbb{C} = \mathbb{H}^l(X/\mathbb{C}, \Omega^\bullet_{X/\mathbb{C}})$$

has a "K-structure".

Since $(\Omega_{X/\mathbb{C}})^{an} = \Omega_{an}$, we have by GAGA, as already noticed,

$$\mathbb{H}^l(X_{\mathbb{C}}, \Omega^\bullet_{X/\mathbb{C}}) \cong \mathbb{H}^l(X^{an}, \Omega^\bullet_{an}).$$

As $\Omega^\bullet_{an}$ is a resolution in the usual topology of the constant sheaf $\underline{\mathbb{C}}$ (this is called the holomorphic Poincaré resolution [36, 8.2.1]), we have

$$\mathbb{H}^l(X^{an}, \Omega^\bullet_{an}) = H^l(X_{cl}, \mathbb{C}) =: H^l(X, \mathbb{C})_B.$$

By way of conclusion,

THEOREM 1. *We have*

$$H^l_{dR}(X/K) \otimes_K \mathbb{C} \ \cong\ H^l(X_{cl}, \mathbb{Q})_B \otimes \mathbb{C}.$$

There is, however, no obvious relation between the K-structure on the left and the rational structure on the right.

Suppose that $L^{an} \to X^{an}$ is a holomorphic line bundle. In the Betti theory,

$$c_1(L^{an}) \in H^2(X, \mathbb{Z})_B$$

is defined using the exponential sequence

$$0 \to \mathbb{Z} \to \mathcal{O}_X \to \mathcal{O}^*_X \to 0,$$

and the resulting long exact sequence

$$\cdots \to H^1(X^{an}, \mathcal{O}) \to H^1(X^{an}, \mathcal{O}^*) \xrightarrow{\delta} H^2(X^{an}, \mathbb{Z}) \to H^2(X^{an}, \mathcal{O}) \to \cdots . \tag{10}$$

The transition functions of L define a Čech class in $H^1(X^{an}, \mathcal{O}^*)$ and one applies the coboundary homomorphism δ to compute $c_1(L^{an})$.

Now suppose that $D = \sum n_i D_i$ is the divisor associated to a meromorphic section of a holomorphic line bundle L.

THEOREM 2 (Lelong formula). $c_1(L^{an}) = \sum n_i [D_i]_B$.

This is [36, Theorem 11.33], whose proof we sketch only briefly.

Proof. By considering the components of D and by using resolution of singularities, one reduces to the case in which $D = D_i$ is smooth and connected. Now L is trivial over $X \setminus D$, so in the long exact sequence

$$\cdots \to H^2(X, X \setminus D, \mathbb{Z}) \to H^2(X, \mathbb{Z}) \to H^2(X \setminus D, \mathbb{Z}) \to \cdots$$

$c_1(L)$ lies in the image of the first space that, by the Thom isomorphism is equal to $H^0(D, \mathbb{Z}) = \mathbb{Z}$. It can be shown that $[D]_B$ is in fact the image of the unit class in $H^0(D, \mathbb{Z})$, so $c_1(L)$ must be an *integral* multiple of that. To verify that the multiple is 1, one can use the usual representative

$$\frac{1}{2\pi i} \partial \bar{\partial} \log h$$

where h is a Hermitian metric on L. The proof is based on applying Stokes' theorem to a tubular neighbourhood of D in V. □

In the algebraic de Rham theory of the line bundle $L \to X$, we take a Zariski open cover $\{U_{ij}\}$ and define L by transition functions $g_{ij} \in \mathcal{O}^*_{U_{ij}}$. Applying the mapping

$$d\log\colon\ g \mapsto \frac{dg}{g}$$

gives us a Čech 1-cocycle $\{dg_{ij}/g_{ij}\}$ of algebraically closed 1-forms over K, relative to this open cover. This defines a class in

$$H^1(X_{\mathrm{Zar}}, \mathcal{O}^*) \to H^1(X_{\mathrm{Zar}}, \Omega^{1,c}_{X/K}) \to H^2_{dR}(X/K)$$

whose image we denote $c_1(L)_{dR}$.

The analogue

$$c_1(L)_{dR} = [D]_{alg}$$

of Lelong's formula holds; this is immediate from the explicit Bloch construction of the cycle class of D.

Let us now compare the first Chern classes of L in both theories. Using the isomorphism

$$H^2(X, \mathbb{C})_B \cong \mathbb{H}^2(X^{an}, \Omega^\bullet_{X^{an}}),$$

the Čech representative of the Betti class $c_1(L)_B$ is given by the 1-cocycle

$$\left\{ \frac{1}{2\pi i} \frac{dg_{ij}}{g_{ij}} \right\}.$$

This shows that there is a discrepancy of $2\pi i$ in the definition of the cycle classes:

$$c_1(L)_{dR} = (2\pi i) c_1(L)_B. \tag{11}$$

We only mentioned above the first Chern class of a line bundle. However, the results immediately extend to higher Chern classes of vector bundles, using the splitting principle [13, Section 3.1, 3.2] and the axiomatic definition of Chern classes.

The generalization of (11) is the following:

THEOREM 3 (Comparison theorem). *With respect to Theorem 1, for $Z \subset X$ a closed algebraic subset of codimension k, we have $[Z]_{dR} = (2\pi i)^k [Z]_B$.*

Proof. The point is that this is a local statement. Assume Z is smooth. Let T be a tubular neighbourhood of Z and use the Leray spectral sequence for the map $p : T \to Z$ to compute $H^{2k}_{Z^{an}}(T, \mathbb{Q})$ as $H^0((R^{2k}p)_{0*}\underline{\mathbb{Q}}) = H^0(Z, \mathbb{Q})$, where the subscript 0 indicates support on the zero section of p. Then $[Z]_B$ is obtained as the image of $1 \in H^0(Z, \mathbb{Q})$, via the map

$$H^{2k}_Z(T, \mathbb{Q}) \to H^{2k}_Z(X, \mathbb{Q}) \to H^{2k}(X, \mathbb{Q}).$$

It is a fact that on $\mathbb{C}^k$,

$$H^{2k}_0(\mathbb{C}^k, \mathbb{C}) = H^{2k-1}(\mathbb{C}^k \setminus \{0\}, \mathbb{C}) \cong \mathbb{C},$$

where the final isomorphism is obtained by taking residues. We can interpret the form

$$\frac{dz_1}{z_1} \wedge \cdots \wedge \frac{dz_k}{z_k}$$

which appeared in the construction of Bloch cycle class as an element of

$$H^{k-1}(\mathbb{C}^k \setminus \{0\}, \Omega^{\bullet \geqslant k}_{an}) \to H^{2k-1}(\mathbb{C}^k \setminus \{0\}, \mathbb{C}),$$

and one can show that it equals $(2\pi i)^k$ times the canonical unit class on the right. □

1.4. Statement of the Hodge conjecture

Let X be a smooth projective complex variety. We saw that cycle classes $[Z]_B$ are Hodge classes on X. The Hodge conjecture states the converse:

CONJECTURE 1. Any Hodge class $\alpha \in \mathrm{Hdg}^{2k}(X)$ is a linear combination with rational coefficients of Betti cycle classes of algebraic subvarieties of X, so

$$\alpha = \sum_{i=1}^{N} a_i [Z_i]_B, \qquad a_i \in \mathbb{Q}.$$

A first observation is that the Hodge conjecture is true for $k = 1$. This is known as the Lefschetz theorem on $(1,1)$-classes, and follows from the exponential sequence and its associated long exact sequence (10). The space $H^2(X^{an}, \mathcal{O}_{X^{an}})$ on the right is isomorphic to $H^{0,2}(X^{an})$, so using (7), any degree 2 integral Hodge class is the image by δ (and therefore c_1) of a class in $H^1(X^{an}, \mathcal{O}^*_{X^{an}})$. The group $H^1(X^{an}, \mathcal{O}^*_{X^{an}})$ is the group of holomorphic line bundles, which by GAGA, is isomorphic to the group $\mathrm{Pic}(X)$ of algebraic line bundles. As algebraic line bundles have meromorphic sections, we conclude using Lelong's formula (Theorem 2) that the Hodge conjecture is true in degree 2, and even with integral coefficients.

REMARK 4. For $k > 1$, the integral Hodge conjecture is false. One possible counterexample arises from the following theorem of Kollár [21]:

Suppose that p is a prime different from 2 or 3, and that $p^3 \mid d$. A generic hypersurface X of degree d in $\mathbb{P}^4$ does not satisfy the integral Hodge conjecture: there exists a class in $H^{2,2} \cap H^4(X, \mathbb{Z})$ that is not algebraic.

More precisely, consider a smooth hypersurface $X \subset \mathbb{P}^4$. By Lefschetz' hyperplane restriction theorem [38, 1.2.2], the restriction $r_i \colon H^i(\mathbb{P}^4, \mathbb{Z}) \to H^i(X, \mathbb{Z})$ is an isomorphism for $i = 0, 1, 2$. Denote by h the hyperplane class in $H^2(\mathbb{P}^4, \mathbb{Z})$, and by $h_X \in H^2(X, \mathbb{Z})$ its restriction. Although $H^4(X, \mathbb{Z})$ and $H^6(X, \mathbb{Z})$ are both isomorphic to $\mathbb{Z}$, they are not generated by powers of h_X. Let α be a generator of H^4 (with positive intersection with h_X). We must have $h_X \cdot \alpha = 1$ by Poincaré duality. Since $\deg h_X^3$ is equal to $d = \deg X$, we deduce that $h_X^2 = d\alpha$. In particular the class $d\alpha$ is algebraic. Kollár's statement is that α itself is not algebraic for very general X. Note that the algebraicity of α is equivalent to the existence of a one-cycle $\sum_i n_i C_i$, of degree 1. It is shown in [30] that the set of points in the moduli space of X where α is algebraic is a countable union of closed algebraic subsets, dense for the usual topology in the moduli space of X.

We conclude this section with the "standard conjectures" that were formulated by Bombieri and Grothendieck [16]. They are in fact particular instances of the Hodge conjecture.

Let X, Y be projective complex manifolds with $\dim X = n$. Suppose that $k + l = 2r$ is even. We apply the Künneth decomposition. Given

$$\alpha \in H^k(X, \mathbb{Q})_B \otimes H^l(Y, \mathbb{Q})_B \subset H^{2r}(X \times Y, \mathbb{Q}),$$

we can, by duality, see α as an element

$$\widetilde{\alpha} \in \mathrm{Hom}(H^{2n-k}(X, \mathbb{Q})_B, H^l(Y, \mathbb{Q})_B).$$

With this terminology, we have:

LEMMA 1. *α is a Hodge class in $X \times Y$ if and only if $\widetilde{\alpha}$ is a morphism of Hodge structures of bidegree $(r - n, r - n)$.*

This is [36, Lemma 11.41].

Let X be a smooth complex algebraic variety, and Δ be the diagonal in $X \times X$. The cycle class $[\Delta]$ belongs to

$$H^{2n}(X \times X, \mathbb{Q}) = \bigoplus_p \mathrm{Hom}(H^p, H^p),$$

where $H^p = H^p(X, \mathbb{Q})_B$, and has accordingly a decomposition

$$[\Delta] = \sum \delta_p,$$

where δ_p stands for the identity map id_{H^p}. But each separate δ_p also provides us with a Hodge class on $X \times X$, by Lemma 1. The question is whether these classes are algebraic:

CONJECTURE 2 (The Künneth conjecture). For each p, the class $[\delta]_p$ is $\mathbb{Q}$-algebraic, i.e., a combination of cycle classes of subvarieties with rational coefficients.

Grothendieck calls this “Conjecture C”.

REMARK 5. Using Lemma 1, one can also define the Künneth components of each algebraic cycle class $[Z]$, where $Z \subset X \times Y$. It is easy to prove the following statement:

If the Künneth conjecture holds for both X *and* Y*, then the Hodge conjecture is true for each Künneth component of each algebraic cycle* $Z \subset X \times Y$.

Let X be a smooth complex algebraic variety of dimension n, and let

$$\ell = c_1(L) \in H^2(X, \mathbb{Q})_B$$

be the first Chern class of a very ample line bundle L on X, so that L is the pullback of the hyperplane line bundle $\mathcal{O}(1)$ for some embedding $X \to \mathbb{P}^N$. The “hard Lefschetz theorem” asserts that multiplication by ℓ^{n-k} is an isomorphism

$$\ell^{n-k}\colon\ H^k(X, \mathbb{Q}) \to H^{2n-k}(X, \mathbb{Q}) \tag{12}$$

The class ℓ has type $(1, 1)$, and (12) is in fact a morphism of Hodge structures of bidegree $(n-k, n-k)$. In this way, we obtain a Hodge class of degree $4n-2k$ on $X \times X$, represented by the homology class of the cycle

$$H_1 \cap \cdots \cap H_{n-k} \subset X = \Delta \subset X \times X,$$

where the last inclusion is the diagonal one. Here the H_i’s are hypersurfaces in the linear system $|L|$ meeting transversally. In particular by Lelong’s formula, the class of H_i is equal to ℓ for any i.

The isomorphism (12) has an inverse

$$\Lambda_{n-k}\colon\ H^{2n-k}(X, \mathbb{Q})_B \xrightarrow{\cong} H^k(X, \mathbb{Q})_B \tag{13}$$

which by Lemma 1 provides a Hodge class on $X \times X$ of degree $2k$. The question is whether this is algebraic. Let us now formulate it in another way.

Assume that $k = 2r$ is even. The group $H^{2r}(X, \mathbb{Q})$ contains the set of $\mathbb{Q}$-algebraic cycles of codimension r, which Grothendieck denotes $C^r(X, \mathbb{Q})$. By definition, an element c belongs to $H^{2r}(X, \mathbb{Q})$ if $c = \sum a_i[Z_i]$ and $a_i \in \mathbb{Q}$.

CONJECTURE 3 (The Lefschetz conjecture). The mapping

$$\ell^{n-2r}\colon\ C^r(X, \mathbb{Q}) \to C^{n-r}(X, \mathbb{Q})$$

is an isomorphism.

Grothendieck calls this "Conjecture A"; it is the assertion that ℓ^{n-2r} is onto. It is obviously implied by the algebraicity of the inverse of ℓ^{n-2r} introduced in (13).

THEOREM 4. *The validity of Conjecture 3 for all r is equivalent to the conjecture that homological equivalence coincides with numerical equivalence.*

Numerical equivalence is defined by saying that $Z \sim_{\text{num}} Z'$ if and only if $[Z - Z'] \cdot [W] = 0$ for all classes $[W]$ with $\dim W = n - k$.

Proof of Theorem 4. First suppose that hom = num. The spaces C^r and C^{n-r} are then dual to each other and thereby have the same dimension. The injective Lefschetz operator between them must therefore be surjective.

