Rend. Sem. Mat. Univ. Politec. Torino
Vol. 68, 4 (2010), 349 – 359

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PERIODIC SOLUTIONS IN TOTALLY NONLINEAR DYNAMIC EQUATIONS WITH FUNCTIONAL DELAY ON A TIME SCALE

Abstract. Let $\mathbb{T}$ be a periodic time scale. The purpose of this paper is to use a modification of Krasnoselskii's fixed point theorem due to T. A. Burton to show that the totally nonlinear dynamic equation with functional delay

$$x^{\triangle}(t) = -a(t)x^3(\sigma(t)) + G\big(t, x^3(t), x^3(t-r(t))\big), \quad t \in \mathbb{T},$$

has a periodic solution. We invert this equation to construct a sum of a compact map and a large contraction, which is suitable for applying the Burton–Krasnoselskii theorem. Finally, an example is given to illustrate our result.

1. Introduction

A time scale is an arbitrary nonempty closed subset of the real numbers. The study of dynamic equations on time scales is a fairly new subject, and research in this area is rapidly growing (see [1, 2, 3, 4, 7, 8, 9] and papers therein). The theory of dynamic equations unifies the theories of differential equations and difference equations. We suppose that the reader is familiar with the basic concepts concerning calculus on time scales for dynamic equations. Most of the material needed to read this paper can in any case be found in the books of Bohner–Peterson [3, 4].

We start by giving some definitions necessary for our work. The notion of periodic time scales is introduced in Atici et. al. [2] and Kaufmann–Raffoul [8]. The following two definitions are borrowed from [2, 8].

DEFINITION 1. *We say that a time scale $\mathbb{T}$ is periodic if there exist a $p > 0$ such that if $t \in \mathbb{T}$ then $t \pm p \in \mathbb{T}$. For $\mathbb{T} \neq \mathbb{R}$, the smallest positive p is called the period of the time scale.*

Below are examples of periodic time scales taken from [8].

EXAMPLE 1. The following time scales are periodic.

(i) $\mathbb{T} = \bigcup_{i=-\infty}^{\infty} [2(i-1)h, 2ih]$, with $h > 0$, has period $p = 2h$.

(ii) $\mathbb{T} = h\mathbb{Z}$ has period $p = h$.

(iii) $\mathbb{T} = \mathbb{R}$.

(iv) $\mathbb{T} = \{t = k - q^n : k \in \mathbb{Z},\ m \in \mathbb{N}_0\}$, with $0 < q < 1$, has period $p = 1$.

REMARK 1 ([8]). All periodic time scales are unbounded above and below.

DEFINITION 2. *Let $\mathbb{T} \neq \mathbb{R}$ be a periodic time scale with the period p. We say that the function $f : \mathbb{T} \to \mathbb{R}$ is periodic with period T if there exists a natural number n such that $T = np$, $f(t \pm T) = f(t)$ for all $t \in \mathbb{T}$ and T is the smallest number such that $f(t \pm T) = f(t)$. If $\mathbb{T} = \mathbb{R}$, we say that f is periodic with period $T > 0$ if T is the smallest positive number such that $f(t \pm T) = f(t)$ for all $t \in \mathbb{T}$.*

REMARK 2 ([8]). If $\mathbb{T}$ is a periodic time scale with period p, then $\sigma(t \pm np) = \sigma(t) \pm np$. Consequently, the graininess function μ satisfies

$$\mu(t \pm np) = \sigma(t \pm np) - (t \pm np) = \sigma(t) - t = \mu(t),$$

and so, is a periodic function with period p.

Let $\mathbb{T}$ be a periodic time scale such that $0 \in \mathbb{T}$ and consider the totally nonlinear dynamic equation with functional delay given by

$$x^{\triangle}(t) = -a(t)x^3(\sigma(t)) + G\big(t, x^3(t), x^3(t - r(t))\big), \quad t \in \mathbb{T}. \tag{1}$$

We shall show that (1) possesses periodic solutions. Toward this end we assume that $r : \mathbb{T} \to \mathbb{R}$ and that $id - r : \mathbb{T} \to \mathbb{T}$ is strictly increasing so that the function $x(t - r(t))$ is well defined over $\mathbb{T}$.

