Rend. Sem. Mat. Univ. Politec. Torino
Vol. 69, 1 (2011), 37 – 49

X. Q. Lin*

FURTHER RESULTS ON UNIQUENESS OF ENTIRE FUNCTIONS SHARING ONE VALUE

Abstract. In this paper, we study the uniqueness problems of entire functions sharing one value with weight ℓ ($\ell = 0,1,2$). The results in this paper improve the related results given by X.Y. Zhang and W.C. Lin, M.L. Fang, C.C. Yang and X.H. Hua, etc.

1. Introduction and main results

In this paper, a meromorphic function means meromorphic in the open complex plane. We shall use the standard notations in Nevanlinna's value distribution theory of meromorphic functions such as $T(r,f)$, $N(r,f)$, $\overline{N}(r,f)$, $m(r,f)$ and so on (see [2, 7]). The notation $S(r,f)$ is defined to be any quantity satisfying $S(r,f) = o\{T(r,f)\}$ as $r \to \infty$ possibly outside a set of r of finite linear measure.

Let f and g be two nonconstant meromorphic functions, a be a complex number. We say that f and g share the value a CM (counting multiplicities) if $f-a$ and $g-a$ have the same zeros with the same multiplicities. Similarly, if $f-a$ and $g-a$ have the same zeros ignoring multiplicities, then we say f and g share the value a IM. Let ℓ be a nonnegative integer or ∞. We denote by $E_\ell(a,f)$ the set of all a-points of f, where an a-point with multiplicity m is counted m times if $m \le \ell$, and $\ell+1$ times if $m > \ell$. If $E_\ell(a,f) = E_\ell(a,g)$, we say that f and g share the value a with weight ℓ. Clearly, if f and g share the value a with weight ℓ, then for any integer p ($0 \le p < \ell$), f and g share the value a with weight p. Also we note that f and g share the value a CM or IM if and only if $\ell = \infty$ or $\ell = 0$ respectively. So, the weight ℓ of a weighted sharing value is used to measure how close the shared value is to being shared CM or to being shared IM (see [3]).

Let m be a positive integer, we also denote by m^* (see [9]) the product $\chi_\mu m$, where

$$\chi_\mu = \begin{cases} 0, & \mu = 0 \\ 1, & \mu \neq 0\,. \end{cases}$$

In 1993, Wang [4] proved the following result.

Theorem A. Let f be a transcendental entire function, and n,k be two positive integers with $n \ge k+1$. Then $(f^n)^{(k)} = 1$ has infinitely many solutions.

Many authors have interest in establishing a uniqueness theorem corresponding to the above result. In 1997, C.C. Yang and X.H. Hua [5] firstly obtained the following:

*The project was supported by NSF of Fujian Province of China (No. S0750013), the Research Foundation of Ningde Normal University (No. 2008J001) and the Special Grant for Scientific Research of Universities under the Leadership of the Provincial Educational Department (No. JK2010062).

Theorem B. Let f and g be two non-constant entire functions and $n > 6$ an integer. If $f^n f'$ and $g^n g'$ share 1 CM, then either $f \equiv dg$ for some $(n+1)th$ root of unity d or $g(z) = c_1 e^{cz}$ and $f(z) = c_2 e^{-cz}$, where c_1, c_2 and c are constants and satisfy $(c_1 c_2)^{n+1} c^2 = -1$.

In 2002, Fang [1] improved Theorem B to the *k-th* derivative and obtained the following theorems.

Theorem C. Let f and g be two non-constant entire functions and let n, k be two positive integers with $n > 2k+4$. If $(f^n)^{(k)}$ and $(g^n)^{(k)}$ share 1 CM, then either $f(z) = c_1 e^{cz}$, $g(z) = c_2 e^{-cz}$, where c_1, c_2 and c are three constants satisfying $(-1)^k (c_1 c_2)^n (nc)^{2k} = 1$, or $f \equiv tg$ for a constant t such that $t^n = 1$.

Theorem D. Let f and g be two non-constant entire functions and let n, k be two positive integers with $n \geq 2k+8$. If $[f^n(f-1)]^{(k)}$ and $[g^n(g-1)]^{(k)}$ share 1 CM, then $f \equiv g$.

In 2008, X.Y. Zhang and W.C. Lin [9] extended the above results by proving the following theorems.

Theorem E. Let f and g be two non-constant entire functions and let n, m and k be three positive integers with $n > 2k + m^* + 4$, and λ, μ be constants such that $|\lambda| + |\mu| \neq 0$. If $[f^n(\mu f^m + \lambda)]^{(k)}$ and $[g^n(\mu g^m + \lambda)]^{(k)}$ share 1 CM, then

(i) when $\lambda\mu \neq 0$, $f \equiv tg$ for a constant t such that $t^n = 1$ and $t^m = 1$;

(ii) when $\lambda\mu = 0$, either $f \equiv tg$, where t is a constant satisfying $t^{n+m^*} = 1$, or $f(z) = c_1 e^{cz}, g(z) = c_2 e^{-cz}$, where c_1, c_2 and c are three constants satisfying

$$(-1)^k \lambda^2 (c_1 c_2)^{n+m^*} [(n+m^*)c)]^{2k} = 1 \ \text{ or } \ (-1)^k \mu^2 (c_1 c_2)^{n+m^*} [(n+m^*)c)]^{2k} = 1.$$

Theorem F. Let f and g be two non-constant entire functions and let n, m and k be three positive integers with $n > 2k+m+4$. If $[f^n(f-1)^m]^{(k)}$ and $[g^n(g-1)^m]^{(k)}$ share 1 CM, then either $f \equiv g$, or f and g satisfy the algebraic equation $R(f, g) \equiv 0$, where $R(w_1, w_2) = w_1^n (w_1 - 1)^m - w_2^n (w_2 - 1)^m$.

