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GLOBAL SPACE-TIME L^p-ESTIMATES FOR THE WAVE OPERATOR ON L^2

Abstract. We prove and disprove several estimates of the following type:

$$\|u\|_* \leq a\|\square u\|_2 + b\|u\|_2$$

where $\square = \frac{\partial^2}{\partial t^2} - \triangle_x$ is the usual wave operator defined on $L^2(\mathbb{R}\times\mathbb{R}^n)$, $n \geq 2$ and where $\|\cdot\|_*$ represents different norms yet to be determined.

1. Introduction

Many authors have worked on Strichartz estimates for the wave equation since the paper of Strichartz [10]. Some of the important papers are [11, 4, 3, 8].

In this work we examine a slightly different type of these estimates. The question asked here is: To what L^p-space (or another space) does a function u belong to if u and $\square u$ already belong to $L^2(\mathbb{R}\times\mathbb{R}^n)$ where $n \geq 2$? This question actually constitutes the follow-up of a work done by the author (see [6]). It was proved in [6] (among others) that the following two inequalities *do* hold:

$$\|u\|_{L^p(\mathbb{R}^2)} \leq a\|\square u\|_{L^2(\mathbb{R}^2)} + b\|u\|_{L^2(\mathbb{R}^2)} \text{ where } 2 \leq p < \infty, \tag{1}$$

$$\operatorname*{ess\,sup}_{t\in\mathbb{R}} \|u(\cdot,t)\|_{L^r(\mathbb{R}^n)} \leq a\|\square u\|_{L^2(\mathbb{R}^{n+1})} + b\|u\|_{L^2(\mathbb{R}^{n+1})}, \tag{2}$$

where $2 < r < \frac{2n}{n-1}$ while the following one *does not* hold:

$$\|u\|_{L^\infty(\mathbb{R}^2)} \leq a\|\square u\|_{L^2(\mathbb{R}^2)} + b\|u\|_{L^2(\mathbb{R}^2)}. \tag{3}$$

Note that Estimate (1) was a consequence of the following important inequality (which also appeared in [6]):

$$\|u\|_{BMO(\mathbb{R}^2)} \leq a\|\square u\|_2 + b\|u\|_2,$$

for some constants a and b.

In this paper, we prove that in $\mathbb{R}\times\mathbb{R}^n$ and for $n \geq 2$, the estimate

$$\|u\|_{L^p(\mathbb{R}^n\times\mathbb{R})} \leq a\|\square u\|_{L^2(\mathbb{R}^n\times\mathbb{R})} + b\|u\|_{L^2(\mathbb{R}^n\times\mathbb{R})}$$

holds if and only if $p \leq \frac{2n+2}{n-1}$.

In the end, we answer some questions left open in [6].

It is worth mentioning that the author has a similar work (see [7]) done for the time dependent Schrödinger operator. See also [1] for some related work for the Airy operator. For literature on PDEs and Fourier transforms, see [2] and [5].

2. Main results

Here is the first main result in the paper.

THEOREM 1. Let $n \geq 2$ and let $p = \frac{2n+2}{n-1}$. Then for all $a > 0$, there exists $b > 0$ such that

$$\|u\|_{L^p(\mathbb{R}\times\mathbb{R}^n)} \leq a\|\Box u\|_{L^2(\mathbb{R}\times\mathbb{R}^n)} + b\|u\|_{L^2(\mathbb{R}\times\mathbb{R}^n)} \tag{4}$$

for all $u \in L^2(\mathbb{R}\times\mathbb{R}^n)$ such that $\Box u \in L^2(\mathbb{R}\times\mathbb{R}^n)$.

Proof. Let the x-Fourier transform of u, i.e. $\hat{u}(t,\xi)$, be supported in the region $|\xi| \leq 2$. Then

$$\|\triangle u\|_{L^2(\mathbb{R}^{n+1})} \lesssim \|u\|_{L^2(\mathbb{R}^{n+1})}$$

and the right hand side will be equivalent to

$$\|\partial_t^2 u\|_{L^2(\mathbb{R}^{n+1})} + \|u\|_{L^2(\mathbb{R}^{n+1})}$$

which controls $\|u\|_{L^p_t L^2_x(\mathbb{R}^{n+1})}$ thanks to the Sobolev embedding theorem (in the t variable). Hence by the frequency localization,

$$\|u(t,\cdot)\|_{L^p_x(\mathbb{R}^n)} \lesssim \|u(t,\cdot)\|_{L^2_x(\mathbb{R}^n)}.$$

