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A LOWER BOUND FOR THE SECOND SECTIONAL GEOMETRIC GENUS OF QUASI-POLARIZED MANIFOLDS AND ITS APPLICATIONS*

Abstract. In our previous papers, we investigated a lower bound for the second sectional geometric genus $g_2(X,L)$ of n-dimensional polarized manifolds (X,L) and by using these, we studied the dimension of global sections of K_X+tL with $t\geq 2$. In this paper, we consider the case where (X,L) is a quasi-polarized manifold. First we will prove $g_2(X,L)\geq h^1(\mathscr{O}_X)$ for the following cases: (a) $n=3$, $\kappa(X)=-\infty$ and $\kappa(K_X+L)\geq 0$. (b) $n\geq 3$ and $\kappa(X)\geq 0$. Moreover, by using this inequality, we will study $h^0(K_X+tL)$ for the case where (X,L) is a quasi-polarized 3-fold.

1. Introduction

Let X be a smooth projective variety of dimension n defined over the field of complex numbers and let L be a line bundle on X. Then (X,L) is called a *quasi-polarized* (resp. *polarized*) *manifold* if L is nef and big (resp. ample). In [14, 15, 16], we defined the ith sectional geometric genus $g_i(X,L)$ of (X,L) for any integer i with $0\leq i\leq n$, and we studied some properties of this invariant. In particular, we proved the inequality $g_2(X,L)\geq h^1(\mathscr{O}_X)$ if (X,L) is a polarized manifold with one of the following cases:

(a) $n=3$, $\kappa(X)=-\infty$ and $\kappa(K_X+L)\geq 0$ (see [16, Theorem 3.3.1 (2)]).

(b) $n\geq 3$ and $\kappa(X)\geq 0$ (see [15, Theorem 2.3.2]).

Using these results, we also studied the dimension of global sections of K_X+tL with $t\geq 2$ for polarized 3-folds (see [17, 18, 19]).

In this paper, we consider the case where (X,L) is a *quasi-polarized* manifold. This generalization is very important. When we investigate a polarized manifold (X,L), we sometimes need to take a birational morphism $\mu:\widetilde{X}\to X$. For example, if (X,L) is a polarized variety such that X has singularities, then, by taking a resolution $\mu:\widetilde{X}\to X$, $(\widetilde{X},\mu^*(L))$ is not a polarized manifold but a quasi-polarized manifold, and investigation of $(\widetilde{X},\mu^*(L))$ makes possible to find some properties of (X,L).

In this paper, first we will study a lower bound for the second sectional geometric genus $g_2(X,L)$ of quasi-polarized manifolds (X,L) for the following cases:

(a) $n=3$, $\kappa(X)=-\infty$ and $\kappa(K_X+L)\geq 0$ (Theorem 3).

(b) $n\geq 3$ and $\kappa(X)\geq 0$ (Corollary 1 and Theorem 6).

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In Section 4, by using these results, we will investigate the dimension of global sections of $K_X + tL$ with $t \geq 2$ for quasi-polarized 3-folds. Specifically, we obtain the following results concerning a quasi-polarized 3-fold (X, L):

(a) A classification of (X, L) with $h^0(K_X + 2L) = 0$ (Theorem 7 (a)).

(b) A classification of (X, L) with $h^0(K_X + 2L) = 1$ (Theorem 8 (a)).

(c) A classification of (X, L) with $h^0(K_X + 3L) = 0$ (Theorem 7 (b)).

(d) A classification of (X, L) with $h^0(K_X + 3L) = 1$ (Theorem 8 (b)).

(e) $h^0(K_X + tL) \geq \binom{t-1}{3}$ for $t \geq 4$ (Theorem 7 (c)).

(f) A classification of (X, L) with $h^0(K_X + tL) = \binom{t-1}{3}$ for some $t \geq 4$ (Theorem 8 (c)).

This paper is the revised version of [11]. After the first version of [11] had been completed, Höring's papers [21, 22] appeared. We note that Theorem 7 (a) is obtained from [21, 1.5 Theorem] and Proposition 2 (ii) below, which is obtained from [21]. But Theorems 7 (b), (c) and 8 in this paper are not in [21] and these will be useful when we study the dimension of the global sections of adjoint bundles for higher dimensional varieties.

We use standard notation in algebraic geometry.

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2. Preliminaries

LEMMA 1. *Let X and C be smooth projective varieties with $\dim X = n \geq 2$ and $\dim C = 1$, and let L be a nef and big line bundle on X. Assume that there exists a fiber space $f : X \to C$ such that $h^0(K_F + L_F) \neq 0$ for a general fiber F of f. Then $f_*(K_{X/C} + L)$ is ample.*

Proof. Note that there exists a natural number m such that $(mL)^n - n(mL)^{n-1}F > 0$. Then by [6, (4.1) Lemma], there exists a natural number k such that $\mathcal{O}_X(k(mL - F))$ has a nontrivial global section. Hence we have an injective map $\mathcal{O}_X(kF) \to \mathcal{O}(kmL)$. On the other hand, there exists a line bundle $\mathcal{N}$ on C such that $\mathcal{O}(kF) = f^*(\mathcal{N})$. Hence by [7, Corollary 1.9] we see that $f_*(K_{X/C} + L)$ is ample. $\square$

LEMMA 2. *Let X be a smooth projective variety of dimension $n \geq 2$ and let V be a normal projective variety of dimension $n \geq 2$ such that V has only $\mathbb{Q}$-factorial terminal singularities. Let $\pi : X \to V$ be a birational morphism such that $X \backslash \pi^{-1}(\mathrm{Sing}(V)) \cong V \backslash \mathrm{Sing}(V)$. Let E be a π-exceptional irreducible and reduced divisor on X, A a line bundle on X and $L_1, \dots, L_{n-2}$ line bundles on V. Then $EA(\pi^*(L_1)) \cdots (\pi^*(L_{n-2})) = 0$.*

Proof. First we assume that $L_1,\dots,L_{n-2}$ are very ample line bundles. Let H_i be a general member of $|L_i|$ for $1 \le i \le n-2$. Then we see that $EA\pi^*(L_1)\cdots\pi^*(L_{n-2}) = 0$ because $H_1\cap\cdots\cap H_{n-2}$ does not meet $\pi(E)$.

Next we assume that $L_1,\dots,L_{n-2}$ are line bundles in general. Then for each i we can show that there exist very ample line bundles B_i and C_i such that $L_i = B_i - C_i$. Therefore we get $EA\pi^*(L_1)\cdots\pi^*(L_{n-2}) = 0$ from above. □

PROPOSITION 1. *Let (X,L) be a quasi-polarized manifold with* $\dim X \le 2$. *Let m be a positive integer. Assume that $\kappa(K_X+mL)\ge 0$. Then $h^0(K_X+mL)>0$.*

Proof. We use the same argument as in the proof of [18, Theorem 2.8]. □

DEFINITION 1. *Let (X_1,L_1) and (X_2,L_2) be quasi-polarized varieties. Then (X_1,L_1) and (X_2,L_2) are said to be birationally equivalent if there is another variety G with birational morphisms $g_i : G \to X_i$ $(i=1,2)$ such that $g_1^*L_1 = g_2^*L_2$.*

Before we state Proposition 2, we note the following.