The converse is less straightforward; it requires the Hodge–Riemann relations [36, 6.3.2]. Suppose that Conjecture 3 holds for all r. It follows that the Lefschetz decomposition holds on the subspaces $C^r(X)$ of algebraic classes. But then, on each piece of this decomposition, which is orthogonal for the intersection pairing

$$(z, z')_\ell := \left\langle \ell^{n-r} \cup z \cup z', [X] \right\rangle,$$

$(\, , \,)_\ell$ is non-degenerate. It follows immediately that the morphism

$$C^r(X, \mathbb{Q}) \to C^r(X, \mathbb{Q}) / \sim_{\text{num}}$$

is injective for $r \leqslant n$. For $r \geqslant n$, we use the Lefschetz isomorphism $\ell^{n-2r} : C^r(X, \mathbb{Q}) \cong C^{n-r}(X, \mathbb{Q})$ to get the same conclusion. Hence hom = num. □

2. The geometry and arithmetic of Hodge loci

We begin by studying Hodge loci à la Griffiths and then a variant, namely the locus of Hodge classes as defined by Cattani–Deligne–Kaplan. We then examine the structure of this locus, as predicted by the Hodge conjecture. After that, we discuss absolute Hodge classes, and formulate a structure statement on the components of this locus. We will also address partially the following question: for which Hodge classes can the Hodge conjecture be "reduced" to the Hodge conjecture on varieties over $\mathbb{Q}$?

The main reference is Milne's notes [12] (see also [33]).

2.1. Hodge locus of a class

Consider a smooth projective morphism

$$p : \mathcal{X} \to B \tag{14}$$

over $\mathbb{C}$. We may regard the total space $\mathcal{X}$ as embedded inside $B \times \mathbb{P}^N$, but we place no assumptions on B at this stage, merely that it be a complex analytic space or manifold.

Fix a point $o \in B$ and let $X_o = p^{-1}(o)$ be the fiber o. Consider a Hodge class

$$\alpha \in \text{Hdg}^{2k}(X_o) = H^{2k}(X_o, \mathbb{Q})_B \cap H^{k,k}(X_o),$$

and recall that we may replace $H^{k,k}$ by F^kH^{2k} in the intersection. We shall be studying the theory of the pair (X_o, α), noting that α must remain locally constant if it is to be defined over $\mathbb{Q}$. Accordingly, we introduce the sheaf

$$\underline{H}^{2k}_{\mathbb{Q}} = R^{2k}p_*\underline{\mathbb{Q}},$$

where $\underline{\mathbb{Q}}$ is the constant sheaf with fiber $\mathbb{Q}$ over $\mathcal{X}$. Since p is smooth, $\underline{H}^{2k}_{\mathbb{Q}}$ is a $\mathbb{Q}$-local system on the base B. There is an associated holomorphic vector bundle

$$\mathcal{H}^{2k} = \underline{H}^{2k}_{\mathbb{Q}} \otimes \mathcal{O}_B \tag{15}$$

over the base B with fibre $H^{2k}(X_t, \mathbb{C})_B$ at the point $t \in B$. This vector bundle contains $F^k\mathcal{H}^{2k}$ as a holomorphic subbundle and $\underline{H}^{2k}_{\mathbb{Q}}$ as the sheaf of locally constant rational sections (cf. [36, 10.2]).

We can "analytically continue" α along a path in the base B starting at o, since the local system $\underline{H}^{2k}_{\mathbb{Q}}$ is canonically trivialised over such a path. Let us study what happens over a ball $U \subset B$ containing the point o. Since U is contractible, there is a section $\widetilde{\alpha} \in H^0(U, \underline{H}^{2k}_{\mathbb{Q}})$ extending α, and we write $\widetilde{\alpha}_t \in H^{2k}(X_t, \mathbb{Q})_B$ for each $t \in U$.

DEFINITION 3. *The set*

$$U_\alpha = \{t \in U : \widetilde{\alpha}_t \in F^kH^{2k}(X_t)\}$$

of those points in U *for which* $\widetilde{\alpha}_t$ *is a Hodge class is called the* Hodge locus of α.

Recall from (7) that this is equivalent to the fact that $\widetilde{\alpha}_t \in H^{k,k}(X_t)$, because $\widetilde{\alpha}_t$ is rational (hence real).

Now consider the projection

$$q\colon\ \mathcal{H}^{2k} \to \mathcal{H}^{2k}/F^k\mathcal{H}^{2k}$$

of holomorphic vector bundles on B. The image $q(\widetilde{\alpha})$ is a section of the quotient bundle, and its zeroes determine the Hodge locus:

$$U_\alpha = \{t \in U : q(\widetilde{\alpha}_t) = 0\}.$$

This shows that the locally defined Hodge locus U_α is a closed analytic subset, but is not convenient for globalization due to the presence of monodromy. To define the Hodge locus globally in B, we could pull back the family $\mathcal{X} \to B$ to the universal cover $\widetilde{B}$ of B, where the local system $\underline{H}^{2k}_{\mathbb{Q}}$ becomes trivial, take the Hodge locus in $\widetilde{B}$, and then push it down to B.

In search of a better definition, we adopt another point of view by considering the total space of the holomorphic vector bundle

$$p : F^k\mathcal{H}^{2k} \to B \tag{16}$$

whose points are pairs (t, α) with $t \in B$ and $\alpha \in F^kH^{2k}(X_t)$. We arrive at the definition of Cattani–Deligne–Kaplan in which the Hodge locus appears as a subset of the total space $\mathcal{H}^{2k}$ rather than of the base:

DEFINITION 4. *The locus of Hodge classes is the subset*

$$\mathcal{H}dg^{2k} = \{(t,\alpha) : \alpha \in F^k H^{2k}(X_t) \cap H^{2k}(X_t,\mathbb{Q})\} \subset F^k\mathcal{H}^{2k}.$$

Defining the Hodge locus in this way, as a set of rational points in $F^k\mathcal{H}^{2k}$, is better suited to our purposes.

Let us consider further the local structure. Over the contractible subset U of B, we can see that $\mathcal{H}dg^{2k}$ is a countable union of analytic subsets. Indeed, over U, there is a holomorphic trivialization of $\mathcal{H}^{2k} \cong U \times H^{2k}(X_o,\mathbb{C})$. Using this, consider

$$F\colon\ F^k\mathcal{H}^{2k} \hookrightarrow \mathcal{H}^{2k} \to H^{2k}(X_o,\mathbb{C}).$$

Then

$$\alpha \in \mathcal{H}dg^{2k} \iff F(\alpha) \in H^{2k}(X_o,\mathbb{Q}).$$

Therefore, over U,

$$\mathcal{H}dg^{2k} = F^{-1}(H^{2k}(X_o,\mathbb{Q}))$$

is a union of fibers of F parametrized by the countable set $H^{2k}(X_o,\mathbb{Q})$.

Fix α as before. We can now define $\mathcal{H}dg^{2k}_\alpha$ to be the connected component of $\mathcal{H}dg^{2k}$ passing through α, and $B_\alpha = p(\mathcal{H}dg^{2k}_\alpha)$ its image by (16):

$$\begin{array}{ccc} \mathcal{H}dg^{2k}_\alpha & \subset & F^k\mathcal{H}^{2k} \\ {\scriptstyle p_\alpha}\downarrow & & {\scriptstyle p}\downarrow \\ B_\alpha & \subset & B \end{array} \qquad (17)$$

By construction, the image B_α contains the Hodge locus U_α previously defined.

REMARK 6. One could try to define, in place of B_α, a subset $T_\alpha \subset B$ that extends U_α by analytic continuation and parallel transport. However, an abstract T_α obtained in this way is a countable union over all transforms of α under monodromy, and thus it could be a countable union of closed analytic subsets even locally in the base. The fact that this situation does not hold, that is the sheets of T_α do not accumulate locally, is true essentially due to the fact that monodromy acts in a finite way on the set of rational classes which remain (k,k)-classes everywhere, see [33].

Now suppose that $p : \mathcal{X} \to B$ is smooth projective, and $\mathcal{X}, B$ are quasi-projective and everything is defined over a field K (that only later will necessarily lie in $\mathbb{C}$). By GAGA, the vector bundle (15) is the analytization of the algebraic vector bundle

$$R^{2k}p^{\mathrm{Zar}}_*((\Omega^\bullet_{\mathcal{X}/B})_{\mathrm{alg}})$$

on B over K, defined using the relative algebraic de Rham complex. In a similar manner, $F^k\mathcal{H}^{2k}$ is obtained as the analytization

$$R^{2k}p^{\mathrm{Zar}}_*((\Omega^{\bullet\geqslant k}_{\mathcal{X}/B})_{\mathrm{alg}}).$$

The total space of these bundles are thus quasi-projective algebraic varieties.

The structure of the set

$$(2\pi i)^k \mathcal{H}dg^{2k} \subset F^k\mathcal{H}^{2k}$$

is predicted by Theorem 3 and the Hodge conjecture. Its points should then be pairs $(t, [Z_t]_{dR})$, where the de Rham class is taken over $\mathbb{C}$ and

$$Z_t = \sum_{i=1}^{n} a_i Z_i$$

with $a_i \in \mathbb{Q}$, and Z_i is a codimension k subvariety of X_t. The Z_i are parametrized by suitable relative Hilbert schemes over B. There are countably many choices; fixing the rational numbers a_i and the Hilbert polynomials of Z_i gives a projective algebraic variety $\mathcal{M}$ over B defined over K (so its irreducible components are defined over $\overline{K}$) parameterizing $(t; Z_1, \ldots, Z_n)$: The relative algebraic cycle map gives then a morphism

$$\begin{array}{ccc} c_{rel} : \mathcal{M} & \longrightarrow & F^k\mathcal{H}^{2k} \\ \downarrow & & \\ B & & \end{array}$$

(c_{rel} is in fact constant along the connected components of the fibres of $\mathcal{M}$ over B).

By the above argument, the Hodge conjecture implies that $(2\pi i)^k \mathcal{H}dg^{2k}$ is a countable union of closed algebraic subsets of $F^k\mathcal{H}^{2k}$ defined over K (or a countable union stable under $\mathrm{Gal}(\overline{K}/K)$ of closed irreducible algebraic subsets of $F^k\mathcal{H}^{2k}$ defined over $\overline{K}$).

It was suggested by A. Weil [41] that one could test the Hodge conjecture by seeing the extent to which the locus of Hodge classes is algebraic. In fact, the best evidence so far for the Hodge conjecture comes from the following deep result of Cattani, Deligne and Kaplan in [7]:

THEOREM 5. *The connected components of $\mathcal{H}dg^{2k}$ are closed algebraic subsets of $F^k\mathcal{H}^{2k}$.*

By a theorem of Hironaka, B admits a compactification $\overline{B}$ such that the complement $B \setminus \overline{B}$ is a divisor with normal crossing. The above theorem is proved by using Schmid's nilpotent orbit theorem (see Looijenga's lectures in this volume or [25]). The main statement is the fact that the Hodge locus can be extended to the boundary to give a closed analytic subset in $\overline{B}$; one then uses Chow's theorem saying that closed analytic subsets of a projective variety are closed algebraic.

Note however that the theorem says nothing about the fields of definition, which will be discussed in the sequel.

2.2. Examples from elliptic curves

We discuss the moduli space of elliptic curves E, and interpret the Hodge locus of a class in $E \times E$ when E is equipped with complex multiplication. The (coarse) moduli space of elliptic curves is the affine line $\mathbb{A}^1$. Given an algebraic family $\mathcal{X} \to B$ of such curves there is the classifying map $j : B \to \mathbb{A}^1$. Consider the case of the Hesse pencil of cubic curves, $X(\mu) := x^3 + y^3 + z^3 - 3\mu xyz = 0$, then $B = (A^1 - \{1, \zeta, \zeta^2\})$ with $\zeta = e^{2\pi i/3}$, and $j(\mu) = 27\mu^3(\mu^3+8)^3(\mu^3-1)^{-3}$. We say that $\mathbb{A}^1$ is a coarse moduli space because there is no universal elliptic curve over $\mathbb{A}^1$

$$\mathcal{E} \to \mathbb{A}^1, \tag{18}$$

the reason for this is monodromy, as we explain presently. On the other hand the universal curve exists over the moduli space of elliptic curves with some level structure (the Hesse pencil is when the level is 3), and therefore (18) is meaningful *up to passing to a finite cover of* $\mathbb{A}^1$; it is with this understanding that occasionally we refer to (18) in what follows.

The Hodge structure of weight 1 on the vector space $H^1(X_t, \mathbb{C})$ is determined by the $\mathbb{Q}$-lattice $H^1(X_t, \mathbb{Q})$ and the complex line $H^0(X_t, \Omega^1)$. If we choose a basis in $H_1(X_o, \mathbb{Z})$ for some point $o \in \mathbb{A}^1$ (and thus for every other point in $\mathbb{A}^1$, since the latter is contractible), the line $H^0(X_t, \Omega^1)$ has a well-defined slope τ, where $1, \tau$ are the periods of the elliptic curve X_t. However, it is well known that τ is merely an analytic (non-algebraic) function of j (indeed, the map $H^+ \to H^+/\mathrm{PSL}_2(\mathbb{Z})$ is of infinite degree, where H^+ is the upper half-plane with coordinate τ).

EXAMPLE 1. Consider an elliptic curve

$$E = \mathbb{C}/\Lambda, \qquad \Lambda = \mathbb{Z} + \mathbb{Z}\tau.$$

For any integer n, multiplication by n gives an isogeny

$$0 \to E_n \to E \xrightarrow{n} E \to 0,$$

where $E_n \cong (\mathbb{Z}/n)^2$. This gives us a ring homomorphism $\mathbb{Z} \to \operatorname{End} E$. Suppose that this is not surjective, and let $\gamma\colon E \to E$ be an element in the cokernel. Then γ lifts to $\mathbb{C}$, and is necessarily a homomorphism $z \mapsto wz$ for some complex number w. For this reason, E is said to admit *complex multiplication*[1].

Since γ preserves the lattice Λ, we have

$$\begin{cases} w \cdot 1 = a + b\tau \\ w \cdot \tau = c + d\tau, \end{cases} \tag{19}$$

for some integers a, b, c, d. From the exact sequence

$$0 \to \ker\gamma \to E \xrightarrow{\gamma} E \to 0,$$

[1] The following treatment of this topic is due to A. Levin

we obtain an exact sequence

$$0 \to H_1(E,\mathbb{Z}) \to H_1(E,\mathbb{Z}) \to \ker\gamma \to 0,$$

and conclude that $\ker\gamma$ is a group of order $\Delta = ad - bc$. From (19),

$$b\tau^2 + (a-d)\tau - c = 0,$$

so that $\tau \in \mathbb{Q}(\sqrt{-\Delta})$, where $\Delta = (a-d)^2 + 4bc$.

