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SHIFT OPERATORS AND STABILITY IN DELAYED DYNAMIC EQUATIONS

Abstract. In this paper, we use what we call the shift operator so that general delay dynamic equations of the form

$$x^{\Delta}(t) = a(t)x(t) + b(t)x(\delta_-(h,t))\delta_-^{\Delta}(h,t), \quad t \in [t_0, \infty) \cap \mathbb{T}$$

can be analyzed with respect to stability and existence of solutions. By means of the shift operators, we define a general delay function opening an avenue for the construction of Lyapunov functional on time scales. Thus, we use the Lyapunov's direct method to obtain inequalities that lead to stability and instability. Therefore, we extend and unify stability analysis of delay differential, delay difference, delay *h*-difference, and delay *q*-difference equations which are the most important particular cases of our delay dynamic equation.

1. Introduction

Lyapunov functionals are widely used in stability analysis of differential and difference equations. However, the extension of utilization of Lyapunov functionals in dynamical systems on time scales has been lacking behind due to the constraints presented by the particular time scale. For example, in delay differential equations, a suitable Lyapunov functional will involve a term with double integrals, in which one of the integral's lower limit is of the form $t+s$. Such a requirement will restrict the time scale that can be considered.

For a few references on the study of stability in differential equations using Lyapunov functionals, we refer the interested reader to [4, 3] and [12]–[24]. The reader may consult Yoshizawa [24, pp. 183–213] (or any book on functional differential equations and Lyapunov's direct method) for definitions of stability and for properties of Lyapunov functionals. For the stability analysis of the delay differential equation

$$x'(t) = a(t)x(t) + b(t)x(t-h), \quad h > 0 \tag{1}$$

we refer to [16]–[18] and [22]. In [2], the authors improved the results of [22] by considering the delay differential equation of the form

$$x'(t) = a(t)x(t) + b(t)x(t-h(t)), \quad 0 < h(t) \le r_0. \tag{2}$$

On the other hand, stability analysis of delay difference equations of the form

$$x(t+1) = a(t)x(t) + b(t)x(t-\tau), \quad \tau \in \mathbb{Z}_+ \tag{3}$$

is treated in [8, 20, 21].

A time scale, denoted $\mathbb{T}$, is a nonempty closed subset of real numbers. The set $\mathbb{T}^{\kappa}$ is derived from the time scale $\mathbb{T}$ as follows: if $\mathbb{T}$ has a left-scattered maximum M, then $\mathbb{T}^\kappa = \mathbb{T} - \{M\}$, otherwise $\mathbb{T}^\kappa = \mathbb{T}$. The delta derivative f^Δ of a function $f : \mathbb{T} \to \mathbb{R}$, defined at a point $t \in \mathbb{T}^\kappa$ by

$$f^\Delta(t) := \lim_{s \to t} \frac{f(\sigma(t)) - f(s)}{\sigma(t) - s}, \quad \text{where } s \to t, \ \ s \in \mathbb{T} \backslash \{\sigma(t)\}, \tag{4}$$

was first introduced by Hilger [19] to unify discrete and continuous analyses. In (4), $\sigma : \mathbb{T} \to \mathbb{T}$ is the forward jump operator defined by $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$. Hereafter, we denote by $\mu(t)$ the step size function $\mu : \mathbb{T} \to \mathbb{R}$ defined by $\mu(t) := \sigma(t) - t$. A point $t \in \mathbb{T}$ is said to be right dense (right scattered) if $\mu(t) = 0$ ($\mu(t) > 0$). A point is said to be left dense if $\sup\{s \in \mathbb{T} : s < t\} = t$. A function $f : \mathbb{T} \to \mathbb{R}$ is called *rd*-continuous if it is continuous at right-dense points and its left sided limits exists (finite) at left dense points. Every *rd*-continuous function $f : \mathbb{T} \to \mathbb{R}$ has an anti-derivative F denoted by

$$F(t) := \int_{t_0}^{t} f(t)\,\Delta t.$$

To indicate the time scale interval $[a,b] \cap \mathbb{T}$ we use the notation $[a,b]_{\mathbb{T}}$. The intervals $[a,b)_{\mathbb{T}}$, $(a,b]_{\mathbb{T}}$, and $(a,b)_{\mathbb{T}}$ are defined similarly. For brevity, we assume the reader is familiar with the basic calculus of time scales. A comprehensive review on dynamic equations on time scales can be found in [10, 11].

In [4, 7], the authors handle the stability analysis of the dynamic equation

$$x^\Delta(t) = a(t)x(t) + b(t)x(\delta(t))\delta^\Delta(t), \tag{5}$$

where the delay function $\delta : [t_0, \infty)_{\mathbb{T}} \to [\delta(t_0), \infty)_{\mathbb{T}}$ is surjective, strictly increasing and is supposed to have the following properties

$$\delta(t) < t, \qquad \delta^\Delta(t) < \infty, \qquad \delta \circ \sigma = \sigma \circ \delta.$$

Afterwards, we point out in [6] that the assumption $\delta \circ \sigma = \sigma \circ \delta$ is redundant whenever the delay function $\delta : [t_0, \infty)_{\mathbb{T}} \to [\delta(t_0), \infty)_{\mathbb{T}}$ is surjective and strictly increasing.

Note that the delta derivative in (4) turns into the ordinary derivative $f'(t)$ and the forward difference $\Delta f(t) := f(t+1) - f(t)$ when $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$, respectively. Hence, (5) is a general equation including the particular cases (1) and (3). However, this paper improves the results of [4].

In this paper, we define the general shift operator and make use of them in the construction of the Lyapunov functional to improve previous results on delay dynamic equations regarding stability and boundedness of solutions. In particular, we improve the results concerning (1), (3) and (5). The main task of this paper can be outlined as follows:

- To create a suitable Lyapunov function that leads to exponential stability of the zero solution.
- To give criteria for instability.
- To compare the results of this paper with ones in the existing literature.

In [22], the author used the following

$$V(t)=\left[x(t)+\int_{t-h}^{t}b(s+h)x(s)ds\right]^2+\lambda\int_{-h}^{0}\int_{t+s}^{t}b^2(z+h)x^2(z)dzds. \tag{6}$$

to study the exponential stability of the zero solution of (1).

We do not adopt this type of Lyapunov functional since it requires the time scale to be additive. An additive time scale is a time scale which is closed under addition. There are many time scales that are not additive. To be more specific, the time scales $\overline{q^{\mathbb{Z}}}=\{0\}\cup\{q^n : n\in\mathbb{Z}\}$, $\sqrt{\mathbb{N}}=\{\sqrt{n}: n\in\mathbb{N}\}$ are not additive. However, $\delta_{\pm}(s,t)=ts^{\pm 1}$ and $\delta_{\pm}(s,t)=\sqrt{t^2\pm s^2}$ are the shift operators defined on $\overline{q^{\mathbb{Z}}}$ and $\sqrt{\mathbb{N}}$, respectively. It turns out that we need the notion of shift operators to avoid the additivity assumption on the time scale. That is, to include more time scales in the investigation. Shift operators were first introduced in [1] to obtain function bounds for convolution type Volterra integro-dynamic equations on time scales. However, the time scales considered in [1] is restricted to the ones having an initial point $t_0\in\mathbb{T}$ so that there exist shift operators defined on $[t_0,\infty)\cap\mathbb{T}$. Afterwards, in [5] the definition of shift operators was extended so that they are defined on the whole time scale $\mathbb{T}$. In this paper, our new and generalized shift operators include positive and negative values.

We end this section by giving some basic definitions and theorems that will be used in further sections.

DEFINITION 1. *A function $h:\mathbb{T}\to\mathbb{R}$ is said to be regressive if $1+\mu(t)h(t)\neq 0$ for all $t\in\mathbb{T}^{\kappa}$, where $\mu(t)=\sigma(t)-t$. The set of all regressive rd-continuous functions $\varphi:\mathbb{T}\to\mathbb{R}$ is denoted by $\mathcal{R}$ while $\mathcal{R}^+=\{h\in\mathcal{R}: 1+\mu(t)\varphi(t)>0 \text{ for all } t\in\mathbb{T}\}$.*

Let $\varphi\in\mathcal{R}$. The *exponential function* on $\mathbb{T}$ is defined by

$$e_{\varphi}(t,s)=\exp\left(\int_s^t \zeta_{\mu(r)}(\varphi(r))\Delta r\right) \tag{7}$$

where $\zeta_{\mu(s)}$ is the cylinder transformation given by

$$\zeta_{\mu(r)}(\varphi(r)):=\begin{cases}\frac{1}{\mu(r)}\mathrm{Log}(1+\mu(r)\varphi(r)) & if\ \mu(r)>0\\ \varphi(r) & if\ \mu(r)=0.\end{cases} \tag{8}$$

It is well known that if $p\in\mathcal{R}^+$, then $e_p(t,s)>0$ for all $t\in\mathbb{T}$. Also, the exponential function $y(t)=e_p(t,s)$ is the solution to the initial value problem $y^{\Delta}=p(t)y$, $y(s)=1$. Other properties of the exponential function are given in the following lemma:

LEMMA 1 ([10, Theorem 2.36]). *Let $p,q\in\mathcal{R}$. Then*

(i) $e_0(t,s)\equiv 1$ *and* $e_p(t,t)\equiv 1$;

(ii) $e_p(\sigma(t),s)=(1+\mu(t)p(t))e_p(t,s)$;

(iii) $\dfrac{1}{e_p(t,s)}=e_{\ominus p}(t,s)$ *where* $\ominus p(t)=-\dfrac{p(t)}{1+\mu(t)p(t)}$;

(iv) $e_p(t,s) = \dfrac{1}{e_p(s,t)} = e_{\ominus p}(s,t)$;

(v) $e_p(t,s)e_p(s,r) = e_p(t,r)$;

(vi) $\left(\dfrac{1}{e_p(\cdot,s)}\right)^{\Delta} = -\dfrac{p(t)}{e_p^{\sigma}(\cdot,s)}$.

THEOREM 1 ([10, Theorem 1.117]). *Let $a \in \mathbb{T}^{\kappa}$, $b \in \mathbb{T}$ and assume that $k : \mathbb{T}\times\mathbb{T}^{\kappa} \to \mathbb{R}$ is continuous at (t,t), where $t \in \mathbb{T}^{\kappa}$ with $t > a$. Also assume that $k^{\Delta}(t,.)$ is rd-continuous on $[a,\sigma(t)]$. Suppose that for each $\varepsilon > 0$ there exists a neighborhood U of t, independent of $\tau \in [t_0,\sigma(t)]$, such that*

$$\left|k(\sigma(t),\tau) - k(s,r) - k^{\Delta}(t,\tau)(\sigma(t)-s)\right| \le \varepsilon|\sigma(t)-s|$$

for all $s \in U$, where k^{Δ} is the derivative of k with respect to the first variable. Then

$$g(t) := \int_a^t k(t,\tau)\Delta\tau \text{ implies } g^{\Delta}(t) = \int_a^t k^{\Delta}(t,\tau)\Delta\tau + k(\sigma(t),t)$$
$$h(t) := \int_t^b k(t,\tau)\Delta\tau \text{ implies } g^{\Delta}(t) = \int_t^b k^{\Delta}(t,\tau)\Delta\tau - k(\sigma(t),t).$$

2. Shift operators

Next, we define generalized shift operators. A limited version can be found in [1].