LEMMA 3. *Let (X,L) be a quasi-polarized manifold of dimension n. Then $K_X+(n-1)L$ is generically nef if and only if $K_X+(n-1)L$ is pseudoeffective.*

Proof. This is proved in [21, 1.2 Theorem]. □

PROPOSITION 2. *Let (X,L) be a quasi-polarized manifold of dimension n.*

(i) *If $K_X+(n-1)L$ is not pseudoeffective, then (X,L) satisfies one of the following:*

(i.1) $g(X,L)=\Delta(X,L)=0$. *Here $g(X,L)$ (resp. $\Delta(X,L)$) denotes the sectional genus (resp. the Δ-genus) of (X,L).*

(i.2) *(X,L) is birationally equivalent to a scroll over a smooth curve.*

(ii) *If $K_X+(n-1)L$ is pseudoeffective, there exists a quasi-polarized variety (V,H) which is birationally equivalent to (X,L) such that V is a normal projective variety with only $\mathbb{Q}$-factorial terminal singularities and $K_V+(n-1)H$ is nef.*

Proof. By Lemma 3 we get (i) from [22, Proposition 1.3]. Moreover we can also prove (ii) by the same argument as Step 1 of Case IV in the proof of [21, 1.2 Theorem]. The argument in [21] consists of running a Minimal Model Program with scaling, which was proved in [4]. □

PROPOSITION 3. *Let (X,L) be a quasi-polarized manifold of dimension n. Then:*

(i) $g(X,L)\ge 0$ *holds.*

(ii) *If $g(X,L)=0$, then $\Delta(X,L)=0$.*

(iii) *If* $g(X,L)=1$*, then there exists a quasi-polarized variety* (V,H) *which is birationally equivalent to* (X,L) *such that* (V,H) *is one of the following two types:*

(iii.1) V *is a normal projective variety with only Gorenstein* $\mathbb{Q}$*-factorial terminal singularities and* $\mathscr{O}(K_V+(n-1)H)=\mathscr{O}_V$ *holds.*

(iii.2) *A scroll over a smooth elliptic curve.*

Proof. For the proof of (i) and (ii), see [22, Theorems 1.1 and 1.2]. Here we prove (iii).

Assume that $K_X+(n-1)L$ is pseudoeffective. Then by Proposition 2 (ii) we see that there exist a quasi-polarized variety (V,H), a smooth projective variety M and birational morphisms $\mu_1 : M \to X$ and $\mu_2 : M \to V$ such that V is a normal projective variety with only $\mathbb{Q}$-factorial terminal singularities, $\mu_1^*(L)=\mu_2^*(H)$ and $K_V+(n-1)H$ is nef. Since $g(V,H)=g(X,L)=1$, we have $(K_V+(n-1)H)(H)^{n-1}=0$. By the base point free theorem, there exists a natural number m such that $m(K_V+(n-1)H)$ is free. So we get $\mathscr{O}(m(K_V+(n-1)H))=\mathscr{O}_V$. Namely $K_V+(n-1)H$ is numerically trivial. By the same argument as in the proof of [8, (3.9) Corollary], we can prove that V is Gorenstein and $\mathscr{O}(K_V+(n-1)H)=\mathscr{O}_V$.

Next we consider the case where $K_X+(n-1)L$ is not pseudoeffective. Then by Proposition 2 (i) we see that (X,L) is birationally equivalent to a scroll over a smooth curve N because $g(X,L)=1$. In this case, we can easily show that $g(X,L)=g(N)$. Hence N is a smooth elliptic curve. This completes the proof. □

THEOREM 1. *Let* X *and* C *be smooth projective varieties with* $\dim X=n\geq 2$ *and* $\dim C=1$*, and let* L *be a nef and big line bundle on* X*. Assume that there exists a fiber space* $f : X \to C$*. Then* $g(X,L)\geq g(C)$*.*

Proof. See [10, Theorem 3.1]. □

DEFINITION 2 ([12, Definition 1.9 (2)]). *Let* (X,L) *be a quasi-polarized surface. We say that* (X,L) *is* L*-minimal if* $LE>0$ *for any* (-1)*-curve* E *on* X*. For any quasi-polarized surface* (X,L)*, there exists a birational morphism* $\rho : (X,L) \to (X_0,L_0)$ *such that* $L=\rho^*L_0$ *and* (X_0,L_0) *is* L_0*-minimal. Then we call* (X_0,L_0) *an* L*-minimalization of* (X,L)*.*

3. A lower bound for the second sectional geometric genus

In this section, we will consider a lower bound for the second sectional geometric genus of quasi-polarized manifolds.

3.1. Review of the sectional geometric genus

Here we will review the ith sectional geometric genus of quasi-polarized varieties (X,L) for every integer i with $0\leq i\leq \dim X$.

Notation. Let (X,L) be a quasi-polarized variety of dimension n, and let $\chi(tL)$ be the Euler–Poincaré characteristic of tL. Then $\chi(tL)$ is a polynomial in t of degree n, and we set

$$\chi(tL)=\sum_{j=0}^{n}\chi_j(X,L)\binom{t+j-1}{j}. \tag{1}$$

DEFINITION 3 ([14, Definition 2.1] and [16, Definition 2.1]). *Let (X,L) be a quasi-polarized variety of dimension n.*

(a) *For any integer i with $0\le i\le n$ the ith sectional H-arithmetic genus $\chi_i^H(X,L)$ of (X,L) is defined by*
$$\chi_i^H(X,L):=\chi_{n-i}(X,L).$$

(b) *For any integer i with $0\le i\le n$ the ith sectional geometric genus $g_i(X,L)$ of (X,L) is defined by*

$$g_i(X,L):=(-1)^i(\chi_i^H(X,L)-\chi(\mathcal{O}_X))+\sum_{j=0}^{n-i}(-1)^{n-i-j}h^{n-j}(\mathcal{O}_X).$$

REMARK 1. We make four comments:

(i) If $i=0$, then $\chi_0^H(X,L)$ and $g_0(X,L)$ are equal to the degree L^n. If $i=1$, then $g_1(X,L)$ is equal to the sectional genus $g(X,L)$ of (X,L). If $i=n$, then $\chi_n^H(X,L)=\chi(\mathcal{O}_X)$ and $g_n(X,L)=h^n(O_X)$.

(ii) Assume that X is smooth, L is very ample and i is an integer with $1\le i\le n-1$. Let $H_1,\dots,H_{n-i}$ be general members of $|L|$, and we put $X_i:=H_1\cap\cdots\cap H_{n-i}$. Then X_i is smooth with $\dim X_i=i$, and we can show that $\chi_i^H(X,L)=\chi(\mathcal{O}_{X_i})$ and $g_i(X,L)=h^i(O_{X_i})=h^0(\Omega^i_{X_i})$. This is the reason why we call $g_i(X,L)$ the ith sectional geometric genus.

(iii) By definition we have

$$\chi_i^H(X,L)=1-h^1(\mathcal{O}_X)+\cdots+(-1)^{i-1}h^{i-1}(\mathcal{O}_X)+(-1)^ig_i(X,L)$$

for every integer i with $1\le i\le n$.

(iv) Using intersection numbers, the second sectional geometric genus can be written as follows:

$$\begin{aligned} g_2(X,L) \;=\; & -1+h^1(\mathcal{O}_X)+\tfrac{1}{12}(K_X+(n-1)L)(K_X+(n-2)L)L^{n-2}\\ & +\tfrac{1}{12}c_2(X)L^{n-2}+\tfrac{n-3}{24}(2K_X+(n-2)L)L^{n-1}. \end{aligned}$$

The following result will be used later.