In this paper, we establish the following theorems which improve the above related results.

THEOREM 1. *Let f and g be two non-constant entire functions, and let n, m and k be three positive integers, and λ, μ be constants such that $|\lambda| + |\mu| \neq 0$. If $E_\ell(1, [f^n(\mu f^m + \lambda)]^{(k)}) = E_\ell(1, [g^n(\mu g^m + \lambda)]^{(k)})$, and one of the following conditions holds:*

(i) $\ell = 2$ *and* $n > 2k + 4 + m^*$*;*

(ii) $\ell = 1$ *and* $n > \dfrac{5k+9+3m^*}{2}$*;*

(iii) $\ell = 0$ *and* $n > 5k + 7 + 4m^*$.

Then the conclusion of Theorem E holds.

THEOREM 2. *Let f and g be two non-constant entire functions and let n,m and k be three positive integers. If $E_\ell(1,[f^n(f-1)^m]^{(k)}) = E_\ell(1,[g^n(g-1)^m]^{(k)})$, and one of the following conditions holds:*

(i) $\ell = 2$ *and* $n > 2k+4+m$*;*

(ii) $\ell = 1$ *and* $n > \dfrac{5k+9+3m}{2}$*;*

(iii) $\ell = 0$ *and* $n > 5k+7+4m$*.*

Then the conclusion of Theorem F holds.

By Theorem 1, we immediately obtain the following results.

COROLLARY 1. *Let f and g be two non-constant entire functions and let n,k be two positive integers with $n > 2k+4$. If $E_2(1,[f^n]^{(k)}) = E_2(1,[g^n]^{(k)})$, then either $f(z) = c_1e^{cz}$, $g(z) = c_2e^{-cz}$, where c_1,c_2 and c are three constants satisfying $(-1)^k(c_1c_2)^n(nc)^{2k} = 1$, or $f \equiv tg$ for a constant t such that $t^n = 1$.*

COROLLARY 2. *Let f and g be two nonconstant entire functions and let n,k be two positive integers with $n > 5k+7$. If $[f^n]^{(k)}$ and $[g^n]^{(k)}$ share 1 IM, then either $f(z) = c_1e^{cz}$, $g(z) = c_2e^{-cz}$, where c_1,c_2 and c are three constants satisfying $(-1)^k(c_1c_2)^n(nc)^{2k} = 1$, or $f \equiv tg$ for a constant t such that $t^n = 1$.*

Moreover, from Theorem 2, we can also deduce the following results.

COROLLARY 3. *Let f and g be two non-constant entire functions and let n,k be two positive integers with $n > 2k+5$. If $E_2(1,[f^n(f-1)]^{(k)}) = E_2(1,[g^n(g-1)]^{(k)})$, then $f \equiv g$.*

COROLLARY 4. *Let f and g be two non-constant entire functions and let n,k be two positive integers with $n > 5k+11$. If $[f^n(f-1)]^{(k)}$ and $[g^n(g-1)]^{(k)}$ share 1 IM, then $f \equiv g$.*

For convenience, we now explain some notation that will be used in the paper.

DEFINITION 1. Suppose f and g share the value 1 IM. Let z_0 be a common 1-point of f and g with multiplicity p and q respectively. We denote by $\overline{N}_L(r,1;f)$ the reduced counting function of those 1-points of f and g where $p > q$, by $\overline{N}_E^{(3}(r,1;f)$ (resp. $\overline{N}_E^{(2}(r,1;f)$) the reduced counting function of those 1-points of f and g where $p = q \geq 3$ (resp. $p = q \geq 2$). In the same way, we define $\overline{N}_L(r,1;g)$, $\overline{N}_E^{(3}(r,1;g)$ and $\overline{N}_E^{(2}(r,1;g)$. In particular, we denote by $\overline{N}_{L2}(r,1;f)$ (resp. $\overline{N}_{L1}(r,1;f)$) the reduced counting function of those 1-points of f and g where $2 = q < p$ (resp. $1 = q < p$). Similarly, we can define $\overline{N}_{L2}(r,1;g)$ and $\overline{N}_{L1}(r,1;g)$. In addition, we denote by $N_E^{1)}(r,1;f)$ (resp. $N_E^{1)}(r,1;g)$) the counting function of those common simple 1-points of f and g, each point is counted only once.

DEFINITION 2. Let p be a positive integer and $a \in \mathcal{C} \cup \{\infty\}$, we denote by $N_{p)}(r,a;f)$ the counting function of the zeros of $f-a$ whose multiplicities are not greater than p, by $N_{(p}(r,a;f)$ the counting function of the zeros of $f-a$ with multiplicities at least p, and by $\overline{N}_{p)}(r,a;f)$, $\overline{N}_{(p}(r,a;f)$ the corresponding reduced counting functions. Also by $\overline{N}(r,a;f)$ the reduced counting function of those a-points of f. Moreover, we set $N_p(r,a;f) = \overline{N}(r,a;f) + \overline{N}_{(2}(r,a;f) + \cdots + \overline{N}_{(p}(r,a;f)$. Clearly, $N_1(r,a;f) = \overline{N}(r,a;f)$.