DEFINITION 2. *Let $\mathbb{T}^*$ be a non-empty subset of the time scale $\mathbb{T}$ and $t_0 \in \mathbb{T}^*$ a fixed number such that there exist operators $\delta_{\pm} : [t_0,\infty)_{\mathbb{T}} \times \mathbb{T}^* \to \mathbb{T}^*$ satisfying the following properties:*

P1 The functions $\delta_{\pm}$ are strictly increasing with respect to their second arguments, i.e., if

$$(T_0,t),(T_0,u) \in \mathcal{D}_{\pm} := \{(s,t) \in [t_0,\infty)_{\mathbb{T}} \times \mathbb{T}^* : \delta_{\pm}(s,t) \in \mathbb{T}^*\},$$

then $T_0 \le t < u$ implies $\delta_{\pm}(T_0,t) < \delta_{\pm}(T_0,u)$;

P2 if $(T_1,u),(T_2,u) \in \mathcal{D}_-$ with $T_1 < T_2$, then $\delta_-(T_1,u) > \delta_-(T_2,u)$, and if $(T_1,u),(T_2,u) \in \mathcal{D}_+$ with $T_1 < T_2$, then $\delta_+(T_1,u) < \delta_+(T_2,u)$;

P3 if $t \in [t_0,\infty)_{\mathbb{T}}$, then $(t,t_0) \in \mathcal{D}_+$ and $\delta_+(t,t_0) = t$. Moreover, if $t \in \mathbb{T}^*$, then $(t_0,t) \in \mathcal{D}_+$ and $\delta_+(t_0,t) = t$;

P4 if $(s,t) \in \mathcal{D}_{\pm}$, then $(s,\delta_{\pm}(s,t)) \in \mathcal{D}_{\mp}$ and $\delta_{\mp}(s,\delta_{\pm}(s,t)) = t$;

P5 if $(s,t) \in \mathcal{D}_{\pm}$ and $(u,\delta_{\pm}(s,t)) \in \mathcal{D}_{\mp}$, then $(s,\delta_{\mp}(u,t)) \in \mathcal{D}_{\pm}$ and

$$\delta_{\mp}(u,\delta_{\pm}(s,t)) = \delta_{\pm}(s,\delta_{\mp}(u,t)).$$

Then the operators δ_- *and* δ_+ *associated with* $t_0 \in \mathbb{T}^*$ *(called the initial point) are said to be backward and forward shift operators on the set* $\mathbb{T}^*$*, respectively. The variable* $s \in [t_0,\infty)_{\mathbb{T}}$ *in* $\delta_\pm(s,t)$ *is called the shift size. The values* $\delta_+(s,t)$ *and* $\delta_-(s,t)$ *in* $\mathbb{T}^*$ *indicate s units translation of the term* $t \in \mathbb{T}^*$ *to the right and left, respectively. The sets* $\mathcal{D}_\pm$ *are the domains of the shift operators* $\delta_\pm$*, respectively.*

EXAMPLE 1. Let $\mathbb{T} = \mathbb{R}$ and $t_0 = 1$. The operators

$$\delta_-(s,t) = \begin{cases} t/s & \text{if } t \geq 0 \\ st & \text{if } t < 0, \end{cases} \qquad \text{for } s \in [1,\infty) \tag{9}$$

and

$$\delta_+(s,t) = \begin{cases} st & \text{if } t \geq 0 \\ t/s & \text{if } t < 0, \end{cases} \qquad \text{for } s \in [1,\infty) \tag{10}$$

are backward and forward shift operators on $\mathbb{T}^* = \mathbb{R} - \{0\}$, respectively. In the table, we state different time scales with their corresponding shift operators:

$\mathbb{T}$	t_0	$\mathbb{T}^*$	$\delta_-(s,t)$	$\delta_+(s,t)$
$\mathbb{R}$	0	$\mathbb{R}$	$t-s$	$t+s$
$\mathbb{Z}$	0	$\mathbb{Z}$	$t-s$	$t+s$
$q^{\mathbb{Z}} \cup \{0\}$	1	$q^{\mathbb{Z}}$	t/s	st
$\mathbb{N}^{1/2}$	0	$\mathbb{N}^{1/2}$	$\sqrt{t^2-s^2}$	$\sqrt{t^2+s^2}$

The proof of the next lemma is a direct consequence of Definition 2.

LEMMA 2. *Let* δ_-, δ_+ *be the shift operators associated with the initial point* t_0*. We have*

(i) $\delta_-(t,t) = t_0$ *for all* $t \in [t_0,\infty)_{\mathbb{T}}$*;*

(ii) $\delta_-(t_0,t) = t$ *for all* $t \in \mathbb{T}^*$*;*

(iii) If $(s,t) \in \mathcal{D}_+$*, then* $\delta_+(s,t) = u$ *implies* $\delta_-(s,u) = t$*. Conversely, if* $(s,u) \in \mathcal{D}_-$*, then* $\delta_-(s,u) = t$ *implies* $\delta_+(s,t) = u$*;*

(iv) $\delta_+(t,\delta_-(s,t_0)) = \delta_-(s,t)$ *for all* $(s,t) \in \mathcal{D}(\delta_+)$ *with* $t \geq t_0$*;*

(v) $\delta_+(u,t) = \delta_+(t,u)$ *for all* $(u,t) \in ([t_0,\infty)_{\mathbb{T}} \times [t_0,\infty)_{\mathbb{T}}) \cap \mathcal{D}_+$*;*

(vi) $\delta_+(s,t) \in [t_0,\infty)_{\mathbb{T}}$ *for all* $(s,t) \in \mathcal{D}_+$ *with* $t \geq t_0$*;*

(vii) $\delta_-(s,t) \in [t_0,\infty)_{\mathbb{T}}$ *for all* $(s,t) \in ([t_0,\infty)_{\mathbb{T}} \times [s,\infty)_{\mathbb{T}}) \cap \mathcal{D}_-$*;*

(viii) If $\delta_+(s,\cdot)$ *is* Δ*-differentiable in its second variable, then* $\delta_+^{\Delta_t}(s,\cdot) > 0$*;*

(ix) $\delta_+(\delta_-(u,s),\delta_-(s,v)) = \delta_-(u,v)$ *for all* $(s,v) \in ([t_0,\infty)_{\mathbb{T}} \times [s,\infty)_{\mathbb{T}}) \cap \mathcal{D}_-$*, and* $(u,s) \in ([t_0,\infty)_{\mathbb{T}} \times [u,\infty)_{\mathbb{T}}) \cap \mathcal{D}_-$*;*

(x) If $(s,t) \in \mathcal{D}_-$ and $\delta_-(s,t) = t_0$, then $s = t$.

Proof. (i) is obtained from P3–P5 since $\delta_-(t,t) = \delta_-(t,\delta_+(t,t_0)) = t_0$ for all $t \in \mathbb{T}^*$. Part (ii) is obtained from P3–P4 since $\delta_-(t_0,t) = \delta_-(t_0,\delta_+(t_0,t)) = t$.

Let $u := \delta_+(s,t)$. By P4 we have $(s,u) \in \mathcal{D}_-$ for all $(s,t) \in \mathcal{D}_+$, and hence

$$\delta_-(s,u) = \delta_-(s,\delta_+(s,t)) = t.$$

The latter part of (iii) can be done similarly. We have (iv) since P3 and P5 yield

$$\delta_+(t,\delta_-(s,t_0)) = \delta_-(s,\delta_+(t,t_0)) = \delta_-(s,t).$$

P3 and P5 guarantee that

$$t = \delta_+(t,t_0) = \delta_+(t,\delta_-(u,u)) = \delta_-(u,\delta_+(t,u))$$

for all $(u,t) \in ([t_0,\infty)_{\mathbb{T}} \times [t_0,\infty)_{\mathbb{T}}) \cap \mathcal{D}_+$. Using (iii) we have

$$\delta_+(u,t) = \delta_+(u,\delta_-(u,\delta_+(t,u))) = \delta_+(t,u).$$

This proves (v). To prove (vi) and (vii) we use P1–P2 to get

$$\delta_+(s,t) \geq \delta_+(t_0,t) = t \geq t_0$$

for all $(s,t) \in ([t_0,\infty) \times [t_0,\infty)_{\mathbb{T}}) \cap \mathcal{D}_+$ and

$$\delta_-(s,t) \geq \delta_-(s,s) = t_0$$

for all $(s,t) \in ([t_0,\infty)_{\mathbb{T}} \times [s,\infty)_{\mathbb{T}}) \cap \mathcal{D}_-$. Since $\delta_+(s,t)$ is strictly increasing in its second variable we have (viii) by [11, Corollary 1.16]. (ix) is proven as follows: from P5 and (v) we have

$$\begin{aligned}
\delta_+(\delta_-(u,s),\delta_-(s,v)) &= \delta_-(s,\delta_+(v,\delta_-(u,s))) \\
&= \delta_-(s,\delta_-(u,\delta_+(v,s))) \\
&= \delta_-(s,\delta_+(s,\delta_-(u,v))) \\
&= \delta_-(u,v)
\end{aligned}$$

for all $(s,v) \in ([t_0,\infty)_{\mathbb{T}} \times [s,\infty)_{\mathbb{T}}) \cap \mathcal{D}_-$ and $(u,s) \in ([t_0,\infty)_{\mathbb{T}} \times [u,\infty)_{\mathbb{T}}) \cap \mathcal{D}_-$. Suppose $(s,t) \in \mathcal{D}_- = \{(s,t) \in [t_0,\infty)_{\mathbb{T}} \times \mathbb{T}^* : \delta_-(s,t) \in \mathbb{T}^*\}$ and $\delta_-(s,t) = t_0$. Then by P4 we have

$$t = \delta_+(s,\delta_-(s,t)) \in \delta_+(s,t_0) = s.$$

This is (x). The proof is complete. □

Notice that the shift operators $\delta_\pm$ are defined once the initial point $t_0 \in \mathbb{T}^*$ is known. For instance, we choose the initial point $t_0 = 0$ to define shift operators $\delta_\pm(s,t) = t \pm s$ on $\mathbb{T} = \mathbb{R}$. However, if we choose $\lambda \in (0,\infty)$ as the initial point, then the new shift operators associated with λ are defined by $\widetilde{\delta}_\pm(s,t) = t \mp \lambda \pm s$. In terms of $\delta_\pm$, the operators $\widetilde{\delta}_\pm$ can be given as

$$\widetilde{\delta}_\pm(s,t) = \delta_\mp(\lambda,\delta_\pm(s,t)).$$

EXAMPLE 2. Here are some particular time scales to show the change in the formula of shift operators as the initial point changes:

	$\mathbb{T}=\mathbb{N}^{1/2}$		$\mathbb{T}=h\mathbb{Z}$		$\mathbb{T}=2^{\mathbb{N}}$	
t_0	0	λ	0	$h\lambda$	1	2^{λ}
$\delta_-(s,t)$	$\sqrt{t^2-s^2}$	$\sqrt{t^2+\lambda^2-s^2}$	$t-s$	$t+h\lambda-s$	t/s	$2^{\lambda}ts^{-1}$
$\delta_+(s,t)$	$\sqrt{t^2+s^2}$	$\sqrt{t^2-\lambda^2+s^2}$	$t+s$	$t-h\lambda+s$	ts	$2^{-\lambda}ts$

where $\lambda\in\mathbb{Z}_+$, $\mathbb{N}^{1/2}:=\{\sqrt{n}:n\in\mathbb{N}\}$, $2^{\mathbb{N}}:=\{2^n:n\in\mathbb{N}\}$, and $h\mathbb{Z}:=\{hn:n\in\mathbb{Z}\}$.

