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SOME APPROXIMATE METHODS FOR COMPUTING ARC LENGTHS BASED ON QUADRATIC AND CUBIC SPLINE INTERPOLATION OR QUASI-INTERPOLATION*

Abstract. This paper presents a comparison of two methods for the computation of arc lengths. The first consists in computing the exact length of a quadratic spline approximant of the initial function. The second is based on approximating values of the first derivatives by those of cubic spline approximants which are then used in quadrature formulas for the computation of the integral giving the length. For each method, several types of interpolants or quasi-interpolants are compared.

1. Introduction

This paper presents a comparison of two families of methods for the computation of arc lengths. The first consists in computing the exact length of a quadratic spline approximant of the initial function or parametric curve. The second is based on approximating values of the first derivatives by those of cubic spline approximants which are then used in quadrature formulas for the computation of the integral giving the arc length. For each method, several types of interpolants or quasi-interpolants are compared.

There exists a vast literature on the subject, which is closely connected with the problem of parameterization of parametric curves, see e.g. references [5, 12, 11]. Recently, this topic has been reconsidered and developed in [1, 8, 7, 9, 23], in particular for quadratic and cubic spline approximants.

Here is an outline of the paper. In Section 2, we recall the formula giving the length of an arc of parabola in Bernstein–Bézier (abbr. BB) form. In Section 3, we describe a list of C^1 quadratic spline approximants: classical quasi-interpolants (abbr. QI), modified QIs, quasi-interpolating projectors, Lagrange interpolants of type 1 (needing the solution of a tridiagonal system) and Lagrange interpolants of type 2 based on Lagrange splines with compact support (abbr. LC-splines). In Section 4, we compare the results of the computation of arc lengths of some functions and parametric curves.

In Section 5, we compute approximate first derivatives on a uniform partition using respectively: finite differences exact on cubics, derivatives of Lagrange C^2 cubic spline interpolants (using the Not a Knot (NAK) and Pretty Good Slopes (PGS) methods described below), classical cubic spline QIs, modified QIs, quasi-interpolating projectors, Lagrange interpolants of type 2, based on LC-splines. In Section 6, we use the preceding derivative estimates for the approximate computation of arc lengths using Simpson's quadrature formula and a companion to the latter described in [19, 20].

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The overall convergence order of both classes of methods is $O(h^4)$. However, there are variations depending on the method and on the function (or the parametric curve) as can be seen on numerical results given in Section 7. In the final Section 8, we give a sketch of proofs of this convergence order.

2. Length of an arc of parabola

Let $\{b_0^2(x) = (1-x)^2, b_1^2(x) = 2x(1-x), b_2^2(x) = x^2\}$ be the Bernstein basis of $\mathbb{P}_2$. Given an arc of parabola written in BB-form whose control polygon has vertices S_0, S_1, S_2,

$$M(x) = S_0 b_0^2(x) + S_1 b_1^2(x) + S_2 b_2^2(x), \quad \text{then} \quad M'(x) = 2(\Delta S_0(1-x) + \Delta S_1 x),$$

with $\Delta S_0 := S_1 - S_0$ and $\Delta S_1 := S_2 - S_1$. Then its length is given by

$$L = \int_0^1 |M'(x)|\, dx = 2\int_0^1 \sqrt{\alpha_0 b_0^2(x) + \alpha_1 b_1^2(x) + \alpha_2 b_2^2(x)}\, dx,$$

where $|M'(x)|$ is the euclidean norm of the vector $M'(x)$ and

$$\alpha_0 := |\Delta S_0|^2, \quad \alpha_1 := \Delta S_0.\Delta S_1, \quad \alpha_2 := |\Delta S_1|^2.$$

Here $\Delta S_0.\Delta S_1$ denotes the scalar product of the two vectors. Then, denoting

$$\Delta\alpha_0 = \Delta S_0.\Delta^2 S_0, \quad \Delta\alpha_1 = \Delta S_1.\Delta^2 S_0, \quad \Delta^2\alpha_0 = |\Delta^2 S_0|^2,$$

the above integral (see e.g [4, 12, 11]) is equal to

$$L = \frac{\alpha_0\alpha_2 - \alpha_1^2}{(\Delta^2\alpha_0)^{3/2}} \ln\left(\frac{\Delta\alpha_1 + \sqrt{\alpha_2\,\Delta^2\alpha_0}}{\Delta\alpha_0 + \sqrt{\alpha_0\,\Delta^2\alpha_0}}\right) + \frac{\Delta\alpha_1\sqrt{\alpha_2} - \Delta\alpha_0\sqrt{\alpha_0}}{\Delta^2\alpha_0}.$$

3. Quadratic spline approximants

In Subsections 3.1 to 3.3, we consider quasi-interpolants on an interval $I := [a,b]$ endowed with the uniform partition $X_n := \{x_i := a + ih, 0 \le i \le n\}$. We denote by $\mathscr{S}_2^1(I, X_n)$ the space of C^1 quadratic splines with knot set X_n and triple knots at the endpoints. It has dimension $n+2$ and the B-splines $\{B_i, 0 \le i \le n+1\}$ with support $[x_{i-2}, x_{i+1}]$ (taking into account knot multiplicities) form a basis of this space. We denote $T_n := \{t_i, 0 \le i \le n+1\}$, with $t_0 := a$, $t_i := (x_{i-1} + x_i)/2$ for $i = 1 \ldots n$, $t_{n+1} := b$. Thus, any quadratic spline $S \in \mathscr{S}_2^1(I, X_n)$ can be expressed as $S := \sum_{i=0}^{n+1} \lambda_i B_i$.

3.1. Classical quasi-interpolant

Denoting $f_i = f(t_i)$ for $0 \le i \le n+1$, the simplest quasi-interpolant is defined by

$$Qf := \sum_{i=0}^{n+1} \lambda_i(f) B_i,$$

where

$$\lambda_i(f) := (-f_{i-1} + 10f_i - f_{i+1})/8, \quad \text{for} \quad i = 2, \ldots, n-1,$$

and the four specific linear functionals for the extreme indices introduced in [18]:

$$\lambda_0(f) := f_0, \qquad \lambda_1(f) := (-2f_0 + 9f_1 - f_2)/6,$$

$$\lambda_n(f) := (-f_{n-1} + 9f_n - 2f_{n+1})/6, \qquad \lambda_{n+1}(f) := f_{n+1}.$$

This choice ensures that Q is exact on $\mathbb{P}_2$, i.e. $Qp = p$ for $p \in \mathbb{P}_2$. By a careful study of the Lebesgue function, one can prove

THEOREM 1 ([18]). *For a uniform partition of I, there holds $\|P\|_\infty \approx 1.47$.*

3.2. Modified quasi-interpolant

A slightly more complex QI is given in [10]. The extreme functionals are modified as follows:

$$\lambda_1(f) := -\frac{2}{5}f_0 + \frac{13}{8}f_1 - \frac{1}{4}f_2 + \frac{1}{40}f_3$$

$$\lambda_n(f) := -\frac{2}{5}f_{n+1} + \frac{13}{8}f_n - \frac{1}{4}f_{n-1} + \frac{1}{40}f_{n-2}.$$

With this choice, there is superconvergence at points of X_n and T_n, i.e. $f - Qf = O(h^4)$ instead of $O(h^3)$ for the overall convergence.