DEFINITION 3. Let F, G be two nonconstant entire functions. We denote by $N_0(r,0;F')$ the counting function of those zeros of F' which are not the zeros of $F(F-1)$, by $\overline{N}_0(r,0;F')$ the corresponding reduced counting function. Similarly, we can define $N_0(r,0;G')$ and $\overline{N}_0(r,0;G')$.

2. Some lemmas

For the proof of our results, we need the following lemmas.

LEMMA 1 ([8]). *Let $f(z)$ be non-constant meromorphic function and k be a positive integer. Then*

$$N_p(r,0;f^{(k)}) \le N_{p+k}(r,0;f) + k\overline{N}(r,\infty;f) + S(r,f).$$

LEMMA 2 ([6]). *Let $f(z)$ be non-constant meromorphic function and k be a positive integer, then*

$$N(r,0;f^{(k)}) \le T(r,f^{(k)}) - T(r,f) + N(r,0;f) + S(r,f), \tag{1}$$

$$N(r,0;f^{(k)}) \le N(r,0;f) + k\overline{N}(r,\infty;f) + S(r,f). \tag{2}$$

LEMMA 3. *Let F and G be two nonconstant entire functions. Suppose that $E_\ell(1,F) = E_\ell(1,G)$.*

(i) *When $\ell = 1$,*

$$\overline{N}_{L2}(r,1;F) \le \tfrac{1}{2}\overline{N}(r,0;F) + S(r,F), \quad \overline{N}_{L2}(r,1;G) \le \tfrac{1}{2}\overline{N}(r,0;G) + S(r,G). \tag{3}$$

(ii) *When $\ell = 0$,*

$$\overline{N}_L(r,1;F) \le \overline{N}(r,0;F) + S(r,F), \quad \overline{N}_L(r,1;G) \le \overline{N}(r,0;G) + S(r,G). \tag{4}$$

Proof. Note that F is a nonconstant entire function, from (2) in Lemma 2, we have

$$N(r,1;F) - \overline{N}(r,1;F) + N(r,0;F) - \overline{N}(r,0;F) \le N(r,0;F') \le N(r,0;F) + S(r,F).$$

This shows that

$$N(r,1;F) - \overline{N}(r,1;F) \le \overline{N}(r,0;F) + S(r,F). \tag{5}$$

(i) When $\ell = 1$,

$$\overline{N}_{L2}(r,1;F) \;\leq\; \tfrac{1}{2}[N(r,1;F) - \overline{N}(r,1;F)].$$

Combining this with (5), we obtain

$$\overline{N}_{L2}(r,1;F) \;\leq\; \tfrac{1}{2}\overline{N}(r,0;F) + S(r,F).$$

Similarly, we have

$$\overline{N}_{L2}(r,1;G) \;\leq\; \tfrac{1}{2}\overline{N}(r,0;G) + S(r,G).$$

(ii) When $\ell = 0$,

$$\overline{N}_{L}(r,1;F) \;\leq\; N(r,1;F) - \overline{N}(r,1;F).$$

Combining this with (5), we obtain

$$\overline{N}_{L}(r,1;F) \;\leq\; \overline{N}(r,0;F) + S(r,F).$$

Similarly, we have

$$\overline{N}_{L}(r,1;G) \;\leq\; \overline{N}(r,0;G) + S(r,G).$$

This completes the proof of Lemma 3. $\square$

LEMMA 4. *Let F and G be two nonconstant entire functions. Set*

$$H = \left[\frac{F''}{F'} - 2\frac{F'}{F-1}\right] - \left[\frac{G''}{G'} - 2\frac{G'}{G-1}\right].$$

If $E_\ell(1,F) = E_\ell(1,G)$ and $H \not\equiv 0$.

(i) *When $\ell = 2$,*

$$T(r,F) + T(r,G) \leq 2[N_2(r,0;F) + N_2(r,0;G)] + S(r,F) + S(r,G). \tag{6}$$

(ii) *When $\ell = 1$,*

$$\begin{aligned} T(r,F) + T(r,G) \quad &\leq \quad 2[N_2(r,0;F) + N_2(r,0;G)] + \tfrac{1}{2}[\overline{N}(r,0;F) + \overline{N}(r,0;G)] \\ &\quad\; + S(r,F) + S(r,G). \end{aligned} \tag{7}$$

(iii) *When $\ell = 0$,*

$$\begin{aligned} T(r,F) + T(r,G) \quad &\leq \quad 2[N_2(r,0;F) + N_2(r,0;G)] + 3[\overline{N}(r,0;F) + \overline{N}(r,0;G)] \\ &\quad\; + S(r,F) + S(r,G). \end{aligned} \tag{8}$$

Proof. By $H \not\equiv 0$ and a simple calculation, we get

$$N_E^{1)}(r,1;F) \leq N(r,0;H) \leq T(r,H) + O(1) \leq N(r,\infty;H) + S(r,F) + S(r,G). \tag{9}$$

Since $E_\ell(1,F) = E_\ell(1,G)$ and F, G are two nonconstant entire functions, we get

$$\begin{aligned} N(r,\infty;H) \quad &\leq \quad \overline{N}_{(2}(r,0;F) + \overline{N}_{(2}(r,0;G) + \overline{N}_L(r,1;F) + \overline{N}_L(r,1;G) \\ &\quad\; + \overline{N}_0(r,0;F') + \overline{N}_0(r,0;G'). \end{aligned} \tag{10}$$