3. Delay function

In this section we introduce the delay function on time scales that will be used for the construction of the Lyapunov functional.

DEFINITION 3. *Let $\mathbb{T}$ be a time scale that is unbounded above and $t_0\in\mathbb{T}^*$ an element such that there exist the shift operators $\delta_\pm:[t_0,\infty)\times\mathbb{T}^*\to\mathbb{T}^*$ associated with t_0. Suppose that $h\in(t_0,\infty)_{\mathbb{T}}$ is a constant such that $(h,t)\in D_\pm$ for all $t\in[t_0,\infty)_{\mathbb{T}}$, the function $\delta_-(h,t)$ is differentiable with an rd-continuous derivative, and $\delta_-(h,t)$ maps $[t_0,\infty)_{\mathbb{T}}$ onto $[\delta_-(h,t_0),\infty)_{\mathbb{T}}$. Then the function $\delta_-(h,t)$ is called the delay function generated by the shift δ_- on the time scale $\mathbb{T}$.*

It is obvious from P2 and (iii) of Lemma 2 that

$$\delta_-(h,t)<\delta_-(t_0,t)=t \quad \text{for all } t\in[t_0,\infty)_{\mathbb{T}}. \tag{11}$$

Notice that $\delta_-(h,\cdot)$ is strictly increasing and it is invertible. Hence, by P4–P5, we have $\delta_-^{-1}(h,t)=\delta_+(h,t)$.

Hereafter, we shall suppose that $\mathbb{T}$ is a time scale with the delay function $\delta_-(h,\cdot):[t_0,\infty)_{\mathbb{T}}\to[\delta_-(h,t_0),\infty)_{\mathbb{T}}$, where $t_0\in\mathbb{T}$ is fixed. Set

$$\mathbb{T}_1=[t_0,\infty)_{\mathbb{T}} \quad \text{and} \quad \mathbb{T}_2=\delta_-(h,\mathbb{T}_1). \tag{12}$$

Evidently, $\mathbb{T}_1$ is closed in $\mathbb{R}$. By definition we have $\mathbb{T}_2=[\delta_-(h,t_0),\infty)_{\mathbb{T}}$. Hence, $\mathbb{T}_1$ and $\mathbb{T}_2$ are both time scales. Let σ_1 and σ_2 denote the forward jump operators on the time scales $\mathbb{T}_1$ and $\mathbb{T}_2$, respectively. By (11)–(12),

$$\mathbb{T}_1\subset\mathbb{T}_2\subset\mathbb{T}.$$

Thus,

$$\sigma(t)=\sigma_2(t) \quad \text{for all } t\in\mathbb{T}_2$$

and

$$\sigma(t)=\sigma_1(t)=\sigma_2(t) \quad \text{for all } t\in\mathbb{T}_1.$$

That is, σ_1 and σ_2 are the restrictions of the forward jump operator $\sigma:\mathbb{T}\to\mathbb{T}$ to the time scales $\mathbb{T}_1$ and $\mathbb{T}_2$, respectively, i.e.,

$$\sigma_1=\sigma|_{\mathbb{T}_1} \text{ and } \sigma_2=\sigma|_{\mathbb{T}_2}.$$

Recall that the Hilger derivatives Δ, Δ_1, and Δ_2 on the time scales $\mathbb{T}$, $\mathbb{T}_1$, and $\mathbb{T}_2$ are defined in terms of the forward jumps σ, σ_1, and σ_2, respectively. Hence, if f is a differentiable function at $t \in \mathbb{T}_2$, then we have

$$f^{\Delta_2}(t) = f^{\Delta_1}(t) = f^{\Delta}(t), \quad \text{for all } t \in \mathbb{T}_1.$$

Similarly, if $a,b \in \mathbb{T}_2$ are two points with $a < b$ and if f is a rd-continuous function on the interval $(a,b)_{\mathbb{T}_2}$, then

$$\int_a^b f(s)\Delta_2 s = \int_a^b f(s)\Delta s.$$

The next result is essential for future calculations.

LEMMA 3. *The delay function $\delta_-(h,t)$ preserves the structure of the points in $\mathbb{T}_1$. That is,*

$$\sigma_1(\widehat{t}) = \widehat{t} \ \textit{ implies } \ \sigma_2(\delta_-(h,\widehat{t})) = \delta_-(h,\widehat{t}).$$
$$\sigma_1(\widehat{t}) > \widehat{t} \ \textit{ implies } \ \sigma_2(\delta_-(h,\widehat{t}) > \delta_-(h,\widehat{t}).$$

Proof. By definition $\sigma_1(t) \geq t$ for all $t \in \mathbb{T}_1$. Thus,

$$\delta_-(h,\sigma_1(t)) \geq \delta_-(h,t).$$

Since $\sigma_2(\delta_-(h,t))$ is the smallest element satisfying

$$\sigma_2(\delta_-(h,t)) \geq \delta_-(h,t),$$

we get

$$\delta_-(h,\sigma_1(t)) \geq \sigma_2(\delta_-(h,t)) \ \text{ for all } \ t \in \mathbb{T}_1. \tag{13}$$

If $\sigma_1(\widehat{t}) = \widehat{t}$, then we have

$$\delta_-(h,\widehat{t}) = \delta_-(h,\sigma_1(\widehat{t})) \geq \sigma_2(\delta_-(h,\widehat{t})).$$

That is,

$$\delta_-(h,\widehat{t}) = \sigma_2(\delta_-(h,\widehat{t})).$$

If $\sigma_1(\widehat{t}) > \widehat{t}$, then

$$(\widehat{t},\sigma_1(\widehat{t}))_{\mathbb{T}_1} = (\widehat{t},\sigma_1(\widehat{t}))_{\mathbb{T}} = \varnothing$$

and

$$\delta_-(h,\sigma_1(\widehat{t})) > \delta_-(h,\widehat{t}).$$

Suppose the contrary. That is $\delta_-(h,\widehat{t})$ is right dense; namely $\sigma_2(\delta_-(h,\widehat{t})) = \delta_-(h,\widehat{t})$. This along with (13) implies

$$(\delta_-(h,\widehat{t}),\delta_-(h,\sigma_1(\widehat{t})))_{\mathbb{T}_2} \neq \varnothing.$$

Pick one element $s \in (\delta_-(h,\widehat{t}),\delta_-(h,\sigma_1(\widehat{t})))_{\mathbb{T}_2}$. Since $\delta_-(h,t)$ is strictly increasing in t and invertible, there should be an element $t \in (\widehat{t},\sigma_1(\widehat{t}))_{\mathbb{T}_1}$ such that $\delta_-(h,t) = s$. This leads to a contradiction. Hence, $\delta_-(h,\widehat{t})$ must be right scattered. □

Using the preceding lemma and applying the fact that $\sigma_2(u) = \sigma(u)$ for all $u \in \mathbb{T}_2$, we arrive at the following result.

COROLLARY 1. *We have*

$$\delta_-(h, \sigma_1(t)) = \sigma_2(\delta_-(h,t)) \ \ \textit{for all} \ \ t \in \mathbb{T}_1.$$

Thus,

$$\delta_-(h, \sigma(t)) = \sigma(\delta_-(h,t)) \ \ \textit{for all} \ \ t \in \mathbb{T}_1. \tag{14}$$

By (14) we have

$$\delta_-(h, \sigma(s)) = \sigma(\delta_-(h,s)) \ \ \text{for all} \ \ s \in [t_0, \infty)_{\mathbb{T}}.$$

Substituting $s = \delta_+(h,t)$ we obtain

$$\delta_-(h, \sigma(\delta_+(h,t))) = \sigma(\delta_-(h, \delta_+(h,t))) = \sigma(t).$$

This and (iv) of Lemma 2 imply

$$\sigma(\delta_+(h,t)) = \delta_+(h, \sigma(t)) \ \ \text{for all} \ \ t \in [\delta_-(h,t_0), \infty)_{\mathbb{T}}.$$

EXAMPLE 3. We next give some time scales with their shift operators:

$\mathbb{T}$	h	$\delta_-(h,t)$	$\delta_+(h,t)$
$\mathbb{R}$	$\in \mathbb{R}_+$	$t-h$	$t+h$
$\mathbb{Z}$	$\in \mathbb{Z}_+$	$t-h$	$t+h$
$q^{\mathbb{Z}} \cup \{0\}$	$\in q^{\mathbb{Z}_+}$	$\frac{t}{h}$	ht
$\mathbb{N}^{1/2}$	$\in \mathbb{Z}_+$	$\sqrt{t^2-h^2}$	$\sqrt{t^2+h^2}$

EXAMPLE 4. There is no delay function $\delta_-(h,\cdot) : [0,\infty)_{\widetilde{\mathbb{T}}} \to [\delta_-(h,0),\infty)_{\mathbb{T}}$ on the time scale $\widetilde{\mathbb{T}} = (-\infty, 0] \cup [1, \infty)$. Suppose to the contrary that there exists such a delay function on $\widetilde{\mathbb{T}}$. Then since 0 is right scattered in $\widetilde{\mathbb{T}}_1 := [0,\infty)_{\widetilde{\mathbb{T}}}$ the point $\delta_-(h,0)$ must be right scattered in $\widetilde{\mathbb{T}}_2 = [\delta_-(h,0),\infty)_{\mathbb{T}}$, i.e., $\sigma_2(\delta_-(h,0)) > \delta_-(h,0)$. Since $\sigma_2(t) = \sigma(t)$ for all $t \in [\delta_-(h,0),0)_{\mathbb{T}}$, we have

$$\sigma(\delta_-(h,0)) = \sigma_2(\delta_-(h,0)) > \delta_-(h,0).$$

That is, $\delta_-(h,0)$ must be right scattered in $\widetilde{\mathbb{T}}$. However, in $\widetilde{\mathbb{T}}$ we have $\delta_-(h,0) < 0$, that is, $\delta_-(h,0)$ is right dense. This leads to a contradiction.