The graph $\Gamma = \Gamma(w)$ of multiplication by w is a divisor in $S = E_1 \times E_2$ (we use subscripts to distinguish between the two copies $E_1 = E_2 = E$). Let (e_1, e_2) be a standard basis in the homology group $H_1(E_1,\mathbb{Z})$ so that $(e_1,e_1) = (e_2,e_2) = 0$ and $(e_1,e_2) = 1$, and let (f_1,f_2) be the corresponding basis in $H_1(E_2,\mathbb{Z})$. Computing intersection indices, one finds that

$$[\Gamma] = c\, e_1 \otimes f_1 - d\, e_1 \otimes f_2 - a\, e_2 \otimes f_1 + b\, e_2 \otimes f_2 + e_1 \wedge e_2 + D\, f_1 \wedge f_2 \tag{20}$$

in the second homology group

$$H_2(S,\mathbb{Z}) \cong H_1(E_1) \otimes H_1(E_2) + H_2(E_1) \otimes H_0(E_2) + H_0(E_1) \otimes H_2(E_2).$$

(In particular, if E has complex multiplication by $w = \tau$ with $\tau^2 = -D$, then (20) becomes $[\Gamma] = -De_1 \otimes f_1 + e_2 \otimes f_2 + e_1 \wedge e_2 + Df_1 \wedge f_2$, where $\Delta = -4D$.)

Now consider the fibered square $Y = \mathcal{E} \times_{\mathbb{A}^1} E$ of the universal curve (18). It is a 3-dimensional (non-compact) variety with a projection $p\colon Y \to \mathbb{A}^1$. Let us fix one particular fiber Y_o of the map p, and consider the cycle $\alpha \in H^2(Y_o,\mathbb{Z})$ determined by (20). Then the Hodge locus U_α in the base $U = \mathbb{A}^1$ parametrizes elliptic curves which have a complex multiplication of a given type (a,b,c,d).

It is known that for a given discriminant D, there are finitely many elliptic curves $E_1,\dots,E_h$ with complex multiplication from the field $K = \mathbb{Q}(\sqrt{-D})$ [6]. The number h is equal to the order of the group $\mathrm{Cl}(K)$ of the class ideals of K. (Every such a curve is of the form $\mathbb{C}/\Lambda$, where Λ is a lattice in $\mathbb{C}$ and is a projective $\mathcal{O}_K$-module of rank 1, which gives a one-to-one correspondence between $\{E_1,\dots,E_h\}$ and the group $\mathrm{Cl}(K)$.) The j-invariants $j_l = j(E_l)$ are algebraic numbers all belonging to the same field

$$K' = \mathbb{Q}(j_1) = \cdots = \mathbb{Q}(j_h),$$

an extension of K of degree h. The Galois group $\mathrm{Gal}(K'/K) \cong \mathrm{Cl}(K)$ is abelian, and K' is the maximal abelian unramified extension of the field K. Thus the Hodge locus U_α is defined over the field K', and the $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ action on this Hodge locus factors through the class ideal group $\mathrm{Cl}(K)$.

2.3. Absolute Hodge classes

In this subsection we introduce a refinement due to Deligne of the notion of Hodge class on smooth complex projective varieties. It takes into account the comparison between Betti cohomology and algebraic de Rham cohomology, where the first one is purely topological, while the second one is algebraic and in particular is defined over the same definition field as X.

The group $\mathrm{Gal}(\mathbb{C}/\mathbb{Q})$ of automorphisms of $\mathbb{C}$ acts coefficient-wise on the ring of polynomials $\mathbb{C}[x_0,\dots,x_n]$. It takes homogeneous ideals to homogeneous ideals and therefore determines an action $\tau\colon R \mapsto R_\tau$ on the category of graded finitely-generated rings over $\mathbb{C}$. If $X = \mathrm{Proj}\, R$ and $X_\tau = \mathrm{Proj}\, R_\tau$, τ produces a bijection on complex points (that is, complex solutions of the defining equations)

$$X(\mathbb{C}) \xrightarrow{\cong} X_\tau(\mathbb{C}).$$

If X is a smooth projective variety X embedded in $\mathbb{P}^N$, then by definition X_τ is obtained by making τ act on the coefficients of the equations defining X. However, τ is not necessarily continuous and, as a consequence, X_{cl} and $X_{\tau,\mathrm{cl}}$ may not be homeomorphic in the classical topology, while they are in the Zariski topology.

On the other hand, the group $\mathrm{Gal}(\mathbb{C}/\mathbb{Q})$ acts naturally on the modules of Kähler differentials, inducing a map $\Omega^1_R \mapsto \Omega^1_{R_\tau}$, which is τ-linear but not $\mathbb{C}$-linear, and a corresponding action

$$\tau_{\mathrm{dR}}\colon\ \mathbb{H}^{2k}(X,\Omega^{\bullet\geqslant k}_{X/\mathbb{C}}) \longrightarrow \mathbb{H}^{2k}(X_\tau,\Omega^{\bullet\geqslant k}_{X_\tau/\mathbb{C}}) \tag{21}$$

on algebraic de Rham cohomology. The groups are isomorphic, by the discussion in Section 1, to the respective Betti cohomology groups and we shall write

$$\begin{array}{ccc} H^*(X,\mathbb{C})_B & \longrightarrow & H^*(X_\tau,\mathbb{C})_B \\ \alpha & \mapsto & \alpha_\tau \end{array} \tag{22}$$

for the resulting bijection. In particular, this shows (as was noticed by Serre in [27]) that X and X_τ have equal Betti numbers. However, Serre [28] constructed an example for which their fundamental groups are different!

Despite the equivalence over $\mathbb{C}$, (22) does not preserve the $\mathbb{Q}$-structures of Betti cohomology. To see this, recall the formula

$$[Z]_{\mathrm{dR}} = (2\pi i)^k\,[Z]_B \tag{23}$$

of Theorem 3 for a codimension k-cycle. If $Z \subset X$ is defined over $\mathbb{Q}$, it follows from the construction of the algebraic de Rham cycle class that the class $[Z]_{\mathrm{dR}}$ is invariant by the action of $\mathrm{Gal}(\mathbb{C}/\mathbb{Q})$ on $H^{2k}_{\mathrm{dR}}(X/\mathbb{Q})$:

$$([Z]_{\mathrm{dR}})_\tau = [Z]_{\mathrm{dR}}. \tag{24}$$

It follows then from Theorem 1 that $\tau \in \mathrm{Gal}(\mathbb{C}/\mathbb{Q})$ acts on $[Z]_B$ by

$$([Z]_B)_\tau = \frac{(2\pi i)^k)}{\tau((2\pi i)^k)}[Z]_B.$$

Hence the left-hand class is not rational in the Betti sense when the coefficient $\frac{(2\pi i)^k}{\tau((2\pi i)^k)}$ is not in $\mathbb{Q}$.

When X is not defined over $\mathbb{Q}$, the construction of the algebraic de Rham cycle class still shows immediately that

$$([Z]_{dR})_\tau = [Z_\tau]_{dR}. \tag{25}$$

It follows from (25) that, if the Hodge conjecture is true, then τ will map

$$(2\pi i)^k \mathrm{Hdg}^{2k}(X) = (2\pi i)^k H^{2k}(X,\mathbb{Q})_B \cap F^k H^{2k}(X)$$

to $(2\pi i)^k \mathrm{Hdg}^{2k}(X_\tau)$. This motivates the following:

DEFINITION 5 (Deligne [12]). *A class* $\alpha \in \mathrm{Hdg}^{2k}(X)$ *is called* absolute Hodge *if the class* $((2\pi i)^k\alpha)_\tau$ *belongs to* $(2\pi i)^k \mathrm{Hdg}^{2k}(X_\tau)$ *for every automorphism* τ *in* $\mathrm{Gal}(\mathbb{C}/\mathbb{Q})$.

In other words, the class $\frac{\tau((2\pi i)^k)}{(2\pi i)^k}\alpha_\tau$ has to be a Hodge class for any τ. Note that if it is rational, the class $\frac{\tau((2\pi i)^k)}{(2\pi i)^k}\alpha_\tau$ will automatically be of type (k,k); this follows because the transport map τ_{dR} in (21) preserves the Hodge filtration, so we already know that $\alpha_\tau \in F^k H^{2k}$, and we use (7).

REMARK 7. Motivation for the definition comes from the fact that Betti cycle classes are absolute Hodge by (25). Letting AH^{2k} denote the set of absolute Hodge classes in H^{2k}, the following inclusions are clear:

$$\{\text{cycle classes}\} \subseteq AH^{2k} \subseteq \mathrm{Hdg}^{2k}_{\mathbb{Q}}.$$

The Hodge conjecture is now equivalent to the validity of the two statements:

(a) every Hodge class is absolute Hodge, and

(b) every absolute Hodge class is algebraic.

Statement (a) was proved by Deligne for abelian varieties in [12, Theorem 2.11]. It also holds for the Hodge classes that feature in the standard conjectures at the end of Section 1.

We next study absolute Hodge classes for a family $p\colon \mathcal{X} \to B$ defined over $\mathbb{Q}$, and the fields of definition of the corresponding Hodge loci. (It is always the case that an interesting variety X defined over $\mathbb{C}$ is included as a fiber in such a family. For example, the Hilbert scheme of subschemes in $\mathbb{P}^n$ with a given Poincaré polynomial and the universal family over it are defined over $\mathbb{Q}$. The Hilbert scheme is not usually geometrically irreducible so B may be not reducible, but that does not matter. Refer to [21] for more details.)

Suppose then that X is a complex projective variety which is a fibre of a family (14) where $\mathcal{X}, B, p$ are defined over $\mathbb{Q}$ (with B not necessarily irreducible). Recall that $\mathcal{H}dg^{2k}_\alpha$ is the connected component of the locus of Hodge classes containing α inside the holomorphic vector bundle $F^k\mathcal{H}^{2k}$.

To formulate the next proposition, recall that the total space of the locally free sheaf $F^k\mathcal{H}^{2k}$ is an algebraic variety defined over $\mathbb{Q}$. Moreover, by Theorem 5, it contains $(2\pi i)^k \mathcal{H}dg^{2k}_\alpha$ as a closed *algebraic* subset Thus $\mathrm{Gal}(\mathbb{C}/\mathbb{Q})$ acts on the set of its complex points (t, α_t) with $\alpha_t \in F^kH^{2k}(X_t, \mathbb{C})$. For $\tau \in \mathrm{Gal}(\mathbb{C}/\mathbb{Q})$, the action of τ is nothing but the map

$$\alpha \in F^kH^{2k}(X_t, \mathbb{C}) \mapsto \alpha_\tau \in F^kH^{2k}(X_{\tau(t)}, \mathbb{C})$$

considered previously.

PROPOSITION 1 ([37, Lemma 1.4]). *The class* $\alpha \in \mathrm{Hdg}^{2k}(X)$ *is absolute Hodge on* X_o *if and only if*

(i) *the set* $(2\pi i)^k \mathcal{H}dg^{2k}_\alpha$ *is defined over* $\overline{\mathbb{Q}}$*;*

(ii) *for every* $\tau \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, *the set* $\tau((2\pi i)^k \mathcal{H}dg^{2k}_\alpha)$ *is contained in* $(2\pi i)^k \mathcal{H}dg^{2k}$.

Proof. The "if" implication holds by definition of absolute Hodge.

For the converse, we claim firstly that if α is absolute Hodge, then all the α' in $\mathcal{H}dg^{2k}_\alpha$ are absolute Hodge. For this statement, we use the algebraicity Theorem 5. Recalling the picture (17), we obtain a tautological section $\widetilde{\alpha}$ of the bundle $p^*F^k\mathcal{H}^{2k}$ pulled back to $\mathcal{H}dg^{2k}_\alpha$:

$$\begin{array}{ccc} \mathcal{H}dg_\alpha & \hookrightarrow & F^k\mathcal{H}^{2k} \\ p_\alpha \downarrow & \swarrow p & \\ B_\alpha & & \end{array}$$

By definition of $\mathcal{H}dg^{2k}_\alpha$, this section is *flat* relative to the *Gauss–Manin* connection

$$\nabla\colon\ \mathcal{H}^{2k} \to \Omega^1_B \otimes \mathcal{H}^{2k},$$

as described in the paper by Katz–Oda [19]. Indeed, it is a Betti rational cohomology class at any point, hence must be locally constant. Therefore, we have $\nabla\widetilde{\alpha} = 0$.

Now ∇ is algebraic and defined over $\mathbb{Q}$. Therefore, if $\widetilde{\alpha}_\tau$ now denotes the section of $p^*_{\alpha,\tau}F^k\mathcal{H}^{2k}$ on $(\mathcal{H}dg^{2k}_\alpha)_\tau$ deduced from $\widetilde{\alpha}$ by applying τ, then we also have

$$\nabla\widetilde{\alpha}_\tau = 0.$$

In other words $\widetilde{\alpha}_\tau$ is also flat, and this implies that if it is $(2\pi i)^k$ times a rational Betti cohomology class at *some* point of $(\mathcal{H}dg^{2k}_\alpha)_\tau$, then it is $(2\pi i)^k$ times a rational class everywhere. Now, $\alpha \in H^{2k}(X_t, \mathbb{C})_B$ being absolute Hodge, where $t = p_\alpha(\alpha)$, it follows that α_τ is $(2\pi i)^k$ times a rational class at the point $\tau(t) = p_{\alpha,\tau}(\alpha_\tau)$. This establishes the claim. The claim tells us that if α is absolute Hodge, so are all the classes in $\mathcal{H}dg^{2k}_\alpha$. It follows easily, using the local structure of $\mathcal{H}dg^{2k}_\alpha$ as a countable union of closed analytic sets, that each translate $((2\pi i)^k \mathcal{H}dg^{2k}_\alpha)_\tau$ in $(2\pi i)^k \mathcal{H}dg^{2k}$ is actually a connected component of the latter. As a consequence, $(2\pi i)^k \mathcal{H}dg^{2k}_\alpha$ has only countably many translates under $\mathrm{Gal}(\mathbb{C}/\mathbb{Q})$, and it follows that it is defined over $\overline{\mathbb{Q}}$ by the Lemma 2, immediately below. This completes the proof of (i) and (ii).

LEMMA 2. *Let* Z *be an algebraic variety defined over* $\mathbb{Q}$ *and let* $Z'_{\mathbb{C}} \subset Z_{\mathbb{C}}$ *be a closed algebraic subset. Assume that* $Z'_{\mathbb{C}}$ *has only countably many Galois translates under* $\mathrm{Gal}(\mathbb{C}/\mathbb{Q})$. *Then* Z' *is defined over* $\overline{\mathbb{Q}}$. □

2.4. A weaker notion

Let X be a fiber of p over a point $o \in B$:

$$\begin{array}{ccc} X & \subset & \mathcal{X} \\ \big\downarrow & & \big\downarrow p \\ o & \in & B \end{array} \tag{26}$$

As before, given a class $\alpha \in \mathrm{Hdg}^{2k}(X)$, let $\mathcal{H}dg^{2k}_{\alpha}$ be the corresponding connected component in $\mathcal{H}^{2k}$, and $B_\alpha = p(\mathcal{H}dg^{2k}_{\alpha})$. Let now look at how $\mathrm{Gal}(\mathbb{C}/\mathbb{Q})$ acts on the base.