Now we consider the following three cases:

(i) For $\ell = 2$, we note that

$$\frac{1}{2}[\overline{N}_E^{(3}(r,1;F)+\overline{N}_L(r,1;F)+\overline{N}_L(r,1;G)] \le \frac{1}{2}N(r,1;F)-\frac{1}{2}N_{1)}(r,1;F)-\overline{N}_{(2}(r,1;F).$$

So

$$\begin{aligned}\frac{1}{2}N_{1)}(r,1;F)&+\overline{N}_{(2}(r,1;F)\\ &\le \frac{1}{2}N(r,1;F)-\frac{1}{2}[\overline{N}_E^{(3}(r,1;F)+\overline{N}_L(r,1;F)+\overline{N}_L(r,1;G)].\end{aligned} \tag{11}$$

By symmetry,

$$\begin{aligned}\frac{1}{2}N_{1)}(r,1;G)&+\overline{N}_{(2}(r,1;G)\\ &\le \frac{1}{2}N(r,1;G)-\frac{1}{2}[\overline{N}_E^{(3}(r,1;G)+\overline{N}_L(r,1;G)+\overline{N}_L(r,1;F)].\end{aligned} \tag{12}$$

Combining with (9)–(12), we deduce that

$$\begin{aligned}
&\overline{N}(r,1;F)+\overline{N}(r,1;G)\\
&\quad= \frac{1}{2}N_{1)}(r,1;F)+\overline{N}_{(2}(r,1;F)+\frac{1}{2}N_{1)}(r,1;G)+\overline{N}_{(2}(r,1;G)+N_E^{1)}(r,1;F)\\
&\quad\le \frac{1}{2}N(r,1;F)+\frac{1}{2}N(r,1;G)-[\overline{N}_E^{(3}(r,1;F)+\overline{N}_L(r,1;F)+\overline{N}_L(r,1;G)]\\
&\qquad +\overline{N}_{(2}(r,0;F)+\overline{N}_{(2}(r,0;G)+\overline{N}_L(r,1;F)+\overline{N}_L(r,1;G)+\overline{N}_0(r,0;F')\\
&\qquad +\overline{N}_0(r,0;G')+S(r,F)+S(r,G)\\
&\quad\le \frac{1}{2}N(r,1;F)+\frac{1}{2}N(r,1;G)+\overline{N}_{(2}(r,0;F)+\overline{N}_{(2}(r,0;G)\\
&\qquad +\overline{N}_0(r,0;F')+\overline{N}_0(r,0;G')+S(r,F)+S(r,G).
\end{aligned} \tag{13}$$

On the other hand, by the second fundamental theorem, we get

$$\begin{aligned}T(r,F)+T(r,G) \le\ & \overline{N}(r,1;F)+\overline{N}(r,1;G)+\overline{N}(r,0;F)+\overline{N}(r,0;G)\\ &-N_0(r,0;F')-N_0(r,0;G')+S(r,F)+S(r,G).\end{aligned}$$

Substituting (13) into this, we get

$$\begin{aligned}
&T(r,F)+T(r,G)\\
&\quad\le \frac{1}{2}N(r,1;F)+\frac{1}{2}N(r,1;G)+\overline{N}_{(2}(r,0;F)+\overline{N}_{(2}(r,0;G)\\
&\qquad +\overline{N}(r,0;F)+\overline{N}(r,0;G)+S(r,F)+S(r,G)\\
&\quad\le \frac{1}{2}[T(r,F)+T(r,G)]+N_2(r,0;F)+N_2(r,0;G)+S(r,F)+S(r,G).
\end{aligned} \tag{14}$$

This implies

$$T(r,F)+T(r,G)\le 2[N_2(r,0;F)+N_2(r,0;G)]+S(r,F)+S(r,G).$$

(ii) For $\ell = 1$, we note that

$$\begin{aligned}\frac{1}{2}[\overline{N}_E^{(3}(r,1;F)&+\overline{N}_L(r,1;F)+\overline{N}_L(r,1;G)-\overline{N}_{L2}(r,1;G)]\\ &\le \frac{1}{2}N(r,1;F)-\frac{1}{2}N_{1)}(r,1;F)-\overline{N}_{(2}(r,1;F)\end{aligned}$$

So

$$
\begin{aligned}
&\tfrac{1}{2}N_{1)}(r,1;F)+\overline{N}_{(2}(r,1;F)\\
(15)\qquad &\leq\ \tfrac{1}{2}N(r,1;F)-\tfrac{1}{2}[\overline{N}_E^{(3}(r,1;F)+\overline{N}_L(r,1;F)+\overline{N}_L(r,1;G)]+\tfrac{1}{2}\overline{N}_{L2}(r,1;G).
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
&\tfrac{1}{2}N_{1)}(r,1;G)+\overline{N}_{(2}(r,1;G)\\
(16)\qquad &\leq\ \tfrac{1}{2}N(r,1;G)-\tfrac{1}{2}[\overline{N}_E^{(3}(r,1;G)+\overline{N}_L(r,1;G)+\overline{N}_L(r,1;F)]+\tfrac{1}{2}\overline{N}_{L2}(r,1;F).
\end{aligned}
$$