In the case $\mathbb{T} = \mathbb{R}$, the authors in [7] used a modification of Krasnoselskii's fixed point theorem to show the existence of periodic solutions of (1) when the delay is some positive continuous and periodic function r.

In Section 2, we present some preliminary material that we will need through the remainder of the paper. We will state some facts about the exponential function on a time scale as well as the modification of Krasnoselskii's fixed point theorem due to T. A. Burton (see [5, Theorem 3] and [6]). For details on Krasnoselskii theorem we refer the reader to [10]. We present our main results on periodicity in Section 3 and provide an example to illustrate our claim.

2. Preliminaries

We begin this section by considering some advanced topics in the theory of dynamic equations on a time scales. Most of the following definitions, lemmas and theorems can be found in [3, 4]. Our first two theorems concern the composition of two functions. The first theorem is the chain rule on time scales [3, Theorem 1.93].

THEOREM 1 (Chain Rule). *Assume $\nu : \mathbb{T} \to \mathbb{R}$ is strictly increasing and $\widetilde{\mathbb{T}} := \nu(\mathbb{T})$ is a time scale. Let $\omega : \widetilde{\mathbb{T}} \to \mathbb{R}$. If $\nu^{\triangle}(t)$ and $\omega^{\widetilde{\triangle}}(\nu(t))$ exist for $t \in \mathbb{T}^k$, then*

$$\big(\omega \circ \nu\big)^{\triangle} = \big(\omega^{\widetilde{\triangle}} \circ \nu\big)\nu^{\triangle}.$$

In the sequel we will need to differentiate and integrate functions of the form $f(t - r(t)) = f(\nu(t))$ where, $\nu(t) := t - r(t)$. Our second theorem is the substitution rule [3, Theorem 1.98].

THEOREM 2 (Substitution). *Assume* $\nu : \mathbb{T} \to \mathbb{R}$ *is strictly increasing and* $\widetilde{\mathbb{T}} := \nu(\mathbb{T})$ *is a time scale. If* $f : \mathbb{T} \to \mathbb{R}$ *is an rd-continuous function and* ν *is differentiable with rd-continuous derivative, then for* $a, b \in \mathbb{T}$,

$$\int_a^b f(t)\nu^{\triangle}(t)\triangle t = \int_{\nu(a)}^{\nu(b)} (f \circ \nu^{-1})(s)\widetilde{\triangle} s.$$

A function $p : \mathbb{T} \to \mathbb{R}$ is said to be *regressive* if $1 + \mu(t)p(t) \neq 0$ for all $t \in \mathbb{T}^k$. The set of all regressive rd-continuous function $f : \mathbb{T} \to \mathbb{R}$ is denoted by $\mathcal{R}$ while the set $\mathcal{R}^+ = \{f \in \mathcal{R} : 1 + \mu(t)f(t) > 0 \text{ for all } t \in \mathbb{T}\}$.

Let $p \in \mathcal{R}$ and $\mu(t) \neq 0$ for all $t \in \mathbb{T}$. The exponential function on $\mathbb{T}$ is defined by

$$e_p(t,s) = \exp\left(\int_s^t \frac{1}{\mu(z)} \mathrm{Log}(1 + \mu(z)p(z))\triangle z\right).$$

It is well known that if $p \in \mathcal{R}^+$, then $e_p(t,s) > 0$ for all $t \in \mathbb{T}$. Also, the exponential function $y(t) = e_p(t,s)$ is the solution to the initial value problem $y^{\triangle} = p(t)y$, $y(s) = 1$. Other properties of the exponential function are given in the following lemmas.