For a uniform partition of I, there holds $\|P\|_\infty \approx 1.52$.

3.3. Quasi-interpolating projector

The preceding QI is not a projector, i.e. $Q(Qf) \neq Qf$. Denoting $f_{2i} := f(x_i)$ for all $0 \le i \le n$ and $f_{2i-1} := f(t_i)$ for $1 \le i \le n$, we now define a QI projector by

$$Pf := \sum_{i=0}^{n+1} \lambda_i(f) B_i,$$

where the linear discrete functionals are defined by:

$$\lambda_0(f) := f_0, \qquad \lambda_{n+1}(f) := f_{2n},$$

$$\lambda_1(f) := 2f_1 - \frac{1}{2}(f_0 + f_2), \qquad \lambda_n(f) := 2f_n + \frac{1}{2}(f_{n-1} + f_{n+1}),$$

and, for $2 \le i \le n-1$, by

$$\lambda_i(f) := \frac{1}{14}f_{2i-4} - \frac{2}{7}f_{2i-3} + \frac{10}{7}f_{2i-1} - \frac{2}{7}f_{2i+1} + \frac{1}{14}f_{2i+2},$$

where the coefficients are deduced from the conditions $\lambda_i(B_j) = \delta_{ij}$ for all pairs (i, j). By a careful study of the Lebesgue function, one can prove

THEOREM 2. *For a uniform partition of I, there holds $\|P\|_\infty \approx 2.28$.*

3.4. Lagrange interpolant of type 1

We now consider the classical Lagrange interpolation problem on T_n. Writing the quadratic spline interpolant of the function f in terms of B-splines

$$S(x) = \sum_{i=0}^{n} a_i B_i(x),$$

we get the following linear equations

$$a_{i-1} + 6a_i + a_{i+1} = 8f_i, \quad \text{for } 2 \le i \le n-1,$$

with four specific equations at the endpoints

$$a_0 = f_0, \quad 2a_0 + 5a_1 + a_2 = 8f_1, \quad a_{n-1} + 5a_n + 2a_{n+1} = 8f_n, \quad a_{n+1} = f_{n+1}.$$

The system can be written in matrix form

$$Aa = c, \quad \text{where } a := [a_1, a_2, \ldots, a_n]^T, \; c := [8f_1 - 2f_0, 8f_2, \ldots, 8f_{n-1}, 8f_n - 2f_{n+1}]^T,$$

$$A := \begin{bmatrix} 5 & 1 & 0 & 0 & \ldots & 0 & 0 \\ 1 & 6 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & 6 & 1 & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & 1 & 6 & 1 & 0 \\ 0 & 0 & \ldots & 0 & 1 & 6 & 1 \\ 0 & 0 & \ldots & 0 & 0 & 1 & 5 \end{bmatrix}.$$

It has been proved in [13] that the infinity norm satisfies $||Q||_\infty \le 2$.

3.5. Lagrange interpolant of type 2

Having endowed the interval $I := [a, b]$ with the partition

$$X_n := \{x_i := a + ih, 0 \le i \le n\},$$

consider the same spline space $\mathscr{S}_2^1(I, X_n)$ as before. Assume further that $n = 2m$ is even and that the set of data points is $\mathscr{X}_m := \{x_{2j}, j \in J_m\}$, $J_m := \{j : 0 \le j \le m\}$. Thus, data values are $f_j := f(x_{2j}), j \in J_m$. Setting $J_m^* := J_m \cup \{-1, m+1\}$, then there exist Lagrange splines with compact support (abbr. LC-splines) $\{L_j, j \in J_m^*\}$ satisfying $L_j(x_k) = \delta_{jk}$ for $j, k \in J_m^*$. They were defined in [15] in the uniform case and generalized in [2] for nonuniform partitions. The support of L_j is the interval $[x_{2j-4}, x_{2j+4}]$. The Lagrange conditions $L_j(x_{2j-2}) = L_j(x_{2j+2}) = 0, L_j(x_{2j}) = 1$ and the fact that the interpolant $\mathscr{I} f(x) := \sum_{j=-1}^{m+1} \alpha_j(f) L_j(x)$ is exact on $\mathbb{P}_2$, leads to the expansion

$$L_j = -\frac{1}{8}B_{2j-2} + \frac{1}{8}B_{2j-1} + B_{2j} + B_{2j+1} + \frac{1}{8}B_{2j+2} - \frac{1}{8}B_{2j+3}$$

(see e.g. [21] for details) where B_k is the quadratic B-spline with support $[x_{k-2}, x_{k+1}]$ where simple knots are taken outside the interval. We have of course $\alpha_j = f_j$ for $0 \le j \le m$. As the outer values f_j for $j = -1, m+1$ are not given, we take instead as coefficients of the LC-splines L_j, for $j = -1, m+1$, the following linear forms

$$\alpha_{-1}(f) := \frac{1}{16}(55 f_0 - 69 f_1 + 37 f_2 - 7 f_3)$$

$$\alpha_{m+1}(f) = \frac{1}{16}(55 f_m - 69 f_{m-1} + 37 f_{m-2} - 7 f_{m-3}).$$

This choice is done in order to get the following superconvergence result

$$f(x_1) - \mathscr{I} f(x_1) = f(x_{n-1}) - \mathscr{I} f(x_{n-1}) = O(h^4),$$

which is already valid for interior knots $x_{2j+1}, j = 3, \ldots, m-2$ (see [21] for details).