Now, consider the case where $\hat{u}(t,\xi)$ is supported in the region $|\xi| \geq 1$. Recall the Strichartz estimate for $\Box u = 0$, that is

$$\|u\|_{L^p(\mathbb{R}^{n+1})} \lesssim \|\partial_t u(0,\cdot)\|_{H^{-\frac{1}{2}}} + \|u(0,\cdot)\|_{H^{\frac{1}{2}}}.$$

One can hence choose any value of s instead of 0 for the initial data time and obviously restrict the left to a finite interval. The Duhamel's principle (in the case $\Box u \neq 0$) then implies that

$$\begin{aligned}\|u\|_{L^p([0,T]\times\mathbb{R}^n)} &\lesssim \|\Box u\|_{L^1_t H^{-\frac{1}{2}}([0,T]\times\mathbb{R}^n)} + \inf_{0\leq s\leq T}(\|\partial_t u(s,\cdot)\|_{H^{-\frac{1}{2}}} + \|u(s,\cdot)\|_{H^{\frac{1}{2}}})\\ &\lesssim T^{\frac{1}{2}}\|\Box u\|_{L^2_t H^{-\frac{1}{2}}([0,T]\times\mathbb{R}^n)} + T^{-\frac{1}{2}}(\|\partial_t u(s)\|_{L^2_t H^{-\frac{1}{2}}([0,T]\times\mathbb{R}^n)} + \|u\|_{L^2_t H^{\frac{1}{2}}([0,T]\times\mathbb{R}^n)})\end{aligned}$$

Applying a Littlewood–Paley decomposition reduces the work to the case $\hat{u}(t,\xi)$ is supported where $|\xi| \in [\lambda, 3\lambda]$, with $\lambda \geq 1$. Taking $T = \lambda$ leads to

$$\|u\|_{L^p([0,\lambda]\times\mathbb{R}^n)} \lesssim \|\Box u\|_{L^2([0,\lambda]\times\mathbb{R}^n)} + \lambda^{-1}\|\partial_t u\|_{L^2([0,\lambda]\times\mathbb{R}^n)} + \|u\|_{L^2([0,\lambda]\times\mathbb{R}^n)}.$$

Summing over a disjoint decomposition of $\mathbb{R}$ into intervals of length λ and remembering that $p \geq 2$ yield

$$\|u\|_{L^p(\mathbb{R}^{n+1})} \lesssim \|\Box u\|_{L^2(\mathbb{R}^{n+1})} + \lambda^{-1}\|\partial_t u\|_{L^2(\mathbb{R}^{n+1})} + \|u\|_{L^2(\mathbb{R}^{n+1})}.$$

The proof will be complete once we show that for $\hat{u}(t,\xi)$ supported where $|\xi| \in [\lambda, 3\lambda]$ we may bound

$$\lambda^{-1}\|\partial_t u\|_{L^2(\mathbb{R}^{n+1})} \lesssim \|\Box u\|_{L^2(\mathbb{R}^{n+1})} + \|u\|_{L^2(\mathbb{R}^{n+1})}.$$

This will be proved once we can show that

$$\lambda^{-2}\eta^2 \lesssim (\eta^2 - |\xi|^2)^2 + 1,\ |\xi| \approx \lambda \geq 1$$

by means of Fourier transforms. But, the previous inequality can easily be verified in the cases $\eta \leq 4\lambda$ and $\eta \geq 4\lambda$.

The proof is complete. $\square$

Another result which generalizes one which appeared in [6] is the following:

PROPOSITION 1. For all $a > 0$, there exists $b > 0$ such that

$$\sum_{k=-\infty}^{\infty} \operatorname{ess}\sup_{k \leq t \leq k+1} \|u(.,.,t)\|^2_{L^2(\mathbb{R}^n)} \leq a\|\Box u\|^2_{L^2(\mathbb{R}\times\mathbb{R}^n)} + b\|u\|^2_{L^2(\mathbb{R}\times\mathbb{R}^n)}$$

for all $u \in L^2(\mathbb{R}\times\mathbb{R}^n)$ such that $\Box u \in L^2(\mathbb{R}\times\mathbb{R}^n)$.

Proof. The proof is based on energy estimates. Let $u \in L^2(\mathbb{R}\times\mathbb{R}^n)$ be such that $\Box u \in L^2(\mathbb{R}\times\mathbb{R}^n)$.