A simple consequence of what we already explained for the $\mathcal{H}dg^{2k}_{\alpha}$'s is that, if the Hodge conjecture is true for $\mathcal{X}$, then

(a) B_α is defined over $\overline{\mathbb{Q}}$, and

(b) for every element $\tau \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, the variety $(B_\alpha)_\tau$ is again a component B_γ of the Hodge locus.

Proposition 1 shows that (a) and (b) together constitute a weaker property than asserting that α is an absolute Hodge class.

Assume that we are again in the situation in which $p\colon \mathcal{X} \to B$ is a family defined over $\mathbb{Q}$, X is the fiber over a point $o \in B$, and $\alpha \in \mathrm{Hdg}^{2k}(X)$.

PROPOSITION 2 ([37, Proposition 0.5]). *If the Hodge locus* B_α *is defined over* $\overline{\mathbb{Q}}$, *then the Hodge conjecture for* (X, α) *is implied by the validity of the Hodge conjecture for some variety* Y *defined over* $\overline{\mathbb{Q}}$.

Thus, if the Hodge conjecture is true for varieties defined over $\overline{\mathbb{Q}}$, and a variety X is a fibre in a rational family for which the Hodge loci are defined over $\overline{\mathbb{Q}}$, then the Hodge conjecture is also true for X. This proposition leads us to ask: "Under what circumstances is B_α defined over $\overline{\mathbb{Q}}$?"

Proof. Consider as before the restriction $p_\alpha : \mathcal{H}dg^{2k}_{\alpha} \to B_\alpha$ of $p\colon \mathcal{X} \to B$ to $\mathcal{H}dg^{2k}_{\alpha}$ and the base-changed family $\mathcal{X}_\alpha := \mathcal{H}dg^{2k}_{\alpha} \times_B \mathcal{X}$. Note that p_α is étale over the smooth locus of B_α, though it could be finite-to-one due to monodromy. By further restriction if necessary, we may assume that B_α is smooth.

Since B_α is defined over $\mathbb{Q}$, being an étale cover of it, $\mathcal{H}dg^{2k}_{\alpha}$ is "abstractly" defined over $\overline{\mathbb{Q}}$. Furthermore we already noticed that α extends naturally to a single-valued section $\tilde{\alpha}$ of $\mathcal{H}^{2k}$ over $\mathcal{H}dg^{2k}_{\alpha}$. Let $\overline{\mathcal{X}}_\alpha$ be a smooth compactification of $\mathcal{X}_\alpha$; we may assume that $\overline{\mathcal{X}}_\alpha$ is also defined over $\overline{\mathbb{Q}}$.

Applying the invariant cycle theorem [38, Theorem 4.24], we see that there is a cohomology class $\beta \in H^{2k}(\overline{\mathcal{X}}_\alpha, \mathbb{Q})$ whose restriction to X is equal to α (observe that X is naturally a fiber of $p_\alpha : \mathcal{X}_\alpha \to \mathcal{H}dg^{2k}_\alpha$). This theorem indeed says that the image of the restriction map

$$H^{2k}(\overline{\mathcal{X}}_\alpha, \mathbb{Q}) \to H^{2k}(X, \mathbb{Q})$$

coincides with the invariant subspace $H^{2k}(X, \mathbb{Q})^{\pi_1(B)}$, which contains the class α by assumption. To complete the proof, we need the semi-simplicity of the category of polarized Hodge structures [36, Lemma 7.26], which has the following consequence:

PROPOSITION 3. *Let $\phi\colon H \to H'$ be a morphism of Hodge structures of even weight, where H is polarized. Let $\alpha \in \operatorname{Im}\phi$ be a Hodge class in H'. Then there is a Hodge class $\beta \in H$ such that $\phi(\beta) = \alpha$.*

It follows from this proposition than there exists a *Hodge class* β on $\overline{\mathcal{X}}_\alpha$ which restricts to α. Now, if the Hodge conjecture is valid for the pair $(\overline{\mathcal{X}}_\alpha, \beta)$, which is defined over $\overline{\mathbb{Q}}$, then β is the class of an algebraic cycle, and so is $\alpha = \beta_{|X}$. We have thus proved that the Hodge conjecture for a variety defined over $\overline{\mathbb{Q}}$ implies the Hodge conjecture for (X, α). □

There is a criterion given in [37] which can ensure that the Hodge locus B_α is actually defined over $\overline{\mathbb{Q}}$:

THEOREM 6 ([37, Theorem 0.6-2]). *In the situation* (26), *assume that any locally defined constant subvariation of Hodge structure $\mathcal{H}^{2k}$ on B_α is of type (k, k). Then B_α is defined over $\overline{\mathbb{Q}}$.*

Proof. As in the previous proof, set $\mathcal{X}_\alpha = \mathcal{X} \times_B \widetilde{\mathcal{H}dg^{2k}}_\alpha$, where $\widetilde{\mathcal{H}dg}^{2k}_\alpha$ is a desingularization of $\mathcal{H}dg^{2k}_\alpha$. We have maps

$$\mathcal{X}_\alpha \to \widetilde{\mathcal{H}dg}^{2k}_\alpha \to B_\alpha. \tag{27}$$

Consider the composition

$$j\colon\ X_t \subset \mathcal{X}_\alpha \subset \overline{\mathcal{X}}_\alpha,$$

and set $\alpha = j^*\beta$ where $\beta \in H^{2k}(\overline{\mathcal{X}}_\alpha, \mathbb{Q})$.

Given $s \in \widetilde{\mathcal{H}dg}^{2k}_\alpha$, the restriction map

$$j_s^*\colon\ H^{2k}(\overline{\mathcal{X}}_\alpha, \mathbb{Q}) \longrightarrow H^{2k}(X_s, \mathbb{Q})$$

provides a locally constant Hodge substructure in $H^{2k}(X_{p_\alpha(s)}, \mathbb{Q})$. It follows then from our assumption that $\operatorname{Im}(j_s^*)$ is a trivial Hodge substructure (meaning, of type (k, k)). Now let $\tau \in \operatorname{Gal}(\mathbb{C}/\mathbb{Q})$. It will suffice to prove that

$$(B_\alpha)_\tau = B_{\alpha'}$$

for some $\alpha' \in \mathrm{Hdg}^{2k}(X)$. Indeed, there are countable many such sets, so this will imply that B_α is defined over $\overline{\mathbb{Q}}$ by Lemma 2.

Now τ acts on everything in the tower (27):

$$(\mathcal{X}_\alpha)_\tau \to (\mathcal{H}dg^{2k}_\alpha)_\tau \to (B_\alpha)_\tau.$$

There is also the inclusion $j_\tau\colon X_\tau \hookrightarrow \overline{\mathcal{X}}_{\alpha,\tau}$, and the restriction map

$$j_\tau^*\colon H^{2k}(\overline{(\mathcal{X}_\alpha)_\tau}, \mathbb{Q})_B \to H^{2k}(X_\tau, \mathbb{Q})_B.$$

Now we have the following fact, which is proved by observing that the property under consideration is purely algebraic and can be checked by looking at the corresponding restriction map in algebraic de Rham cohomology:

If $\mathrm{Im}(j^*)$ *is of type* (k,k), *then* $\mathrm{Im}(j_\tau^*)$ *is of type* (k,k).

But $\mathrm{Im}(j^*) = j^*(F^k H^{2k})$ and $\mathrm{Im}(j_\tau^*)$ contains α_τ on X_τ. "Locally" $(B_\alpha)_\tau$ is defined by the condition that $\alpha_\tau \in F^k H^{2k}$. Now replace α_τ by any generically chosen

$$\alpha' \in j_\tau^*(H^{2k}(\overline{\mathcal{X}}_\alpha, \mathbb{Q})).$$

It follows easily that $(B_\alpha)_\tau = B_{\alpha'}$ (equality here as sets, but not scheme-theoretically), as required. □

Let us give one corollary:

COROLLARY 1. *Assume that the Hodge structure on* $H^{2k}(X,\mathbb{Q})_B/\mathrm{Hdg}^{2k}(X)$ *is simple and that the variation of Hodge structure on* $H^{2k}(X_t,\mathbb{Q})_B$ *is maximal (that is, the Hodge structure does not remain constant on positive-dimensional subsets). Then for any Hodge class* α *on* X, $\mathcal{H}dg^{2k}_\alpha$ *is defined over* $\overline{\mathbb{Q}}$.

Soulé asked whether the Hodge conjecture for varieties defined over $\overline{\mathbb{Q}}$ implies the Hodge conjecture in general. Corollary 1 gives an example of a situation when the answer to his question is positive.

EXAMPLE 2 (Complex multiplication revisited). Let $j\colon \mathcal{E} \to \mathbb{A}^1$ be the universal elliptic curve, and $D \geqslant 1$ be an integer. It was shown in Subsection 2.2 that the set T_D of points $t \in \mathbb{A}^1$ such that the curve E_t admits a complex multiplication with discriminant D is a Hodge locus in $\mathbb{A}^1$. It is known from class field theory that the order of T_D is equal to $h(D)$, the class number of the field $\mathbb{Q}(\sqrt{-D})$. The field of definition K of each point in the locus T_D (and thus of T_D itself) is the maximal *unramified abelian* extension K of the field $\mathbb{Q}(\sqrt{-D})$, with the degree $\deg K/\mathbb{Q}(\sqrt{-D}) = h(D)$ and the Galois group isomorphic to the group of class ideals $\mathrm{Cl}(K)$. Here the base $\mathbb{A}^1$ can be substituted with any of the modular curves $X(N)$.

3. The generalized Hodge conjecture

In this section, we shift gear by returning to more obviously geometrical applications of Hodge decompositions.

We first introduce the concept of *coniveau* that allows one to extend attention to Hodge classes "surrounding" those of diagonal type. Such Hodge substructures arise naturally from subvarieties as images of the Gysin map, and the generalized Hodge conjecture postulates a converse to this process.

3.1. Coniveau and the Gysin sequence

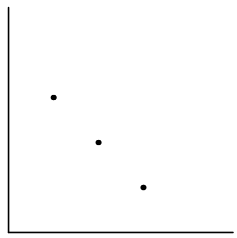
Suppose that L is a lattice or $\mathbb{Q}$-vector space endowed with a Hodge structure of weight k, so that

$$L_{\mathbb{C}} = \bigoplus_{\substack{p,q\geqslant 0\\ p+q=k}} L^{p,q}.$$

DEFINITION 6. *We say that the Hodge structure on* L *has* coniveau r *if* $L^{p,q}$ *vanishes for* $p<r$ *and* $q<r$ *but* $L^{r,k-r}\neq 0$*, so*

$$L_{\mathbb{C}} = L^{k-r,r}\oplus\cdots\oplus L^{r,k-r}.$$

Note that the direct sum includes the summand $L^{r,r}$ if and only if $k=2r$ is even. The figure helps one imagine a Hodge structure of weight 4 and coniveau 1:



Given a Hodge structure of coniveau r, we can define a new Hodge structure $L' = L(r)$ of weight $k-2r$ by setting

$$(L')^{p,q} = L^{p+r,q+r}, \qquad p+q=k-2r, \tag{28}$$

with the same underlying lattice L. This modification is elementary, but has a deep motivation, which will come from the study of Gysin morphisms in algebraic geometry.

Let X be a smooth projective variety, and $Y\subset X$ a smooth subvariety of complex codimension r. We shall see that Y provides more cohomology classes on X than just its fundamental class $[Y]\in H^{2r}(X,\mathbb{Q})$.

Consider the inclusions $i\colon Y\subset X$ and $j\colon Y\setminus X\subset X$, as displayed in (9) from Subsection 1.2. The Gysin exact sequence reads

$$\cdots\to H^{k-2r}(Y)\xrightarrow{i_!} H^k(X)\xrightarrow{j^*} H^k(X\setminus Y)\xrightarrow{\delta} H^{k-2r+1}(Y)\to\cdots$$

It can be constructed from the exact sequence of the pair $(X, X\setminus Y)$ combined with the Thom isomorphism

$$H^k(X,X\setminus Y)\cong H^{k-2r}(Y)$$

(see [36, Section 7.3.3]). The Gysin map $j_!$ coincides with $D_X \circ i_* \circ D_Y$, where D_X, D_Y are the Poincaré duality isomorphisms and $i_*\colon H_{2n-k}(Y) \to H_{2n-k}(X)$, where $n = \dim X$.

Another way to construct the Gysin sequence is via the Leray spectral sequence. It is more subtle, but applies when Y is singular, which is the context of the next result.

THEOREM 7. *Let X be a smooth projective variety, and $Y \subset X$ a closed algebraic subset of codimension r. The kernel of the map*

$$H^k(X,\mathbb{Q}) \xrightarrow{j^*} H^k(X \setminus Y,\mathbb{Q})$$

is a Hodge substructure of $H^k(X,\mathbb{Q})$ of coniveau at least r.

Note that some Hodge numbers of Y may be zero, so the coniveau might be strictly bigger than r.

Proof. Choose a desingularization $\tau\colon \widetilde{Y} \to Y$, and assume that $\widetilde{Y}$ has pure complex dimension $n-r$.

We need the fact that morphisms of mixed Hodge structures are strict for the weight filtration (cf. [10], [38, 4.3.2]). We already know (via Poincaré duality for open varieties) that the kernel of j^* is the same as the image of

$$i_*\colon\ H_{2n-k}(Y,\mathbb{Q}) \to H_{2n-k}(X,\mathbb{Q}) \overset{D_X}{\cong} H^k(X,\mathbb{Q}).$$

There is a mixed Hodge structure on both sides. The composition is a morphism of mixed Hodge structures of bidegree (r,r), with a *pure* Hodge structure on the right (cf. [10]). Thus its image is the same as the image of the map

$$i_*\colon W_{2n-k}H_{2n-k}(Y,\mathbb{Q}) \to H_{2n-k}(X,\mathbb{Q}) \cong H^k(X,\mathbb{Q}).$$

But by construction, the pure part $W_{2n-k}H_{2n-k}(Y,\mathbb{Q})$ of the mixed Hodge structure on $H_{2n-k}(Y,\mathbb{Q})$ coincides with the image of the map

$$\tau_*\colon\ H_{2n-k}(\widetilde{Y},\mathbb{Q}) \to H_{2n-k}(Y,\mathbb{Q}).$$

That concludes the proof, because, applying Poincaré duality on $\widetilde{Y}$, we proved that $\operatorname{Ker} j_* = \operatorname{Im}(i \circ \tau)_* : H^{k-2r}(\widetilde{Y},\mathbb{Q})_B \to H^k(X,\mathbb{Q})_B$ and this morphism is a morphism of Hodge structures of bidegree (r,r). □

3.2. Statement of the generalized Hodge conjecture

In his original question, Hodge asked which cohomology classes on a smooth complex projective variety X are supported on subvarieties. Theorem 7 provides strong restrictions on them. Grothendieck corrected the original formulation in a way which takes Theorem 7 into account, asking the converse question:

CONJECTURE 4. Suppose that X is a smooth projective variety defined over $\mathbb{C}$, and that $L \subset H^k(X, \mathbb{Q})$ is a Hodge substructure of coniveau r. Then there exists a closed algebraic set $Y \subset X$ of codimension at least r such that L is contained in $\ker(j^*: H^k(X, \mathbb{Q}) \to H^k(X \setminus Y, \mathbb{Q}))$ in the notation of (9).