THEOREM 2 (Substitution [10, Theorem 1.98]). *Assume $\nu : \mathbb{T} \to \mathbb{R}$ is strictly increasing and $\tilde{\mathbb{T}} := \nu(\mathbb{T})$ is a time scale. If $f : \mathbb{T} \to \mathbb{R}$ is an rd-continuous function and ν is differentiable with rd-continuous derivative, then for $a, b \in \mathbb{T}$,*

$$\int_a^b g(t,s)\nu^{\Delta}(s)\,\Delta s = \int_{\nu(a)}^{\nu(b)} g(t,\nu^{-1}(s))\,\tilde{\Delta} s. \tag{15}$$

First, since the operator $\delta : [t_0,\infty)_{\mathbb{T}} \to [\delta(t_0),\infty)_{\mathbb{T}}$ is strictly increasing, it is bijection. If we substitute $\nu(t) = \delta_-(h,t)$ and $f(t,s) = g(t,\delta_-^{-1}(h,s)) = g(t,\delta_+(h,s))$ into (15), we obtain

$$\int_a^b f(t,\delta_-(h,s))\delta_-^{\Delta_1}(h,s)\,\Delta_1 s = \int_{\delta_-(h,a)}^{\delta_-(h,b)} f(t,s)\,\Delta_2 s \tag{16}$$

for $a,b \in \mathbb{T}_1$. For any $t \in \mathbb{T}_1$, we have $[\delta_-(h,t_0),t)_{\mathbb{T}_1} \subset \mathbb{T}_2$. This and (16) yield

$$\begin{aligned} \int_{\delta_-(h,t)}^t f(t,s)\Delta s &= \int_{\delta_-(h,t)}^t f(t,s)\Delta_2 s \\ &= \int_{\delta_-(h,t)}^{\delta_-(h,t_0)} f(t,s)\Delta_2 s + \int_{\delta_-(h,t_0)}^t f(t,s)\Delta_2 s \\ &= \int_t^{t_0} f(t,\delta_-(h,s))\delta_-^{\Delta_1}(h,s)\,\Delta_1 s + \int_{\delta_-(h,t_0)}^t f(t,s)\Delta s \\ &= \int_t^{t_0} f(t,\delta_-(h,s))\delta_-^{\Delta}(h,s)\,\Delta s + \int_{\delta_-(h,t_0)}^t f(t,s)\Delta s. \end{aligned} \tag{17}$$

The formula

$$\left[\int_{\delta_-(h,t)}^t f(t,s)\Delta s\right]^{\Delta} = f(\sigma(t),t) - f(\sigma(t),\delta_-(h,t))\delta_-^{\Delta}(h,t) + \int_{\delta_-(h,t)}^t f^{\Delta}(t,s)\Delta s \tag{18}$$

follows from (17) and Theorem 1.

THEOREM 3. *Let k be an rd-continuous function. Then*

$$\int_{\delta_-(h,t)}^t \Delta s \int_s^t k(u)\Delta u = \int_{\delta_-(h,t)}^t \Delta u \int_{\delta_-(h,t)}^{\sigma(u)} k(u)\Delta s. \tag{19}$$

Proof. Substituting

$$f(s) = s - \delta_-(h,t), \qquad g(s) = \int_s^t k(u)\Delta u$$

into the formula

$$\int_a^z f(\sigma(x))g(x)\Delta x = [f(x)g(x)]_a^z - \int_a^z f^{\Delta}(x)g(x)\Delta x$$

(see [10, Theorem 1.77]) and using Lemma 2, we get

$$\begin{aligned} \int_{\delta_-(h,t)}^t \Delta s \int_s^t k(u)\Delta u &= \int_{\delta_-(h,t)}^t [\sigma(s) - \delta_-(h,t)]\,k(s)\Delta s \\ &= \int_{\delta_-(h,t)}^t \Delta u \int_{\delta_-(h,t)}^{\sigma(u)} k(u)\Delta s. \end{aligned} \tag{20}$$

□

4. Stability analysis using Lyapunov's method

Let $\mathbb{T}$ be a time scale having a delay function $\delta_-(h,t)$, where $h \geq t_0$ and $t_0 \in \mathbb{T}$ is nonnegative and fixed. In this section we consider the equation

$$x^\Delta(t) = a(t)x(t) + b(t)x(\delta_-(h,t))\delta_-^\Delta(h,t), \qquad t \in [t_0,\infty)_\mathbb{T} \tag{21}$$

and assume that

$$\left|\delta_-^\Delta(h,t)\right| \leq M < \infty \text{ for all } t \in [t_0,\infty)_\mathbb{T}. \tag{22}$$

Let ψ: $[\delta_-(h,t_0),t_0]_\mathbb{T} \to \mathbb{R}$ be *rd*-continuous and let $x(t) := x(t,t_0,\psi)$ be the solution of (21) on $[t_0,\infty)_\mathbb{T}$ with $x(t) = \psi(t)$ on $[\delta_-(h,t_0),t_0]_\mathbb{T}$. Let

$$\|\varphi\| = \sup\{|\varphi(t)| : t \in [\delta_-(h,t_0),t_0)_\mathbb{T}\}.$$

Observe that using (18), equation (21) can be rewritten as follows

$$x^\Delta(t) = Q(t)x(t) - \left[\int_{\delta_-(h,t)}^t b(\delta_+(h,s))x(s)\Delta s\right]^{\Delta_t}, \tag{23}$$

where

$$Q(t) := a(t) + b(\delta_+(h,t))$$

and Δ_t indicates the delta derivative with respect to t.

LEMMA 4. *Let*

$$A(t) := x(t) + \int_{\delta_-(h,t)}^{t} b(\delta_+(h,s))x(s)\Delta s \tag{24}$$

and

$$\beta(t) := t - \delta_-(h,t). \tag{25}$$

Assume that there exists a $\lambda > 0$ *such that*

$$-\frac{\lambda\delta_-^\Delta(h,t)}{\beta(t) + \lambda\left[\beta(t) + \mu(t)\right]} \leq Q(t) \leq -\lambda\left[\beta(t) + \mu(t)\right]b(\delta_+(h,t))^2 - \mu(t)Q^2(t) \tag{26}$$

for all $t \in [t_0,\infty)_\mathbb{T}$. *If*

$$V(t) = A(t)^2 + \lambda \int_{\delta_-(h,t)}^{t} \Delta s \int_s^t b(\delta_+(h,u))^2 x(u)^2 \Delta u \tag{27}$$

then, along the solutions of (21) we have

$$V^\Delta(t) \leq Q(t)V(t) \;\; \textit{for all} \;\; t \in [t_0,\infty)_\mathbb{T}. \tag{28}$$

Proof. It is obvious from (23) and (24) that

$$A^{\Delta}(t) = Q(t)x(t).$$

Then by (18) and the formula $A(\sigma(t)) = A(t) + \mu(t)A(t)$ we have

$$\begin{aligned} V^{\Delta}(t) &= [A(t)+A(\sigma(t))]A^{\Delta}(t) + \lambda\int_{t}^{\sigma(t)} b(\delta_{+}(h,u))^2 x(u)^2\Delta u \\ &\quad - \lambda\delta_{-}^{\Delta}(h,t)\int_{\delta_{-}(h,t)}^{\sigma(t)} b(\delta_{+}(h,u))^2 x(u)^2\Delta u + \lambda\,(t-\delta_{-}(h,t))\,b(\delta_{+}(h,t))^2 x(t)^2 \\ &= [2A(t)+\mu(t)Q(t)x(t)]\,Q(t)x(t) - \lambda\delta_{-}^{\Delta}(h,t)\int_{\delta_{-}(h,t)}^{\sigma(t)} b(\delta_{+}(h,u))^2 x(u)^2\Delta u \\ &\quad + \lambda\,[\beta(t)+\mu(t)]\,b(\delta_{+}(h,t))^2 x(t)^2. \end{aligned}$$

Using the identity

$$2A(t)x(t) = x^2(t) + A^2(t) - \Bigg(\int\limits_{\delta_{-}(h,t)}^{t} b(\delta_{+}(h,s))x(s)\Delta s\Bigg)^2 \tag{29}$$

and condition (26) we have

$$\begin{aligned} V^{\Delta}(t) &= Q(t)V(t) + R(t) + x^2(t)\left[\lambda\,(\beta(t)+\mu(t))\,b(\delta_{+}(h,t))^2 + Q(t) + \mu(t)Q^2(t)\right] \\ &\leq Q(t)V(t) + R(t), \end{aligned} \tag{30}$$

where

$$\begin{aligned} R(t) = -Q(t)\Bigg(\int\limits_{\delta_{-}(h,t)}^{t} b(\delta_{+}(h,s))x(s)\Delta s\Bigg)^2 - \lambda\delta_{-}^{\Delta}(h,t)\int_{\delta_{-}(h,t)}^{\sigma(t)} b(\delta_{+}(h,u))^2 x(u)^2\Delta u \\ - \lambda Q(t)\int\limits_{\delta_{-}(h,t)}^{t}\Delta s\int\limits_{s}^{t} b(\delta_{+}(h,u))^2 x(u)^2\Delta u. \end{aligned} \tag{31}$$

Hereafter, we will show that (26) implies $R(t) \leq 0$. This and (30) will enable us to derive the desired inequality (28). First we have

$$\int_{\delta_{-}(h,t)}^{\sigma(t)} b(\delta_{+}(h,u))^2 x(u)^2\Delta u \geq \int_{\delta_{-}(h,t)}^{t} b(\delta_{+}(h,u))^2 x(u)^2\Delta u. \tag{32}$$

From Hölder's inequality [10, Theorem 6.13] we get

$$\Bigg(\int\limits_{\delta_{-}(h,t)}^{t} b(\delta_{+}(h,s))x(s)\Delta s\Bigg)^2 \leq \beta(t)\int\limits_{\delta_{-}(h,t)}^{t} b(\delta_{+}(h,s))^2 x(s)^2\Delta s. \tag{33}$$

On the other hand, (19) yields

$$\int_{\delta_-(h,t)}^{t}\Delta s\int_{s}^{t} b(\delta_+(h,u))^2 x(u)^2\Delta u = \int_{\delta_-(h,t)}^{t}\Delta u\int_{\delta_-(h,t)}^{\sigma(u)} b(\delta_+(h,u))^2 x(u)^2\Delta s$$

$$(34)\qquad = \int_{\delta_-(h,t)}^{t}\left[\sigma(u)-\delta_-(h,t)\right] b(\delta_+(h,u))^2 x(u)^2\Delta u$$

$$\leq \left[\beta(t)+\mu(t)\right]\int_{\delta_-(h,t)}^{t} b(\delta_+(h,u))^2 x(u)^2\Delta u.$$

Substituting (33) and (34) into (31) and using (32) together with $\delta_-^{\Delta}(h,t)>0$ we deduce

$$R(t)\leq -\left\{(\beta(t)+[\lambda\beta(t)+\mu(t)])\,Q(t)+\lambda\delta_-^{\Delta}(h,t)\right\}\int_{\delta_-(h,t)}^{t} b(\delta_+(h,s))^2 x(s)^2\Delta s.$$

Hence, using the left-hand side of (26), we arrive at the inequality $R(t)\leq 0$. The proof is complete. □

In preparation for the proof of the next theorem we state the following lemma.