LEMMA 1 ([3]). *Let* $p, q \in \mathcal{R}$. *Then*

(i) $e_0(t,s) = 1$ *and* $e_p(t,t) = 1$;

(ii) $e_p(\sigma(t),s) = (1 + \mu(t)p(t))e_p(t,s)$;

(iii) $\dfrac{1}{e_p(t,s)} = e_{\ominus p}(t,s)$, *where* $\ominus p(t) = -\dfrac{p(t)}{1 + \mu(t)p(t)}$;

(iv) $e_p(t,s) = \dfrac{1}{e_p(s,t)} = e_{\ominus p}(s,t)$;

(v) $e_p(t,s)e_p(s,r) = e_p(t,r)$;

(vi) $e_p^{\triangle}(\cdot,s) = pe_p(.,s)$ *and* $\left(\dfrac{1}{e_p(\cdot,s)}\right)^{\triangle} = -\dfrac{p(t)}{e_p^{\sigma}(\cdot,s)}$.

LEMMA 2 ([1]). *If* $p \in \mathcal{R}^+$, *then*

$$0 < e_p(t,s) \leq \exp\left(\int_s^t p(u)\triangle u\right), \quad \forall t \in \mathbb{T}.$$

COROLLARY 1 ([1]). *If* $p \in \mathcal{R}^+$ *and* $p(t) < 0$ *for all* $t \in \mathbb{T}$, *then for all* $s \in \mathbb{T}$ *with* $s \leq t$ *we have*

$$0 < e_p(t,s) \leq \exp\left(\int_s^t p(u)\triangle u\right) < 1.$$

In the analysis, we employ a fixed point theorem in which the notion of a large contraction is required as one of the sufficient conditions. The following definition, due to T. A. Burton, can be found in [5, 6].

DEFINITION 3 (Large Contraction). *Let (M,d) be a metric space and consider $B : M \to M$. Then B is said to be a large contraction if given $\phi, \varphi \in M$ with $\phi \neq \varphi$ then $d(B\phi, B\varphi) \leq d(\phi, \varphi)$ and if for all $\varepsilon > 0$, there exists a $\delta < 1$ such that*

$$[\phi, \varphi \in M, d(\phi, \varphi) \geq \varepsilon] \Rightarrow d(B\phi, B\varphi) \leq \delta d(\phi, \varphi).$$

The next theorem is also a result of T. A. Burton. This captivating theorem, which constitutes a basis for our main result, is a reformulated version of Krasnoselskii's fixed point theorem (see [5, Theorem 3] and [6]).

THEOREM 3 (Burton–Krasnoselskii). *Let M be a bounded convex nonempty subset of a Banach space $(\mathbb{B}, \|\cdot\|)$. suppose that A and B map M into $\mathbb{B}$ such that*

(i) *$x, y \in M$, implies $Ax + By \in M$;*

(ii) *A is continuous and AM is contained in a compact subset of M;*

(iii) *B is a large contraction mapping.*

Then there exists $z \in M$ with $z = Az + Bz$.

It is obvious that if we want to apply the above theorem we need to construct two mappings, one a large contraction and the other compact.

3. Existence of periodic solutions

We will state and prove our main result in this section. After we provide an example to illustrate our results. Let $T > 0$, $T \in \mathbb{T}$ be fixed and if $\mathbb{T} \neq \mathbb{R}$, $T = np$ for some $n \in \mathbb{N}$. By the notation $[a,b]$, we mean

$$[a,b] = \{t \in \mathbb{T} : a \leq t \leq b\}$$

unless otherwise specified. The intervals $[a,b)$, $(a,b]$ and (a,b) are defined similarly.

Define $C_T = \{\varphi : \mathbb{T} \to \mathbb{R} \mid \varphi \in C \text{ and } \varphi(t+T) = \varphi(t)\}$ where C is the space of continuous real-valued functions on $\mathbb{T}$. Then $(C_T, \|\cdot\|)$ is a Banach space with the supremum norm

$$\|\varphi\| = \sup_{t \in \mathbb{T}} |\varphi(t)| = \sup_{t \in [0,T]} |\varphi(t)|.$$

We will need the following lemmas whose proof can be found in [8].