Combining (15) and (16) with (9) and (10), we obtain

$$
\begin{aligned}
&\overline{N}(r,1;F)+\overline{N}(r,1;G)\\
&\quad= \tfrac{1}{2}N_{1)}(r,1;F)+\overline{N}_{(2}(r,1;F)+\tfrac{1}{2}N_{1)}(r,1;G)+\overline{N}_{(2}(r,1;G)+N_E^{1)}(r,1;F)\\
&\quad\leq \tfrac{1}{2}N(r,1;F)+\tfrac{1}{2}N(r,1;G)-[\overline{N}_E^{(3}(r,1;F)+\overline{N}_L(r,1;F)+\overline{N}_L(r,1;G)]\\
&\qquad +\tfrac{1}{2}[\overline{N}_{L2}(r,1;G)+\overline{N}_{L2}(r,1;F)]+[\overline{N}_{(2}(r,0;F)+\overline{N}_{(2}(r,0;G)+\overline{N}_L(r,1;F)\\
&\qquad +\overline{N}_L(r,1;G)+\overline{N}_0(r,0;F')+\overline{N}_0(r,0;G')+S(r,F)+S(r,G)]\\
&\quad\leq \tfrac{1}{2}[N(r,1;F)+N(r,1;G)]+\tfrac{1}{2}[\overline{N}_{L2}(r,1;G)+\overline{N}_{L2}(r,1;F)]\\
&\qquad +\overline{N}_{(2}(r,0;F)+\overline{N}_{(2}(r,0;G)+\overline{N}_0(r,0;F')+\overline{N}_0(r,0;G')+S(r,F)+S(r,G).
\end{aligned}
$$

Since $\ell=1$, substituting (3) into the above inequality, we get

$$
\begin{aligned}
&\overline{N}(r,1;F)+\overline{N}(r,1;G)\\
&\quad\leq \tfrac{1}{2}[N(r,1;F)+N(r,1;G)]+\tfrac{1}{4}[\overline{N}(r,0;F)+\overline{N}(r,0;G)]\\
&\qquad +\overline{N}_{(2}(r,0;F)+\overline{N}_{(2}(r,0;G)+\overline{N}_0(r,0;F')+\overline{N}_0(r,0;G')+S(r,F)+S(r,G).
\end{aligned}
$$

Combining this with the second fundamental theorem, similar to the proof of (14), we can deduce that

$$
\begin{aligned}
&T(r,F)+T(r,G)\\
&\quad\leq \tfrac{1}{2}[N(r,1;F)+N(r,1;G)]+\tfrac{1}{4}[\overline{N}(r,0;F)+\overline{N}(r,0;G)]\\
&\qquad +\overline{N}_{(2}(r,0;F)+\overline{N}_{(2}(r,0;G)+\overline{N}(r,0;F)+\overline{N}(r,0;G)+S(r,F)+S(r,G)\\
&\quad\leq \tfrac{1}{2}[T(r,F)+T(r,G)]+\tfrac{1}{4}[\overline{N}(r,0;F)+\overline{N}(r,0;G)]+N_2(r,0;F)+N_2(r,0;G)\\
&\qquad +S(r,F)+S(r,G).
\end{aligned}
$$

This yields

$$
\begin{aligned}
T(r,F)+T(r,G)\ \leq\ &\tfrac{1}{2}[\overline{N}(r,0;F)+\overline{N}(r,0;G)]+2[N_2(r,0;F)+N_2(r,0;G)]\\
&+S(r,F)+S(r,G).
\end{aligned}
$$

(iii) For $\ell=0$, we note that

$$
(17)\qquad \overline{N}(r,1;F)\ =\ N_E^{1)}(r,1;F)+\overline{N}_E^{(2}(r,1;F)+\overline{N}_L(r,1;F)+\overline{N}_L(r,1;G).
$$

On the other hand, we have

$$\overline{N}_E^{(2}(r,1;G)+\overline{N}_L(r,1;G)+\overline{N}_L(r,1;F)-\overline{N}_{L1}(r,1;F)\leq N(r,1;G)-\overline{N}(r,1;G).$$

This and (17) yield

$$\begin{aligned}
&\overline{N}(r,1;F)+\overline{N}(r,1;G)\\
&\quad\leq\ N_E^{1)}(r,1;F)+\overline{N}_E^{(2}(r,1;F)+\overline{N}_L(r,1;F)+\overline{N}_L(r,1;G)+N(r,1;G)\\
&\qquad -[\overline{N}_E^{(2}(r,1;G)+\overline{N}_L(r,1;G)+\overline{N}_L(r,1;F)-\overline{N}_{L1}(r,1;F)]\\
&\quad\leq\ N_E^{1)}(r,1;F)+\overline{N}_{L1}(r,1;F)+T(r,G)+O(1).
\end{aligned}$$

Combining this with (9) and (10) and applying (4) in Lemma 3, we deduce

$$\begin{aligned}
&\overline{N}(r,1;F)+\overline{N}(r,1;G)\\
&\quad\leq\ \overline{N}_{(2}(r,0;F)+\overline{N}_{(2}(r,0;G)+\overline{N}_L(r,1;F)+\overline{N}_L(r,1;G)+\overline{N}_{L1}(r,1;F)\\
&\qquad +\overline{N}_0(r,0;F')+\overline{N}_0(r,0;G')+T(r,G)+S(r,F)+S(r,G)\\
&\quad\leq\ \overline{N}_{(2}(r,0;F)+\overline{N}_{(2}(r,0;G)+2\overline{N}(r,0;F)+\overline{N}(r,0;G)\\
&\qquad +\overline{N}_0(r,0;F')+\overline{N}_0(r,0;G')+T(r,G)+S(r,F)+S(r,G).
\end{aligned}$$