THEOREM 2. *Let X be a projective variety with $\dim X = n$ and let L be a nef and big line bundle on X. Then:*
(i) *For any integer i with $0 \le i \le n-1$, we have*

$$g_i(X,L) = \sum_{j=0}^{n-i-1} (-1)^{n-j} \binom{n-i}{j} \chi(-(n-i-j)L) + \sum_{k=0}^{n-i} (-1)^{n-i-k} h^{n-k}(\mathcal{O}_X).$$

(ii) *Assume that X is smooth. Then for any integer i with $0 \le i \le n-1$, we have*

$$g_i(X,L) = \sum_{j=0}^{n-i-1} (-1)^{j} \binom{n-i}{j} h^0(K_X + (n-i-j)L) + \sum_{k=0}^{n-i} (-1)^{n-i-k} h^{n-k}(\mathcal{O}_X).$$

Proof. For (i) (resp. (ii)), see [17, Theorem 1.1 (1)] (resp. [14, Theorem 2.3]). □

In the next subsection, we need the following lemma.

LEMMA 4. *Let X be a normal projective variety of dimension n and let $\delta : X' \to X$ be a resolution of X such that $X' \backslash \delta^{-1}(\mathrm{Sing}(X)) \cong X \backslash \mathrm{Sing}(X)$. Let L be a nef and big line bundle on X and let i be an integer with $0 \le i \le n$. If $\dim \mathrm{Sing}(X) \le n-i-1$, then for every integer k with $0 \le k \le i$ we have $\chi_k^H(X', \delta^*(L)) = \chi_k^H(X,L)$.*

Proof. Here we put $\mathscr{F}_q := R^q\delta_*\mathcal{O}_{X'}$. Then $\mathscr{F}_0 = \mathcal{O}_X$ and if $1 \le q \le n-1$, then by [20, (4.2.2) in III] (see also [9, (1.9) Fact in Chapter 0])

$$\begin{aligned} \dim \mathrm{Supp}\,\mathscr{F}_q &\le \dim\{x \in X \mid \dim \delta^{-1}(x) \ge q\} \\ &\le \min\{\dim \mathrm{Sing}(X), n-q-1\}. \end{aligned} \tag{2}$$

If $q \ge n$, then $\mathrm{Supp}\,\mathscr{F}_q = \varnothing$.

By the Leray spectral sequence we have

$$\chi(X', (\delta^*(L))^{\otimes t}) = \sum_q (-1)^q \chi(X, \mathscr{F}_q(L^{\otimes t})). \tag{3}$$

Here we use Notation (1). If $i = n$, then $\mathrm{Sing}(X) = \varnothing$ and the assertion holds. So we assume $i \le n-1$. By the assumption that $\dim \mathrm{Sing}(X) \le n-i-1$, we see from (2) and (3) that $\chi_l(X,L) = \chi_l(X', \delta^*(L))$ for every integer l with $n-i \le l \le n$. Therefore by the definition of $\chi_k^H(X,L)$ we get the assertion. □

3.2. A lower bound for the second sectional geometric genus

In this subsection, we will investigate a lower bound for the second sectional geometric genus of quasi-polarized manifolds. First we will prove the following.

THEOREM 3. *Let (X,L) be a quasi-polarized 3-fold.*

(i) *Assume that $h^0(K_X) = 0$. Then $g_2(X,L) \ge h^2(\mathcal{O}_X) \ge 0$ holds.*

(ii) *Assume that $\kappa(X) = -\infty$ and $\kappa(K_X + L) \ge 0$. Then $g_2(X,L) \ge h^1(\mathcal{O}_X)$ holds.*

Proof. (i) By Theorem 2 (ii), we have

$$g_2(X,L) = h^0(K_X + L) - h^0(K_X) + h^2(\mathscr{O}_X) = h^0(K_X + L) + h^2(\mathscr{O}_X) \geq h^2(\mathscr{O}_X) \geq 0.$$

(ii) First we note that $h^3(\mathscr{O}_X) = 0$ in this case. If $h^1(\mathscr{O}_X) = 0$, then by Theorem 2 (ii) we have $g_2(X,L) = h^0(K_X + L) + h^2(\mathscr{O}_X) \geq 0 = h^1(\mathscr{O}_X)$. Hence we may assume that $h^1(\mathscr{O}_X) > 0$. Let $\alpha : X \to \mathrm{Alb}(X)$ be the Albanese map of X.

If $\dim \alpha(X) = 2$, then by the same method as in the proof of [16, Theorem 3.3.1], we get the assertion. So we may assume that $\dim \alpha(X) = 1$. Then $\alpha(X)$ is a smooth curve and $\alpha : X \to \alpha(X)$ is a fiber space, that is, a surjective morphism with connected fibers. Set $C := \alpha(X)$. Then $g(C) \geq 1$. Since $\kappa(K_X + L) \geq 0$, we have $\kappa(K_F + L_F) \geq 0$, where F is a general fiber of α. Therefore $h^0(K_F + L_F) \neq 0$ by Proposition 1 and $\alpha_*(K_{X/C} + L)$ is ample by Lemma 1. By the same argument as in the proof of [16, Theorem 3.3.1], we get $h^0(K_X + L) > h^0(K_F + L_F)(g(C) - 1)$. Therefore

$$g_2(X,L) = h^0(K_X + L) + h^2(\mathscr{O}_X) > h^0(K_F + L_F)(g(C) - 1) \geq g(C) - 1 = h^1(\mathscr{O}_X) - 1.$$

This completes the proof of Theorem 3. $\square$

In order to prove Theorem 5, we use the following.

THEOREM 4. *Let V be a normal projective variety of dimension $n \geq 3$ such that V has only $\mathbb{Q}$-factorial terminal singularities. Let X be a smooth projective variety of dimension n. Assume that a birational morphism $\pi : X \to V$ satisfies $X \setminus \pi^{-1}(\mathrm{Sing}(V)) \cong V \setminus \mathrm{Sing}(V)$. Let H be a nef and big line bundle on V such that $K_V + sH$ is nef and the $\mathbb{Q}$-twisted sheaf $\Omega_V \left\langle \frac{s}{n} H \right\rangle$ is generically nef for a positive integer s. Let $H_1, \ldots, H_{n-2}$ be nef and big line bundles on V. Then*

$$\begin{aligned} c_2(X)\pi^*(H_1)\cdots\pi^*(H_{n-2}) \quad \geq \quad & -\tfrac{(n-1)}{n} K_X \pi^*(sH)\pi^*(H_1)\cdots\pi^*(H_{n-2}) \\ & -\tfrac{1}{n^2}\tbinom{n}{2}(\pi^*(sH))^2\pi^*(H_1)\cdots\pi^*(H_{n-2}). \end{aligned}$$

Proof. See [21, 2.11 Corollary]. $\square$

THEOREM 5. *Let (X,L) be a quasi-polarized manifold of dimension $n \geq 3$. Assume that $K_X + (n-1)L$ is pseudoeffective and (X,L) is not birationally a scroll*[1]. *Let (V,H) be as in Proposition 2 (ii) and let $\pi : X' \to V$ be a resolution of V such that $X' \setminus \pi^{-1}(\mathrm{Sing}(V)) \cong V \setminus \mathrm{Sing}(V)$. Then the following inequality holds.*

$$\begin{aligned} g_2(X,L) \quad \geq \quad & -1 + h^1(\mathscr{O}_X) + \tfrac{1}{12}\pi^*(K_V)(\pi^*(K_V + (2n-5)H))(\pi^*(H))^{n-2} \\ & + \tfrac{n^2-5n+5}{12}(\pi^*(H))^n. \end{aligned}$$

Proof. Then there exist a quasi-polarized manifold (M,A), birational morphisms $\pi_1 : M \to X$ and $\pi_2 : M \to V$ such that $A = \pi_1^*(L) = \pi_2^*(H)$. Since $\dim \mathrm{Sing}(V) \leq n-3$, we have $\chi_2^H(X,L) = \chi_2^H(M,A) = \chi_2^H(V,H)$ by Lemma 4. By assumption, V has only