The B-spline expansion of the interpolant is then

$$\mathscr{I} f = \sum_{i=0}^{n+1} \lambda_i B_i,$$

with

$$\lambda_{2j} = \frac{1}{8} f_{j-1} + f_j - \frac{1}{8} f_{j+1}, \ \lambda_{2j+1} = -\frac{1}{8} f_{j-1} + f_j + \frac{1}{8} f_{j+1}, \ 1 \le j \le m-1,$$

and for extreme coefficients

$$\lambda_0 = \frac{1}{8}\alpha_{-1} + f_0 - \frac{1}{8} f_1, \qquad \lambda_1 = -\frac{1}{8}\alpha_{-1} + f_0 + \frac{1}{8} f_1,$$

$$\lambda_{2m} = -\frac{1}{8}\alpha_{m+1} + f_m + \frac{1}{8} f_{m-1}, \qquad \lambda_{2m+1} = \frac{1}{8}\alpha_{m+1} + f_m - \frac{1}{8} f_{m-1}.$$

Taking now the B-splines $B_i, 0 \le i \le n+1$ with multiple knots at the endpoints, we get the expansion

$$\mathscr{I} f = \sum_{i=0}^{n+1} \mu_i B_i,$$

with the coefficients:

$$\mu_0 = \frac{1}{2}(\lambda_0 + \lambda_1) = f_0, \qquad \mu_{n+1} = \frac{1}{2}(\mu_n + \mu_{n+1}) = f_m,$$

$$\mu_1 = \frac{1}{2}(\lambda_1 - \lambda_0) = \frac{1}{8}(f_1 - \alpha_{-1}) = \frac{1}{128}(-55 f_0 + 85 f_1 - 37 f_2 + 7 f_3),$$

$$\mu_n = \frac{1}{2}(\lambda_n - \lambda_{n+1}) = \frac{1}{8}(f_{m-1} - \alpha_{m+1}) = \frac{1}{128}(-55 f_m + 85 f_{m-1} - 37 f_{m-2} + 7 f_{m-3}).$$

This allows one to give the following upper bound for the infinity norm of the operator: $\|\mathscr{I}\|_\infty \le \max\{\|\mu_i\|_\infty, 0 \le i \le n+1\} = \|\mu_1\|_\infty = \frac{23}{16} \approx 1.44$.

4. Comparison of lengths of quadratic spline approximants

We have compared the lengths of the following arcs of functions $f(x)$ and parametric curves $(f(t), g(t))$.

$$f_1(x) := \frac{1}{1+5x^2}, \quad x \in [0,1], \qquad f_2(x) := \frac{x}{8} + e^{-x/4}\sin(3x), \quad x \in [0,6],$$

$$f_3(t) = \cos(\pi t), \quad g_3(t) := \sin(\pi t), \quad t \in [0,1].$$

$$f_4(t) = (\cos(\pi t/2))^3, \quad g_4(t) := (\sin(\pi t/2))^3, \quad t \in [0,1].$$

Columns 3,5,7,9 in tables below give the numerical convergence order (abbr. NCO).

4.1. Classical quadratic QI

n	1	NCO	2	NCO	3	NCO	4	NCO
64	3.9(-7)	15.1	5.2(-4)	15.3	3.4(-7)	15.6	-1.8(-7)	11.2
128	2.5(-8)	15.6	3.3(-5)	15.8	2.2(-8)	15.8	-1.4(-8)	12.3
256	1.6(-9)	15.8	2.1(-6)	15.9	1.4(-9)	15.9	-1.1(-9)	13.0
512	9.8(-11)	15.9	1.3(-7)	16	8.6(-11)	15.9	-8.1(-11)	13.5
1024	6.2(-12)	16	8.1(-9)	16	5.3(-12)	16	-5.9(-12)	13.8

4.2. Modified QI

n	1		2		3		4	
64	3.0(-7)	11.4	5.3(-4)	15.3	3.4(-7)	15.5	1.2(-9)	–
128	2.2(-8)	13.8	3.3(-5)	15.8	2.2(-8)	15.7	-1.5(-9)	–
256	1.5(-9)	14.9	2.1(-6)	15.9	1.4(-9)	15.9	-1.9(-10)	7.7
512	9.6(-11)	15.5	1.3(-7)	16	8.5(-11)	15.9	-1.8(-11)	10.5
1024	6.1(-12)	15.7	8.1(-9)	16	5.3(-12)	16	-1.5(-12)	11.9

4.3. QI projector

n	1		2		3		4	
64	6.8(-8)	19.5	6.7(-5)	18.2	4.8(-8)	16.8	-1.2(-7)	12.4
128	3.6(-9)	18.7	4.0(-6)	16.7	2.9(-9)	16.4	-8.8(-9)	13.1
256	2.1(-10)	17.5	2.5(-7)	16.2	1.8(-10)	16.2	-6.5(-10)	13.6
512	1.2(-11)	16.8	1.6(-8)	16	1.1(-11)	16.1	-4.7(-11)	13.9
1024	7.5(-13)	16.4	9.6(-10)	16	7.1(-12)	16	-3.3(-12)	14.1

4.4. Lagrange interpolant of type 1

n	1		2		3		4	
64	6.6(-8)	14.7	9.5(-5)	16	6.6(-8)	15.9	-1.1(-7)	12.3
128	4.3(-9)	15.4	5.9(-6)	16	4.1(-9)	15.9	-8.3(-9)	13
256	2.7(-10)	15.7	3.7(-7)	16	2.6(-10)	16	-6.2(-10)	13.5
512	1.7(-11)	15.8	2.3(-8)	16	1.6(-11)	16	-4.5(-11)	13.8
1024	1.1(-12)	15.9	1.4(-9)	16	1.0(-12)	16	-3.2(-12)	14.1

4.5. Lagrange interpolant of type 2

n	1		2		3		4	
64	1.5(-7)	13.6	2.5(-4)	15.4	2.0(-7)	15.3	-4.1(-7)	13.6
128	1.1(-8)	14.4	1.6(-5)	15.8	1.3(-8)	15.6	-3.0(-8)	13.7
256	7.1(-10)	15.2	9.8(-7)	15.9	8.1(-10)	15.8	-2.2(-9)	13.9
512	4.5(-11)	15.6	6.2(-8)	16	5.1(-11)	15.9	-1.5(-10)	14.1
1024	2.9(-12)	15.8	3.8(-9)	16	3.2(-12)	15.9	-1.1(-11)	14.3

4.6. Comparison of various approximants on the same example

The final table gives the results of 4 quadratic approximants on the same example, a quartic curve in parametric form:

$$y^2 = (1-x)x^3, \quad x = (\cos t)^2, \ y = (\sin t)(\cos t)^3$$

n	Class QI	Proj QI	Lag T1	Lag T2
64	6.8(-6)	-6.6(-7)	3.0(-9)	-2.3(-6)
128	3.5(-7)	-7.9(-8)	-3.5(-8)	-2.3(-7)
256	1.7(-8)	-7.3(-9)	-4.5(-9)	-2.1(-8)
512	7.8(-10)	-6.0(-10)	-4.3(-10)	-1.7(-9)
1024	3.0(-11)	-4.7(-11)	-3.6(-11)	-1.3(-10)