Combining this with the second fundamental theorem, we get

$$\begin{aligned}
T(r,F)\ &\leq\ \overline{N}(r,0;F)+\overline{N}_{(2}(r,0;F)+\overline{N}(r,0;G)+\overline{N}_{(2}(r,0;G)\\
&\qquad +2\overline{N}(r,0;F)+\overline{N}(r,0;G)+S(r,F)+S(r,G)\\
&=\ N_2(r,0;F)+N_2(r,0;G)+2\overline{N}(r,0;F)+\overline{N}(r,0;G)+S(r,F)+S(r,G).
\end{aligned}$$

Similarly, we have

$$T(r,G)\ \leq\ N_2(r,0;G)+N_2(r,0;F)+2\overline{N}(r,0;G)+\overline{N}(r,0;F)+S(r,F)+S(r,G).$$

The above two inequalities show

$$\begin{aligned}
T(r,F)+T(r,G)\ \leq\ 2[N_2(r,0;F)+N_2(r,0;G)]+3[\overline{N}(r,0;F)+\overline{N}(r,0;G)]\\
+S(r,F)+S(r,G).
\end{aligned}$$

This completes the proof of Lemma 4. □

LEMMA 5 ([7]). *Let f be a nonconstant meromorphic function and let n be a positive integer. Suppose that $P(f)=a_nf^n+a_{n-1}f^{n-1}+...+a_1f+a_0$, where a_i are meromorphic functions such that $T(r,a_i)=S(r,f)$ $(i=0,1,2,\ldots,n)$ and $a_n\not\equiv 0$. Then*

$$T(r,P(f))=nT(r,f)+S(r,f).$$

3. Proof of Theorem 1

Proof. Set

$$\widetilde{F} = f^n(\mu f^m + \lambda), \qquad F = \widetilde{F}^{(k)}, \tag{18}$$

$$\widetilde{G} = g^n(\mu g^m + \lambda), \qquad G = \widetilde{G}^{(k)}, \tag{19}$$

and let H be defined as in Lemma 4. If

$$E_\ell(1, [f^n(\mu f^m + \lambda)]^{(k)}) = E_\ell(1, [g^n(\mu g^m + \lambda)]^{(k)}),$$

then $E_\ell(1,F) = E_\ell(1,G)$.

By Lemma 5 and (18), we have

$$\begin{aligned} T(r,\widetilde{F}) &= (n+m^*)T(r,f) + S(r,f), \\ N(r,0;\widetilde{F}) &\leq nN(r,0;f) + m^*T(r,f) + S(r,f). \end{aligned} \tag{20}$$

By (18) and applying (1) in Lemma 2 to $\widetilde{F}$, we get

$$\begin{aligned} T(r,\widetilde{F}) &\leq T(r,\widetilde{F}^{(k)}) - N(r,0;\widetilde{F}^{(k)}) + N(r,0;\widetilde{F}) + S(r,F) \\ &= T(r,F) - N(r,0;F) + N(r,0;\widetilde{F}) + S(r,f). \end{aligned} \tag{21}$$

(20) and (21) yield

$$(n+m^*)T(r,f) \leq T(r,F) - N(r,0;F) + nN(r,0;f) + m^*T(r,f) + S(r,f).$$

This shows

$$nT(r,f) \leq T(r,F) - N(r,0;F) + nN(r,0;f) + S(r,f).$$

Similarly, we have

$$nT(r,g) \leq T(r,G) - N(r,0;G) + nN(r,0;g) + S(r,g).$$

So

$$\begin{aligned} n[T(r,f) + T(r,g)] &\leq T(r,F) + T(r,G) - [N(r,0;F) + N(r,0;G)] \\ &\quad + n[N(r,0;f) + N(r,0;g)] + S(r,f) + S(r,g). \end{aligned} \tag{22}$$

On the other hand, note that if z_0 is a zero of f with multiplicity p then z_0 is a zero of $[f^n(\mu f^m + \lambda)]^{(k)}$ with multiplicity at least 3 since $np - k > (k+2)p - k \geq 2$, so we get

$$N_{(3}(r,0;F) - 2\overline{N}_{(3}(r,0;F) \quad \geq \quad (n-k-2)N(r,0;f).$$

From this we deduce that

$$\begin{aligned} N_2(r,0;F) &= \overline{N}(r,0;F) + \overline{N}_{(2}(r,0;F) \\ &= N(r,0;F) - [N_{(3}(r,0;F) - 2\overline{N}_{(3}(r,0;F)] \\ &\leq N(r,0;F) - (n-k-2)N(r,0;f). \end{aligned}$$

Similarly, we have

$$N_2(r,0;G) \quad \le \quad N(r,0;G)-(n-k-2)N(r,0;g).$$

Therefore

$$\begin{aligned} & N_2(r,0;F)+N_2(r,0;G) \\ (23) \qquad & \quad \le \quad N(r,0;F)+N(r,0;G)-(n-k-2)[N(r,0;f)+N(r,0;g)]. \end{aligned}$$