[1] For the definition of "birationally a scroll", see [21, 1.3 Definition].

rational singularities. Hence $h^j(\mathscr{O}_V) = h^j(\mathscr{O}_M)$ for every j. Therefore $g_2(X,L) = g_2(M,A) = g_2(V,H)$ by the definition of the second sectional geometric genus. Since $g_2(X',\pi^*(H)) = g_2(V,H)$ by the same argument as above, we have

$$
\begin{aligned}
(4)\quad & g_2(X,L) \\
&= -1 + h^1(\mathscr{O}_{X'}) + \tfrac{1}{12}(K_{X'} + (n-1)\pi^*(H))(K_{X'} + (n-2)\pi^*(H))(\pi^*(H))^{n-2} \\
&\qquad + \tfrac{1}{12}c_2(X')(\pi^*(H))^{n-2} + \tfrac{n-3}{24}(2K_{X'} + (n-2)\pi^*(H))(\pi^*(H))^{n-1}
\end{aligned}
$$

from Remark 1 (iv). Here we note that (V,H) is not birationally a scroll because (X,L) is not birationally a scroll. Hence by [21, 1.4 Theorem], we see that $\Omega_V\langle H\rangle$ is generically nef. Since $K_V + (n-1)H$ is nef, so is $K_V + nH$. Therefore by Theorem 4, we have

$$
(5)\qquad c_2(X')(\pi^*(H))^{n-2} \geq -(n-1)K_{X'}(\pi^*(H))^{n-1} - \tfrac{n(n-1)}{2}(\pi^*(H))^n.
$$

Here we note that

$$
K_{X'}\pi^*(H)^{n-1} = \pi^*(K_V)\pi^*(H)^{n-1} \ \text{ and } \ (K_{X'})^2\pi^*(H)^{n-2} = (\pi^*(K_V))^2\pi^*(H)^{n-2}
$$

hold by Lemma 2. So we get the assertion by using (4) and (5). □

In particular, for $n \geq 4$, we get the following corollary from Proposition 2 and Theorem 5.

COROLLARY 1. *Let (X,L) be a quasi-polarized n-fold with $n \geq 4$ and $\kappa(X) \geq 0$. Then $g_2(X,L) \geq h^1(\mathscr{O}_X)$ holds.*

Proof. Since $\kappa(X) \geq 0$, we see that (X,L) satisfies assumptions in Theorem 5. By Proposition 2, there exist a normal projective variety V of dimension n and a nef and big line bundle H on V such that V has only $\mathbb{Q}$-factorial terminal singularities, (V,H) is birationally equivalent to (X,L) and $K_V + (n-1)H$ is nef. Let $\pi : X' \to V$ be a resolution of V such that $X' \setminus \pi^{-1}(\mathrm{Sing}(V)) \cong V \setminus \mathrm{Sing}(V)$. Then we have

$$
\pi^*(K_V)(\pi^*(K_V + (2n-5)H))(\pi^*(H))^{n-2} \geq 0
$$

because $\kappa(X) \geq 0$, $K_V + (n-1)H$ is nef and $2n-5 \geq n-1$.

On the other hand, since $n \geq 4$ we have

$$
\frac{n^2 - 5n + 5}{12} = \frac{(n - \frac{5}{2})^2 - \frac{5}{4}}{12} > 0.
$$

Therefore we get the assertion because $g_2(X,L)$ is an integer. □

If $n = 3$, then we cannot prove the inequality $g_2(X,L) \geq h^1(\mathscr{O}_X)$ by using the argument of the proof of Corollary 1 because $n^2 - 5n + 5 = -1 < 0$ in the case of $n = 3$. So we consider the case where $\dim X = 3$ and $\kappa(X) \geq 0$. In this case, the following lemma is very important.

LEMMA 5. *Let* (X,L) *be a quasi-polarized manifold with* $\dim X = 3$*. Assume that* $\kappa(K_X+L)\geq 0$*. Then there exists a quasi-polarized variety* (X^+,L^+) *with* $\dim X^+ = 3$ *such that* X^+ *is a normal projective variety with only* $\mathbb{Q}$*-factorial terminal singularities,* X^+ *is birationally equivalent to* X*,* $g_i(X,L)=g_i(X^+,L^+)$ *for* $i=1,2$ *and* $K_{X^+}+L^+$ *is nef.*

Proof. (A) By a result of Fujita [8, (4.2) Theorem], there exist a normal projective variety M of dimension three with only $\mathbb{Q}$-factorial terminal singularities and a nef and big line bundle A on M such that (X,L) and (M,A) are birationally equivalent and K_M+2A is nef.

(B) Assume that there exists an irreducible curve C on M such that $(K_M+2A)C=0$ and $AC>0$. Then there exists an extremal ray R on M such that $(K_M+2A)R=0$ and $AR>0$. Let $\rho: M\to M'$ be the contraction morphism of R. Assume that ρ is not birational. Then $\dim M'\leq 2$ and there exists a $\mathbb{Q}$-Cartier divisor B on M' such that $K_M+2A=\rho^*(B)$. Hence $K_M+A=\rho^*(B)-A$. But this is impossible because $\kappa(K_M+A)=\kappa(K_X+L)\geq 0$ by assumption. Hence ρ is birational. By [1, Theorem 3.1] we see that ρ is a blowing up of a smooth point of M'. Let E be its exceptional divisor and $A':=\rho_*(A)$. Then A' is a nef and big Cartier divisor on M' with $A=\rho^*(A')-E$ and $K_M+2A=\rho^*(K_{M'}+2A')$.

(C) Let $\{E_i\}_i$ be the set of all the exceptional divisors E_i of the contraction morphism of the extremal ray R_i as in (B). Then by the same argument as in the proof of [3, Lemma 4.2.17] (and by using [21, 4.3 Lemma]), we can prove that all E_i are disjoint.

(D) By contracting all these extremal rays, we get a normal projective variety Y with only $\mathbb{Q}$-factorial terminal singularities, a nef and big Cartier divisor H on Y and a surjective morphism $\mu: M\to Y$ such that $K_M+2A=\mu^*(K_Y+2H)$ and $K_M+A=\mu^*(K_Y+H)+E_\mu$, where E_μ is an effective μ-exceptional divisor. In particular, we see that $\kappa(K_M+A)=\kappa(K_Y+H)$.

(E) Next we will prove that K_Y+H is nef. Let τ be the nef value of (Y,H). Assume that $\tau>1$. Here we note that there exists an extremal ray R on Y such that $(K_Y+\tau H)R=0$ and $HR>0$. Let $\psi: Y\to Y'$ be the contraction morphism of R. Then H is ψ-ample and by [2, Theorem (5.6)] (see also [24, Theorem 2.1]) and (D) above, we see that ψ is not birational. In particular $\dim Y'\leq 2$ and there exists a $\mathbb{Q}$-Cartier divisor B' on Y' such that $K_Y+\tau H=\psi^*(B')$. Hence $K_Y+H=\psi^*(B')-(\tau-1)H$. But since we assume that $\tau>1$, this is impossible because $\kappa(K_Y+H)=\kappa(K_M+A)=\kappa(K_X+L)\geq 0$ by assumption. Therefore we get $\tau\leq 1$ and K_Y+H is nef.