The Hodge conjecture implies the generalized Hodge conjecture in two particular cases that we consider next. The most obvious occurs when $k = 2r$. In this case we have $L_{\mathbb{C}} = L^{r,r}$ and L consists of Hodge classes. The Hodge conjecture provides subvarieties $Z_1, \dots, Z_N$ of X all of codimension r such that

$$L = \langle [Z_1], \dots, [Z_N] \rangle \otimes \mathbb{Q}.$$

But then L vanishes on $X \setminus \langle Z_1 \cup \cdots \cup Z_n \rangle$, as required.

The next, but more sophisticated, case is that in which $k = 2r + 1$, so that

$$L_{\mathbb{C}} = L^{r,r+1} \oplus L^{r+1,r}. \tag{29}$$

In this case, referring to (28), $L' = L(-r)$ is a polarized Hodge structure of weight 1. We get such Hodge structures on the first cohomology groups of curves, though not every Hodge structure of weight 1 is actually a Hodge structure of a curve. However,

LEMMA 3. *Any polarized Hodge structure of weight 1 arises as $H^1(A, \mathbb{Q})$ for some abelian variety A.*

The key point is the existence of a polarization (arising from the intersection form), but without polarizations, the lemma remains true with “abelian varieties” replaced by “complex tori”. The lemma is a reformulation of the fact that the category of weight 1 rational polarized Hodge structures is the same as the category of abelian varieties up to isogeny (see [36, Section 7.2.2]).

Using this lemma, we shall prove

THEOREM 8. *The Hodge conjecture implies Conjecture 4 when $k = 2r + 1$ as in* (29)*.*

Proof. We start with a Hodge substructure $L \subset H^{2r+1}(X, \mathbb{Q})$ of coniveau r. It is polarized, because the Hodge structure on $H^{2r+1}(X, \mathbb{Q})$ is polarized (note however that the polarizations are not canonical). There exists then a polarized Hodge structure L' of weight 1 and a morphism $\phi : L' \cong L$ of bidegree (r, r). By Lemma 3 we may assume $L' = H^1(A, \mathbb{Q})$ as Hodge structures. Having an abelian variety A and a morphism of Hodge structures $H^1(A, \mathbb{Q}) \to H^{2r+1}(X, \mathbb{Q})$, we can choose a curve C that is a complete intersection of ample hypersurfaces in A. Then, by the Lefschetz theorem on hyperplane sections [38, 1.2.2], there is a monomorphism

$$i^*: \ H^1(A, \mathbb{Q}) \to H^1(C, \mathbb{Q}), \tag{30}$$

which is a morphism of Hodge structures of pure weight 1. Since the category of polarized Hodge structures of weight 1 is semisimple, (30) always splits, so $H^1(A, \mathbb{Q})$

is a direct summand, as a Hodge substructure, of $H^1(C,\mathbb{Q})$. We thus get a morphism of Hodge structures $\psi : H^1(C,\mathbb{Q}) \to H^{2r+1}(X,\mathbb{Q})$ of bidegree (r,r). By Lemma 1, ψ determines a Hodge class $\tilde{\psi}$ of degree $2r+2$ on $C \times X$. If the usual Hodge conjecture is true on $C \times X$, then this class is induced by a cycle $Z = \sum_i a_i Z_i$ on $C \times X$:

$$\tilde{\psi} = [Z] = \sum a_i [Z_i], \qquad a_i \in \mathbb{Q}.$$

The resulting correspondence induces maps

$$\begin{array}{ccc} H^1(\operatorname{Supp} Z,\mathbb{Q}) & \xleftarrow{p_1^*} & H^1(C,\mathbb{Q}) \\ \sum_i a_i p_{2,i*} \searrow & & \Big\downarrow [Z]_* \\ & & H^{2r+1}(X,\mathbb{Q}) \end{array}$$

In truth, one needs to desingularize the components Z_i of Z and to replace $\operatorname{Supp} Z$ by $\bigsqcup \widetilde{Z}_i$. Furthermore, we may assume that all Z_i have the property that $\operatorname{codim} p_2(Z_i) = r$, since otherwise $Z_{i*} = 0$. In that case, it follows that $\operatorname{Im}([Z]_*) = \operatorname{Im}\psi = L$ vanishes away from $p_2(\operatorname{Supp} Z)$, which is of codimension r. Thus, L satisfies the generalized Hodge conjecture. □

In conclusion, the usual Hodge conjecture for varieties which are products with a curve implies generalized Hodge conjecture for weight $n = 2r+1$ and coniveau r. In order to tackle the general case, a part could be generalized but there is an important missing point, namely, the adequate generalization of Lemma 3. This leads to the following question:

Given a weight k *Hodge structure* $L \subset H^k(X,\mathbb{Q})$ *of coniveau* r*, so that*

$$L_{\mathbb{C}} = L^{k-r} \oplus \cdots \oplus L^{r,k-r},$$

consider the weight $k-2r$ *Hodge structure* $L' = L(-r)$*. Does there exist a smooth projective variety* Y *admitting* L' *as a Hodge substructure of* $H^{k-2r}(Y,\mathbb{Q})$*?*

3.3. Coniveau 1 hypersurfaces

Let X be a smooth hypersurface of $\mathbb{P}^n$ of degree d. The Lefschetz theorem gives an isomorphism

$$H^k(X,\mathbb{Q}) \xleftarrow{\cong} H^k(\mathbb{P}^n,\mathbb{Q}), \qquad k \leqslant n-2.$$

From this point of view, the only interesting part of the cohomology of X is the primitive subspace $H^{n-1}(X.\mathbb{Q})_0$. Let us denote by $\operatorname{con}(X)$ the coniveau of the Hodge structure on $H^{n-1}(X)_0$.

THEOREM 9 (Griffiths, [14]). *With the above terminology,*

$$\operatorname{con}(X) \geqslant r \iff n \geqslant dr.$$

Sketch proof. We refer to [38, 6.1.2] for a complete proof. It is based on Griffiths' description of the Hodge filtration on $H^{n-1}(X)_0$ for X as above. Griffiths proves that $F^kH^{n-1}(X)_0$ is generated by the residue $\mathrm{Res}_X \frac{P\Omega}{f^{n-k}}$, where f is the defining equation of X, Ω is the section (unique up to a scalar) of $K_{\mathbb{P}^n}(n+1)$ and P is a polynomial on $\mathbb{P}^n$ satisfying

$$\deg P + n + 1 = d(n-k).$$

This degree condition guarantees that $\frac{P\Omega}{f^{n-k}}$ is a meromorphic form on $\mathbb{P}^n$ with poles along X. In any case, we conclude from this degree condition that if $n \geqslant dr$, and $k \geqslant n-r$, then $\deg P < 0$, so $P = 0$. Hence $F^{n-r}H^{n-1}(X)_0 = 0$ and thus $\mathrm{con}(X) \geqslant r$. $\square$

For $r = 1$, the coniveau assertion amounts to

$$0 = H^{n-1,0}(X) = H^0(X, K_X) = H^0(X, \mathcal{O}(-n-1+d)).$$

This means that X is a *Fano hypersurface*; it is equivalent to the condition $d \leqslant n$. The generalized Hodge conjecture for coniveau 1 now says that if $d \leqslant n$ then $H^{n-1}(X)_0$ vanishes away from an algebraic set $Y \subset X$ of codimension 1. In fact,

THEOREM 10. *The generalized Hodge conjecture is true for coniveau* 1 *hypersurfaces.*

Proof. We will prove this statement later on for any Fano or rationally connected variety, but here we give an ad hoc proof, *assuming the stronger condition that* $d < n$. This condition implies that, for all $x \in X$, there exists a line $\mathbb{P}^1 \subset X$ containing x. This is true because the set of lines in X through a point $x \in X$ is defined in $\mathbb{P}^{n-1}$ by d equations, of successive degrees $1, \ldots, d$. Hence it is nonempty if $n-1 \geqslant d$.

Let F be the set of lines in X, thus a subvariety of $\mathrm{Gr}_2(\mathbb{C}^{n+1})$. Let P denote the incidence variety, consisting of pairs (ℓ, x) with $x \in X$ and ℓ a line in X containing x:

$$\begin{array}{ccc} & P & \\ {}^{p}\swarrow & & \searrow^{q} \\ F & & X \end{array}$$

By assumption, q is a surjective map between compact Kähler manifolds, and so $q^*\colon H^*(X) \hookrightarrow H^*(P)$ is injective (cf. [36, Lemma 7.28]). Now we have to add the following argument: As $p : P \to F$ is a $\mathbb{P}^1$-bundle, its cohomology decomposes as

$$H^*(P) = p^*H^*(F) \oplus q^*h \cup p^*H^{*-2}(F),$$

where $h = c_1(\mathcal{O}_X(1))$ (cf. [36, 7.3.3]). In this decomposition, the first term is annihilated by p_* while p_* on the second term gives the identity map of $H^{*-2}(F)$. If we start from a primitive cohomology class α on X, it satisfies $h \cup \alpha = 0$ hence $q^*h \cup q^*\alpha = 0$. But then $q^*\alpha$ does not belong to the first term $p^*H^*(F)$ unless it is 0. Thus we conclude that the morphism of Hodge structures

$$p_* \circ q^* \colon\ H^{n-1}(X)_0 \hookrightarrow H^{n-3}(F, \mathbb{Q}),$$

is injective as well. Finally, if we choose $F' \subset F$ to be an intersection of ample hypersurfaces of dimension $n-3$, the Lefschetz hyperplane section theorem tells that

$$H^{n-3}(F, \mathbb{Q}) \hookrightarrow H^{n-3}(F', \mathbb{Q}).$$

The picture reduces to

$$\begin{array}{ccc} & P' & \\ {}^{p'}\swarrow & & \searrow^{q'} \\ F' & & X \end{array}$$

inducing the injective morphism of Hodge structures

$$(p')_* \circ (q')^* \colon\ H^{n-1}(X)_0 \hookrightarrow H^{n-3}(F'),$$

Dualizing,

$$(q')_* \circ (p')^* \colon\ H^{n-3}(F', \mathbb{Q}) \to H^{n-1}(X)/\mathcal{A},$$

where $\mathcal{A}$ is the space of algebraic classes arising from $\mathbb{P}^n$. Thus,

$$H^{n-1}(X) = \mathrm{Im}(q')_* + \mathcal{A}.$$

But $\dim P' = n-3$, so $q'(P')$ is a hypersurface in X and $\mathrm{Im}(q')_*$ is made of classes supported on this hypersurface. This completes the proof. □

4. Rational equivalence and the Bloch conjecture

At the end of the previous section, we discussed the generalized Hodge conjecture for coniveau 1 hypersurfaces.

In this section, we shall take up another approach that actually works for any Fano variety (i.e. one for which $-K$ is ample), or more generally any rationally connected variety X (i.e. one for which any two points can be joined by a chain of rational curves). This is the case of course if X is a unirational variety, that is, rationally dominated by projective space. However a more appropriate assumption, much less restrictive geometrically, is that the Chow group $CH_0(X)$ be "trivial" (i.e. equal to $\mathbb{Z}$).

4.1. Chow groups

Suppose for simplicity that X is a smooth projective variety. Let $\mathcal{Z}_k(X)$ denote the group of algebraic cycles, i.e. the free abelian group with generators closed irreducible reduced algebraic subsets of dimension k. Thus a typical element of $\mathcal{Z}_k(X)$ has the form $\sum_{i=1}^{N} n_i Z_i$ with $n_i \in \mathbb{Z}$, $Z_i \subset X$ closed of dimension k. An algebraic cycle Z in

$\mathcal{Z}_k(X)$ is *rationally equivalent to zero* if there exists a smooth variety W of dimension $k+1$, a rational function $f \in \mathbb{C}(W)^*$, and a map

$$\phi : W \to X$$

such that $Z = \phi_*(\operatorname{div} f)$ (taking account only of components of dimension k), or if Z is a finite sum of such cycles.

DEFINITION 7. *With the above preliminaries,*

$$\mathrm{CH}_k(X) = \frac{\mathcal{Z}_k(X)}{\text{cycles rationally equivalent to } 0}.$$

For cycles of codimension 1, the equivalence relation is akin to setting $D \sim 0$ when D is the divisor of a meromorphic function. In general, two cycles Z_1, Z_2 are *rationally equivalent* if their difference is rationally equivalent to 0 and we write $Z_1 \sim Z_2$. Thus, $\mathrm{CH}_k(X)$ is the group of all k-cycles in X modulo rational equivalence. It is called the k-th *Chow group* of X, and sometimes denoted A_k [13, Chapters 1–2].

If X is connected, the natural degree homomorphism $\mathrm{CH}_0(X) \to H_0(X,\mathbb{Z}) = \mathbb{Z}$ is given by

$$\sum_{i=1}^{N} n_i Z_i \;\mapsto\; \sum_{i=1}^{N} n_i.$$

We shall denote its kernel by $\widetilde{\mathrm{CH}}_0(X)$; this is generated by cycles of degree equal to 0. The trivial situation is that in which $\widetilde{\mathrm{CH}}_0(X) = 0$, or equivalently $\mathrm{CH}_0(X) = \mathbb{Z}$.

If X is rationally connected (or, in particular, rational) then $\mathrm{CH}_0(X) = \mathbb{Z}$. To show that any two points x, y are rationally equivalent, we choose a chain of rational curves connecting them, with marked points $x = x_0,\ x_1, \dots, x_{n-1},\ x_n = y$ such that x_i belongs to the intersection of two curves if $0 < i < n$. Each curve C has a desingularization $\mathbb{P}^1 \to C \subset X$, and $x_i - x_{i-1}$ is the divisor of a rational function on $\mathbb{P}^1$ for $i > 0$. So x is rationally equivalent to y.

If X is smooth projective and we take $k = n-1$, then the Chow group $\mathrm{CH}_{n-1}(X)$ coincides with the Picard group $\mathrm{Pic}(X)$ consisting of divisors modulo linear equivalence. In the complex case, we can identify the subgroup $\mathrm{Pic}^0(X)$ with the abelian variety $H^1(X, \mathcal{O}_X)/H^1(X,\mathbb{Z})$. The quotient

$$\mathrm{NS}(X) = \mathrm{Pic}(X)/\mathrm{Pic}^0(X)$$

is the Néron–Severi group of X, and becomes a subgroup of $H^2(X,\mathbb{Z})$ in the exact sequence (10). It follows that $\mathrm{NS}(X)$ is finitely generated. Much less is known about the groups $\mathrm{CH}_k(X)$ for $k < n-1$.