Remark. When comparing the above results, we see that there is no clear winner. Here are the best results according to the examples

curve	approximant/smaller error
Rational function f_1	QI projector
Exp-sin function f_2	QI projector
Half circle (f_3, g_3)	Lagrange type 1
1/4 astroid (f_4, g_4)	Modified QI
Quartic curve	Classical QI

5. Cubic splines and approximation of derivatives

In the following subsections, we consider an interval $I := [a,b]$ endowed with the uniform partition $X_n := \{x_i := a + ih, 0 \le i \le n\}$. We denote by $\mathscr{S}_3^2(I, X_n)$ the space of C^2 cubic splines with knot set X_n and multiple knots at the endpoints. It has dimension $n+3$ and the B-splines $\{B_i, 0 \le i \le n+2\}$ with support $[x_{i-3}, x_{i+1}]$ (taking into account knot multiplicities) form a basis of this space. Any cubic spline $S \in \mathscr{S}_3^2(I, X_n)$ has an expansion of type $S := \sum_{i=0}^{n+2} \lambda_i B_i$. For the computation of arc lengths in approximating the integral $\int_a^b \sqrt{1 + S'(t)^2}\, dt$ by quadrature rules, we need the expressions of derivatives in function of B-spline coefficients. Their values on the set X_n are the following:

$$S'(a) = \frac{3}{h}(\lambda_1 - \lambda_0), \qquad S'(b) = \frac{3}{h}(\lambda_{n+2} - \lambda_{n+1}),$$

$$S'(x_1) = \frac{1}{4h}(-3\lambda_1 + \lambda_2 + 2\lambda_3), \qquad S'(x_{n-1}) = \frac{1}{4h}(3\lambda_{n+1} - \lambda_n - 2\lambda_{n-1}),$$

$$S'(x_i) = \frac{1}{2h}(\lambda_{i+2} - \lambda_i) \quad \text{for } 2 \le i \le n-2.$$

They are needed in Section 6 for Simpson's rule. Similarly, their values on the set T_n of midpoints (defined in Section 3) are the following:

$$S'(t_1) = \frac{1}{16h}(-12\lambda_0 - 3\lambda_1 + 13\lambda_2 + 2\lambda_3),$$

$$S'(t_2) = \frac{1}{16h}(-3\lambda_1 - 9\lambda_{+} 10\lambda_3 + 2\lambda_4),$$

$$S'(t_i) = \frac{1}{8h}(-\lambda_i - 5\lambda_i + 5\lambda_i + \lambda_i), \quad \text{for } 3 \le i \le n-2,$$

$$S'(t_{n-1}) = \frac{1}{16h}(-3\lambda_n - 9\lambda_{n-1} + 10\lambda_{n-2} + 2\lambda_{n-3}),$$

$$S'(t_n) = \frac{1}{16h}(-12\lambda_{n+1} - 3\lambda_n + 13\lambda_{n-1} + 2\lambda_{n-2}).$$

These values are needed in Section 6 for the companion formula to Simpson's rule.

5.1. Finite differences

Classical approximations of the first derivatives $y_i' := \varphi'(x_i)$, $0 \le i \le n$, of a univariate function φ defined on $[a,b]$ by its values $y_i := \varphi(x_i)$, are given by finite differences (abbr. FD). As those given in this section are in $O(h^4)$, we take the following, which is exact on $\mathbb{P}_4$ (the space of polynomials of degree at most 4), for $2 \le i \le n-2$:

$$y_i' \approx d_i := \frac{1}{h}\left(\frac{1}{12} y_{i-2} - \frac{2}{3} y_{i-1} + \frac{2}{3} y_{i+1} - \frac{1}{12} y_{i+2}\right).$$

Using Taylor's expansions at x_i, we obtain a fourth order approximation of the first derivative:

$$d_i = y_i' - \frac{1}{30}h^4 y_i^{(5)} + O(h^5).$$

Near endpoints, there are two specific formulas:

$$y_0' \approx d_0 = \frac{1}{h}\left(-\frac{25}{12}y_0 + 4y_1 - 3y_2 + \frac{4}{3}y_3 - \frac{1}{4}y_4\right),$$

$$y_1' \approx d_1 = \frac{1}{h}\left(-\frac{1}{4}y_0 - \frac{5}{6}y_1 + \frac{3}{2}y_2 - \frac{1}{2}y_3 + \frac{1}{12}y_4\right).$$

Taylor's expansions at x_0 and x_1 give respectively

$$d_0 = y_0' - \frac{1}{5}h^4 y_0^{(5)} + 0(h^5), \qquad d_1 = y_1' + \frac{1}{20}h^4 y_1^{(5)} + 0(h^5).$$

In a similar way, we take

$$y_{n-1}' \approx d_{n-1} = \frac{1}{h}\left(\frac{1}{4}y_n + \frac{5}{6}y_{n-1} - \frac{3}{2}y_{n-2} + \frac{1}{2}y_{n-3} - \frac{1}{12}y_{n-4}\right),$$

$$y_n' \approx d_n = \frac{1}{h}\left(\frac{25}{12}y_n - 4y_{n-1} + 3y_{m-2} - \frac{4}{3}y_{n-3} + \frac{1}{4}y_{n-4}\right),$$

whose Taylor's expansions at x_{n-1} and x_n give respectively

$$d_{n-1} = y_{n-1}' + \frac{1}{20}h^4 y_{m-1}^{(5)} + 0(h^5), \qquad d_n = y_n' - \frac{1}{5}h^4 y_n^{(5)} + 0(h^5).$$

Remark. These quantities can be obtained as first derivatives of a specific cubic spline quasi-interpolant described in Section 5.3 below.