(F) Since A and H are nef and big, by the Kawamata–Viehweg vanishing theorem ([23, Theorem 1-2-5]) and the Serre duality we have $\chi(-2A)=-h^0(K_M+2A)$, $\chi(-A)=-h^0(K_M+A)$, $\chi(-2H)=-h^0(K_Y+2H)$ and $\chi(-H)=-h^0(K_Y+H)$. We also note that $h^0(K_M+2A)=h^0(K_Y+2H)$, $h^0(K_M+A)=h^0(K_Y+H)$ and $h^i(\mathscr{O}_X)=h^i(\mathscr{O}_M)=h^i(\mathscr{O}_Y)$ for every $i\geq 0$. Therefore by Theorem 2 (i) we see that $g_1(M,A)=g_1(Y,H)$ and $g_2(M,A)=g_2(Y,H)$. By Lemma 4 we have $g_1(X,L)=g_1(M,A)$ and $g_2(X,L)=g_2(M,A)$. Therefore $g_1(X,L)=g_1(Y,H)$ and $g_2(X,L)=g_2(Y,H)$.

(G) By setting $X^+:=Y$ and $L^+:=H$, we get the assertion. $\square$

THEOREM 6. *Let (X,L) be a quasi-polarized manifold with $\dim X = 3$. Assume that $\kappa(X) \geq 0$. Then $g_2(X,L) \geq h^1(\mathscr{O}_X)$.*

Proof. By Lemma 5 we see that there exist a quasi-polarized variety (X^+,L^+) of dimension three such that X^+ is a normal variety with only $\mathbb{Q}$-factorial terminal singularities, X^+ is birationally equivalent to X, $g_2(X,L) = g_2(X^+,L^+)$ and $K_{X^+} + L^+$ is nef. Let $\nu : \widetilde{X^+} \to X^+$ be a resolution of X^+ such that $\widetilde{X^+} \backslash \nu^{-1}(\mathrm{Sing}(X^+)) \cong X^+ \backslash \mathrm{Sing}(X^+)$. Here we note that $\dim \mathrm{Sing}(X^+) \leq 0$ and $h^j(\mathscr{O}_{X^+}) = h^j(\mathscr{O}_{\widetilde{X^+}})$ for $j = 0,1$. Then by Lemma 4 and Remark 1 (iii) we have $g_2(\widetilde{X^+}, \nu^*(L^+)) = g_2(X^+,L^+)$. Since $g_2(X^+,L^+) = g_2(X,L)$, we see that $g_2(X,L) = g_2(\widetilde{X^+}, \nu^*(L^+))$. Here we use Theorem 4. Since Ω_{X^+} is generically nef by [25, Corollary 6.4], we see that $\Omega_{X^+}\langle \frac{1}{3}L^+\rangle$ is also generically nef. Hence by Theorem 4 the following inequality holds:

$$c_2(\widetilde{X^+})(\nu^*(L^+)) \geq -\frac{2}{3}K_{\widetilde{X}}(\nu^*(L^+))^2 - \frac{1}{3}(\nu^*(L^+))^3. \tag{6}$$

Therefore by (6), Remark 1 (iv) and Lemma 2, we have

$$\begin{aligned}
&g_2(\widetilde{X^+}, \nu^*(L^+)) \\
&\geq -1 + h^1(\mathscr{O}_{\widetilde{X^+}}) + \frac{1}{12}((K_{\widetilde{X^+}})^2 + 3K_{\widetilde{X^+}}\nu^*(L^+) + 2\nu^*(L^+)^2)(\nu^*(L^+)) \\
&\qquad - \frac{1}{18}K_{\widetilde{X^+}}\nu^*(L^+)^2 - \frac{1}{36}(\nu^*(L^+))^3 \\
&= -1 + h^1(\mathscr{O}_{\widetilde{X^+}}) + \frac{1}{12}(K_{X^+} + L^+)K_{X^+}L^+ + \frac{1}{9}(K_{X^+} + L^+)(L^+)^2 + \frac{1}{36}(L^+)^3.
\end{aligned}$$

Since $K_{X^+} + L^+$ is nef, we obtain $g_2(\widetilde{X^+}, \nu^*(L^+)) > h^1(\mathscr{O}_{\widetilde{X^+}}) - 1$. So we have

$$g_2(X,L) = g_2(\widetilde{X^+}, \nu^*(L^+)) \geq h^1(\mathscr{O}_{\widetilde{X^+}}) = h^1(\mathscr{O}_X),$$

establishing the assertion. □

4. The dimension of global sections of adjoint bundles for quasi-polarized 3-folds

In this section, we will investigate the dimension of global sections of adjoint bundles for quasi-polarized 3-folds by using the sectional geometric genus.

First in 4.1, we will introduce the notion of the *i th Hilbert coefficient* of (X,L), and we will study lower bounds for this. In 4.2, we will study the dimension of global sections of adjoint bundles for quasi-polarized 3-folds by using results in 4.1.

4.1. The *i* th Hilbert coefficient of quasi-polarized manifolds

DEFINITION 4 (([19, Definitions 3.1 and 3.2])). *Let (X,L) be a quasi-polarized manifold of dimension n and let t be a positive integer. Let*

$$\begin{aligned}
F_0(t) &:= h^0(K_X + tL) \\
F_i(t) &:= F_{i-1}(t+1) - F_{i-1}(t) \ \text{ for every integer } i \text{ with } 1 \leq i \leq n.
\end{aligned}$$

For every integer i with $0 \leq i \leq n$, let $A_i(X,L) := F_{n-i}(1)$. Then we call this $A_i(X,L)$ the i th Hilbert coefficient of (X,L).

In [19], we assumed that L is ample. But the following results are true for the case where L is nef and big by the same argument as [19].

REMARK 2. (A) ([19, Remark 3.2 (A)]) The following hold: $A_0(X,L) = L^n \geq 1$, $A_n(X,L) = h^0(K_X + L)$.
(B) ([19, Proposition 3.2]) For every integer i with $1 \leq i \leq n$ we have

$$A_i(X,L) = g_i(X,L) + g_{i-1}(X,L) - h^{i-1}(\mathscr{O}_X).$$

(C) ([19, Theorem 3.1 and Corollary 3.1]) Let t be a positive integer. The following equality holds:

$$h^0(K_X + tL) = \sum_{j=0}^{n} \binom{t-1}{n-j} A_j(X,L).$$

Next we will investigate the i th Hilbert coefficient of quasi-polarized manifolds for $i \leq 2$. Propositions 4 and 5 are useful when we study the dimension of global sections of adjoint bundles. First we will study the case where $i = 1$.

PROPOSITION 4. *Let (X,L) be a quasi-polarized manifold of dimension n. Then we obtain:*

(i) $A_1(X,L) \geq 0$ *holds.*

(ii) *If $A_1(X,L) = 0$, then there exists a birational morphism $\pi : X \to \mathbb{P}^n$ such that $L = \pi^*(\mathscr{O}_{\mathbb{P}^n}(1))$.*

(iii) *If $A_1(X,L) = 1$, then (X,L) satisfies one of the following:*

(a) *(X,L) is birationally equivalent to a polarized variety (Y,B) which is one of the following types:*

(a.1) *Y is Gorenstein with $K_Y = -(n-1)B$ and $B^n = 1$.*

(a.2) *A scroll over a smooth elliptic curve with $B^n = 1$.*

(b) *There exist a polarized variety (V,H) and a birational morphism $\pi : X \to V$ such that V is a (possibly singular) quadric hypersurface in $\mathbb{P}^{n+1}$, $H = \mathscr{O}_V(1)$ and $L = \pi^*(H)$.*

Proof. (i) We get $A_1(X,L) \geq 0$ by Remarks 1 (i), 2 (B) and Proposition 3 (i).
(ii) If $A_1(X,L) = 0$, then we see that $g_1(X,L) = 0$ and $L^n = 1$. By Proposition 3 (ii), we have $\Delta(X,L) = 0$. Hence by [8, (1.1) Theorem], we infer that there exist a polarized variety (V,H) and a birational morphism $\pi : X \to V$ such that H is very ample, $L = \pi^*(H)$ and $\Delta(V,H) = 0$. By [9, (5.1)], we find that $(V,H) \cong (\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(1))$ because $H^n = 1$. Therefore we get the assertion of (ii).
(iii) If $A_1(X,L) = 1$, then (X,L) satisfies one of the following:
(iii.1) $g_1(X,L) = 1$ and $L^n = 1$.
(iii.2) $g_1(X,L) = 0$ and $L^n = 2$.