LEMMA 3. *Let $x \in C_T$. Then $\|x^\sigma\| = \|x \circ \sigma\|$ exists and $\|x^\sigma\| = \|x\|$.*

In this paper we assume that $a \in \mathcal{R}^+$ is continuous, $a(t) > 0$ for all $t \in \mathbb{T}$ and

$$a(t+T) = a(t), \quad (id - r)(t+T) = (id - r)(t), \tag{2}$$

where *id* is the identity function on $\mathbb{T}$. We also require that $G(t,x,y)$ is continuous and periodic in t and Lipschitz continuous in x and y. That is

$$G(t+T,x,y) = G(t,x,y), \tag{3}$$

and there are positive constants k_1, k_2 such that

$$|G(t,x,y) - G(t,z,w)| \le k_1 |x-z| + k_2 |y-w|, \text{ for } x,y,z,w \in \mathbb{R}. \tag{4}$$

LEMMA 4. *Suppose* (2) *and* (3) *hold. If $x \in C_T$, then x is a solution of equation* (1) *if and only if*

$$x(t) = \big(1 - e_{\ominus a}(t,t-T)\big)^{-1} \Bigg[\int_{t-T}^{t} a(s)\big(x(\sigma(s)) - x^3(\sigma(s))\big) e_{\ominus a}(t,s) \triangle s + \int_{t-T}^{t} G\big(s, x^3(s), x^3(s-r(s))\big) e_{\ominus a}(t,s) \triangle s \Bigg]. \tag{5}$$

Proof. Let $x \in C_T$ be a solution of (1). First we write this equation as

$$x^{\triangle}(t) + a(t)x(\sigma(t)) = a(t)x(\sigma(t)) - a(t)x^3(\sigma(t)) + G\big(t, x^3(t), x^3(t-r(t))\big).$$

Multiply both sides of the above equation by $e_a(t,0)$ and then integrate from $t-T$ to t to obtain

$$\int_{t-T}^{t} \big(e_a(s,0)x(s)\big)^{\triangle} \triangle s = \int_{t-T}^{t} \Big[a(s)\left\{x(\sigma(s)) - x^3(\sigma(s))\right\} + G\big(s, x^3(s), x^3(s-r(s))\big)\Big] e_a(s,0) \triangle s.$$

As a consequence, we arrive at

$$e_a(t,0)x(t) - e_a(t-T,0)x(t-T) = \int_{t-T}^{t} \Big[a(s)\left\{x(\sigma(s)) - x^3(\sigma(s))\right\} + G\big(s, x^3(s), x^3(s-r(s))\big)\Big] e_a(s,0) \triangle s.$$

Now, the lemma follows by dividing both sides of the above equation by $e_a(t,0)$ and using the fact that $x(t) = x(t-T)$. ☐

To apply Theorem 3, we need to define a Banach space $\mathbb{B}$, a bounded convex subset M of $\mathbb{B}$ and construct two mappings, one a large contraction and the other compact. So, we let $(\mathbb{B}, \|\cdot\|) = (C_T, \|\cdot\|)$ and $M = \{\varphi \in \mathbb{B} \mid \|\varphi\| \le L\}$, where $L = \sqrt{3}/3$. We express equation (5) as

$$\varphi(t) = (B\varphi)(t) + (A\varphi)(t) := (H\varphi)(t),$$

where $A,B : M \to \mathbb{B}$ are defined by

$$(A\varphi)(t) = \left(1 - e_{\ominus a}(t,t-T)\right)^{-1} \int_{t-T}^{t} G\big(s, \varphi^3(s), \varphi^3(s-r(s))\big)\, e_{\ominus a}(t,s) \triangle s, \tag{6}$$

and

$$(B\varphi)(t) = \left(1 - e_{\ominus a}(t,t-T)\right)^{-1} \int_{t-T}^{t} a(s)\big(\varphi(\sigma(s)) - \varphi^3(\sigma(s))\big)\, e_{\ominus a}(t,s) \triangle s. \tag{7}$$

We suppose an additional condition, namely that there is $J \geq 3$ with

$$J((k_1+k_2)L^3 + |G(t,0,0)|) \leq La(t), \quad \forall t \in \mathbb{T}. \tag{8}$$

We shall prove that the mapping H has a fixed point which solves (1), whenever its derivative exists.