LEMMA 5. *If* $\varphi\in\mathcal{R}^+$, *then*

$$(35)\qquad 0<e_{\varphi}(t,s)\leq \exp\left(\int_s^t \varphi(r)\Delta r\right)$$

for all $t\in[s,\infty)_{\mathbb{T}}$.

THEOREM 4. *Let* $a\in\mathcal{R}^+$ *and* $Q\in\mathcal{R}$. *Suppose the hypothesis of Lemma 4. If there exists an* $\alpha\in(t_0,h)_{\mathbb{T}}$ *such that*

$$(36)\qquad (a,t)\in\mathcal{D}_{\pm}\quad \textit{for all}\ \ t\in[t_0,\infty)_{\mathbb{T}}$$

and

$$(37)\qquad \delta_-(h,t)\leq\frac{\delta_-(\alpha,t)+\delta_-(h,\delta_-(\alpha,t))}{2}\ \ \textit{for all}\ \ t\in[\alpha,\infty)_{\mathbb{T}},$$

then any solution $x(t)=x(t,t_0,\varphi)$ *of (21) satisfies the exponential inequalities*

$$(38)\qquad |x(t)|\leq\sqrt{\frac{2}{\left(1-\frac{1}{\xi(t)}\right)}V(t_0)\,e^{\frac{1}{2}\int_{t_0}^{\delta_-(\alpha,t)}Q(s)\Delta s}}$$

for all $t \in [\alpha,\infty)_{\mathbb{T}}$ *and*

$$|x(t)| \leq \|\psi\| e^{\int_{t_0}^{t} a(s)\Delta s} \left[1 + M \int_{t_0}^{t} \left| \frac{b(s)}{1+\mu(s)a(s)} \right| e^{-\int_{t_0}^{s} a(u)\Delta u} \Delta s \right] \tag{39}$$

for all $t \in [t_0,\alpha)_{\mathbb{T}}$, *where* M *is as defined by (22),*

$$\xi(t) := 1 + \frac{\lambda\Lambda(t)}{\beta(t)} > 1,$$

and $\Lambda(t) := \delta_-(h,t) - \delta_-(h,\delta_-(\alpha,t))$.

Proof. Since $t_0 < \alpha < h$ the condition (37) implies

$$\delta_-(h,t) < \delta_-(\alpha,t) \quad \text{for all} \quad t \in [\alpha,\infty)_{\mathbb{T}} \tag{40}$$

and

$$0 < \Lambda(t) \leq \delta_-(\alpha,t) - \delta_-(h,t) \quad \text{for all} \quad t \in [\alpha,\infty)_{\mathbb{T}}, \tag{41}$$

Let $V(t)$ be defined by (27). First we get by (20), (27) and (40), (41) that

$$\begin{aligned}
V(t) &\geq \lambda \int_{\delta_-(h,t)}^{t} \Delta s \int_{s}^{t} b(\delta_+(h,u))^2 x(u)^2 \Delta u \\
&= \lambda \int_{\delta_-(h,t)}^{t} [\sigma(u) - \delta_-(h,t)]\, b(\delta_+(h,u))^2 x(u)^2 \Delta u \\
&\geq \lambda \int_{\delta_-(\alpha,t)}^{t} [\sigma(u) - \delta_-(h,t)]\, b(\delta_+(h,u))^2 x(u)^2 \Delta u \\
&\geq \lambda [\delta_-(\alpha,t) - \delta_-(h,t)] \int_{\delta_-(\alpha,t)}^{t} b(\delta_+(h,u))^2 x(u)^2 \Delta u.
\end{aligned}$$

This along with (41) yields

$$V(t) \geq \lambda\Lambda(t) \int_{\delta_-(\alpha,t)}^{t} b(\delta_+(h,u))^2 x(u)^2 \Delta u \tag{42}$$

for all $t \in [\alpha,\infty)_{\mathbb{T}}$. Similarly, we get

$$\begin{aligned}
V(\delta_-(\alpha,t)) &\geq \lambda \int_{\delta_-(h,\delta_-(\alpha,t))}^{\delta_-(\alpha,t)} [\sigma(u) - \delta_-(h,\delta_-(\alpha,t))]\, b(\delta_+(h,u))^2 x(u)^2 \Delta u \\
&\geq \lambda \int_{\delta_-(h,t)}^{\delta_-(\alpha,t)} [\sigma(u) - \delta_-(h,\delta_-(\alpha,t))]\, b(\delta_+(h,u))^2 x(u)^2 \Delta u \\
&\geq \lambda\Lambda(t) \int_{\delta_-(h,t)}^{\delta_-(\alpha,t)} b(\delta_+(h,u))^2 x(u)^2 \Delta u
\end{aligned} \tag{43}$$

for all $t \in [\alpha,\infty)_{\mathbb{T}}$ since $\delta_-(\alpha,t) \le \delta_-(t_0,t) = t$. Utilizing (27), (42), and (43), we obtain

$$\begin{aligned} V(t)+V(\delta_-(\alpha,t)) &\ge A(t)^2 + \lambda\Lambda(t)\int_{\delta_-(\alpha,t)}^{t} b(\delta_+(h,u))^2 x(u)^2\Delta u \\ &\quad + \lambda\Lambda(t)\int_{\delta_-(h,t)}^{\delta_-(\alpha,t)} b(\delta_+(h,u))^2 x(u)^2\Delta u \\ &\ge A(t)^2 + \lambda\Lambda(t)\int_{\delta_-(h,t)}^{t} b(\delta_+(h,u))^2 x(u)^2\Delta u \end{aligned} \tag{44}$$

for all $t \in [\alpha,\infty)_{\mathbb{T}}$. Substituting (33) and (24) into (44) we find

$$\begin{aligned} V(t)+V(\delta_-(\alpha,t)) &\ge \left(1-\frac{1}{\xi(t)}\right)x^2(t) \\ &\quad + \left[\frac{1}{\sqrt{\xi(t)}}x(t) + \sqrt{\xi(t)}\left(\int_{\delta_-(h,t)}^{t} b(\delta_+(h,u))x(u)\Delta u\right)\right]^2 \\ &\ge \left(1-\frac{1}{\xi(t)}\right)x^2(t) \end{aligned} \tag{45}$$

for all $t \in [\alpha,\infty)_{\mathbb{T}}$. Since $V^{\Delta}(t) \le 0$, we get by (45) that

$$\left(1-\frac{1}{\xi(t)}\right)x^2(t) \le V(t)+V(\delta_-(\alpha,t)) \le 2V(\delta_-(\alpha,t)) \tag{46}$$

for all $t \in [\alpha,\infty)_{\mathbb{T}}$. Multiplying (28) by $e_{\ominus Q}(\sigma(s),t_0)$ and integrating the resulting inequality from t_0 to t, we derive

$$\begin{aligned} 0 &\ge \int_{t_0}^{t}\left[V^{\Delta}(s) - Q(s)V(s)\right] e_{\ominus Q}(\sigma(s),t_0)\Delta s \\ &= \int_{t_0}^{t}[V(s)e_{\ominus Q}(s,t_0)]^{\Delta}\Delta s \\ &= V(t)e_{\ominus Q}(t,t_0) - V(t_0). \end{aligned} \tag{47}$$

That is,

$$V(t) \le V(t_0)e_Q(t,t_0) \quad \text{for all} \quad t \in [t_0,\infty)_{\mathbb{T}}. \tag{48}$$

Combining (46) and (48), we arrive at

$$x^2(t) \le \frac{2}{\left(1-\frac{1}{\xi(t)}\right)} V(t_0)e_Q(\delta_-(\alpha,t),t_0)$$

for all $t \in [\alpha,\infty)_{\mathbb{T}}$. The hypothesis $Q \in \mathcal{R}$ and the condition (26) guarantee that $Q(t) \in \mathcal{R}^+$. Thus, (35) implies

$$|x(t)| \le \sqrt{\frac{2}{\left(1-\frac{1}{\xi(t)}\right)} V(t_0)\, e^{\frac{1}{2}\int_{t_0}^{\delta_-(\alpha,t)} Q(s)\Delta s}}$$

for all $t \in [\alpha,\infty)_{\mathbb{T}}$.

Multiplying (21) by $e_{\ominus a}(\sigma(t),t_0)$ and integrating the resulting equation from t_0 to t we have

$$x(t) = x(t_0)e_a(t,t_0) + \int_{t_0}^{t} \frac{b(s)}{1+\mu(s)a(s)} e_a(t,s)x(\delta_-(h,s))\delta_-^{\Delta s}(h,s)\Delta s. \tag{49}$$

Since $\delta_-(h,t) < \delta_-(\alpha,t) \le \delta_-(\alpha,\alpha) = t_0$ for all $t \in [t_0,\alpha)_{\mathbb{T}}$, (35) along with (49) yields

$$\begin{aligned}
|x(t)| &= e_a(t,t_0)\left[\psi(t_0) + \int_{t_0}^{t} \frac{b(s)}{1+\mu(s)a(s)} e_a(t_0,s)\psi(\delta_-(h,s))\delta_-^{\Delta s}(h,s)\Delta s\right] \\
&\le \|\psi\| \left[e^{\int_{t_0}^{t} a(s)ds} + M\int_{t_0}^{t}\left|\frac{b(s)}{1+\mu(s)a(s)}\right| e^{\int_s^t a(u)\Delta u}\Delta s\right] \\
&\le \|\psi\| e^{\int_{t_0}^{t} a(s)ds}\left[1 + M\int_{t_0}^{t}\left|\frac{b(s)}{1+\mu(s)a(s)}\right| e^{-\int_{t_0}^{s} a(u)\Delta u}\Delta s\right].
\end{aligned}$$

The proof is complete. □

Notice that Theorem 4 does not work for the time scales in which

$$(t_0,h)_{\mathbb{T}} = \varnothing.$$

For instance, let $\mathbb{T} = \mathbb{Z}$, $t_0 = 0$, $\delta_-(h,t) = t-h$ and $h = 1$. It is obvious that $(t_0,h)_{\mathbb{Z}} = (0,1)_{\mathbb{Z}} = \varnothing$. That is, there is no α so that (36) and (37) hold. In preparation for the proof of the next theorem we give the following lemma.

LEMMA 6. *Let $\mathbb{T}$ be a time scale and t_0 a fixed point. Suppose that the shift operators $\delta_\pm(h,t)$ associated with the initial point t_0 are defined on $\mathbb{T}$. Suppose also that there is a delay function $\delta_-(h,t)$ defined on $\mathbb{T}$. If $(t_0,h)_{\mathbb{T}} = \varnothing$, then the time scale $\mathbb{T}$ is isolated (i.e., $\mathbb{T}$ consists only of right-scattered points). Moreover,*

$$\sigma(t) = \delta_+(h,t) \tag{50}$$

for all $t \in [\delta_-(h,t_0),\infty)_{\mathbb{T}}$ or equivalently

$$\sigma(\delta_-(h,t)) = t \tag{51}$$

for all $t \in [t_0,\infty)_{\mathbb{T}}$.