In addition, by Lemma 2, Lemma 5 and (18), we get

$$N(r,0;F) \;\le\; N(r,0;\widetilde{F})+S(r,\widetilde{F}) \le nN(r,0;f)+m^*T(r,f)+S(r,f).$$

Similarly, we have

$$N(r,0;G) \;\le\; N(r,0;\widetilde{G})+S(r,\widetilde{G}) \le nN(r,0;g)+m^*T(r,g)+S(r,g).$$

Therefore

$$\begin{aligned} & N(r,0;F)+N(r,0;G) \\ (24) \qquad & \quad \le \quad n[N(r,0;f)+N(r,0;g)]+m^*[T(r,f)+T(r,g)]+S(r,f)+S(r,g). \end{aligned}$$

Suppose that $H \not\equiv 0$. Now we consider the following three cases:

Case (i) $\ell = 2$ and $n > 2k+4+m^*$. By Lemma 4, we have

$$(25) \qquad T(r,F)+T(r,G) \;\le\; 2[N_2(r,0;F)+N_2(r,0;G)]+S(r,F)+S(r,G).$$

Combined with (22), (25), (23) and (24), we deduce that

$$\begin{aligned} & n[T(r,f)+T(r,g)] \\ & \quad \le \quad 2[N_2(r,0;F)+N_2(r,0;G)]-[N(r,0;F)+N(r,0;G)] \\ & \qquad\quad +n[N(r,0;f)+N(r,0;g)]+S(r,f)+S(r,g) \\ & \quad \le \quad 2[N(r,0;F)+N(r,0;G)-(n-k-2)(N(r,0;f)+N(r,0;g))] \\ & \qquad\quad -[N(r,0;F)+N(r,0;G)]+n[N(r,0;f)+N(r,0;g)] \\ & \qquad\quad +S(r,f)+S(r,g) \\ & \quad = \quad N(r,0;F)+N(r,0;G)-(n-2k-4)[N(r,0;f)+N(r,0;g)] \\ & \qquad\quad +S(r,f)+S(r,g) \\ & \quad \le \quad (2k+4)[N(r,0;f)+N(r,0;g)]+m^*[T(r,f)+T(r,g)] \\ (26) \qquad & \qquad\quad +S(r,f)+S(r,g). \end{aligned}$$

This contradicts the assumption $n > 2k+4+m^*$.

Case (ii) $\ell = 1$ and $n > (5k+9+3m^*)/2$. By Lemma 4, we have

$$T(r,F)+T(r,G) \leq 2[N_2(r,0;F)+N_2(r,0;G)]+\frac{1}{2}[\overline{N}(r,0;F)+\overline{N}(r,0;G)] +S(r,F)+S(r,G). \tag{27}$$

Combining with (22), (27), (23) and (24), similar to the proof of (26), we deduce

$$n[T(r,f)+T(r,g)] \leq (2k+4)[N(r,0;f)+N(r,0;g)]+m^*[T(r,f)+T(r,g)] +\frac{1}{2}[\overline{N}(r,0;F)+\overline{N}(r,0;G)]+S(r,f)+S(r,g). \tag{28}$$

Note that

$$\overline{N}(r,0;F) = N_1(r,0;\widetilde{F}^{(k)}),\ \overline{N}(r,0;G) = N_1(r,0;\widetilde{G}^{(k)}). \tag{29}$$

Applying Lemma 1, Lemma 5 and combining (29) with (18), we obtain

$$\begin{aligned}
&\overline{N}(r,0;F)+\overline{N}(r,0;G)\\
&\quad = N_1(r,0;\widetilde{F}^{(k)})+N_1(r,0;\widetilde{G}^{(k)})\\
&\quad \leq N_{1+k}(r,0;\widetilde{F})+N_{1+k}(r,0;\widetilde{G})+S(r,\widetilde{F})+S(r,\widetilde{G})\\
&\quad \leq (1+k)\overline{N}(r,0;f)+N(r,0;\mu f^m+\lambda)+(1+k)\overline{N}(r,0;g)\\
&\qquad +N(r,0;\mu g^m+\lambda)+S(r,f)+S(r,g)\\
&\quad \leq (1+k)[\overline{N}(r,0;f)+\overline{N}(r,0;g)]+m^*[T(r,f)+T(r,g)]+S(r,f)+S(r,g). \qquad (30)
\end{aligned}$$

Substituting (30) into (28), we get

$$\begin{aligned}
&n[T(r,f)+T(r,g)]\\
&\quad \leq (2k+4)[N(r,0;f)+N(r,0;g)]+m^*[T(r,f)+T(r,g)]\\
&\qquad +\frac{1}{2}(1+k)[\overline{N}(r,0;f)+\overline{N}(r,0;g)]+\frac{1}{2}m^*[T(r,f)+T(r,g)]+S(r,f)+S(r,g)\\
&\quad \leq \frac{1}{2}(5k+9)[N(r,0;f)+N(r,0;g)]+\frac{3}{2}m^*[T(r,f)+T(r,g)]+S(r,f)+S(r,g).
\end{aligned}$$

This is impossible, since $n > (5k+9+3m^*)/2$.