LEMMA 5. *For A defined in* (6), *suppose that* (2), (3), (4) *and* (8) *hold. Then $A : M \to M$ is continuous in the supremum norm and maps M into a compact subset of M.*

Proof. We first show that $A : M \to M$. Clearly, if φ is continuous, so is $A\varphi$. Evaluating (6) at $t+T$ gives

$$(A\varphi)(t+T) = \left(1 - e_{\ominus a}(t+T,t)\right)^{-1} \int_{t}^{t+T} G\big(s, \varphi^3(s), \varphi^3(s-r(s))\big)\, e_{\ominus a}(t+T,s) \triangle s. \tag{9}$$

Use Theorem 2 with $u = s - T$ and conditions (2), (3) to get

$$\begin{aligned} (A\varphi)(t+T) \;&=\; \left(1 - e_{\ominus a}(t+T,t)\right)^{-1} \\ &\times \int_{t-T}^{t} G\big(u+T, \varphi^3(u+T), \varphi^3(u+T-r(u+T))\big)\, e_{\ominus a}(t+T,u+T) \triangle u. \end{aligned} \tag{10}$$

From Theorem 2, we have $e_{\ominus a}(t+T,u+T) = e_{\ominus a}(t,u)$ and $e_{\ominus a}(t+T,t) = e_{\ominus a}(t,t-T)$. Thus (9) becomes

$$\begin{aligned} (A\varphi)(t+T) &= \left(1 - e_{\ominus a}(t,t-T)\right)^{-1} \int_{t-T}^{t} G\big(u, \varphi^3(u), \varphi^3(u-r(u))\big)\, e_{\ominus a}(t,u) \triangle u \\ &= (A\varphi)(t). \end{aligned}$$

That is, $A : C_T \to C_T$. In view of (4), we arrive at

$$\begin{aligned} |G(t,x,y)| &= |G(t,x,y) - G(t,0,0) + G(t,0,0)| \\ &\leq |G(t,x,y) - G(t,0,0)| + |G(t,0,0)| \\ &\leq k_1 \|x\| + k_2 \|y\| + |G(t,0,0)|. \end{aligned}$$

Note that from Corollary 1, we have $1 - e_{\ominus a}(t, t-T) > 0$. So, for any $\varphi \in M$, we have

$$
\begin{aligned}
|(A\varphi)(t)| &\leq \big(1 - e_{\ominus a}(t,t-T)\big)^{-1} \int_{t-T}^{t} \big|G\big(s, \varphi^3(s), \varphi^3(s-r(s))\big)\big| e_{\ominus a}(t,s)\triangle s \\
&\leq \big(1 - e_{\ominus a}(t,t-T)\big)^{-1} \int_{t-T}^{t} \Big((k_1+k_2)L^3 + |G(s,0,0)|\Big) e_{\ominus a}(t,s)\triangle s \\
&\leq \big(1 - e_{\ominus a}(t,t-T)\big)^{-1} \int_{t-T}^{t} \frac{La(s)}{J} e_{\ominus a}(t,s)\triangle s \\
&= \frac{L}{J} < L.
\end{aligned}
$$

Thus, $A\varphi \in M$. Consequently, we have $A : M \to M$.

We show that A is continuous in the supremum norm. Toward this, let $\varphi, \psi \in M$, and set

$$
\alpha = (1 - e_{\ominus a}(t,t-T))^{-1}, \quad \gamma = \max_{t\in[0,T]} \{a(t)\}, \quad \rho = \max_{t\in[0,T]} |G(t,0,0)|. \tag{11}
$$

Note that from $a(t) > 0$ we have $\max_{s\in[t-T,t]} e_{\ominus a}(t,s) \leq 1$. So,

$$
\begin{aligned}
&|(A\varphi)(t) - (A\psi)(t)| \\
&\leq \big(1 - e_{\ominus a}(t,t-T)\big)^{-1} \int_{t-T}^{t} \Big| G\big(s, \varphi^3(s), \varphi^3(s-r(s))\big) \\
&\qquad\qquad - G\big(s, \psi^3(s), \psi^3(s-r(s))\big)\Big| e_{\ominus a}(t,s)\triangle s \\
&\leq (k_1+k_2)\left\|\varphi^3 - \psi^3\right\| \big(1 - e_{\ominus a}(t,t-T)\big)^{-1} \int_{t-T}^{t} e_{\ominus a}(t,s)\triangle s \\
&\leq 3(k_1+k_2)T\alpha L^2 \left\|\varphi - \psi\right\|.
\end{aligned}
$$