Proof. Suppose that $(t_0,h)_{\mathbb{T}} = \varnothing$. Define $\delta_+^0(h,t_0) = t_0$ and $\delta_+^k(h,t_0) = \delta_+(h,\delta_+^{k-1}(h,t_0))$ for $k \in \mathbb{Z}_+$. Since $\delta_+(h,t)$ is surjective and strictly increasing we have

$$\left(\delta_+^{k-1}(h,t_0),\delta_+^k(h,t_0)\right)_{\mathbb{T}} = \delta_-\left(h,\left(\delta_+^{k-2}(h,t_0),\delta_+^{k-1}(h,t_0)\right)_{\mathbb{T}}\right), \text{ for } k = 2,3,\ldots$$

Thus, one can show by induction that

$$\left(\delta_+^{k-1}(h,t_0),\delta_+^k(h,t_0)\right)_{\mathbb{T}} = \varnothing \text{ for all } k \in \mathbb{Z}_+. \tag{52}$$

That is, $\sigma\big(\delta_+^{k-1}(h,t_0)\big) = \delta_+^k(h,t_0)$ for $k \in \mathbb{Z}_+$. On the other hand, we can write

$$[t_0,\infty)_{\mathbb{T}} = \cup_{k=1}^{\infty}[\delta_+^{k-1}(h,t_0),\delta_+^k(h,t_0))_{\mathbb{T}}.$$

Hence, for any $t \in [t_0,\infty)_{\mathbb{T}}$ there is a $k_0 \in \mathbb{Z}_+$ so that $t \in [\delta_+^{k_0-1}(h,t_0),\delta_+^{k_0}(h,t_0))_{\mathbb{T}}$. By (52) we have $t = \delta_+^{k_0-1}(h,t_0)$. This shows that

$$\sigma(t) = \sigma(\delta_+^{k_0-1}(h,t_0)) = \delta_+^{k_0}(h,t_0) = \delta_+(h,\delta_+^{k_0-1}(h,t_0)) = \delta_+(h,t)$$

for all $t \in [\delta_-(h,t_0),\infty)_{\mathbb{T}}$. This along with $\sigma(\delta_-(h,t)) = \delta_-(h,\sigma(t))$ yields (51). The proof is complete. □

THEOREM 5. *Let $a \in \mathcal{R}^+$, $Q \in \mathcal{R}$. Assume the hypothesis of Lemma 4. If $(t_0,h)_{\mathbb{T}} = \varnothing$, then any solution $x(t) = x(t,t_0,\varphi)$ of (21) satisfies the exponential inequality*

$$|x(t)| \le \sqrt{\left(1+\frac{1}{\lambda}\right)V(t_0)\,e^{\frac{1}{2}\int_{t_0}^{t} Q(s)\Delta s}}$$

for all $t \in [t_0,\infty)_{\mathbb{T}}$.

Proof. Let H be defined by

$$H(t) = \int_{\delta_-(h,t)}^{t} \Delta s \int_{s}^{t} b(\delta_+(h,u))^2 x(u)^2 \Delta u. \tag{53}$$

From (20), (51), and (33) we get

$$\begin{aligned} H(t) &= \int_{\delta_-(h,t)}^{t} [\sigma(u)-\delta_-(h,t)]\, b(\delta_+(h,u))^2 x(u)^2 \Delta u \\ &\ge [\sigma(\delta_-(h,t))-\delta_-(h,t)] \int_{\delta_-(\alpha,t)}^{t} b(\delta_+(h,u))^2 x(u)^2 \Delta u \\ &= \beta(t) \int_{\delta_-(\alpha,t)}^{t} b(\delta_+(h,u))^2 x(u)^2 \Delta u \\ &\ge \left(\int_{\delta_-(h,t)}^{t} b(\delta_+(h,u)) x(u) \Delta u \right)^2. \end{aligned}$$

Hence, by (27) we have

$$\begin{aligned} V(t) &= A^2(t) + \lambda H(t) \\ &\geq \Big(x(t) + \int_{\delta_-(h,t)}^{t} b(\delta_+(h,s))x(s)\Delta s \Big)^2 + \lambda \Big(\int_{\delta_-(h,t)}^{t} b(\delta_+(h,u))x(u)\Delta u \Big)^2 \\ &= \Big(1 - \frac{1}{1+\lambda}\Big)x^2(t) + \Big[\frac{1}{\sqrt{1+\lambda}} x(t) + \sqrt{1+\lambda}\Big(\int_{\delta_-(h,t)}^{t} b(\delta_+(h,u))x(u)\Delta u \Big) \Big]^2 \\ &\geq \Big(1 - \frac{1}{1+\lambda}\Big)x^2(t). \end{aligned}$$

This along with (48) yields

$$|x(t)| \leq \sqrt{\Big(1 + \frac{1}{\lambda}\Big) V(t_0)}\, e^{\frac{1}{2}\int_{t_0}^{t} Q(s)\Delta s}.$$

The proof is complete. □

In the next corollary, we summarize the results obtained from Theorem 4 and Theorem 5.

COROLLARY 2. *Assume the hypothesis of Lemma 4. Let $a \in \mathcal{R}^+$ and $Q \in \mathcal{R}$. Suppose that there exists a $\lambda > 0$ such that (26) holds for all $t \in [t_0, \infty)_{\mathbb{T}}$.*

1. *If there exists an $\alpha \in (t_0, h)_{\mathbb{T}}$ such that (36) and (37) hold, then any solution $x(t) = x(t, t_0, \varphi)$ of (21) satisfies*

$$|x(t)| \leq \sqrt{\frac{2}{\Big(1 - \frac{1}{\xi(t)}\Big)} V(t_0)}\, e^{-\frac{1}{2}\int_{t_0}^{\delta_-(\alpha,t)} \left[\lambda(\beta(s)+\mu(s))b(\delta_+(h,s))^2 + \mu(s)Q^2(s)\right]\Delta s}.$$

   Thus, if

$$\lim_{t\to\infty} \int_{t_0}^{\delta_-(\alpha,t)} \left[\lambda(\beta(s)+\mu(s))\,b(\delta_+(h,s))^2 + \mu(s)Q^2(s)\right]\Delta s = \infty,$$

   then the zero solution of (21) is exponentially stable.

2. *If $(t_0, h)_{\mathbb{T}} = \varnothing$, then any solution $x(t) = x(t, t_0, \varphi)$ of (21) satisfies*

$$|x(t)| \leq \sqrt{\Big(1 + \frac{1}{\lambda}\Big) V(t_0)}\, e^{-\frac{1}{2}\int_{t_0}^{t} \left[\lambda(\beta(s)+\mu(s))b(\sigma(s))^2 + \mu(s)Q^2(s)\right]\Delta s}.$$

   Thus, if

$$\lim_{t\to\infty} \int_{t_0}^{t} \left[\lambda(\beta(s)+\mu(s))\,b(\sigma(s))^2 + \mu(s)Q^2(s)\right]\Delta s = \infty,$$

   then the zero solution of (21) is exponentially stable.

Let $q > 1$, $\mathbb{T} = \overline{q^{\mathbb{Z}}} = \{0\} \cup \{q^n : n \in \mathbb{Z}\}$, $\delta_-(h,t) = q^{-h}t$, and $h \in \mathbb{Z}_+$. Then, (21) turns into the q-difference equation

$$D_q x(t) = a(t)x(t) + b(t)x(q^{-h}t)q^{-h}, \quad t \in \left\{1, q, q^2, \dots\right\}, \tag{54}$$

where $D_q x(t) = \frac{x(qt)-x(t)}{(q-1)t}$. Next, we use Corollary 2 to derive a stability criteria for the q-difference equation (54).

EXAMPLE 5. Suppose that $1 + \mu(t)a(t) > 0$, $1 + \mu(t)Q(t) \neq 0$, and

$$-\frac{\lambda q^{-h}}{\varpi(t) + \lambda(\varpi(t) + \mu(t))} \leq Q(t) \leq -\lambda(\varpi(t) + \mu(t))\, b(\delta_+(h,t))^2 - \mu(t)Q^2(t)$$

for all $t \in \left\{1, q, q^2, \dots\right\}$, where $\varpi(t) := t\left(1 - q^{-h}\right)$ and $\mu(t) = t(q-1)$.

1. If $(1, q^h)_{q^{\mathbb{Z}}} \neq \varnothing$, then then condition (37) holds. By Corollary 2, we conclude that any solution $x(t) = x(t, t_0, \varphi)$ of the q-difference equation (54) satisfies the exponential inequalities

$$|x(t)| \leq \sqrt{\frac{2}{\left(1 - \frac{1}{\xi(t)}\right)} V(t_0)}\, \exp\left(\frac{1}{2} \sum_{s \in [1, q^{-\alpha}t)_{q^{\mathbb{Z}}}} \mu(s)Q(s)\right)$$

for all $t \in [q^\alpha, \infty)_{q^{\mathbb{Z}}}$ and

$$|x(t)| \leq \|\psi\| \exp\left(\sum_{s \in [1,t)_{q^{\mathbb{Z}}}} \mu(s)a(s)\right) \left[1 + \sum_{s \in [1,t)_{q^{\mathbb{Z}}}} G(s) \exp\left(-\sum_{u \in [1,s)_{q^{\mathbb{Z}}}} \mu(u)a(u)\right)\right]$$

for all $t \in [1, q^\alpha)_{q^{\mathbb{Z}}}$, where

$$G(s) := q^{-h}\mu(s) \left|\frac{b(s)}{1 + \mu(s)a(s)}\right|.$$

Hence, if

$$\lim_{t \to \infty} \sum_{s \in [1, q^{-\alpha}t)_{q^{\mathbb{Z}}}} s^2 \left[\lambda(q - q^{-h}) b(q^h s)^2 + (q-1)Q^2(s)\Delta s\right] = \infty,$$

then the zero solution of (54) is exponentially stable.

2. If $(1, q^h)_{q^{\mathbb{Z}}} = \varnothing$, then $h = 1$ and

$$|x(t)| \leq \sqrt{\left(1 + \frac{1}{\lambda}\right) V(t_0)}\, \exp\left(\frac{1}{2} \sum_{s \in [1,t)_{q^{\mathbb{Z}}}} \mu(s)Q(s)\right).$$

Hence, if

$$\lim_{t \to \infty} \sum_{s \in [1,t)_{q^{\mathbb{Z}}}} s^2 \left[\lambda(q - q^{-1}) b(qs)^2 + (q-1)Q^2(s)\Delta s\right] = \infty,$$

then the zero solution of (54) is exponentially stable.

In the next result, we will display a Lyapunov functional that involves $|x|^{\Delta}$. Thus, in preparation we have the following.