First we consider the case of (iii.1). Then by Proposition 3 (iii), we get type (a) above.

Next we consider the case of (iii.2). Then since $L^n = 2$, by [8, (1.1) Theorem] and [9, (5.1)] we obtain the type (b) above. Therefore we get the assertion. $\square$

Next we will study $A_2(X,L)$ for quasi-polarized 3-folds (X,L).

PROPOSITION 5. *Let (X,L) be a quasi-polarized manifold with* $\dim X = 3$. *Then the following hold.*

(i) *If $\kappa(X) \geq 0$, then $A_2(X,L) \geq 2$.*

(ii) *Assume that $\kappa(X) = -\infty$.*

- (ii.1) *If $h^1(\mathscr{O}_X) = 0$, then $A_2(X,L) = g_2(X,L) + g_1(X,L) \geq 0$.*
- (ii.2) *If $h^1(\mathscr{O}_X) > 0$ and the dimension of the image of the Albanese map of X is one, then $A_2(X,L) \geq g_2(X,L) \geq 0$.*
- (ii.3) *If $h^1(\mathscr{O}_X) > 0$ and the dimension of the image of the Albanese map of X is two, then $A_2(X,L) \geq g_1(X,L) - 1 + \chi(\mathscr{O}_S) \geq 0$, where S is a resolution of the image of the Albanese map of X.*

Proof. (i) By Theorem 6, we have $g_2(X,L) \geq h^1(\mathscr{O}_X)$. On the other hand, since $\kappa(X) \geq 0$, we see that $g_1(X,L) \geq 2$. Hence $A_2(X,L) = g_2(X,L) + g_1(X,L) - h^1(\mathscr{O}_X) \geq 2$.

(ii) Assume that $\kappa(X) = -\infty$.

(ii.1) *The case* $h^1(\mathscr{O}_X) = 0$. Then $A_2(X,L) = g_2(X,L) + g_1(X,L)$ by Remark 2 (B). Since $g_2(X,L) \geq 0$ by Theorem 3 (i) and $g_1(X,L) \geq 0$ by [8, (4.8) Corollary] or Proposition 3 (i), we have $A_2(X,L) = g_2(X,L) + g_1(X,L) \geq 0$.

Next we shall consider the case where $h^1(\mathscr{O}_X) > 0$. Let $\alpha : X \to \mathrm{Alb}(X)$ be the Albanese map of X.

(ii.2) *The case* $\dim \alpha(X) = 1$. Then $\alpha(X)$ is a smooth curve and $\alpha : X \to \alpha(X)$ is a surjective morphism with connected fibers. Let $C := \alpha(X)$. Then since $g(C) = h^1(\mathscr{O}_X)$, by Theorem 1 we have $g_1(X,L) \geq h^1(\mathscr{O}_X)$. Hence by using Theorem 3 (i) we have $A_2(X,L) = g_2(X,L) + g_1(X,L) - h^1(\mathscr{O}_X) \geq g_2(X,L) \geq 0$.

(ii.3) *The case* $\dim \alpha(X) = 2$. Then there exist a smooth projective 3-fold X', a smooth projective surface S, birational morphisms $\mu : X' \to X$ and $\nu : S \to \alpha(X)$ and a surjective morphism $f : X' \to S$ such that $\alpha \circ \mu = \nu \circ f$. Then we note that $h^2(\mathscr{O}_X) = h^2(\mathscr{O}_{X'}) \geq h^2(\mathscr{O}_S)$ and $h^1(\mathscr{O}_X) = h^1(\mathscr{O}_{X'}) = h^1(\mathscr{O}_S)$ hold. Therefore $1 - h^1(\mathscr{O}_X) + h^2(\mathscr{O}_X) \geq \chi(\mathscr{O}_S)$. On the other hand $g_2(X,L) \geq h^2(\mathscr{O}_X)$ holds by Theorem 3 (i). Hence $g_2(X,L) \geq h^2(\mathscr{O}_X) \geq h^1(\mathscr{O}_X) - 1 + \chi(\mathscr{O}_S)$. Therefore we have

$$A_2(X,L) = g_2(X,L) + g_1(X,L) - h^1(\mathscr{O}_X) \geq g_1(X,L) - 1 + \chi(\mathscr{O}_S). \tag{7}$$

Here we note that $\chi(\mathscr{O}_S) \geq 0$ since $\kappa(S) \geq 0$. We also note that $g_1(X,L) \geq 1$ because $h^1(\mathscr{O}_X) = 0$ holds if $g_1(X,L) = 0$ by [8, (4.8) Corollary and (1.1) Theorem] or Proposition 3 (ii). Therefore we get the assertion of (ii.3) and these complete the proof of Proposition 5. $\square$

4.2. The dimension of global sections of adjoint bundles for 3-folds

In this subsection, first we will prove the following theorem.

THEOREM 7. *Let (X,L) be a quasi-polarized manifold of dimension three. Then the following hold.*

(a) $h^0(K_X+2L)=0$ *if and only if (X,L) is birationally equivalent to a scroll over a smooth curve or a quasi-polarized variety (V,H) such that V is a normal projective variety with only $\mathbb{Q}$-factorial terminal singularities and $\Delta(V,H)=0$. In particular, if $\kappa(K_X+2L)\geq 0$, then $h^0(K_X+2L)\geq 1$ holds.*

(b) $h^0(K_X+3L)=0$ *if and only if there exists a birational morphism $f: X\to\mathbb{P}^3$ such that $L=f^*(\mathscr{O}_{\mathbb{P}^3}(1))$. In particular, if $\kappa(K_X+3L)\geq 0$, then $h^0(K_X+3L)\geq 1$.*

(c) *For $t\geq 4$, then $h^0(K_X+tL)\geq\binom{t-1}{3}$.*

Proof. (a) First we consider the dimension $h^0(K_X+2L)$. The "if" part is trivial. So we will prove the "only if" part. We assume that $h^0(K_X+2L)=0$. Then by Lemma 3 and Proposition 2, it suffices to show that $\kappa(K_X+2L)=-\infty$. First we note that $A_3(X,L)\geq 0$.

If $\kappa(X)\geq 0$, then by Proposition 5 (i) and Remark 2 (C) we have $h^0(K_X+2L)=A_3(X,L)+A_2(X,L)\geq 2$ and this is impossible.

Next we consider the case $\kappa(X)=-\infty$. We assume that $\kappa(K_X+2L)\geq 0$.

If $h^1(\mathscr{O}_X)>0$, then we take the Albanese map $\alpha: X\to\mathrm{Alb}(X)$. By taking its Stein factorization, if necessary, we make a fiber space $\alpha: X\to Y$ over a normal projective variety Y. Let F be a general fiber of the map α. Then $1\leq\dim F\leq 2$, and $\kappa(K_F+2L_F)\geq 0$ since $\kappa(K_X+2L)\geq 0$. By Proposition 1, $h^0(K_F+2L_F)>0$. Therefore we have $h^0(K_X+2L)>0$ by [5, Lemma 4.1], and this contradicts the assumption.