4.2. Mumford's approach

Rational equivalence was used extensively by Severi and his school in the first half of the 20th century. The groups CH_k were subsequently named after W. L. Chow who wrote an important review paper in 1956 explaining the Italian proof of the moving lemma in Weil's language. (We refer to Chevalley's account in [8].)

Mumford proved that if S is a complex surface S with $p_g(S) > 0$ then $CH_0(S)$ is (in an appropriate sense) infinite-dimensional, contradicting assumptions that Severi had taken for granted. However, Mumford claims to have used Severi's own methods to prove this fact, and writes "One must admit that in this case the *technique* of the Italians was superior to their vaunted intuition" [22].

Fix a point $p_o \in S$, and consider the mapping

$$\begin{array}{ccc} \mathrm{Sym}^d S & \xrightarrow{a_d} & \widetilde{CH}_0(S), \\ \xi & \mapsto & [\xi - dp_o]. \end{array}$$

One wants to show that if $p_g(S) \neq 0$, this fails to be surjective for any $d \geqslant 1$. The idea is the following. Having chosen a non-zero $(2,0)$-form $\omega \in H^0(S, \Omega^2)$, the symmetric product $\mathrm{Sym}^d S$ inherits a holomorphic nondegenerate form on a dense open set, and the restriction of this form to each fiber of a_d is zero. The generic fiber dimension is therefore at most one half that of $\mathrm{Sym}^d S$, so $\mathrm{Im}(a_d)$ has dimension at least d. For more details, see [38, Section 10.2.2].

For a smooth complex projective surface S, we get the following conclusion:

THEOREM 11.

$$CH_0(S) = \mathbb{Z} \implies q(S) = 0, \ \ p_g(S) = 0.$$

Proof. We just explained why $p_g(S) = 0$ under these assumptions. To see also that $q(S) = 0$, consider the Albanese variety $\mathrm{Alb}(S) = \mathrm{Jac}^3(S)$. Let $\widetilde{\mathcal{Z}}_0(S)$ denote the set of 0-cycles on S of degree 0. The induced group morphism $\widetilde{\mathcal{Z}}_0(S) \to \mathrm{Alb}(S)$ is surjective (cf. [36, Lemma 12.11]) and factors through rational equivalence, because there are no non constant rational maps from $\mathbb{P}^1$ to an abelian variety. Hence, if $\widetilde{CH}_0(S) = 0$ then $\mathrm{Alb}(S)$ is trivial, and $q(S) = 0$. □

The next theorem is the natural generalization of Mumford's theorem, first due to Roitman.

THEOREM 12. *Let X be a smooth projective variety with $CH_0(X) = \mathbb{Z}$. Then $H^{k,0}(X) = 0$ for all $k > 0$.*

By Hodge symmetry, the conclusion is obviously equivalent to asserting that the Hodge structure of $H^k(X, \mathbb{Q})$ has coniveau at least 1. This theorem provided the first known relationship known between Hodge theory and Chow groups in arbitrary dimensions.

The following stronger statement was proved by Bloch–Srinivas. The references are [5] and [38, Chapter 10].

THEOREM 13. *Let X be a smooth projective variety with $CH_0(X) = \mathbb{Z}$. Then the cohomology of X in positive degree is supported on an hypersurface of X. Hence, it has coniveau $\geqslant 1$ and the generalized Hodge conjecture holds for $H^k(X, \mathbb{Q})$ in coniveau 1.*

Bloch–Srinivas deduced Theorem 13 from a version of their "decomposition of the diagonal" argument:

THEOREM 14. *Let* X *be defined over* $\mathbb{C}$ *with* $CH_0(X) = \mathbb{Z}$. *Then, given* $p \in X$, *there exists an integer* N *such that*

$$\Gamma := N\big(\Delta_X - (X \times \{p\})\big) \ \in \ CH_n(X \times X), \tag{31}$$

is rationally equivalent to a cycle supported on $Y \times X$ *for some hypersurface* $Y \subset X$.

Let us postpone the proof, and show how this implies Theorem 13.

Proof of Theorem 13. A cycle rationally equivalent to 0 is homologous to 0, so we get from (31)

$$N[\Delta_X - X \times \{p\}] = [\Gamma] \quad \text{in} \ \ H^{2n}(X \times X, \mathbb{Q})_B.$$

This element defines a morphism of Hodge structures $[\Gamma]^* : H^*(X) \to H^*(X)$. Take a desingularization

$$\widetilde{i}\colon \ \widetilde{Y} \to Y \to X$$

and lift Γ to $\widetilde{\Gamma}$ supported on $\widetilde{Y} \times X$.

Choose $\alpha \in H^k(X)$ with $k = \deg(\alpha) \geqslant 1$. We need to show that α maps to zero in $H^k(X \setminus Y, \mathbb{Q})$. We have

$$N([\Delta_X] - [X \times \{p\}])^* \alpha = [\Gamma]^* \alpha = \widetilde{i}_*([\widetilde{\Gamma}]^* \alpha).$$

Now $N[\Delta_X]^* \alpha = N\alpha$; moreover $[X \times \{p\}]^* \alpha = 0$ since $k \geqslant 1$. Therefore we get $N\alpha = \widetilde{i}_*([\widetilde{\Gamma}]^* \alpha$, which shows that α vanishes on $X \setminus Y$. □

Proof of Theorem 14. We give the idea. The assumption $CH_0 = \mathbb{Z}$ means that, for all $x \in X$, there exists

$$\phi\colon C \to X$$

and a rational function $f \in \mathbb{C}(C)^*$ such that $\phi_*(\operatorname{div} f) = x - p$. Such data (C, ϕ, x) are parametrized by countably many algebraic varieties (using either Chow varieties or Hilbert scheme methods).

Since X is by hypothesis exhausted by the images (via the third projection) of these countably many algebraic varieties, a Baire category argument then shows that one can find a generically finite covering $r\colon \widetilde{X} \to X$ of degree N, a dense Zariski open set $U \subset \widetilde{X}$ such that $Y' = \widetilde{X} \setminus U$ is a hypersurface in $\widetilde{X}$, and with the following properties. First of all, there is a family of curves $\Phi : \mathcal{C} \to X$ on X parameterized by $\widetilde{X}$, illustrated by the diagram

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\Phi} & X \\ {\scriptstyle\rho}\big\downarrow & & \\ \widetilde{X} & & \end{array}$$

Secondly, there exists $F \in \mathbb{C}(\mathcal{C})^*$ such that

$$(\rho, \Phi)_*(\operatorname{div} F) = (r, \mathrm{id})^*(\Delta_X - \widetilde{X} \times p),$$

at least over $U \times X$. Thus,

$$(\rho, \Phi)_*(\operatorname{div} F) = (r, \mathrm{id})^*(\Delta_X - \widetilde{X} \times p) + \Gamma'',$$

where Γ'' is supported over $Y' \times X$. Now apply $(r, \mathrm{id})_*$ and put $Y = r(Y')$. The left-hand side being by definition rationally equivalent to 0, we get the desired decomposition. Note that the integer N which appears in this decomposition is the degree of the map r above. □

Note that the proof above shows more generally (cf. [5]):

THEOREM 15. *Suppose that Z is a closed algebraic subset of X for which $CH_0(X \setminus Z) = 0$. Then for some $N \neq 0$, $N\Delta$ is rationally equivalent to the sum of a cycle supported on $Y \times X$ for some hypersurface $Y \subset X$ and of a cycle supported on $X \times Z$.*

Theorem 14 is the particular case $Z = \{p\}$. Theorem 15 has many Hodge and cycle theoretic consequences for which we refer to [5] or [38, Chapter 10].

4.3. Bloch's conjectures

The following postulates a converse to the generalized Mumford theorem.

CONJECTURE 5 (generalized Bloch conjecture). Suppose that $H^{k,0}(X) = 0$ for all $k > 0$. Then $CH_0(X) = \mathbb{Z}$.

A refinement of this conjecture, which is natural in view of Theorem 15, is the following:

CONJECTURE 6 (Generalized Bloch conjecture). Suppose that $H^{k,0}(X) = 0$ for all $k > r$. Then $CH_0(X)$ is supported on a closed algebraic subset Z of X of dimension at most r.

By this, we mean that any point is rationally equivalent to a cycle supported on Z.

Let us consider the case of a smooth surface X. Thus we need to prove that $H^{2,0}(X) = H^{1,0}(X) = 0$ implies that $CH_0(X)$ is trivial. For Conjecture 6, we need to prove that $H^{2,0}(X) = 0$ implies that $CH_0(X)$ is supported on a curve. We can appeal to Kodaira's classification of surfaces to achieve the result at least for Kodaira dimension less than 2:

If $\kappa(X) = -\infty$ then X is ruled over a curve C. As all points in the fibers are rationally equivalent in X, $CH_0(X)$ is supported on any section of the fibration $X \to C$.

If $\kappa(X) = 0, 1$ and $q = 0 = p_g$, then X admits an elliptic fibration $f : X \to B$. In general q equals the genus of B, so here $B = \mathbb{P}^1$. Suppose first that $f \colon X \to \mathbb{P}^1$ has a section.

Then

$$K_X = f^*\mathcal{O}_{\mathbb{P}^1}(-m), \quad m > 0$$

(see [3, Corollary 5.12.3]), so $h^0(X, K^2) = 0$ and X is rational by Castelnuovo's theorem. A nice argument has been given by Bloch–Kac–Liebermann to complete this case: If X does not have a section, then the surface $X' = \mathrm{Jac}(X/B)$ has a section and the same irregularity and geometric genus as X, although the two are not in general isomorphic. On the other hand, it is an easy lemma that $CH_0(X) = CH_0(X')$, so $CH_0(X) = \mathbb{Z}$.

Bloch also gave a generalization of Conjecture 5 for correspondences between surfaces. We can formulate this as follows.

Let X, X' be two surfaces. We make no assumption on the geometric genera p_g, p'_g or irregularities q, q'. Let

$$\Gamma \subset X \times X'$$

be a 2-cycle. Then Γ induces a correspondence

$$\begin{array}{rcl} [\Gamma]^* : H^2(X', \mathbb{Z}) & \to & H^2(X, \mathbb{Z}), \\ c & \mapsto & p_{1*}(p_2^*(c) \cdot [\Gamma]), \end{array}$$

where p_1, p_2 are the respective projections $X \times X' \to X, X'$. This is a morphism of Hodge structures by Lemma 1. In particular, $[\Gamma]^*$ induces a mapping

$$[\Gamma]^{*2,0}: \ H^{2,0}(X') \longrightarrow H^{2,0}(X). \tag{32}$$

Denoting by $CH_0(X)_{\mathrm{ab}}$ the subgroup of zero-cycles homologous to 0 and annihilated by the Albanese map, our 2-cycle Γ also gives rise, by functoriality of the Albanese maps with respect to correspondences, to a homomorphism

$$\Gamma_*: \ CH_0(X)_{\mathrm{ab}} \to CH_0(X')_{\mathrm{ab}}, \quad z \mapsto p_{2*}(p_1^* z \cdot \Gamma). \tag{33}$$

(We refer to [13] for the basic results on functoriality properties of Chow groups and intersection theory).

CONJECTURE 7 (Generalized Bloch conjecture). If $[\Gamma]^* = 0$ in (32) then $\Gamma_* = 0$ in (33).

REMARK 8. We point out that this conjecture implies Bloch's conjecture for surfaces in its strong form Conjecture 6. Take $X' = X$ and $\Gamma = \Delta_X$ the diagonal, so that both maps (32) and (33) are the identity. Since $p_g(S) = 0$, we have $[\Gamma]^{*2,0} = 0$, hence Conjecture 7 says that $\Gamma_* = \mathrm{id}$ acts as zero on $CH_0(X)_{\mathrm{ab}}$. It follows that $CH_0(X)_{\mathrm{ab}} = 0$. On the other hand, the Lefschetz theorem on hyperplane sections shows that if $j : C \hookrightarrow X$ is a smooth ample curve, the map $j_* : \mathrm{Alb}\, C \to \mathrm{Alb}\, X$ is surjective. Combining both statements, we conclude that $j_* : CH_0(C) \to CH_0(X)$ is surjective.

4.4. Surfaces with trivial CH_0

This section is mainly devoted to a study of quintic Godeaux surfaces, and a proof of the fact that they satisfy the Bloch conjecture. The references are [38, Section 11.1.4] and [34]. A separate example covers more briefly the case of Barlow's surface.

Having chosen coordinates on $\mathbb{P}^3$, the group $G = \mathbb{Z}/5\mathbb{Z}$ of order 5 acts by

$$(x_0, x_1, x_2, x_3) \mapsto (x_0,\, x_1\zeta,\, x_2\zeta^2,\, x_3\zeta^3),$$

where $\zeta = e^{2\pi i/5}$. Choose a polynomial $F \in H^0(\mathbb{P}^3, \mathcal{O}(5))$ invariant by ζ. If F is generic, it defines a smooth surface $S = V(F)$ on which G acts without fixed points:

$$\begin{array}{c} S \\ G\big\downarrow \\ X = S/G \end{array}$$

An obvious candidate for S is the Fermat quintic defined by the equation

$$x_0^5 + x_1^5 + x_2^5 + x_3^5 = 0.$$

The quotient $X = S/G$ is a smooth surface with ample canonical bundle because S has ample canonical bundle and the action is fixed point free. It is known as the quintic Godeaux surface (Godeaux was a Belgian mathematician and student of Enriques). Since $K_S = \mathcal{O}_S(1)$, it follows that

$$H^0(X, K_X) = H^0(S, K_S)^G = 0.$$

Moreover, $q(S) = 0$ by the Lefschetz hyperplane section theorem, and thus we also have $q(X) = 0$.

THEOREM 16 (Voisin [34]). *The surface* $X = S/G$ *has* $\mathrm{CH}_0(X) = \mathbb{Z}$.

This result was previously known for particular cases, such as the Fermat surface itself (Inose–Mizukami [18]). Also Kimura [20] proves it when X is rationally dominated by a product of curves (this also works for the Fermat surface).

Proof. We know by [26] that $\mathrm{CH}_0(X)$ has no torsion, since $\mathrm{Alb}\, X = 0$. The statement of the theorem is thus equivalent (by taking the pullback $q^* : \mathrm{CH}_0(X) \to \mathrm{CH}_0(S)$ by the quotient map $q : S \to X$, which satisfies $q_* \circ q^* = 5\,\mathrm{Id}_{\mathrm{CH}_0(X)}$) to the assertion

$$\mathrm{CH}_0(S)^G = \mathbb{Z}.$$

The group $\widetilde{\mathrm{CH}}_0(S)^G$ of invariant cycles of degree 0 is generated by 0-cycles of the form $G[x] - G[y]$ for pairs of points $x, y \in S$, where $G[x]$ denotes the orbit $\sum_{i=0}^4 [\zeta^i x]$, a 0-cycle of degree 5. So we need to show that $G[x] = G[y]$ in $\mathrm{CH}_0(S)$.