Using the product rule $(fg)^{\Delta} = f^{\Delta}g^{\sigma} + fg^{\Delta}$and differentiating both sides of $x^2(t) = |x(t)|^2$ we obtain the derivative $|x(t)|^{\Delta}$ as follows:

$$|x|^{\Delta} = \frac{x + x^{\sigma}}{|x| + |x^{\sigma}|}x^{\Delta} \text{ for } x \neq 0. \tag{55}$$

Thus $|x|^{\Delta}$ depends on $\frac{x(t)}{|x(t)|}$ and $\frac{x^{\sigma}(t)}{|x^{\sigma}(t)|}$ (i.e., signs of x and x^{σ}, respectively). Given $x : \mathbb{T} \to \mathbb{R}$, let the sets $\mathbb{T}_x^+$ and $\mathbb{T}_x^-$ be defined by

$$\mathbb{T}_x^+ = \{t \in \mathbb{T} : x(t)x^{\sigma}(t) \geq 0\},$$
$$\mathbb{T}_x^- = \{t \in \mathbb{T} : x(t)x^{\sigma}(t) < 0\},$$

respectively. The set $\mathbb{T}_x^-$ consists only of right-scattered points of $\mathbb{T}$. Since the time scale $\mathbb{T} = \mathbb{R}$ has no any right-scattered points, we have $\mathbb{T}_x^- = \varnothing$. Thus for all differentiable functions $x : \mathbb{R} \to \mathbb{R}$, the formula (55) turns into $|x|^{\Delta} = \frac{x}{|x|}x^{\Delta}$. However, for an arbitrary time scale (e.g. $\mathbb{T} = \mathbb{Z}$) the set $\mathbb{T}_x^-$ may not be empty. For simplicity, we need to have a formula for $|x|^{\Delta}$ which does not include x^{σ}. The next result provides a relationship between $|x|^{\Delta}$ and $\frac{x}{|x|}x^{\Delta}$. Its proof can be found in [3].

LEMMA 7 ([3, Lemma 5]). *Let $x \neq 0$ be Δ-differentiable. Then*

$$|x(t)|^{\Delta} = \begin{cases} \frac{x(t)}{|x(t)|}x^{\Delta}(t) & \text{if } t \in \mathbb{T}_x^+ \\ -\frac{2}{\mu(t)}|x(t)| - \frac{x(t)}{|x(t)|}x^{\Delta}(t) & \text{if } t \in \mathbb{T}_x^-. \end{cases} \tag{56}$$

THEOREM 6. *Define a continuous function $\eta(t) \geq 0$ by*

$$\eta(t) := \frac{e_a(t,t_0)}{1 + \lambda \int_{\delta_-(h,t)}^{t} e_a(\delta_+(h,s),t_0)\Delta s}. \tag{57}$$

Suppose that $a \in \mathcal{R}^+$ and that

$$|b(t)| - \lambda\eta^{\sigma}(t)\delta_-^{\Delta}(h,t) \leq 0 \tag{58}$$

holds for all $t \in [t_0,\infty)_{\mathbb{T}}$. Then any solution of (21) satisfies the inequality

$$|x(t)| \leq V(t_0,x_{t_0})e_{\gamma}(t,t_0) \text{ for all } t \in [t_0,\infty)_{\mathbb{T}}, \tag{59}$$

where

$$V(t_0,x_{t_0}) := |x(t_0)| + \lambda\eta(t_0)\int_{\delta_-(h,t_0)}^{t_0} |x(s)|\Delta s,$$

$\gamma(t) := a(t) + \lambda\widetilde{M}\eta^{\sigma}(t)$, $\widetilde{M} = \max\{1,M\}$, *and M is as in (22).*

Proof. For convenience define

$$\zeta(t) := 1+\lambda\int_{\delta_-(h,t)}^{t} e_a(\delta_+(h,s),t_0)\Delta s.$$

Then, by (18),

$$\begin{aligned}\zeta^\Delta(t) &= \lambda e_a(\delta_+(h,t),t_0) - e_a(t,t_0)\delta_-^\Delta(h,t)\\ &= \lambda e_a(t,t_0)\left[e_a(\delta_+(h,t),t) - \delta_-^\Delta(h,t)\right]. \end{aligned} \tag{60}$$

This, and a differentiation of (57), yield

$$\begin{aligned}\eta^\Delta(t) &= \frac{e_a(t,t_0)}{\zeta(t)}\left(\frac{a\zeta(t)-\zeta^\Delta(t)}{\zeta^\sigma(t)}\right)\\ &= \eta(t)\left(\frac{a\zeta(t)+a\mu(t)\zeta^\Delta(t)-a\mu(t)\zeta^\Delta(t)-\zeta^\Delta(t)}{\zeta(t)+\mu(t)\zeta^\Delta(t)}\right)\\ &= a(t)\eta(t) - \left[(1+\mu(t)a(t))\eta(t)\frac{\zeta(t)}{\zeta^\sigma(t)}\right]\frac{\zeta^\Delta(t)}{\zeta(t)}\\ &= a(t)\eta(t) - \eta^\sigma(t)\frac{\zeta^\Delta(t)}{\zeta(t)}\\ &= a(t)\eta(t)+\lambda\eta^\sigma(t)\eta(t)\delta_-^\Delta(h,t) - \lambda\eta^\sigma(t)e_a(\delta_+(h,t),t)\\ &\le \eta(t)\left[a(t)+\lambda\widetilde{M}\eta^\sigma(t)\right], \end{aligned} \tag{61}$$

where we also used $\zeta^\sigma(t) = \zeta(t)+\mu(t)\zeta^\Delta(t)$ and

$$(1+\mu(t)a(t))\eta(t)\frac{\zeta(t)}{\zeta^\sigma(t)} = \eta^\sigma(t).$$

Define

$$V(t,x_t) := |x(t)| + \lambda\eta(t)\int_{\delta_-(h,t)}^{t}|x(s)|\Delta s. \tag{62}$$

Let $t\in\mathbb{T}_x^+\cap[t_0,\infty)_{\mathbb{T}}$. Then by (56) we have $|x(t)|^\Delta = \frac{x(t)}{|x(t)|}x^\Delta(t)$. Differentiating (62) and utilizing (58) and (61), we arrive at

$$\begin{aligned}V^\Delta(t,x_t) &= |x(t)|^\Delta + \lambda\eta^\Delta(t)\int_{\delta_-(h,t)}^{t}|x(s)|\Delta s + \lambda\eta^\sigma(t)\left[|x(t)| - |x(\delta_-(h,t))|\,\delta_-^\Delta(h,t)\right]\\ &\le \frac{x(t)}{|x(t)|}x^\Delta(t) + \lambda\eta(t)\left[a(t)+\lambda\widetilde{M}\eta^\sigma(t)\right]\int_{\delta_-(h,t)}^{t}|x(s)|\Delta s\\ &\qquad + \lambda\eta^\sigma(t)\left[|x(t)| - |x(\delta_-(h,t))|\,\delta_-^\Delta(h,t)\right]\\ &= \left(a(t)+\lambda\widetilde{M}\eta^\sigma(t)\right)|x(t)| + \left(|b(t)| - \lambda\delta_-^\Delta(h,t)\eta^\sigma(t)\right)|x(\delta_-(h,t))|\\ &\qquad + \lambda\eta(t)\left[a(t)+\widetilde{M}\eta^\sigma(t)\right]\int_{\delta_-(h,t)}^{t}|x(s)|\Delta s\\ &\le \gamma(t)V(t,x_t). \end{aligned}$$

Similarly, if $t \in \mathbb{T}_x^- \cap [t_0,\infty)_{\mathbb{T}}$, then $|x(t)|^\Delta = -\frac{2}{\mu(t)}|x(t)| - \frac{x(t)}{|x(t)|}x^\Delta(t)$ by (56). Hence,

$$\begin{aligned}
V^\Delta(t,x_t) &\le |x(t)|^\Delta + \eta(t)\left[a(t)+\lambda\widetilde{M}\eta^\sigma(t)\right]\int_{\delta_-(h,t)}^t |x(s)|\,\Delta s \\
&\qquad + \lambda\eta^\sigma(t)\left[|x(t)| - |x(\delta_-(h,t))|\,\delta_-^\Delta(h,t)\right] \\
&\le \left(-\frac{2}{\mu(t)} - a(t) + \lambda\widetilde{M}\eta^\sigma(t)\right)|x(t)| \\
&\qquad + \left(|b(t)| - \lambda\delta_-^\Delta(h,t)\eta^\sigma(t)\right)|x(\delta_-(h,t))| \\
&\qquad + \lambda\eta(t)\left[a(t)+\widetilde{M}\eta^\sigma(t)\right]\int_{\delta_-(h,t)}^t |x(s)|\,\Delta s \\
&\le \left(a(t)+\lambda\widetilde{M}\eta^\sigma(t)\right)|x(t)| + \lambda\eta(t)\left[a(t)+\lambda\widetilde{M}\eta^\sigma(t)\right]\int_{\delta_-(h,t)}^t |x(s)|\,\Delta s \\
&= \gamma(t)V(t,x_t).
\end{aligned}$$

since $1+\mu(t)a(t) > 0$ implies that

$$-\frac{2}{\mu(t)} - a(t) < a(t).$$

Thus,

$$V^\Delta(t,x_t) \le \gamma(t)V(t,x_t) \quad \text{for all } \ t \in [t_0,\infty)_{\mathbb{T}}. \tag{63}$$

Integrating (63) and applying the fact that $V(t,x_t) \ge |x(t)|$, we arrive at the desired result. □

In the next section we give a criteria for instability.

5. A criteria for instability

THEOREM 7. *Suppose there exists a positive constant D such that*

$$\beta(t) < D \le \frac{Q(t)}{b(\delta_+(h,t))^2} \tag{64}$$

for all $t \in [t_0,\infty)_{\mathbb{T}}$*, where* $\beta(t)$ *is as defined in (25). Let the function A be defined by (24). If*

$$V(t) = A(t)^2 - D\int_{\delta_-(h,t)}^t b(\delta_+(h,s))^2 x(s)^2 \Delta s, \tag{65}$$

then along the solutions of (21) we have

$$V^\Delta(t) \ge Q(t)V(t) \quad \textit{for all } \ t \in [t_0,\infty)_{\mathbb{T}}. \tag{66}$$

Proof. Let V be defined by (65). Using (29) and (33) we obtain

$$\begin{aligned}
V^{\Delta}(t) &= [A(t)+A(\sigma(t))]A^{\Delta}(t) - Db(\delta_+(h,t))^2x(t)^2 + Db(t)^2x(\delta_-(h,t))^2\delta_-^{\Delta}(h,t) \\
&\geq [2A(t)+\mu(t)Q(t)x(t)]\,Q(t)x(t) - Db(\delta_+(h,t))^2x(t)^2 \\
&\geq 2Q(t)A(t)x(t) - Db(\delta_+(h,t))^2x(t)^2 \\
&= Q(t)\left[x^2(t)+A^2(t) - \left(\int\limits_{\delta_-(h,t)}^{t} b(\delta_+(h,s))x(s)\Delta s\right)^2\right] - Db(\delta_+(h,t))^2x(t)^2 \\
&\geq Q(t)V(t) + \left[Q(t) - Db(\delta_+(h,t))^2\right]x(t)^2.
\end{aligned}$$