Next we consider the case of $h^1(\mathscr{O}_X)=0$. Then

$$h^0(K_X+2L)=A_2(X,L)+A_3(X,L)\geq A_2(X,L).$$

On the other hand, since $\kappa(K_X+2L)\geq 0$, we have $g_1(X,L)\geq 1$. Hence by Theorem 3 (i) $A_2(X,L)=g_2(X,L)+g_1(X,L)\geq 1$ holds. Therefore we get $h^0(K_X+2L)\geq 1$ and this also contradicts the assumption. Therefore we get (a) in Theorem 7.

(b) Next we consider the dimension $h^0(K_X+3L)$. We can easily check the 'if' part. So we will prove the 'only if' part. Then by Remark 2 (C) we have $h^0(K_X+3L)=A_1(X,L)+2A_2(X,L)+A_3(X,L)$.

Assume that $\kappa(X)\geq 0$. Then by Proposition 5 (i) we have $A_2(X,L)\geq 2$. We also note that $A_3(X,L)\geq 0$ and $A_1(X,L)=g_1(X,L)+L^3-1\geq 2$ because $g_1(X,L)\geq 2$. Hence $h^0(K_X+3L)\geq 6$ and this is impossible. So we may assume that $\kappa(X)=-\infty$. By Remark 2 (A), Propositions 4 and 5 (ii), we see that if $h^0(K_X+3L)=0$, then $A_1(X,L)=0$ and this implies $g_1(X,L)=0$ and $L^3=1$. Therefore by the same argument as in the proof of Proposition 4 (ii), we obtain (b) in Theorem 7.

(c) If $t \geq 4$, then by Remark 2 (C) we have $h^0(K_X+tL)=\sum_{k=0}^{3}\binom{t-1}{3-k}A_k(X,L)$. By Propositions 4, 5 and Remark 2 (A), we get the assertion. □

REMARK 3. (i) Theorem 7 (a) shows that [13, Conjecture NB] for the case of $\dim X=3$ is true. This is a quasi-polarized manifold version of a conjecture of Beltrametti and Sommese [3, Conjecture 7.2.7].

(ii) Theorem 7 (a) is also obtained from [21, 1.5 Theorem] and Proposition 2 (ii).

THEOREM 8. *Let (X,L) be a quasi-polarized manifold with* $\dim X=3$.

(a) *Assume that $h^0(K_X+2L)=1$ holds. Then (X,L) satisfies one of the following:*

(a.1) *(X,L) is birationally equivalent to (V,H), where V is a normal projective variety having only Gorenstein $\mathbb{Q}$-factorial terminal singularities, $\mathcal{O}(K_V+2H)\cong\mathcal{O}_V$ and $\Delta(V,H)=1$.*

(a.2) *There exist an Abelian surface S' and a surjective morphism with connected fibers $f':X\to S'$ such that a general fiber F' of f' is isomorphic to $\mathbb{P}^1$ and $L_{F'}=\mathcal{O}_{\mathbb{P}^1}(1)$.*

(a.3) *There exist a smooth elliptic curve C and a surjective morphism with connected fibers $f:X\to C$ such that the L_F-minimalization[2] of (F,L_F) is isomorphic to either $(\mathbb{P}^2,\mathcal{O}_{\mathbb{P}^2}(2))$ or a scroll over a smooth curve.*

(b) *Assume that $h^0(K_X+3L)=1$ holds. Then (X,L) satisfies one of the following types:*

(b.1) *(X,L) is birationally equivalent to a scroll over a smooth elliptic curve and $L^3=1$.*

(b.2) *There exist a normal projective variety W, a very ample line bundle H and a birational morphism $\mu:X\to W$ such that $L=\mu^*(H)$, $\Delta(W,H)=0$ and $H^3=2$.*

(c) *Assume that $h^0(K_X+tL)=\binom{t-1}{3}$ holds for some $t\geq 4$. Then there exists a birational morphism $f:X\to\mathbb{P}^3$ such that $L=f^*(\mathcal{O}_{\mathbb{P}^3}(1))$.*

Proof. (a) First we assume that $h^0(K_X+2L)=1$. If $\kappa(X)\geq 0$, then $h^0(K_X+2L)\geq 2$ by the proof of Theorem 7 (a). So we may assume that $\kappa(X)=-\infty$.

If $\kappa(K_X+L)\geq 0$, then by Theorem 3 (ii) we have

$$A_2(X,L)=g_1(X,L)+g_2(X,L)-h^1(\mathcal{O}_X)\geq g_1(X,L).$$

On the other hand, $\kappa(K_X+L)\geq 0$ implies that $g_1(X,L)\geq 2$. Hence we have $A_2(X,L)\geq 2$ and by Remark 2 (A) and (C) we get $h^0(K_X+2L)\geq 2$ and this contradicts the assumption that $h^0(K_X+2L)=1$. Therefore we see that

$$\kappa(K_X+L)=-\infty. \tag{8}$$

[2] see Definition 2

In particular $h^0(K_X+L)=0$. On the other hand by Remark 2 (A), we have $A_3(X,L)=h^0(K_X+L)$. Hence

$$1 \quad = \quad h^0(K_X+2L)=A_2(X,L)+A_3(X,L)=A_2(X,L). \tag{9}$$

(a.1) *The case* $h^1(\mathscr{O}_X)=0$. We see that

$$1=A_2(X,L)=g_2(X,L)+g_1(X,L)\geq g_1(X,L)$$

by Remark 2 (B) and Theorem 3 (i). Therefore we have $g_1(X,L)\leq 1$. Since we have $\kappa(K_X+2L)\geq 0$ by assumption, $g_1(X,L)=1$. Since $h^1(\mathscr{O}_X)=0$ in this case, by [8, (4.9) Corollary] or Proposition 3 (iii) we get the type (a.1) in Theorem 8 (a).

(a.2) *The case* $h^1(\mathscr{O}_X)>0$. Let $\alpha: X\to \mathrm{Alb}(X)$ be the Albanese map of X.

(a.2.1) *The case* $\dim\alpha(X)=2$. Then there exist a smooth projective 3-fold X', a smooth projective surface S, birational morphisms $\mu: X'\to X$ and $\nu: S\to\alpha(X)$ and a surjective morphism $f: X'\to S$ such that $\alpha\circ\mu=\nu\circ f$. Then by Proposition 5 (ii.3) we have $A_2(X,L)\geq g_1(X,L)-1+\chi(\mathscr{O}_S)\geq 0$. Since $\kappa(S)\geq 0$, we have $\chi(\mathscr{O}_S)\geq 0$, that is, $A_2(X,L)\geq g_1(X,L)-1$. Therefore we have $g_1(X,L)\leq 2$ from (9). Here we note that $g_1(X,L)\geq 1$ because $h^0(K_X+2L)=1$.

We assume that $g_1(X,L)=1$. Then since $h^1(\mathscr{O}_X)>0$, by [8, (4.9) Corollary] or Proposition 3 (iii) we see that (X,L) is birationally equivalent to a scroll over a smooth elliptic curve. But this is impossible because we assume that $h^0(K_X+2L)=1$.

Therefore we have $g_1(X,L)=2$. In this case we see that $\chi(\mathscr{O}_S)=0$. Since $\chi(\mathscr{O}_S)=0$, we get $\kappa(S)=0$ or 1.

CLAIM 1. $h^1(\mathscr{O}_X)=2$.

Proof. Since $\dim\alpha(X)=2$, it suffices to show that $h^1(\mathscr{O}_X)\leq 2$. We also note that $h^1(\mathscr{O}_X)=h^1(\mathscr{O}_S)$.