Let $\varepsilon > 0$ be arbitrary. Define $\eta = \varepsilon/K$ with $K = 3(k_1+k_2)T\alpha L^2$, where k_1 and k_2 are given by (4). Then, for $\|\varphi - \psi\| < \eta$ we obtain

$$
\|A\varphi - A\psi\| \leq K\|\varphi - \psi\| < \varepsilon.
$$

This proves that A is continuous.

It remains to show that A is compact. Let $\varphi_n \in M$, where n is a positive integer. Then, as above, we see that

$$
\|(A\varphi_n)^\sigma\| = \|A\varphi_n\| \leq L.
$$

Moreover, a direct calculation shows that

$$
(A\varphi_n)^\triangle(t) = G\big(t, \varphi_n^3(t), \varphi_n^3(t-r(t))\big) - a(t)(A\varphi_n)^\sigma(t).
$$

Consequently, by invoking (4) and (11), we obtain

$$\begin{aligned}\left|(A\varphi_n)^{\triangle}(t)\right| &\le \left|G\big(t,\varphi_n^3(t),\varphi_n^3(t-r(t))\big)\right| + a(t)\left|(A\varphi_n)^{\sigma}(t)\right| \\ &\le (k_1+k_2)L^3 + |G(t,0,0)| + \gamma\|(A\varphi_n)^{\sigma}\| \\ &\le (k_1+k_2)L^3 + \rho + \gamma L \\ &\le D,\end{aligned}$$

for some positive constant D. Hence the sequence $(A\varphi_n)$ is uniformly bounded and equicontinuous. The Ascoli–Arzela theorem implies that a subsequence $(A\varphi_{n_k})$ of $(A\varphi_n)$ converges uniformly to a continuous T-periodic function. Thus A is continuous and AM is a compact set. ☐

LEMMA 6. *Let B be defined by* (7) *and that* (2) *holds. Then $B : M \to M$ is a large contraction.*

Proof. We first show that $B : M \to M$. Clearly, if φ is continuous, so is $B\varphi$. Evaluate (7) at $t+T$ to have

$$\begin{aligned}&(B\varphi)(t+T) \\ &= \big(1-e_{\ominus a}(t+T,t)\big)^{-1}\int_t^{t+T} a(s)\big(\varphi(\sigma(s))-\varphi^3(\sigma(s))\big)\,e_{\ominus a}(t+T,s)\triangle s.\end{aligned} \tag{12}$$

Use Theorem 2 with $u=s-T$ and conditions (2), (3) to get

$$\begin{aligned}&(B\varphi)(t+T) \\ &= \big(1-e_{\ominus a}(t+T,t)\big)^{-1}\int_{t-T}^{t} a(u)\big(\varphi(\sigma(u+T))-\varphi^3(\sigma(u+T))\big)\,e_{\ominus a}(t+T,u+T)\triangle u.\end{aligned}$$

From Theorem 2, we have $\sigma(u+T)=\sigma(u)+T$, $e_{\ominus a}(t+T,u+T)=e_{\ominus a}(t,u)$ and $e_{\ominus a}(t+T,t)=e_{\ominus a}(t,t-T)$. Thus (12) becomes

$$\begin{aligned}(B\varphi)(t+T) &= \big(1-e_{\ominus a}(t,t-T)\big)^{-1}\int_{t-T}^{t} a(u)\big(\varphi(\sigma(u))-\varphi^3(\sigma(u))\big)\,e_{\ominus a}(t,u)\triangle u \\ &= (B\varphi)(t).\end{aligned}$$

That is, $B : C_T \to C_T$.