5.2. Lagrange interpolation: NAK and PGS

The unknown derivative values are solutions of the system of $n-1$ linear equations deduced from the C^2 continuity of S at points x_i to which one must add two conditions at the endpoints, in order to get $n+1$ equations for the $n+1$ unknowns $\{d_i, 0 \leq i \leq n\}$. Setting $p_i := (y_i - y_{i-1})/h$, the main equations can be written

$$d_{i-1} + 4d_i + d_{i+1} = 3(p_i + p_{i+1}), \quad 1 \leq i \leq n-1.$$

De Boor [3] suggests to take as additional conditions the C^3 continuity of S at knots $x = x_1$ and $x = x_{n-1}$, therefore the corresponding NAK (Not a Knot) cubic spline has no discontinuity at that point (whence the name "Not a Knot"). Denoting $d := [d_1, \ldots, d_{n-1}]^T \in \mathbb{R}^{n-1}$,

$$c := [(p_1 + 5p_2)/2, 3(p_2 + p_3) \ldots, 3(p_{n-2} + p_{n-1}), (5p_{n-1} + p_n)/2]^T \in \mathbb{R}^{n-1},$$

and introducing the tridiagonal and positive definite matrix

$$K := \begin{bmatrix} 2 & 1 & 0 & & \dots & 0 & 0 \\ 1 & 4 & 1 & 0 & \dots & 0 & 0 \\ \dots & 1 & 4 & 1 & & \dots & \\ \dots & & 1 & 4 & 1 & & \dots \\ \dots & & & & \dots & \dots & \\ 0 & 0 & \dots & 0 & 1 & 4 & 1 \\ 0 & 0 & \dots & & 0 & 1 & 2 \end{bmatrix}$$

then d is solution of the system of linear equations

$$Kd = c.$$

The PGS (Pretty Good Slopes) method introduced by the author in [22] consists in choosing the coefficients in such a way that the computed derivatives have an overall error in $O(h^4)$. The new system is $K'd = c'$ with the new matrix of size $n-1$:

$$K' := \begin{bmatrix} 1 & 1 & 0 & & \dots & 0 & 0 \\ 1 & 4 & 1 & 0 & \dots & 0 & 0 \\ \dots & 1 & 4 & 1 & & \dots & \\ \dots & & 1 & 4 & 1 & & \dots \\ \dots & \dots & & & & & \\ 0 & 0 & \dots & 0 & 1 & 4 & 1 \\ 0 & 0 & \dots & & 0 & 1 & 1 \end{bmatrix}$$

and the new vector in the right-hand side

$$c' = [c'_1, 3(p_2+p_3), \dots, 3(p_{n-2}+p_{n-1}), c'_{n-1}]^T,$$

where

$$c'_1 := \frac{1}{6}(p_1 + 10p_2 + p_3), \quad c'_{n-1} = \frac{1}{6}(p_1 + 10p_2 + p_3).$$

5.3. Cubic quasi-interpolants

Classical cubic quasi-interpolant. The simplest quasi-interpolant is defined by

$$Qf := \sum_{i=0}^{n+2} \lambda_i(f) B_i,$$

where $f_i := f(x_i)$ and

$$\lambda_i(f) := (-f_{i-2} + 8f_{i-1} - f_i)/6, \quad \text{for} \quad i = 2, \dots, n.$$

There are four specific linear functionals for the extreme indices

$$\lambda_0(f) := f_0, \qquad \lambda_1(f) := \frac{7}{18} f_0 + f_1 - \frac{1}{2} f_2 + \frac{1}{9} f_3,$$

$$\lambda_{n+1}(f) := \frac{7}{18} f_n + f_{n-1} - \frac{1}{2} f_{n-2} + \frac{1}{9} f_{n-3}, \qquad \lambda_{n+2}(f) := f_n.$$

Using the array of B-spline coefficients of monomials $m_r(x) = x^r, r = 0,1,2,3$:

	λ_0	λ_1	λ_2	λ_3	λ_4
m_0	1	1	1	1	1
m_1	0	1/3	1	2	3
m_2	0	0	2/3	11/3	26/3
m_3	0	0	0	6	24

it is easy to verify that this QI is exact on $\mathbb{P}_3$. For a uniform partition of I, there holds $\|Q\|_\infty \approx 1.64$.

Modified cubic quasi-interpolant. The two first and last coefficients being modified in such a way that $Qf(x_i) - f(x_i) = O(h^4)$ for all indices, the new functionals at the endpoints are now

$$\lambda_0(f) := \frac{1}{36}(35f_0 + 4f_1 - 6f_2 + 4f_3 - f_4)$$

$$\lambda_1(f) := \frac{1}{18}(5f_0 + 26f_1 - 21f_2 + 10f_3 - 2f_4)$$

$$\lambda_{n+1}(f) := \frac{1}{18}(5f_n + 26f_{n-1} - 21f_{n-2} + 10f_{n-3} - 2f_{n-4})$$

$$\lambda_{n+2}(f) := \frac{1}{36}(35f_n + 4f_{n-1} - 6f_{n-2} + 4f_{n-3} - f_{n-4}).$$

5.4. Cubic quasi-interpolating projector

The preceding QI is not a projector, i.e. $Q(Qf) \neq Qf$. Denote $f_{2i} := f(x_i)$ for all $0 \le i \le n$ and $f_{2i-1} := f(t_i)$ for $1 \le i \le n$, we now define a QI projector [17] by

$$Pf := \sum_{i=0}^{n+2} \lambda_i(f) B_i,$$

where the linear discrete functionals are defined by:

$$\lambda_0(f) := f_0, \quad \lambda_{n+1}(f) := f_{2n},$$

$$\lambda_1(f) := -\frac{5}{18}f_0 + \frac{20}{9}f_1 - \frac{4}{3}f_2 + \frac{4}{9}f_3 - \frac{1}{18}f_4,$$

$$\lambda_n(f) := -\frac{5}{18}f_{2n} + \frac{20}{9}f_{2n-1} - \frac{4}{3}f_{2n-2} + \frac{4}{9}f_{2n-3} - \frac{1}{18}f_{2n-4},$$

and, for $2 \le i \le n-1$, by

$$\lambda_i(f) := \frac{1}{6}f_{2i-4} - \frac{4}{3}f_{2i-3} + \frac{10}{3}f_{2i-2} - \frac{4}{3}f_{2i-1} + \frac{1}{6}f_{2i},$$

The coefficients are deduced from the conditions $\lambda_i(B_j) = \delta_{ij}$ for all pairs (i, j). By a careful study of the Lebesgue function, one can prove

THEOREM 3. *For a uniform partition of I, there holds $\|P\|_\infty \approx 3.86$.*

5.5. Cubic Lagrange interpolants of type 2

As in Section 3.5, the term LC-spline denotes a Lagrange spline with compact support. For univariate splines with integer knots in $\mathbb{R}$, the first construction of such LC-splines has been given by Qi [15] in 1981. The interpolation points are $x_j := 2j$, $j \in \mathbb{Z}$ and the points $t_j := 2j+1$ are auxiliary knots. More specifically, for cubic splines, let B denote the B-spline with support $[x_{-1}, x_1] = [-2, 2]$, then there exists a LC-spline

$$L(x) = \ell_0 B(x) + \sum_{j=1}^{4} \ell_j (B(x+j) + B(x-j)), \quad \text{with support} \quad [-6, 6].$$