This along with (64) implies (66). □

To prove the next theorem we will need to use the following lemma:

LEMMA 8 ([9, Remarks 2]). *If φ is rd-continuous and nonnegative, then*

$$1+\int_s^t \varphi(u)\Delta u \leq e_{\varphi}(t,s) \leq \exp\left(\int_s^t \varphi(u)\Delta u\right) \quad \textit{for all } t \geq s. \tag{67}$$

THEOREM 8. *Suppose all hypotheses of Theorem 7 hold. Suppose also that $\beta(t)$ is bounded above by β_0 with $0<\beta_0<D$. Then the zero solution of (21) is unstable, provided that*

$$\lim_{t\to\infty}\int_{t_0}^{t} b(\delta_+(h,s))^2\Delta s = \infty.$$

Proof. As we did in (47), an integration of (66) from t_0 to t gives

$$V(t) \geq V(t_0)e_Q(t,t_0) \quad \text{for all } t\in[t_0,\infty)_{\mathbb{T}}. \tag{68}$$

Let $V(t)$ be given by (65). Then

$$\begin{aligned}
V(t) &= x(t)^2 + 2x(t)\int_{\delta_-(h,t)}^{t} b(\delta_+(h,s))x(s)\Delta s + \left(\int_{\delta_-(h,t)}^{t} b(\delta_+(h,s))x(s)\Delta s\right)^2 \\
&\quad - D\int_{\delta_-(h,t)}^{t} b(\delta_+(h,s))^2x(s)^2\Delta s.
\end{aligned} \tag{69}$$

Let $C := D-\beta_0$. Then from

$$\left(\sqrt{\frac{\beta_0}{C}}K - \sqrt{\frac{C}{\beta_0}}L\right)^2 \geq 0,$$

we have

$$2KL \leq \frac{\beta_0}{C}K^2 + \frac{C}{\beta_0}L^2.$$

With this in mind, we arrive at

$$2\,|x(t)|\int_{\delta_-(h,t)}^{t}|b(\delta_+(h,s))|\,|x(s)|\,\Delta s\le\frac{\beta_0}{C}x^2(t)+\frac{C}{\beta_0}\left(\int_{\delta_-(h,t)}^{t}b(\delta_+(h,s))x(s)\Delta s\right)^2.$$

A substitution of the above inequality into (69) yields

$$\begin{aligned}V(t)&\le\left(1+\frac{\beta_0}{C}\right)x(t)^2+\left(1+\frac{C}{\beta_0}\right)\left(\int_{\delta_-(h,t)}^{t}b(\delta_+(h,s))x(s)\Delta s\right)^2\\&\qquad-D\int_{\delta_-(h,t)}^{t}b(\delta_+(h,s))^2x(s)^2\Delta s\\&=\frac{D}{C}x(t)^2+\frac{D}{\beta_0}\left(\int_{\delta_-(h,t)}^{t}b(\delta_+(h,s))x(s)\Delta s\right)^2-D\int_{\delta_-(h,t)}^{t}b(\delta_+(h,s))^2x(s)^2\Delta s.\end{aligned}$$

Using (33) we find

$$V(t)\le\frac{D}{C}x(t)^2.$$

By (64), (67), and (68) we get

$$\begin{aligned}|x(t)|&\ge\sqrt{\frac{C}{D}V(t_0)e_Q(t,t_0)}\\&\ge\sqrt{\frac{C}{D}V(t_0)\left(1+\int_{t_0}^{t}Q(s)\Delta s\right)}\\&\ge\sqrt{CV(t_0)\left(\int_{t_0}^{t}b(\delta_+(h,s))^2\Delta s\right)}.\end{aligned}$$

This completes the proof. □

We end this paper by comparing our results to the existing ones.

6. Some applications

In [4], by means of Lyapunov's direct method the authors investigated the stability analysis of the delay dynamic equation

$$x^{\Delta}(t)=a(t)x(t)+b(t)x(\delta(t))\delta^{\Delta}(t), \tag{70}$$

where $a:\mathbb{T}\to\mathbb{R}$ and $b:\mathbb{T}\to\mathbb{R}$ are functions and $a\in\mathcal{R}^+$. Moreover, the delay function $\delta:[t_0,\infty)_{\mathbb{T}}\to[\delta(t_0),\infty)_{\mathbb{T}}$ is surjective, strictly increasing and is supposed to have the following properties

$$\delta(t)<t,\qquad \delta^{\Delta}(t)<\infty,\qquad \delta\circ\sigma=\sigma\circ\delta.$$

It was concluded in [4, Theorem 6] that

$$|b(t)|\le N\ \text{ and }\ a(t)<-N \tag{71}$$

are the sufficient conditions guaranteeing stability of the zero solution of (70). Next, we furnish an example to show that Theorem 4 allows us to relax condition (71) that leads to exponential stability of zero solution (70).

EXAMPLE 6. Let $\mathbb{T}=\mathbb{R}$, $a(t)=1$, $b(t)=-\frac{3}{2}$, $\delta(t)=t-\frac{1}{3}$, and $N=1$. It is obvious that the condition (71) does not hold. So, [4, Theorem 6] does not imply the stability of the zero solution of the delayed differential equation

$$x'(t)=x(t)-\frac{3}{2}x\left(t-\frac{1}{3}\right). \tag{72}$$

On the other hand, setting $\mathbb{T}=\mathbb{R}$, $\lambda=\frac{1}{3}$, and $\delta_-(h,t)=t-\frac{1}{3}$. Equation (21) turns into (72) and the condition (26) becomes

$$-\frac{3}{4}\le Q(t)\le -\frac{1}{9}b(\delta_+(h,t))^2,$$

which holds for all $t\in[0,\infty)$ since $Q(t)=a(t)+b(\delta_+(h,t))=-\frac{1}{2}$. One may easily verify that condition (37) is satisfied for $\delta_-(\alpha,t)=t-\frac{1}{6}$ and $\delta_-(h,t)=t-\frac{1}{3}$. Thus, we conclude the exponential stability of the zero solution of (72) by Corollary 2.

Now, let us consider the equation

$$x^\Delta(t)=b(t)x(\delta_-(h,t))\delta_-^\Delta(h,t),\qquad t\in[t_0,\infty)_\mathbb{T}. \tag{73}$$

We observe the following by combining Corollary 2 and Theorem 6.

REMARK 1. Let $b\in\mathcal{R}$. Suppose that there exists a $\lambda>0$ such that

$$-\frac{\lambda\delta_-^\Delta(h,t)}{\beta(t)+\lambda[\beta(t)+\mu(t)]}\le b(\delta_+(h,t))\le -b(\delta_+(h,t))^2\left[\lambda\beta(t)+(1+\lambda)\mu(t)\right], \tag{74}$$

holds for all $t\in[t_0,\infty)_\mathbb{T}$.

1. If there exists an $\alpha\in(t_0,h)_\mathbb{T}$ such that (36) and (37) hold and if

$$\lim_{t\to\infty}\int_{t_0}^{\delta_-(\alpha,t)}[\lambda\beta(s)+(1+\lambda)\mu(s)]\,b(\delta_+(h,s))^2\Delta s=\infty, \tag{75}$$

then the zero solution of (73) is exponentially stable.

2. If $(t_0,h)_\mathbb{T}=\varnothing$ and if

$$\lim_{t\to\infty}\int_{t_0}^{t}[\lambda\beta(s)+(1+\lambda)\mu(s)]\,b(\sigma(s))^2\Delta s=\infty,$$

then the zero solution of (73) is exponentially stable.

3. Suppose that $a \in \mathcal{R}^+$ and that

$$|b(t)| - \lambda\eta^{\sigma}(t)\delta_-^{\Delta}(h,t) \leq 0$$

holds for all $t \in [t_0,\infty)_{\mathbb{T}}$, where

$$\eta(t) := \frac{1}{1+\lambda\beta(t)}.$$

Then any solution of (21) satisfies the inequality

$$|x(t)| \leq V(t_0,x_{t_0})e^{\frac{1}{2}\int_{t_0}^{t}\gamma(s)\Delta s}e_{\gamma}(t,t_0) \ \text{ for all } \ t \in [t_0,\infty)_{\mathbb{T}},$$

where

$$V(t_0,x_{t_0}) := |x(t_0)| + \lambda\eta(t_0)\int_{\delta_-(h,t_0)}^{t_0}|x(s)|\Delta s,$$

$\gamma(t) := \lambda\widetilde{M}\eta^{\sigma}(t)$, $\widetilde{M} = \max\{1,M\}$, and M is as in (22).

In [4, Theorem 7], the authors utilized fixed point theory and deduced that the conditions

$$p(t) := b(\delta_+(h,t)) \neq 0 \ \text{ for all } \ t \in [t_0,\infty)_{\mathbb{T}}, \tag{76}$$

$$\lim_{t\to\infty} e_p(t,t_0) = 0,$$

and

$$\int_{\delta_-(h,t)}^{t}|p(s)|\Delta s + \int_{t_0}^{t}|\ominus p(s)|e_p(t,s)\left(\int_{\delta_-(h,s)}^{s}|p(u)|\Delta u\right)\Delta s \leq N < 1 \tag{77}$$

lead to stability of solution $x(t,t_0;\psi)$ of (73). Notice that [4] generalizes all the results of [20].

Moreover, Wang (see [22, Corollary 1]) proposed the inequality

$$-\frac{1}{2h} \leq a(t) + b(t+h) \leq -hb^2(t+h) \tag{78}$$

as a sufficient condition for uniform asymptotic stability of the zero solution of the delay differential equation

$$x'(t) = a(t) + b(t)x(t-h), \ \ h > 0.$$

It can be easily seen that the conditions (77),(78) are not satisfied for the data given in the following example.

EXAMPLE 7. Let $a(t) = 0$, $\mathbb{T} = \mathbb{R}$, $\delta_-(h,t) = t-h$, and $p < 0$ be fixed. Then (21) becomes

$$x'(t) = b(t)x(t-h).$$

We can simplify condition (77) as follows

$$h|p|(2-e^{pt}) \leq N < 1. \tag{79}$$

If $h = \frac{2}{3}$, and $b(t) = -\frac{9}{10}$, then (76) implies

$$h|p|(2-e^{pt}) = \frac{3}{5}\left(2 - e^{-\frac{9}{10}t}\right) \geq 1$$

for all $t \geq -\frac{10}{9}\ln\left(\frac{1}{3}\right) \cong 1.22$. Thus, the condition (79) does not hold. On the other hand, for $h = \frac{2}{3}$ and $\lambda = \frac{3}{2}$, condition (74) turns into

$$-\frac{9}{10} \leq b(\delta_+(h,t)) \leq -b(\delta_+(h,t))^2.$$

The last inequality holds for $b(t) = -\frac{9}{10}$. In addition, setting $\delta_-(\alpha,t) = t - \frac{1}{3}$ one may easily verify that conditions (36), (37) and (75) are satisfied. Hence, the first part of Remark 1 yields exponential stability while [4, Theorem 7] and [22, Corollary 1] cannot.

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