One can check that $\dim H^0(\mathbb{P}^3, \mathcal{O}(5))^G = 12$, so for any two points $x, y \in S$ there is an invariant polynomial F' of degree 5 that vanishes at both x and y, and (by the Bertini theorem) that the curve

$$C\colon\ F = F' = 0$$

is smooth. Let $j\colon C \hookrightarrow S$ be the inclusion. It is sufficient to show that

$$j_*^G\colon\ \widetilde{CH}_0(C)^G \longrightarrow CH_0(S)^G$$

is zero, since $G[x] - G[y]$ belongs to $\operatorname{Im} j_*^G$ for $x, y \in C$. Recall that $\widetilde{CH}_0 = CH_0^0$ denotes 0-cycles of degree 0, so

$$\widetilde{CH}_0(C) = \mathrm{Jac}(C) = \mathrm{Pic}^0(C).$$

Now consider the pencil of G-invariant surfaces $\{S_t : t \in \mathbb{P}^1\}$, where

$$S_t = V(F + tF').$$

One has $H^{2,0}(S_t)^G = 0$ since each S_t is defined by a G-invariant polynomial of degree 5, so we have a Hodge substructure

$$H^2(S_t, \mathbb{Q})^G \subset H^2(S_t, \mathbb{Q})$$

in which the left-hand side is generated by classes of G-invariant divisors on S_t (of degree 0, because we are dealing with primitive cohomology). This indeed follows from the fact that $H^{2,0}(S_t)^G = 0$ and from the Lefschetz theorem on $(1,1)$-classes.

It now follows that there exists a branched covering

$$r\colon D \to \mathbb{P}^1,$$

where $d \in D$ parameterizes pairs (S_t, L_d) with $t = r(d)$ and L_d is a G-invariant degree 0 line bundle on S_t, and satisfying the following property. Consider the family $p\colon \mathcal{S} \to \mathbb{P}^1$, with fibre $p^{-1}(t) = S_t$. *The morphism of local systems*

$$\begin{array}{ccc} R^0 r_* \mathbb{Q} & \longrightarrow & R^2 p_* \mathbb{Q}_0^G \\ 1_d & \mapsto & c_1(L_d) \end{array}$$

is surjective (at least on the Zariski open set where both r and p are submersive).

As we know that the invariant rational cohomology of degree 2 of S_t is generated by classes of G-invariant curves in S_t, this is proved using a Hilbert scheme argument (for curves in fibers S_t), together with the fact that $R^2 p_* \mathbb{Q}_0^G$ is finitely generated over $\mathbb{Q}$.

Let j_t denote the inclusion $C \hookrightarrow S_t$, the correspondence $d = (S_t, L) \mapsto j_t^* L$ determines a morphism ψ, i.e.

$$\begin{array}{ccc} \mathrm{Jac}(D) & \xrightarrow{\psi} & \mathrm{Jac}(C)^G \\ d = (S_t, L) & \mapsto & j_t^* L. \end{array}$$

The main point of the proof, for which we refer to [34] or [38, 11.1.4], is the following:

LEMMA 4 ([38, Lemma 11.17]). $\psi(D)$ *generates* $\mathrm{Jac}(C)^G$ *as a group modulo the images* $j_{t_i}^* \,\mathrm{Pic}(\widetilde{S_{t_i}})_0^G$ *of the Picard groups of the desingularizations of the singular fibers in the pencil.*

To conclude the proof of Theorem 16, recall that we want to show that

$$j_*: \ \mathrm{Jac}(C)^G \longrightarrow \mathrm{CH}_0(S)$$

is zero. We know by Lemma 4 that the left-hand side is generated by the image of $\mathrm{Pic}(D)$ in $\mathrm{Jac}(C)^G$. Therefore, we must show that every element of the set

$$\left\{ j_t^* L : L \in \mathrm{Pic}(S_t)_0^G = \mathrm{CH}_1(S_t)_0^G \right\},$$

maps to zero in $\mathrm{CH}_0(S)$. (The subscript 0 indicates classes homologous, and therefore rationally equivalent, to zero in $\mathbb{P}^3$.) With reference to the diagram

$$\begin{array}{ccccccc} \mathbb{P}^3 & \overset{k}{\hookleftarrow} & S & & S_t & \overset{k_t}{\hookrightarrow} & \mathbb{P}^3 \\ & & & j \nwarrow \quad \nearrow j_t & & & \\ & & & C & & & \end{array}$$

we have the following equality in the Chow group $\widetilde{\mathrm{CH}}_0(S)$ (cf. [13]):

$$j_*(j_t^* L) = k^*(k_{t*} L).$$

But the right-hand side is zero because $k_{t*}L$, being homologous to 0, vanishes in $\mathrm{CH}_1(\mathbb{P}^3)$. It follows that $j_*(j_t^* L) = 0$ in $\mathrm{CH}_0(S)$, which finishes the proof. □

The proof above uses an ad hoc argument. The Bloch conjecture is still open for fake $\mathbb{P}^2$'s, which are also surfaces of general type with $q = p_g = 0$.

EXAMPLE 3 (The Barlow surface). R. Barlow constructed an example of a simply-connected surface of general type with geometric genus $p_g = 0$ and trivial Chow group $\mathrm{CH}_0 = \mathbb{Z}$ (see [1, 2]).

Let $K = \mathbb{Q}(\sqrt{23})$, and let Γ be the principal congruence-subgroup in the group $SL_2(\mathcal{O}_K)$, and $\Gamma(2)$ be the principal congruence-subgroup of level 2. Then $\mathbb{F}_4 = \mathcal{O}_K / 2\mathcal{O}_K$ is a field of 4 elements, and $\Gamma/\Gamma(2) \cong SL_2(\mathbb{F}_4)$. Factoring $SL_2(\mathbb{F}_4)$ by its center, we see that $SL_2(\mathbb{F}_4)$ is the binary icosahedral group, i.e., a central extension

$$1 \to \mathbb{Z}/2 \to SL_2(\mathbb{F}_4) \to A_5 \to 1.$$

Let $S = H^+/\Gamma$, where H^+ is the upper half-plane with respect to the fractional-linear action of Γ corresponding to two complex embeddings of the field K. Then S is called the Hilbert modular surface (cf. Van der Geer's book [31]), and has a (singular) compactification $Y = \widehat{S}$. The surface Y has an action of the group $\Gamma/\Gamma(2) \cong A_5$. One can show that $p_g(\widetilde{Y}) = 4$ (where $\widetilde{Y}$ is the resolution of singularities of Y, easily constructed), and the canonical map $Y \to \mathbb{P}^3$ factors as

$$\widetilde{Y} \to Y \to Q \to \mathbb{P}^3,$$

where $Q = Y/(\sigma)$ and σ is the central element in $SL_2(\mathbb{F}_4)$. Using modular forms, one can show that the image of Q in $\mathbb{P}^3$ has model in $\mathbb{P}^4$ given by the equations

$$n_1 = 0, \qquad 4n_5 - 5n_2 n_3 = 0,$$

where $n_i = \sum x^i$ is the Newton's i-th symmetric function of the coordinates in $\mathbb{P}^4$. The symmetry of this equation reflects the A_5 action on Y and Q.

There is a "twisted" embedding of D_5 into the binary icosahedral group $\tilde{A}_5$, and the factor $B = Y/D_5$ (or its smooth model) is the Barlow surface:

$$\begin{array}{ccc} & Y & \\ {\scriptstyle \mathbb{Z}/2}\swarrow & & \searrow{\scriptstyle D_5} \\ Q & & B \end{array}$$

One can show that $p_g(B') = 0 = \pi_1(B)$ (a computation of O. Shvartsman.). More interestingly, one can prove using the same method as Inose and Mizukami that

$$\widetilde{CH}_0(B) = 0.$$

Barlow's proof proceeds as follows. The surface B is the quotient $B = Y/D_5$, and thus $\widetilde{CH}_0(B) \otimes \mathbb{Q} \cong (\widetilde{CH}_0(Y) \otimes \mathbb{Q})^{D_5}$ [13]; we thus need to study the action of $G = D_5$ on $\widetilde{CH}_0(Y) \otimes \mathbb{Q}$. There is a natural map

$$a\colon\ \mathbb{Q}[G] \to \operatorname{End} \widetilde{CH}_0(Y)_{\mathbb{Q}},$$

and the vanishing of $\widetilde{CH}_0(B)_{\mathbb{Q}}$ is clearly equivalent to the statement $a(z(G)) = 0$, where $z(G) = \sum_{g \in G} g$. By studying the ideal generated by $z(D_5)$ in $\mathbb{Q}[G]$, Barlow proves that there are two other subgroups G_1 and G_2 in the group $\Gamma/\Gamma(2)$ acting on Y such that the ideal generated by $z(G)$ is contained in the ideals generated by the elements $z(G_1)$ and $z(G_2)$. But then the surfaces Y/G_1 and Y/G_2 are special (not of general type), and the Chow groups can be computed in a straightforward way, and turn out to be trivial.

5. Further topics

Let us summarize material from the last two sections, before introducing some more advanced research material.

The generalized Hodge conjecture (Conjecture 4). This states that, given a Hodge substructure L of $H^k(X, \mathbb{Q})$ of coniveau r, it is supported on some closed algebraic set of codimension at least r (meaning the classes vanish on $X \setminus Y$).

Projective hypersurfaces. Given a hypersurface X of degree d in $\mathbb{P}^n$, the Hodge structure $L = H^{n-1}(X)_0$ has coniveau 1 if and only if X is a Fano hypersurface. We gave two proofs of the generalized Hodge conjecture. One using lines as in Subsection 3.3. The second using Theorem 13 saying that any X with $CH_0(X) = \mathbb{Z}$ has $h^{k,0} = 0$ for $k > 0$, and satisfies the generalized Hodge conjecture in coniveau 1.

The Bloch conjecture. This states conversely that $H^{k,0} = 0$ for all $k > 0$ implies that $CH_0(X) = \mathbb{Z}$. We focussed on surfaces, giving a proof for any quintic Godeaux surface.

In Section 5.1, we shall give an alternative approach to the Bloch conjecture for surfaces, which works for surfaces with $p_g = q = 0$ that are rationally dominated by curves. We shall begin by stating the nilpotence conjecture that says that correspondences in $X \times X$ homologous to zero are nilpotent for the composition of correspondences in $CH(X \times X)_{\mathbb{Q}}$.

We shall prove the nilpotence conjecture for cycles algebraically equivalent to zero. We discuss a kind of converse to the Mumford theorem, relying on validity of the generalized Hodge conjecture for coniveau 1 and the nilpotence conjecture.

Section 5.3 will be devoted to explaining a strategy to attack the generalized Hodge conjecture for coniveau 2 hypersurfaces, which is a wide-open problem.

5.1. Nilpotence conjecture and Kimura's theorem

Let X be smooth projective, and $\Gamma \subset X \times X$ a correspondence. Given a second such "self" correspondence Γ', we can construct the composition $\Gamma' \circ \Gamma$ using the diagram

$$\begin{array}{ccccc} & & X \times X & & \\ & & \uparrow p_{13} & & \\ & & X \times X \times X & & \\ & p_{12} \swarrow & & \searrow p_{23} & \\ X \times X & & & & X \times X \end{array}$$

to set

$$\Gamma' \circ \Gamma := (p_{13})_*\big(p_{12}^*\Gamma \cdot p_{23}^*\Gamma'\big).$$

It is a fact that

$$(\Gamma' \circ \Gamma)_* = \Gamma'_* \circ \Gamma_*,$$

as maps $CH(X) \to CH(X)$.

The following essential conjecture is proved by Kimura to be a consequence of his "finite dimensionality conjecture", and remains open.

CONJECTURE 8 (Nilpotence conjecture). Suppose that $\Gamma \in CH(X \times X)$ is homologous to zero. Then there exists a positive integer N such that $\Gamma^{\circ N} = 0$ in the group $CH(X \times X)_{\mathbb{Q}}$.

The following result is proved independently by Voevodsky and Voisin.

THEOREM 17 (See [32, 35]). *The nilpotence conjecture* is *known to hold for cycles algebraically equivalent to* 0.

Proof. Let $\Gamma \in CH^d(X \times X)$, $d = \dim X$, be algebraically equivalent to zero. This means that there is a curve C that we may assume to be smooth, a zero cycle z in $CH_0(C)$ homologous to 0, and a correspondence $Z \in CH^d(C \times X \times X)$ such that

$$\Gamma = Z_*(z) \text{ in } CH^d(X \times X).$$

For any integer k, we can construct a correspondence $Z_k \in CH^d(C^k \times X \times X)$ using the composition of the cycles Z_t, $t \in C$. Namely, we define Z_k by the formula

$$Z_k(t_1, \ldots, t_k) = Z(t_1) \circ \ldots \circ Z(t_k), \quad t_1, \ldots, t_k \in C.$$

By definition, we get

$$\Gamma^{\circ k} = Z_{k*}(z^k),$$

where the product $z^k \in CH_0(C^k)$ is defined as $pr_1^* z \cdot \ldots \cdot pr_k^* z$.

The proof concludes with the following easy fact (see [38, Lemma 11.33]):

LEMMA 5. *For a zero cycle* z *homologous to* 0 *on a smooth curve* C*, the cycle* z^k *vanishes in* $CH_0(C^k)$ *for* k *large enough.*

$\square$

Returning to the general case, Kimura established the following result.

THEOREM 18 (Kimura's theorem). *If* X *is dominated by a product of curves then* X *satisfies Conjecture 8.*

The proof of this theorem is rather tricky, and would be out of place in here. We refer the reader to [20]. If X is a surface, it is in fact sufficient that X be *rationally* dominated by the product of two curves to conclude the validity of Conjecture 8.

We prove next the following beautiful application:

THEOREM 19. *Conjecture 8 implies Bloch's Conjecture 5 for surfaces with* $p_g = q = 0$. *In particular, Bloch's conjecture is valid for surfaces with* $p_g = q = 0$ *that are rationally dominated by curves.*

Proof. By Lefchetz's theorem, since $p_g = 0$, we know that

$$H^2(X, \mathbb{Z}) = \langle [C_i] \rangle$$

is generated by classes of curves. Since $q = 0$, the Künneth decomposition of the diagonal takes the following simple form:

$$[\Delta] \in (H^0 \otimes H^4) \oplus (H^2 \otimes H^2) \oplus (H^4 \otimes H^0),$$

and as $H^2(X,\mathbb{Q})$ is generated by the classes $[C_i]$, we may write

(34) $$[\Delta_X] \;=\; [X\times\{x\}] \;+\; \sum n_{ij}[C_i\times C_j] \;+[\{x\}\times X].$$

Consider the cycle

$$\Gamma \;=\; \Delta_X - (X\times\{x\}) - \sum n_{ij} C_i\times C_j - (\{x\}\times X)$$

that is a linear combination of *algebraic* cycles. Then by (34), $[\Gamma]=0$. The nilpotence conjecture implies that there is a positive integer N such that $\Gamma^{\circ N}=0$ in $CH_2(X\times X)_{\mathbb{Q}}$. Hence,

$$(\Gamma_*)^{\circ N} = (\Gamma^{\circ N})_*\colon\;\; CH_0(X)_{\mathbb{Q}} \to CH_0(X)_{\mathbb{Q}}$$

is zero. But Γ_* acts as the identity on the group $\widetilde{CH}_0(X)$ of cycles of degree 0, since Δ_X acts as the identity, but other terms act trivially. (For example, $\gamma = X\times\{x\}$ acts trivially on $\widetilde{CH}_0(X)$ because $\gamma_*(z)=(\deg z)x$.) Thus $\widetilde{CH}_0(X)=0$. □

REMARK 9. Looking more closely at the proof and introducing Murre–Chow's Künneth decomposition [23], one sees that the proof above would show as well that Conjecture 8 implies Bloch's conjecture 6 for surfaces with $p_g=0$.