The Lagrange interpolatory conditions

$$L(0) = 1 \quad \text{et} \quad L(-4) = L(-2) = L(2) = L(4) = 0,$$

give a first system of linear equations. Then, writing that the interpolant $\mathscr{I} f(x) := \sum_{k\in\mathbb{Z}} f(2k)L(x-2k)$ is exact on $\mathbb{P}_3$, i.e. on monomials $m_r(x) = x^r$ for $r = 0,1,2,3$ and using the B-spline representation of the latter, we get another system of linear equations. The global system of equations has a unique solution [21]:

$$[\ell_0, \ell_1, \ell_2, \ell_3, \ell_4] = \left[\frac{29}{24}, \frac{7}{12}, -\frac{1}{8}, -\frac{1}{12}, \frac{1}{48}\right].$$

On a bounded interval, the first coefficients are

$$\lambda_1(f) := 5f(0) - 10f(2h) + 10f(4h) - 5f(6h) + f(8h)$$

$$\lambda_2(f) := 15f(0) - 40f(2h) + 45f(4h) - 24f(6h) + 5f(8h)$$

and the corresponding errors on derivatives are respectively

$$f'(0) - S'(0) = \frac{16}{5} h^4 f^{(5)}(0) + O(h^5)$$

$$f'(h) - S'(h) = -\frac{71}{120} h^4 f^{(5)}(h) + O(h^5)$$

$$f'(2h) - S'(2h) = -\frac{4}{5} h^4 f^{(5)}(2h) + O(h^5)$$

$$f'(3h) - S'(3h) = \frac{3}{40} h^4 f^{(5)}(3h) + O(h^5).$$

A similar study can be done for the four last points of the interval.

For $2 \le i \le n-2$, the derivatives at data sites have the expressions

$$S'(x_i) = \frac{1}{24h} \left(f(x_{i-2}) - 8f(x_{i-1}) + 8f(x_{i+1}) - f(x_{i+2}) \right),$$

and the derivatives at intermediate knots have the expressions

$$S'(t_i) = \frac{1}{96h} \left(-f(x_{i-2}) + 7f(x_{i-1}) - 64f(x_i) + 64f(x_{i+1}) - 7f(x_{i+2}) + f(x_{i+3}) \right).$$

The corresponding errors are respectively

$$f'(x_i) - S'(x_i) = \frac{8}{15} h^4 f^{(5)}(x_i) + O(h^5),$$

$$f'(t_i) - S'(t_i) = -\frac{31}{120} h^4 f^{(5)}(t_i) + O(h^5).$$

Therefore, there is superconvergence in the approximation of derivatives on subsets X and T. For a uniform partition of I, there holds $\|P\|_\infty \approx 5.4$.

6. Approximate lengths based on knot values

6.1. Quadrature formulas

We assume that $n = 2m$ is even, with $m \geq 3$, and that the interval $I := [a,b]$ is endowed with a uniform partition in $2n$ subintervals, $h := (b-a)/2n$ and $x_i = a + ih, 0 \leq i \leq 2n$. We first use Simpson's formula based on the $n+1$ points $x_{2i}, 0 \leq i \leq n$:

$$\mathscr{S}_n(g) := \frac{2h}{3}\left(g(a) + 4\sum_{i=1}^{m} g(x_{4i-2}) + 2\sum_{i=1}^{m-1} g(x_{4i}) + g(b)\right).$$

We compare it with its companion formula associated with quadratic spline QIs [20], which is based on the $n+2$ points a, b and $x_{2i-1}, 1 \leq i \leq n$:

$$\mathscr{Q}_n(g) := 2h\left(\frac{1}{9}g(a) + \frac{7}{8}g(x_1) + \frac{73}{72}g(x_3) + \sum_{i=3}^{n-2} g(x_{2i-1}) + \frac{73}{72}g(x_{2n-3}) + \frac{7}{8}g(x_{2n-1}) + \frac{1}{9}g(b)\right).$$

For a smooth function g, the error estimates are respectively

$$\int_a^b (g - \mathscr{S}_n(g)) = -\frac{2h^4}{15} f^{(4)}(c_1), \int_a^b (g - \mathscr{Q}_n(g)) = \frac{23h^4}{240} f^{(4)}(c_2) + O(h^5).$$

6.2. Comparison of approximate lengths

Finite Differences

n	1		2		3		4	
64	6.8(-7)	10.9	8.1(-4)	14.2	5.8(-7)	15.1	6.1(-7)	13.3
128	4.4(-8)	15.3	5.4(-5)	15	3.7(-8)	15.6	4.1(-8)	14.9
256	2.8(-9)	15.9	3.3(-6)	16.2	2.3(-9)	15.8	2.6(-9)	15.6
512	1.7(-10)	16	2.1(-7)	15.7	1.5(-10)	15.9	1.7(-10)	15.8
1024	1.1(-11)	16	1.3(-8)	15.9	9.2(-12)	15.9	1.0(-11)	15.9

NAK = Not a Knot

n	1		2		3		4	
64	-2.0(-7)	38.1	1.7(-4)	8.2	6.7(-8)	7.9	-1.7(-6)	15.9
128	-2.8(-9)	70.4	1.2(-5)	14.2	5.3(-9)	12.7	-1.1(-7)	16
256	1.4(-10)	20	6.3(-7)	18.7	3.6(-10)	14.5	-6.8(-9)	15.9
512	1.9(-11)	7.5	3.9(-8)	15.9	2.4(-11)	15.3	-4.3(-10)	16
1024	1.5(-12)	12.6	2.5(-9)	16.1	1.5(-12)	15.7	-2.7(-11)	16

PGS = Pretty Good Slopes

n	1		2		3		4	
64	1.1(-7)	–	1.8(-4)	12.4	1.0(-7)	15.9	1.9(-8)	39.5
128	7.4(-9)	14.4	1.2(-5)	14.8	6.3(-9)	15.9	3.0(-9)	6.2
256	4.7(-10)	15.8	6.4(-7)	18.9	3.9(-10)	15.9	2.4(-10)	12.4
512	2.9(-11)	16	3.9(-8)	16	2.5(-11)	16	1.7(-11)	14.5
1024	1.8(-12)	15.9	2.5(-9)	16.1	1.5(-12)	16	1.1(-12)	15.3

Classical cubic QI

n	1		2		3		4	
64	1.1(-7)	54.9	9.3(-4)	14.2	5.4(-7)	14.1	-2.3(-6)	15.9
128	2.5(-8)	4.5	6.0(-5)	15.4	3.6(-8)	15.1	-1.4(-7)	16
256	2.2(-9)	11.6	3.7(-6)	16.4	2.3(-9)	15.6	-8.9(-9)	16
512	1.5(-10)	14.0	2.3(-7)	16	1.5(-10)	15.8	-5.5(-10)	16
1024	1.0(-11)	15.1	1.4(-8)	16.3	9.2(-12)	15.9	-3.5(-11)	16