For any $s \in [a,b]$, we have

$$\operatorname{ess}\sup_{a\leq t\leq b} \|u(\cdot,t)\|_{L^2(\mathbb{R}^n)} \leq \int_a^b \|\Box u(\cdot,t)\|_{L^2(\mathbb{R}^n)} dt + \|u(\cdot,s)\|_{L^2(\mathbb{R}^n)}.$$

Hence

$$\operatorname{ess}\sup_{a\leq t\leq b} \|u(\cdot,t)\|_{L^2(\mathbb{R}^n)} \leq \int_a^b \|\Box u(\cdot,t)\|_{L^2(\mathbb{R}^n)} dt + \frac{1}{b-a}\int_a^b \|u(\cdot,t)\|_{L^2(\mathbb{R}^n)} dt.$$

Thus

$$\operatorname{ess}\sup_{a\leq t\leq b} \|u(\cdot,t)\|_{L^2(\mathbb{R}^n)} \leq (b-a)^{\frac{1}{2}}\|\Box u\|_{L^2([a,b]\times\mathbb{R}^n)} + (b-a)^{-\frac{1}{2}}\|u\|_{L^2([a,b]\times\mathbb{R}^n)}.$$

Finally, we easily get:

$$\operatorname{ess}\sup_{k\leq t\leq k+1} \|u(\cdot,t)\|_{L^2(\mathbb{R}^n)} \leq c^{\frac{1}{2}}\|\Box u\|_{L^2([a,b]\times\mathbb{R}^n)} + c^{-\frac{1}{2}}\|u\|_{L^2([a,b]\times\mathbb{R}^n)}.$$

The square-summation over k yields the desired result. The proof is over. $\square$

3. Counterexamples

The next result tells us that $p = \frac{2n+2}{n-1}$ is best possible in Theorem 1.

PROPOSITION 2. Let $n \geq 2$ and let $p > \frac{2n+2}{n-1}$. Then there are no constants a and b such that the estimate of Theorem 1 holds for all $u \in L^2(\mathbb{R}\times\mathbb{R}^n)$ such that $\Box u \in L^2(\mathbb{R}\times\mathbb{R}^n)$.

Proof. There are two different ways to establish this proposition. One can be found in [1] and one in [7]. The operators considered in the quoted papers are different from the wave operator but the method can be adapted to this case. $\square$

We finish this paper by answering an open question raised in [6] about the self-adjointness of $\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} + V$ where V is *positive* and belongs to $L^2_{loc}(\mathbb{R}^2)$. We then insisted on the positivity of V since it was already known back then that $\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} + V$ was not essentially self-adjoint for some V which was not positive.

The proof exploits the non essential self-adjointness of the one-dimensional Laplacian perturbed by some *negative* potential.

PROPOSITION 3. There exists a positive V belonging to $L^2_{loc}(\mathbb{R}^2)$ such that $\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} + V$ is not essentially self-adjoint on $C_0^\infty(\mathbb{R}^2)$.

Proof. The operator $-\frac{d^2}{dt^2} - t^4$ is not essentially self-adjoint on $C_0^\infty(\mathbb{R})$ (details can be found in [9]) meaning that

$$\left(-\frac{d^2}{dt^2} - t^4\right) f(t) = -if(t) \text{ or } \left(\frac{d^2}{dt^2} + t^4\right) f(t) = if(t)$$

has a non-zero solution in $L^2(\mathbb{R})$. Now the perturbed one-dimensional Laplacian by x^2 has as eigenvalue $\frac{1}{2}$, i.e. there is a non-zero $g \in L^2(\mathbb{R})$ such that

$$\left(-\frac{d^2}{dx^2} + x^2\right) g(x) = \frac{1}{2} g(x)$$

Now by adding up the last two displayed equations we get

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right) f(x)g(t) + (t^4 + x^2) f(x)g(t) = \left(i + \frac{1}{2}\right) f(x)g(t). \tag{5}$$

Take $\varphi(x,t) = f(x)g(t)$. Since f,g are both in $L^2(\mathbb{R})$, φ will be in $L^2(\mathbb{R}^2)$ and Equation 5 will have a non-zero solution in $L^2(\mathbb{R}^2)$ and yet $V(x,t) = t^4 + x^2$ is *nonnegative* and it belongs to $\in L^2_{loc}(\mathbb{R}^2)$. $\square$

A conjecture

We ask the interested reader the following rather hard question which is true in the case of the Laplacian (see [9]) but it remains open for general operators (not known for the Wave and the time-dependent Schrödinger operators at least).

CONJECTURE. Let P be an essentially self-adjoint partial differential operator with domain $C_0^\infty(\mathbb{R}^n)$, $n \geq 2$. If there *do not* exist two positive constants a and b such that for all $C_0^\infty(\mathbb{R}^n)$

$$\|f\|_\infty \leq a\|Pf\|_2 + b\|f\|_2,$$

then $P+V$ is *not* essentially self-adjoint on $C_0^\infty(\mathbb{R}^n)$ for real-valued V belonging to $L^2(\mathbb{R}^n)$.

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