Case (iii) $\ell = 0$ and $n > 5k+7+4m^*$. By Lemma 4, we have

$$T(r,F)+T(r,G) \leq 2[N_2(r,0;F)+N_2(r,0;G)] +3[\overline{N}(r,0;F)+\overline{N}(r,0;G)]+S(r,F)+S(r,G). \tag{31}$$

Combining with (22), (31), (23) and (24), similarly to the proof of (26), we deduce

$$n[T(r,f)+T(r,g)] \leq (2k+4)[N(r,0;f)+N(r,0;g)]+m^*[T(r,f)+T(r,g)] +3[\overline{N}(r,0;F)+\overline{N}(r,0;G)]+S(r,f)+S(r,g). \tag{32}$$

Substituting (30) into (32), we get

$$\begin{aligned}
&n[T(r,f)+T(r,g)]\\
&\quad \leq (2k+4)[N(r,0;f)+N(r,0;g)]+m^*[T(r,f)+T(r,g)]\\
&\qquad +3(1+k)[\overline{N}(r,0;f)+\overline{N}(r,0;g)]+3m^*[T(r,f)+T(r,g)]+S(r,f)+S(r,g)\\
&\quad \leq (5k+7)[N(r,0;f)+N(r,0;g)]+4m^*[T(r,f)+T(r,g)]+S(r,f)+S(r,g).
\end{aligned}$$

This contradicts the assumption $n > 5k+7+4m^*$.

Therefore, $H \equiv 0$, i.e.,

$$\frac{F''}{F'} - 2\frac{F'}{F-1} = \frac{G''}{G'} - 2\frac{G'}{G-1}.$$

By integrating, we get from above equality that

$$\frac{F'}{(F-1)^2} = c\frac{G'}{(G-1)^2},$$

where c is a nonzero constant. It follows that F and G share 1 CM, that is to say, $[f^n(\mu f^m+\lambda)]^{(k)}$ and $[g^n(\mu g^m+\lambda)]^{(k)}$ share 1 CM. This together with $n > 2k+4+m^*$, by Theorem E, we obtain the conclusion of Theorem 1.

This completes the proof of Theorem 1. $\square$

4. Proof of Theorem 2

Proof. Set

$$\widetilde{F} = f^n(f-1)^m, \quad F = \widetilde{F}^{(k)},$$
$$\widetilde{G} = g^n(g-1)^m, \quad G = \widetilde{G}^{(k)},$$

and let H be defined as in Lemma 4. If

$$E_\ell(1,[f^n(f-1)^m]^{(k)}) = E_\ell(1,[g^n(g-1)^m]^{(k)}),$$

then $E_\ell(1,F) = E_\ell(1,G)$. Using the discussion in the proof of Theorem 1, we can verify that if one of the three conditions in Theorem 2 holds, then $H \equiv 0$. Next, by integrating this, we get F and G share 1 CM, i.e., $[f^n(z)(f(z)-1)^m]^{(k)}$ and $[g^n(z)(g(z)-1)^m]^{(k)}$ share 1 CM. This together with $n > 2k+4+m$, by Theorem F, we obtain the conclusion of Theorem 2.

This completes the proof of Theorem 2. $\square$

5. Proof of Corollary 3 and Corollary 4

Proof. In fact, let $m = 1$ and $\ell = 2$ ($\ell = 0$), by Theorem 2, if $E_\ell(1,[f^n(f-1)]^{(k)}) = E_\ell(1,[g^n(g-1)]^{(k)})$ and $n > 2k+5$ ($n > 5k+11$) then either $f \equiv g$ or $f^n(f-1) = g^n(g-1)$. Suppose $f \not\equiv g$, then we get $g = \frac{h^n-1}{h^{n+1}-1}$, where $h = \frac{f}{g}$. From this and by Picard's theorem, we can deduce that h is a constant since g is an entire function. So, g is a constant, a contradiction. Hence $f \equiv g$.

This completes the proof of Corollary 3 (Corollary 4). $\square$

References

[1] FANG M. L. Uniqueness and value-sharing of entire functions. *Comput. Math. Appl. 44* (2002), 823–831.

[2] HAYMAN W. K. *Meromorphic Functions*. Clarendon Press, Oxford, 1964.

[3] LAHIRI I. Weighted sharing and uniqueness of meromorphic functions. *Nagoya Math. J. 29* (2001), 193–206.

[4] WANG Y. F. On Mues conjecture and Picard values. *Sci. China 36*, 1 (1993), 28–35.

[5] YANG C. C. AND HUA X. H. Uniqueness and value-sharing of meromorphic functions. *Ann. Acad. Sci. Fenn. Math. 22*, 2 (1997), 395–406.

[6] YI H. X. Uniqueness of meromorphic functions and a question of C.C.Yang. *Complex Var. Theory Appl. 14* (1990), 169–176.

[7] YI H. X. AND YANG C. C. *Uniqueness Theory of Meromorphic Functions*. Science Press, Beijing, 1995.

[8] ZHANG Q. C. Meromorphic function that shares one small function with its derivative. *JIPAM. J. Inequal. Pure Appl. Math. 6*, 4 (2005), Art.116.

[9] ZHANG X. Y. AND LIN W. C. Uniqueness and value-sharing of entire functions. *J. Math. Anal. Appl. 343* (2008), 938–950.

AMS Subject Classification: 30D35, 30D30

Xiu-Qing LIN
Department of Mathematics, Ningde Normal University
Ningde 352100, Fujian Province, CHINA
e-mail: lxqnd4407@yahoo.cn

Lavoro pervenuto in redazione il 25.01.2011 e, in forma definitiva, il 03.04.2011