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NON-EXISTENCE OF CONTACT TOTALLY UMBILICAL PROPER SLANT SUBMANIFOLDS OF A KENMOTSU MANIFOLD

Abstract. In this paper, we prove that there do not exist totally contact umbilical proper slant submanifolds of a Kenmotsu manifold and Kenmotsu space form $\overline{M}(c)$ for $c = -1$.

1. Introduction

The notion of a slant submanifold of an almost Hermitian manifold was introduced by Chen [5, 4]. On the other hand, A. Lotta [11] has defined and studied slant submanifolds of an almost contact metric manifold. He has also studied the intrinsic geometry of 3-dimensional non-anti-invariant slant submanifolds of K-Contact manifolds [12]. Later, L. Cabrerizo et al. [3] investigated slant submanifolds of a Sasakian manifold and obtained many interesting results. Afterwards, we have also studied slant submanifolds of Kenmotsu manifolds and trans-Sasakian manifolds [8, 6, 7]. Recently, V. Khan et al [10] have studied slant and semi-slant submanifolds of a Kenmotsu manifold and obtained that a totally contact umbilical semi-slant submanifold of a Kenmotsu manifold is totally contact geodesic if the invariant distribution is integrable. In this paper, we prove that a contact totally umbilical proper slant submanifold of a Kenmotsu manifold with structure $\{\varphi,\xi,\eta,g\}$ is necessarily totally geodesic, provided ξ is tangent to submanifold.

2. Preliminaries

An odd-dimensional Riemannian manifold $\overline{M}$ is said to be an almost contact metric manifold if there exist structure tensors $\{\varphi,\xi,\eta,g\}$, where φ is a (1,1) tensor field, ξ a vector field, η a 1-form and g is a Riemannian metric on $\overline{M}$ satisfying

$$\varphi^2 X = -X + \eta(X)\xi, \quad \varphi\xi = 0, \quad \eta(\xi) = 1, \quad \eta(\varphi X) = 0$$

$$g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\eta(Y), \quad \eta(X) = g(X,\xi)$$

for any $X,Y \in T\overline{M}$, where $T\overline{M}$ denotes the Lie algebra of vector fields on $\overline{M}$. An almost contact metric manifold is called a Kenmotsu manifold (see [9]) if

$$(\overline{\nabla}_X \varphi)(Y) = g(\varphi X, Y)\xi - \eta(Y)\varphi X$$

where $\overline{\nabla}$ denotes the Levi-Civita connection on $\overline{M}$. On a Kenmotsu manifold we also have the formula: $\overline{\nabla}_X \xi = X - \eta(X)\xi$.

Let M be an m-dimensional Riemannian manifold with induced metric g isometrically immersed in an n-dimensional Kenmotsu manifold $\overline{M}$. We denote by TM the Lie algebra of vector fields on M and by $TM^{\perp}$ the set of all vector fields normal to M. For any $X \in TM$ and $N \in TM^{\perp}$, we write

$$\varphi X = PX + FX, \quad \varphi N = tN + fN \tag{1}$$

where PX (resp. FX) denotes the tangential (resp. normal) component of φX, and tN (resp. fN) denotes the tangential (resp. normal) component of φN. In view of (1), we have

$$T_x\overline{M} = T_xM \oplus F(T_xM) \oplus \mu_x,$$

where μ_x is orthogonal complement to $F(T_xM)$ in $T_xM^{\perp}$.

In what follows, we suppose that the structure vector field ξ is tangent to M. Hence if we denote by D the orthogonal distribution to ξ in TM, we can consider the orthogonal direct decomposition $TM = D \oplus \xi$.

For each non zero X tangent to M at x and not proportional to ξ_x, we denote by $\theta(X)$ the Wirtinger angle of X, that is, the angle between φX and T_xM.

The submanifold M is called slant if the Wirtinger angle $\theta(X)$ is a constant, which is independent of the choice of $x \in M$ and X [11]. The Wirtinger angle θ of a slant submanifold is called the slant angle of the submanifold. Invariant and anti-invariant submanifolds are slant submanifolds with slant angle θ equal to 0 and $\frac{\pi}{2}$, respectively. A slant submanifold which is neither invariant nor anti-invariant is called a proper slant submanifold.

Let ∇ be be the Riemannian connection on M. Then the Gauss and Weingarten formulae are

$$\overline{\nabla}_X Y = \nabla_X Y + h(X,Y)$$

$$\overline{\nabla}_X N = -A_N X + \nabla^{\perp}_X N$$

for $X,Y \in TM$ and $N \in TM^{\perp}$; h and A_N are the second fundamental forms related by

$$g(A_N X,Y) = g(h(X,Y),N)$$

where $\nabla^{\perp}$ is the connection in the normal bundle $TM^{\perp}$.

Similar to the concept of contact totally umbilical submanifold of a Sasakian manifold introduced in the book of Yano and Kon (cf. [13, page 374]), we define:

DEFINITION 1. *If the second fundamental form h of a submanifold M, tangent to the structure vector field ξ, of a Kenmotsu manifold , is of the form*

$$h(X,Y) = \left[g(X,Y) - \eta(X)\eta(Y)\right]\alpha \tag{2}$$

for any $X,Y \in TM$, where α is a vector field normal to M, then M is called contact totally umbilical. Furthermore M is called totally geodesic if α = 0.

The mean curvature vector H is defined by $H = \frac{1}{m}\,\mathrm{trace}\,h$. We say that M is minimal if H vanishes identically.