If $\kappa(S)=0$, then by the classification theory of surfaces we have $h^1(\mathscr{O}_S)\leq 2$. So we may assume that $\kappa(S)=1$. Then there exist a smooth projective curve B and an elliptic fibration $\sigma: S\to B$. In this case $h^1(\mathscr{O}_S)\leq h^1(\mathscr{O}_B)+1$ holds. If $g(B)=0$, then $h^1(\mathscr{O}_S)\leq 1$. So we assume that $g(B)\geq 1$. We consider the map $\psi:=\sigma\circ f: X'\to S\to B$. By taking the Stein factorization, if necessary, we may assume that ψ is a surjective morphism with connected fibers. Let F_ψ be a general fiber of ψ. Since $h^0(K_{X'}+2\mu^*(L))=h^0(K_X+2L)=1$ by assumption, we have $h^0(K_{F_\psi}+2\mu^*(L)_{F_\psi})>0$. Therefore $\psi_*(K_{X'/B}+2\mu^*(L))\neq 0$ and by the Riemann–Roch theorem we have

$$\begin{aligned} h^0(K_{X'}+2\mu^*(L)) &= h^0(\psi_*(K_{X'}+2\mu^*(L)))\\ &= h^1(\psi_*(K_{X'}+2\mu^*(L)))+\deg\psi_*(K_{X'/B}+2\mu^*(L))\\ &\quad +h^0(K_{F_\psi}+2\mu^*(L)_{F_\psi})(g(B)-1)\\ &\geq \deg\psi_*(K_{X'/B}+2\mu^*(L))+h^0(K_{F_\psi}+2\mu^*(L)_{F_\psi})(g(B)-1). \qquad (10)\end{aligned}$$

Since $1=h^0(K_X+2L)=h^0(K_{X'}+2\mu^*(L))$, we get $g(B)=1$ by Lemma 1. Hence $h^1(\mathscr{O}_S)\leq 2$.

This completes the proof of Claim 1. $\square$

By this claim, we see that α is surjective. By [26, Lemma 10.1 and Corollary 10.6] we have $h^2(\mathscr{O}_S) > 0$ and $\kappa(S) = 0$. We also note that $\chi(\mathscr{O}_S) = 0$. Hence S is birationally equivalent to an Abelian surface. Let $\tau : S \to S'$ be the minimalization of S. Then S' is an Abelian surface. Here we note that there exists a rational map $\tau \circ f \circ \mu^{-1} : X \dashrightarrow S'$. By [26, Lemma 9.11], this map is a morphism. We set $f' := \tau \circ f \circ \mu^{-1}$, and let F' be a general fiber of f'. Then we see that $F' \cong \mathbb{P}^1$. If $h^0(K_{F'} + L_{F'}) > 0$, then $h^0(K_X + L) > 0$ by [5, Lemma 4.1]. But this contradicts (8). Hence $h^0(K_{F'} + L_{F'}) = 0$, and we have $(F', L_{F'}) = (\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(1))$. This is the type (a.2) in Theorem 8 (a).

(a.2.2) *The case* $\dim \alpha(X) = 1$. Then $\alpha(X)$ is a smooth curve and α is a surjective morphism with connected fibers. Let $C := \alpha(X)$. Here we note that $h^1(\alpha_*(K_X + 2L)) \leq h^1(K_X + 2L) = 0$ by the Leray spectral sequence. Moreover $h^0(K_X + 2L) > 0$ implies $h^0(K_{F_\alpha} + 2L_{F_\alpha}) > 0$ for a general fiber F_α of α. So we get $\alpha_*(K_{X/C} + 2L) \neq 0$. On the other hand, by Lemma 1 we see that $\deg \alpha_*(K_{X/C} + 2L) > 0$. Therefore by the Riemann–Roch theorem we have

$$\begin{aligned} h^0(K_X + 2L) &= h^0(\alpha_*(K_X + 2L)) \\ &= h^1(\alpha_*(K_X + 2L)) + \deg \alpha_*(K_{X/C} + 2L) + h^0(K_F + 2L_F)(g(C) - 1) \\ &\geq g(C) \geq 1. \end{aligned}$$

Since $h^0(K_X + 2L) = 1$, we have $g(C) = 1$, that is, $h^1(\mathscr{O}_X) = 1$. Hence $A_2(X, L) = g_2(X, L) + g_1(X, L) - 1$. Since $g_2(X, L) \geq 0$ by Theorem 3 (i), we have $g_1(X, L) \leq 2$ from (9). By the same argument as (a.2.1) above, we get $g_1(X, L) = 2$.

By [5, Lemma 4.1] we see from (8) that $h^0(K_{F_\alpha} + L_{F_\alpha}) = 0$. Since $\dim F_\alpha = 2$ and $\kappa(F_\alpha) = -\infty$, we have

$$h^0(K_{F_\alpha} + L_{F_\alpha}) = g(F_\alpha, L_{F_\alpha}) - h^1(\mathscr{O}_{F_\alpha})$$

by the Riemann–Roch theorem and the Kawamata–Viehweg vanishing theorem, where $g(F_\alpha, L_{F_\alpha})$ is the sectional genus of (F_α, L_{F_α}). Hence $h^0(K_{F_\alpha} + L_{F_\alpha}) = 0$ implies that $g(F_\alpha, L_{F_\alpha}) = h^1(\mathscr{O}_{F_\alpha})$. By [12, Theorem 3.1], the L_{F_α}-minimalization of (F_α, L_{F_α}) is either $(\mathbb{P}^2, \mathscr{O}_{\mathbb{P}^2}(1))$, $(\mathbb{P}^2, \mathscr{O}_{\mathbb{P}^2}(2))$ or a scroll over a smooth curve. But if $(F_\alpha, L_{F_\alpha}) \cong (\mathbb{P}^2, \mathscr{O}_{\mathbb{P}^2}(1))$, then $h^0(K_{F_\alpha} + 2L_{F_\alpha}) = 0$ and this is a contradiction. Therefore we get the type (a.3) in Theorem 8 (a).

(b) Assume that $h^0(K_X + 3L) = 1$. Then by the proof of (b) in Theorem 7, we see that $\kappa(X) = -\infty$, and by Proposition 4 (i) and (ii) we have $A_1(X, L) \geq 1$. But since $h^0(K_X + 3L) = A_1(X, L) + 2A_2(X, L) + A_3(X, L)$, $A_2(X, L) \geq 0$ and $A_3(X, L) \geq 0$, we see that $A_1(X, L) = 1$, $A_2(X, L) = 0$ and $A_3(X, L) = 0$. Then $A_1(X, L) = 1$ implies $(g_1(X, L), L^3) = (1, 1)$ or $(0, 2)$. If $g_1(X, L) = 1$ and $h^1(\mathscr{O}_X) = 0$, then $\kappa(X) = -\infty$, and by Theorem 3 (i) we have $A_2(X, L) = g_2(X, L) + 1 \geq 1$ and this is impossible. Hence we see that $h^1(\mathscr{O}_X) \geq 1$ if $(g_1(X, L), L^3) = (1, 1)$. Therefore if $(g_1(X, L), L^3) = (1, 1)$ (resp. $(0, 2)$), then by [8, Corollaries (4.8) and (4.9)] we see that (X, L) is the type (b.1) (resp. (b.2)) in Theorem 8 (b).

(c) Assume that $h^0(K_X + tL) = \binom{t-1}{3}$ for some $t \geq 4$. Then by the proof of (c) in Theorem 7, we see that $A_0(X,L) = 1$ and $A_1(X,L) = 0$. Hence $g_1(X,L) = 0$ and $L^3 = 1$. So we get the assertion (c) by the same argument as in the proof of Proposition 4 (ii).

These complete the proof of Theorem 8. □

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