5.2. A converse to Mumford's theorem

The next theorem provides a converse to the generalized Mumford Theorem 12, under the generalized Hodge conjecture and the nilpotence conjecture:

THEOREM 20. *Let X be a smooth projective variety with $h^{k,0}(X)=0$ for $k>0$. Assume that:*

(i) *The generalized Hodge conjecture holds for X in coniveau 1.*

(ii) *The Hodge conjecture is true for $Y\times X$ with* $\dim Y<\dim X$.

(iii) X *satisfies the nilpotence conjecture.*

Then $CH_0(X)=\mathbb{Z}$.

The list of hypotheses may seem over the top, but this is the natural generalization of Theorem 19.

Proof. Here is a sketch. We work again with the diagonal $\Delta_X\subset X\times X$ and its cycle class

$$[\Delta_X] \;\in\; (H^0(X,\mathbb{Q})\otimes H^{2n}(X,\mathbb{Q}))\oplus\cdots$$

We have

$$[\Delta_X]=[X\times\{x\}] \quad \text{mod} \; \bigoplus_{k>0} H^k\otimes H^{2n-k}.$$

The generalized Hodge conjecture implies that there exist Y of codimension 1 and a resolution $\tilde{\imath}\colon \widetilde{Y}\to Y\to X$, so that

$$\tilde{\imath}_*\colon H^{k-2}(\widetilde{Y}) \longrightarrow H^k(X).$$

is surjective, as previously explained. Hence

$$[\Delta_X \setminus (X \times \{x\}] \in \operatorname{Im}\Big((\widetilde{i}, \mathrm{id})_* \colon H^{2n-2}(\widetilde{Y} \times X) \to H^{2n}(X \times X)\Big).$$

On the right we have a Gysin map and thus a morphism of Hodge structures.

Semisimplicity gives an orthogonal decomposition of Hodge structures in the space $H^{2n-2}(\widetilde{Y} \times X)$, and this implies that

$$[\Delta_X - (X \times \{x\})] = (\widetilde{i}, \mathrm{id})_* \beta, \tag{35}$$

where β is a Hodge class on $\widetilde{Y} \times X$. We now finish the argument as before: the Hodge conjecture on $\widetilde{Y} \times X$ implies that β equals a class $[Z]$ with Z a cycle in $\widetilde{Y} \times X$. Put

$$\Gamma = \Delta_X - (X \times \{x\}) - (\widetilde{i}, \mathrm{id})_* Z,$$

so that $[\Gamma] = 0$ by (35). It follows that Γ_* is nilpotent, yet Γ_* acts as the identity on $\widetilde{CH}_0(X)$, since Z_* is zero on $CH_0(X)$ (as $Z \subset Y \times X$ with $Y \subsetneq X$) and for $z \in CH_0(X)$, $(X \times \{x\})_*(z) = (\deg z)x$ in $CH_0(X)$. Therefore $CH_0(X) = \mathbb{Z}$. □

5.3. Coniveau 2 hypersurfaces

This concluding lecture follows closely [40]. The generalized Hodge conjecture for smooth complex projective varieties X has been already stated as Conjecture 4. Given a Hodge substructure $L \subset H^k(X, \mathbb{Q})$ of coniveau r, there should be a codimension r closed algebraic subset $Y \subset X$ such that L vanishes under restriction to $X \setminus Y$. There is the following more precise formulation, which is equivalent if one knows the Lefschetz standard conjecture:

CONJECTURE 9. Given a Hodge substructure $L \subset H^k(X, \mathbb{Q})$ of coniveau r, there should exist a smooth projective variety Z of dimension $n - 2r$, where $n = \dim X$, and a cycle $\Gamma \subset Z \times X$ of dimension $n - r$, such that the image of the morphism of Hodge structures

$$\Gamma_* : H^{k-2r}(Z, \mathbb{Q}) \to H^k(X, \mathbb{Q}) \tag{36}$$

contains L.

This formulation is interesting as it shows that solving the generalized Hodge conjecture for coniveau r Hodge substructures has to do with the study of r-cycles on X.

The precise relationship between Conjectures 9 and 4 is as follows: Conjecture 9 clearly implies Conjecture 4, since the image of Γ_* in (36) is supported on $pr_2(\operatorname{Supp} \Gamma)$ which has dimension $\leqslant n - r$. In the other direction, assume Conjecture 4 holds for L. As we saw there, there is then a smooth projective variety $\widetilde{Y}$ of dimension $n - r$, and a morphism $j : \widetilde{Y} \to X$, such that L is contained in $\operatorname{Im}(j_* : H^{k-2r}(\widetilde{Y}, \mathbb{Q}) \to H^k(X, \mathbb{Q}))$. The Lefschetz standard conjecture for $\widetilde{Y}$ implies now that there exists a smooth projective variety Z of dimension $n-2r$, and a cycle $\Gamma \subset Z \times Y$ of dimension $n-r$, such that the morphism of Hodge structures

$$\Gamma_* : H^{k-2r}(Z,\mathbb{Q}) \to H^{k-2r}(Y,\mathbb{Q}) \tag{37}$$

is surjective. The cycle $(Id, j)(\Gamma) \subset Z \times X$ then satisfies the conclusion of Conjecture 9.

We consider now coniveau 2 hypersurfaces X of degree d in $\mathbb{P}^n$. The Griffiths condition (Theorem 9) is $n \geqslant 2d$. In view of the above reasoning, it is tempting to solve the generalized Hodge conjecture for them by constructing interesting 2-cycles. If we look at the first proof we gave for coniveau 1 (which involved lines in X), it is even tempting to look at planes contained in X. This however does not work at all, by a simple dimension count which shows that a general hypersurface of degree d in $\mathbb{P}^n$ contains a plane only if $3(n-2) \geqslant \frac{(d+1)(d+2)}{2}$. As the right hand side is quadratic in d, this certainly does not help to deal with the general situation $n \geqslant 2d$. Note however that in the range

$$3(n-2)-(n-5) \geqslant \frac{(d+1)(d+2)}{2},$$

planes in X sweep out a codimension 2 subvariety of X, and one can use them to conclude that the generalized Hodge conjecture holds for coniveau 2, and also to prove that $CH_1(X)_{\mathbb{Q}} = \mathbb{Q}$ (cf. [24], [38, 9.3.4]).

The paper [40] proposes an alternative way to attack the generalized Hodge conjecture for coniveau 2 hypersurfaces, and shows that it would be a consequence of Conjecture 10 concerning cones of effective cycles.

5.4. Big classes

Let Y be a smooth projective variety. For any $k \leqslant \dim Y$, one can consider the real vector space $A^{2k}(Y) \subset H^{2k}(Y,\mathbb{R})$ generated by classes of algebraic cycles of codimension k. It contains a convex cone $\mathrm{Eff}^{2k}(Y)$ which is the convex cone generated by classes of effective cycles $\sum_i n_i Z_i$, with $n_i \geqslant 0$.

DEFINITION 8. *A codimension* k *cycle class* $[Z] \in A^{2k}(Y)$ *is said to be* big *if it belongs to the interior of the effective cone* $\mathrm{Eff}^{2k}(Y)$.

Let now $Z \subset Y$ be a smooth subvariety of codimension k. We will say that Z is *very movable* if the following holds: for a generic point $y \in Y$ and a generic vector subspace $W \subset T_{Y,y}$ of codimension k, there is a deformation Z' of Z in Y which passes through y with tangent space $T_{Z',y} = W$. The following conjecture is formulated in [40]:

CONJECTURE 10. The class $[Z] \in A^{2k}(Y)$ of a very movable subvariety $Z \subset Y$ is big.

5.5. Application to coniveau 2 hypersurfaces

Let $X \subset \mathbb{P}^n$ be a generic smooth degree d hypersurface. For a generic polynomial $g \in H^0(X, \mathcal{O}_X(n-d-1))$, we get a smooth hypersurface $X_g \subset X$, and the corresponding subvariety of lines $F_g \subset F$, where $F_g \subset \mathrm{Grass}(2, n+1)$ is the variety of lines contained in X_g and $F \subset \mathrm{Grass}(2, n+1)$ is the variety of lines contained in X. We have $\dim F = 2n-d-3$ and $\dim F_g = n-3$. The class $[F_g] \in A^{(n-d)}(F)$ is easy to compute as this is $c_{n-d}(S^{n-d-1}\mathcal{E})$, where $\mathcal{E}$ is the (restriction to F of the) rank 2 bundle on $\mathrm{Grass}(2, n+1)$ with fiber $H^0(\mathcal{O}_\Delta(1))$ over the point parameterizing the line Δ. This however does not tell us whether this class is big or not, since the class $c_{n-d}(S^{n-d-1}\mathcal{E})$ is not big on the Grassmannian.

Recall the incidence diagram

$$\begin{array}{ccc} P & \xrightarrow{q} & X \\ \downarrow{\scriptstyle p} & & \\ F & & \end{array} \tag{38}$$

inducing for $n \geqslant d$ the injective morphism of Hodge structures

$$p_* \circ q^* : H^{n-1}(X, \mathbb{Q})_0 \to H^{n-3}(F, \mathbb{Q}).$$

LEMMA 6. (i) *For any* $\alpha \in H^{n-1}(X, \mathbb{Q})_0$, $\eta := p_* \circ q^* \alpha$ *satisfies* $\eta_{|F_g} = 0$ *in* $H^{n-3}(F_g, \mathbb{Q})$.
(ii) *For any* $\alpha \in H^{n-1}(X, \mathbb{Q})_0$, $\eta := p_* \circ q^* \alpha$ *is primitive with respect to the Plücker polarization.*

Proof. Indeed, consider the incidence diagram for X_g:

$$\begin{array}{ccc} P_g & \xrightarrow{q_g} & X_g \\ \downarrow{\scriptstyle p_g} & & \\ F_g & & \end{array} \tag{39}$$

One immediately concludes that $\eta_{|F_g} = p_g * \circ q_g^*(\alpha_{|X_g})$. On the other hand, α being primitive, it vanishes under restriction to X_g. This proves (i).

Statement (ii) is proved in [29]. □

The main observation made in [40] concerning the geometry of $F_g \subset F$ is the following:

PROPOSITION 4. *Assume that* $n \geqslant 2d$. *Then the subvarieties* $F_g \subset F$ *are very movable.*

We prove now the following:

THEOREM 21. *Assume $n \geqslant 2d$ and Conjecture 10 is satisfied by $F_g \subset F$, that is the class $[F_g]$ is big. Then the generalized Hodge conjecture for coniveau 2 is satisfied by coniveau $\geqslant 2$ hypersurfaces, that is, their primitive cohomology vanishes away from a codimension 2 closed algebraic subset.*

Proof. By assumption $[F_g]$ is big, which is equivalent to the fact that it can be written

$$[F_G] = ml^{n-d} + [E], \tag{40}$$

where $m > 0$ and $E = \sum_i m_i E_i$, with $m_i > 0$ (m and m_i are real numbers). By Lemma 6, for $\alpha \in H^{n-1}(X)_{\text{prim}}$, $\eta = p_* q^* \alpha \in H^{n-3}(F)$ is primitive with respect to l and furthermore vanishes on F_G, with $\dim F_G = n-3$. Let us assume that $\alpha \in H^{p,q}(X)_{\text{prim}}$ and integrate $(-1)^{\frac{k(k-1)}{2}} i^{p-q} \eta \cup \overline{\eta}$, $k = p+q-2 = n-3$, over both sides in (40). We thus get

$$0 = m \int_F (-1)^{\frac{k(k-1)}{2}} i^{p-q} l^{n-\sum_i d_i} \cup \eta \cup \overline{\eta} + \int_E (-1)^{\frac{k(k-1)}{2}} i^{p-q} \eta \cup \overline{\eta}.$$

As η is primitive, and non-zero if α is non-zero, by the second Hodge–Riemann bilinear relations (cf. [36, 6.3.2]), we have $\int_F (-1)^{\frac{k(k-1)}{2}} i^{p-q} l^{n-\sum_i d_i} \cup \eta \cup \overline{\eta} > 0$. It thus follows that

$$\int_E (-1)^{\frac{k(k-1)}{2}} i^{p-q} \eta \cup \overline{\eta} < 0.$$

Let $\widetilde{E} = \sqcup \widetilde{E_j}$ be a desingularization of the support of $E = \sum_j m_j E_j$, $m_j > 0$. We thus have $\sum_j m_j \int_{\widetilde{E_j}} (-1)^{\frac{k(k-1)}{2}} i^{p-q} \eta \cup \overline{\eta} < 0$, and it follows that there exists at least one E_j such that

$$\int_{\widetilde{E_j}} (-1)^{\frac{k(k-1)}{2}} i^{p-q} \eta \cup \overline{\eta} < 0. \tag{41}$$

By the second Hodge–Riemann bilinear relations, the inequality (41) implies that $\eta_{|\widetilde{E_j}}$ is not primitive with respect to any polarization, where $\widetilde{E_j}$ is a desingularization of E_j. In particular, $\eta_{|H_j} \neq 0$, where H_j is an ample divisor on $\widetilde{E_j}$.

In conclusion, we have proved that the composed map

$$H^{n-r}(X)_{\text{prim}} \stackrel{p_* q^*}{\to} H^{n-r-2}(F) \to \bigoplus H^{n-r-2}(H_j)$$

is injective, where the second map is given by restriction. If we dualize this injectivity result using Poincaré duality, we conclude that $H^{n-1}(X, \mathbb{Q})_{\text{prim}}$ is supported on the $n-3$-dimensional variety $\cup_j q(P_j)$, where $P_j \to H_j$ is the restriction to H_j of the tautological $\mathbb{P}^1$-bundle on F. The result is proved. □

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Claire VOISIN
Institut de Mathématiques de Jussieu, Projet Topologie et Géométrie Algébriques
Case 247, 4 Place Jussieu, 75005 Paris, FRANCE
e-mail: voisin@math.jussieu.fr

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