Note that from Lemma 3 and Corollary 1, we have $\left\|\varphi(\sigma)-\varphi^3(\sigma)\right\| = \left\|\varphi-\varphi^3\right\|$ and $1-e_{\ominus a}(t,t-T)>0$. So, for any $\varphi\in M$, we have

$$\begin{aligned}|(B\varphi)(t)| &\le \big(1-e_{\ominus a}(t,t-T)\big)^{-1}\int_{t-T}^{t} a(s)\left|\varphi(\sigma(s))-\varphi^3(\sigma(s))\right| e_{\ominus a}(t,s)\triangle s \\ &\le \big(1-e_{\ominus a}(t,t-T)\big)^{-1}\int_{t-T}^{t} a(s)\left\|\varphi(\sigma)-\varphi^3(\sigma)\right\| e_{\ominus a}(t,s)\triangle s \\ &= \left\|\varphi-\varphi^3\right\|.\end{aligned}$$

From [5], we have $\left\|\varphi - \varphi^3\right\| \le (2\sqrt{3})/9 < L$. So, for any $\varphi \in M$, we obtain

$$\|B\varphi\| < L.$$

Thus $B\varphi \in M$. Consequently, we have $B : M \to M$.

It remains to show that B is large contraction. Let $\varphi, \psi \in M$, with $\varphi \neq \psi$. Using $\varphi^2(\sigma(s)) + \psi^2(\sigma(s)) < 1$, we have

$$\begin{aligned}
&|(B\varphi)(t) - (B\psi)(t)| \\
&\le \left(1 - e_{\ominus a}(t, t-T)\right)^{-1} \int_{t-T}^{t} a(s) \left|\varphi(\sigma(s)) - \varphi^3(\sigma(s)) - \psi(\sigma(s)) + \psi^3(\sigma(s))\right| e_{\ominus a}(t,s) \triangle s \\
&\le \|\varphi(\sigma) - \psi(\sigma)\| \left(1 - \inf_{s\in[t-T,t]} \frac{\varphi^2(\sigma(s)) + \psi^2(\sigma(s))}{2}\right) \\
&\qquad\qquad \times \left(1 - e_{\ominus a}(t, t-T)\right)^{-1} \int_{t-T}^{t} a(s) e_{\ominus a}(t,s) \triangle s \\
&\le \|\varphi - \psi\|.
\end{aligned}$$

Then $\|B\varphi - B\psi\| \le \|\varphi - \psi\|$.

Thus B is a large pointwise contraction. But B is still a large contraction for the supremum norm. To show this, let $\varepsilon \in (0,1)$ be given and suppose that $\varphi, \psi \in M$ satisfy $\|\varphi - \psi\| \ge \varepsilon$. From [6] we have

$$\|B\varphi - B\psi\| \le \left(1 - \frac{\varepsilon^2}{16}\right) \|\varphi(\sigma) - \psi(\sigma)\| = \left(1 - \frac{\varepsilon^2}{16}\right) \|\varphi - \psi\|.$$

Consequently, B is a large contraction. □

THEOREM 4. *Let $(C_T, \|\cdot\|)$ be the Banach space of continuous T-periodic real valued functions on $\mathbb{T}$ and $M = \{\varphi \in C_T \mid \|\varphi\| \le L\}$, where $L = \sqrt{3}/3$. Suppose* (2), (3), (4) *and* (8) *hold. Then equation* (1) *has a T-periodic solution φ in the subset M.*

Proof. By Lemma 5, $A : M \to M$ is continuous and AM is contained in a compact set. Also, from Lemma 6, the mapping $B : M \to M$ is a large contraction. Next, we show that if $\varphi, \psi \in M$, we have

$$\|A\varphi + B\psi\| \le \|A\varphi\| + \|B\psi\| \le L/J + (2\sqrt{3})/9 \le L.$$

Thus $A\varphi + B\psi \in M$.