We mention the following results for later use:

THEOREM A ([3]). *Let M be a submanifold of an almost contact metric manifold $\overline{M}$ such that $\xi \in TM$. Then M is slant if and only if there exists a constant $\lambda \in [0,1]$ such that*

$$P^2 = -\lambda(I - \eta \otimes \xi).$$

Furthermore, if θ is the slant angle of M, then $\lambda = \cos^2\theta$.

COROLLARY B ([3]). *Let M be a submanifold of an almost contact metric manifold $\overline{M}$ with slant angle θ. Then for any $X,Y \in TM$, we have*

$$g(PX,PY) = \cos^2\theta(g(X,Y) - \eta(X)\eta(Y)), \tag{3}$$

$$g(FX,FY) = \sin^2\theta(g(X,Y) - \eta(X)\eta(Y)). \tag{4}$$

Let M be an m-dimensional proper slant submanifold of an n-dimensional Kenmotsu manifold $\overline{M}$. Then $F(T_xM)$ is a subspace of $T_xM^{\perp}$. Then for $x \in M$, there exists an invariant subspace μ_x of $T_x\overline{M}$ for which we have

$$T_x\overline{M} = T_xM \oplus F(T_xM) \oplus \mu_x,$$

in view of (1).

3. Main result

THEOREM 1. *Every contact totally umbilical proper slant submanifold M of a Kenmotsu manifold $\overline{M}$ such that the structure vector field ξ is tangent to M is totally geodesic, provided $\nabla^{\perp}_X\alpha \in \mu$ for every $X \in TM$.*

Proof. First, a formula for covariant derivative of F is already known, see for instance [10, (2.19)]:

$$(\nabla_X F)Y = h(X,PY) + fh(X,Y) - \eta(Y)FX$$

for $X,Y \in TM$, following the notations as in (1). Now, assuming (2) and using (3), this yields, for any (local) unit vector field X tangent to M and orthogonal to ξ:

$$(\nabla_{PX}F)X = -g(PX,PX)\alpha = -(\cos^2\theta)\alpha \tag{5}$$

The left hand side of this equation is orthogonal to FX as a consequence of (4); hence, $g(\alpha,FX) = 0$. It follows that $\alpha \in \mu$. Finally, taking the inner product of both sides of (5) with α, one gets

$$g(\nabla^{\perp}_{PX}FX,\alpha) = -(\cos^2\theta)g(\alpha,\alpha) \tag{6}$$

because $g(F\nabla_{PX}X,\alpha) = 0$ since $\alpha \in \mu$. Now assuming also that $\nabla^{\perp}_X\alpha \in \mu$ for every $X \in TM$, the left hand side of (6) must vanish identically, and result follows. □

THEOREM 2. *Every contact totally umbilical proper slant submanifold M of a Kenmotsu space form $\overline{M}(c)$ such that the structure vector field ξ is tangent to M is totally geodesic, provided $c = -1$.*

Proof. The curvature tensor $\overline{R}$ of Kenmotsu space form $\overline{M}(c)$ with constant curvature $c = -1$ is given (see [9]) by

$$\overline{R}(X,Y)Z = g(X,Z)Y - g(Y,Z)X$$

for any X, Y and Z vector fields on $\overline{M}$. Then, we have the Codazzi equation

$$(\nabla_X h)(Y,Z) - (\nabla_Y h)(X,Z) = 0. \tag{7}$$

Taking the covariant derivative of (2), we get

$$(\nabla_X h)(Y,Z) = -\alpha\{\eta(Z)g(\phi X,\phi Y) + \eta(Y)g(\phi X,\phi Z)\} + g(\phi Z,\phi Y)\nabla^{\perp}_X\alpha. \tag{8}$$

Using (8) in (7), and taking the inner product of both sides with FY, we find that

$$g(\phi Z,\phi Y)g(\nabla^{\perp}_X\alpha,FY) - g(\phi Z,\phi X)g(\nabla^{\perp}_Y\alpha,FY) = 0. \tag{9}$$

On the other hand, using (2) and $g(\alpha,FY) = 0$, it is easy to prove that

$$g(\nabla^{\perp}_X\alpha,FY) = g(X,PY)g(\alpha,\alpha). \tag{10}$$

From (10), we have $g(\nabla^{\perp}_Y\alpha,FY) = 0$. Using this fact in (9), we get

$$g(\phi Z,\phi Y)g(\nabla^{\perp}_X\alpha,FY) = 0.$$

Since $g(\phi Z,\phi Y) \neq 0$ in general, one obtains $g(\nabla^{\perp}_X\alpha,FY) = 0$. Consequently, the result follows from (10). □

REMARK 1. It is easy to see that an invariant submanifold of a Kenmotsu manifold $\overline{M}$ with structure vector field tangent to M is minimal. This is because $\theta = 0$ for an invariant submanifold M of $\overline{M}$; from (6), it follows that $\alpha = 0$ which implies that M is minimal. Thus, from Theorem 1, we can say that every contact totally umbilical invariant submanifold of a Kenmotsu manifold is totally geodesic. We can also see that if M is $(m+1)$-dimensional proper slant submanifold of $(2m+1)$-dimensional Kenmotsu manifold $\overline{M}$, then μ={0}, which shows that $F(T_pM) = T_pM^{\perp}$. We get that $\alpha = 0$, which implies that the proof of Theorem 1 is valid in this case as well.

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