Modified cubic QI

n	1		2		3		4	
64	6.8(-7)	10.9	8.1(-4)	14.2	5.8(-7)	15.1	6.1(-7)	13.3
128	4.4(-8)	15.3	5.4(-5)	15.0	3.7(-8)	15.6	4.1(-8)	14.9
256	2.8(-9)	15.9	3.3(-6)	16.1	2.3(-9)	15.8	2.6(-9)	15.5
512	1.7(-10)	16	2.1(-7)	15.8	1.5(-10)	15.9	1.7(-10)	15.8
1024	1.1(-11)	16	1.3(-8)	15.9	9.2(-12)	15.9	1.0(-11)	15.9

QI cubic projector

n	1		2		3		4	
64	1.9(-7)	8.62	3.7(-4)	13.9	2.2(-7)	15.4	-8.7(-8)	16
128	1.4(-8)	13.4	2.4(-5)	15.3	1.4(-8)	15.7	-5.4(-9)	16
256	9.7(-10)	14.8	1.4(-6)	17.3	8.8(-10)	15.9	-3.4(-10)	16
512	6.3(-11)	15.4	8.8(-8)	16	5.5(-11)	15.9	-2.1(-11)	16
1024	4.0(-12)	15.7	5.5(-9)	16	3.5(-12)	16	-1.3(-12)	15.9

Cubic Lagrange interpolant of type 2

n	1		2		3		4	
64	6.8(-7)	10.9	8.1(-4)	14.2	5.8(-7)	15.1	6.1(-7)	13.3
128	4.4(-8)	15.3	5.4(-5)	15	3.7(-8)	15.6	4.1(-8)	14.9
256	2.8(-9)	15.9	3.3(-6)	16.1	2.3(-8)	15.8	2.6(-9)	15.5
512	1.7(-10)	16	2.1(-7)	15.8	1.5(-10)	15.9	1.7(-10)	15.8
1024	1.1(-11)	16	1.3(-8)	15.9	9.2(-12)	15.9	1.0(-11)	15.9

6.3. Comparison of approximate lengths on the same example by various methods

The derivatives are approximated by finite differences (FD), Not a Knot (NK) and Pretty Good Slope (PG). The integrals are computed by Simpson's rule (S) or by its companion rule (C).

Example 1: Rational function f_1

n	FD/S	NK/S	PG/S	FD/C	NK/C	PG/C
64	6.7(-7)	-2.0(-7)	1.1(-7)	-2.1(-7)	4.8(-8)	7.6(-8)
128	4.4(-8)	-2.8(-9)	7.4(-9)	-1.7(-8)	8.1(-11)	9.8(-10)
256	2.8(-9)	1.4(-10)	4.7(-10)	-1.2(-9)	-9.2(-11)	-6.4(-11)
512	1.7(-10)	1.9(-11)	2.9(-11)	-8.0(-11)	-8.8(-12)	-8.0(-12)
1024	1.1(-11)	1.5(-12)	1.8(-12)	-5.1(-12)	-6.5(-13)	-6.2(-13)

Example 2: Exp-sin function f_2

n	FD/S	NK/S	PG/S	FD/C	NK/C	PG/C
64	8.1(-4)	1.7(-4)	1.8(-4)	-3.8(-4)	-6.6(-5)	-7.4(-5)
128	5.4(-5)	1.2(-5)	1.2(-5)	-2.6(-5)	-4.4(-6)	-4.8(-6)
256	3.3(-6)	6.3(-7)	6.4(-7)	-1.6(-6)	-2.7(-7)	-2.9(-7)
512	2.1(-7)	3.9(-8)	4.0(-8)	-1.0(-7)	-1.7(-8)	-1.8(-8)
1024	1.3(-8)	2.5(-9)	2.5.(-9)	-6.5(-9)	-1.1(-9)	-1.1(-9)

Example 3: Half circle (f_3, g_3)

n	FD/S	NK/S	PG/S	FD/C	NK/C	PG/C
64	5.8(-7)	6.7(-8)	1.0(-7)	-2.8(-7)	-4.3(-8)	-4.2(-8)
128	3.7(-8)	5.3(-9)	6.3(-9)	-1.8(-8)	-2.6(-9)	-2.6(-9)
256	2.3(-9)	3.6(-10)	3.9(-10)	-1.1(-9)	-1.6(-10)	-1.6(-10)
512	1.5(-10)	2.4(-11)	2.5(-11)	-7.1(-11)	-1.0(-11)	-1.0(-11)
1024	9.2(-12)	1.5(-12)	1.5(-12)	-4.5(-12)	-6.3(-13)	-6.3(-13)

Example 4: $1/4$ *Astroid* (f_4, g_4)

n	FD/S	NK/S	PG/S	FD/C	NK/C	PG/C
64	6.1(-7)	-1.7(-6)	1.9(-8)	-3.5(-7)	-6.5(-7)	-3.5(-8)
128	4.1(-8)	-1.1(-7)	3.0(-9)	-2.1(-8)	-4.1(-8)	-1.6(-9)
256	2.6(-9)	-6.8(-9)	2.4(-10)	-1.3(-9)	-2.5(-9)	-7.8(-11)
512	1.7(-10)	-4.3(-10)	1.7(-11)	-8.0(-11)	-1.6(-10)	-4.3(-12)
1024	1.0(-11)	-2.7(-11)	1.1(-12)	-5.0(-12)	-9.9(-12)	-2.5(-13)

7. Closed curves

It is easy to adapt the preceding algorithms to closed curves. We have selected for both types of splines the classical QI, the QI projector and the type 2 Lagrange interpolant. As examples, we take the circle, an ellipse, a lemniscate and a prolate epicycloid.

Example 1. Circle: $x = \cos(2\pi t)$, $y = \sin(2\pi t)$.

Example 2. Ellipse: $x = 2\cos(2\pi t)$, $y = \sin(2\pi t)$.

Example 3. Lemniscate: $x = \sin(2\pi t)$, $y = \sin(2\pi t)\cos(2\pi t)$.

Example 4. Epicycloid: $x = 4\cos(\pi t) - 4\cos(4\pi t)$, $y = 4\sin(\pi t) - 4\sin(4\pi t)$.