Clearly, all the hypotheses of Burton–Krasnoselskii's Theorem 3 are satisfied. Thus there exists a fixed point $\varphi \in M$ such that $\varphi = A\varphi + B\varphi$. Hence the equation (1) has a T-periodic solution in M. □

EXAMPLE 2. Let $\mathbb{T}$ be a periodic time scale. We consider the totally nonlinear dynamic equation with functional delay

$$x^{\triangle}(t) = -8x^3(\sigma(t)) + \sin(x^3(t)) + \cos(x^3(t - r(t))), \quad t \in \mathbb{T}, \tag{13}$$

where
$$(id-r)(t+T)=(id-r)(t).$$
Also, we assume that the function r is continuous with $id-r:\mathbb{T}\to\mathbb{T}$ strictly increasing. So, we have
$$a(t)=8,\qquad G\big(t,x^3(t),x^3(t-r(t))\big)=\sin(x^3(t))+\cos(x^3(t-r(t))).$$

Define $M=\{\varphi\in C_T \mid \|\varphi\|\le L\}$, where $L=\sqrt{3}/3$. Then for $\varphi,\psi\in M$, we have
$$\gamma=8,\qquad \rho=1.$$
Clearly, $G(t,x,y)$ is continuous and periodic in t Lipschitz continuous in x and y. That is to say
$$G(t+T,x,y)=G(t,x,y),$$
and
$$\begin{aligned}|G(t,x,y)-G(t,z,w)|&=|\sin(x)-\sin(z)+\cos(y)-\cos(w)|\\&\le|\sin(x)-\sin(z)|+|\cos(y)-\cos(w)|\\&\le|x-z|+|y-w|.\end{aligned}$$
Note that if $J=3$ we have
$$\begin{aligned}J\big((k_1+k_2)L^3+|G(t,0,0)|\big)&=3\big(2(\sqrt{3}/3)^3+1\big)\\&\le(\sqrt{3}/3)8\\&=La(t),\qquad \forall t\in\mathbb{T}.\end{aligned}$$
Hence (13) has a T-periodic solution in M, by Theorem 4.

Acknowledgement. The authors would like to thank the editorial board and the anonymous referee for their suggestions and valuable advice.

References

[1] ADIVAR M. AND RAFFOUL Y. N. Existence of periodic solutions in totally nonlinear delay dynamic equations. *Electron. J. Qual. Theory Differ. Equ. Special Edition I*, 1 (2009), 20 pp.

[2] ATICI F. M., GUSEINOV G. S. AND KAYMAKÇALAN B. Stability criteria for dynamic equations on time scales with periodic coefficients. In *Dynamic Systems and Applications, Vol. 3 (Atlanta, GA, 1999)*. 2001, pp. 43–48.

[3] BOHNER M. AND PETERSON A. *Dynamic Equations on Time Scales, An Introduction with Applications*. Birkhäuser, Boston, 2001.

[4] BOHNER M. AND PETERSON A. *Advances in Dynamic Equations on Time Scales*. Birkhäuser, Boston, 2003.

[5] BURTON T. A. Liapunov functionals, fixed points and stability by Krasnoselskii's theorem. *Nonlinear Stud. 9*, 2 (2002), 181–190.

[6] BURTON T. A. *Stability by Fixed Point Theory for Functional Differential Equations*. Dover, New York, 2006.

[7] DEHAM H. AND DJOUDI A. Periodic solutions for nonlinear differential equation with functional delay. *Georgian Math. J. 15*, 4 (2008), 635–642.

[8] KAUFMANN E. R. AND RAFFOUL Y. N. Periodic solutions for a neutral nonlinear dynamical equation on a time scale. *J. Math. Anal. Appl. 319*, 1 (2006), 315–325.

[9] KAUFMANN E. R. AND RAFFOUL Y. N. Periodicity and stability in neutral nonlinear dynamic equation with functional delay on a time scale. *Electron J. Differential Equations 27* (2007), 12 pp.

[10] SMART D. R. *Fixed Point Theorems*. Cambridge Tracts in Mathematics 66. Cambridge University Press, London-New York, 1974.

AMS Subject Classification: 34K13, 34A34; 34K30, 34L30

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Lavoro pervenuto in redazione il 24.09.2010 e, in forma definitiva, il 18.01.2011