7.1. Classical quadratic QI

n	1		2		3		4	
64	1.1(-5)	15.9	1.7(-5)	15.9	9.0(-5)	15.8	4.4(-2)	15.1
128	7.0(-7)	16	1.1(-6)	16	5.7(-6)	15.9	2.8(-3)	15.8
256	4.4(-8)	16	6.8(-8)	16	3.5(-7)	16	1.8(-4)	15.9
512	2.7(-9)	16	4.2(-9)	16	2.2(-8)	16	1.1(-5)	16
1024	1.7(-10)	16	2.6(-10)	16	1.4(-9)	16	6.9(-7)	16

7.2. Quadratic QI projector

n	1		2		3		4	
64	1.5(-6)	16.3	2.3(-6)	16.3	1.2(-5)	17	6.4(-3)	19.6
128	9.2(-8)	16.1	1.4(-7)	16.1	7.2(-7)	16.3	3.8(-4)	17.1
256	5.8(-9)	16	8.9(-9)	16	4.5(-8)	16.1	2.3(-5)	16.3
512	3.6(-10)	16	5.6(-10)	16	2.8(-9)	16	1.4(-6)	16.1
1024	2.2(-11)	16	3.5(-10)	16	1.7(-10)	16	9.0(-8)	16

7.3. Quadratic Lagrange interpolant of type 2

n	1		2		3		4	
64	6.7(-6)	15.9	1.0(-5)	15.9	4.9(-5)	15.8	2.6(-2)	15
128	4.2(-7)	16	6.4(-7)	16	3.1(-6)	15.9	1.6(-3)	15.7
256	2.6(-8)	16	4.0(-8)	16	1.9(-7)	16	-1.0(-4)	15.9
512	1.6(-9)	16	2.5(-9)	16	1.2(-8)	16	6.5(-6)	16
1024	1.0(-10)	16	1.6(-10)	16	6.9(-10)	17.6	4.0(-7)	16

7.4. Classical cubic QI

Ex	1		2		3		4	
64	1.9(-5)	15.9	2.9(-5)	15.1	1.5(-4)	15.8	7.7(-1)	15.1
128	1.2(-6)	16	1.9(-6)	15.6	9.9(-6)	15.7	4.9(-3)	15.8
256	7.6(-8)	16	1.2(-7)	15.8	6.2(-7)	16	3.1(-4)	15.9
512	4.7(-9)	16	7.3(-9)	15.8	3.9(-8)	16	1.9(-5)	16
1024	2.9(-10)	16	4.6(-10)	15.8	2.4(-9)	16	1.2(-6)	16

7.5. Cubic QI projector

n	1		2		3		4	
64	7.3(-6)	16	1.1(-5)	17.1	5.7(-5)	16.1	2.9(-2)	16.2
128	4.6(-7)	16	7.0(-7)	16	3.7(-6)	15.5	1.8(-3)	16
256	2.8(-8)	16	4.4(-8)	16	2.3(-7)	16	1.1(-4)	16
512	1.8(-9)	16	2.7(-9)	16	1.4(-8)	16	7.2(-6)	16
1024	1.1(-10)	16	1.7(-10)	16	9.0(-10)	16	4.5(-7)	16

7.6. Cubic Lagrange interpolant of type 2

n	1		2		3		4	
64	6.8(-7)	10.9	3.0(-5)	15.5	1.6(-4)	15.7	7.7(-2)	15.1
128	4.4(-8)	15.3	1.9(-6)	16	9.9(-6)	16.1	4.9(-3)	15.8
256	2.8(-9)	15.9	1.2(-7)	16	6.2(-7)	16	3.1(-4)	15.9
512	1.7(-10)	16	7.3(-9)	16	3.9(-8)	16	1.9(-5)	16
1024	1.1(-11)	16	4.6(-10)	16	2.4(-9)	16	1.2(-6)	16

8. Error estimates

8.1. First method: quadratic splines

Let us consider a parametric planar curve (C) defined by $x = f(t), y = g(t), t \in I := [a,b]$. Let $X = \varphi(t), Y = \gamma(t)$ be a spline approximant (S) of this curve and let (P) be the C^0 quadratic spline formed by parabolas interpolating C at midpoints and endpoints of each subinterval of length h. Then, denoting the lengths of the corresponding curves by $L(C), L(S)$ and $L(P)$ and according to [9] (Lemma 6), we have $L(C) - L(P) = O(h^5)$. As (S) is such that $S(t) - C(t) = O(h^4)$ at the interpolation points (superconvergence of quadratic approximants), then it is enough to prove that the difference between the lengths of local parabolas of (P) and (S) is also of order $O(h^4)$. This has been checked directly on the exact expression of the length given in Section 2 by using a computer algebra system (Maple). More specifically, we have chosen two parabolas defined on the same interval whose ordinates of control points differ by an order $O(h^4)$ and we have verified that the lengths also differ by an order $O(h^4)$. Another proof is given in [23].

8.2. Second method: cubic splines

Using the composite *Simpson's rule* $SL(C) := \sum A_i \sqrt{f'(x_i)^2 + g'(x_i)^2}$, we get

$$L(C) - L(S) = (L(C) - SL(C)) + (SL(C) - SL(S)) + (SL(S) - L(S)).$$

We first have (see e.g. [9]):

$$L(C) - SL(C) = O(h^4) \quad \text{and} \quad L(S) - SL(S) = O(h^4).$$

As the approximation order of f' by φ' (and of g' by γ') is $O(h^4)$ *at knots* (all of them for PGS, except near endpoints for NAK), the global error is in $O(h^4)$ thanks to the following majoration:

$$|SL(C) - SL(S)| \leq \sum A_i \left| \sqrt{f'(x_i)^2 + g'(x_i)^2} - \sqrt{\varphi'(x_i)^2 + \gamma'(x_i)^2} \right|$$

$$\leq K_3 \sum A_i \left(\left| f'(x_i)^2 - \varphi'(x_i)^2 \right| + \left| g'(x_i)^2 - \gamma'(x_i)^2 \right| \right)$$

$$\leq K_4 \left(\| f' \|_\infty \max \left| f'(x_i) - \varphi'(x_i) \right| + (\| g' \|_\infty \max \left| g'(x_i) - \gamma'(x_i) \right| \right) = O(h^4).$$

9. Final remarks and open questions

- Comparing the results obtained by the two classes of methods, it is very difficult to decide which method is the best. However, the various examples show that interpolating projectors are quite good and also, in the second class of methods, the use of NAK and PGS methods followed by Simpson's or companion's formulas. We have not tested Gauss quadrature rules because we where mainly interested in the cases where the curves are known by points defined on uniform partitions.

- We refer to the book [6] and references therein for the interesting applications of Pythagorean hodograph curves to the computation of arc lengths.

- For piecewise cubics, in theory it is possible to compute exact arc lengths using elliptic functions. However it is rather complicated in practice.

- See [1] and [23] for applications of chord length parameterization of plane curves to the approximate computation of areas and various moments associated with closed curves, and also in [24] to the numerical solution of boundary integral equations.

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