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THE SECOND GAUSSIAN MAP FOR CURVES: A SURVEY*

Abstract. We give an overview of results on the second Gaussian map for curves in relation to the second fundamental form of the period map. We will also concentrate on the case in which the curve is contained in an abelian surface, or in a K3 surface. This is an expanded version of the talk given at the "Workshop on Hodge Theory and Algebraic Geometry" in Povo (Trento) on 4–5 September 2009.

1. Introduction

Let A_g be the moduli space of principally polarized abelian varieties of dimension g and let $j : M_g \to A_g$ be the period map sending a curve to its Jacobian. It is an interesting and classical problem to understand the geometry of the image of M_g in A_g.

On A_g there is a natural metric coming from the unique (up to scalar) $Sp(2g,\mathbb{R})$ invariant metric on the Siegel space $H_g \simeq Sp(2g,\mathbb{R})/U(g)$ of which A_g is the quotient by $Sp(2g,\mathbb{Z})$. In [12] we studied the metric on M_g induced by this metric via the period map, which we call the Siegel metric. In [14] an explicit expression for the second fundamental form of the immersion j is given and it is proven that the second fundamental form lifts the second Gaussian map $\gamma^2_C : I_2(K_C) \to H^0(C,4K_C)$, as stated in an unpublished paper of Green and Griffiths (cf. [16]).

This paper is a survey of results that were obtained in collaboration with Elisabetta Colombo and Giuseppe Pareschi in [10, 12, 11, 13] on the second Gaussian map γ^2_C of a curve C and its relation to the second fundamental form of the period map.

More precisely, in Section 2 we recall the definition of the Gaussian (or Wahl) maps, while in Section 3 we describe some results of [12] concerning the computation of the holomorphic sectional curvature of M_g, endowed with the Siegel metric, along the tangent directions given by the Schiffer variations, in terms of the second Gaussian map γ^2_C.

In Section 4 we first explain some results of [10], namely the computation of the rank of γ^2_C on the hyperelliptic and trigonal loci. Then from these results and from the fact (proven in [10]) that for a non-hyperelliptic and non-trigonal curve of genus $g \geq 5$, the image of γ^2_C is base point free, we derive some properties of the holomorphic sectional curvature of M_g. In particular along a Schiffer variation ξ_P the holomorphic sectional curvature $H(\xi_P)$ of M_g is strictly smaller than the holomorphic sectional curvature of A_g unless P is either a Weierstrass point of a hyperelliptic curve or a ramification point of the g^1_3 on a trigonal curve. In these last cases, $H(\xi_P) = -1$.

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In Section 5 we restrict our attention to the hyperelliptic locus and we show that the holomorphic sectional curvature of the hyperelliptic locus, computed along the Schiffer variations at the Weierstrass points of the curve is identically equal to -1.

In Section 6 we describe some results of [11] on the second Gaussian map of a curve in a K3 surface. More precisely we explain the strategy of the proof of the main result of [11], which asserts that for a general hyperplane section of a general polarized K3 surface of genus $g > 280$, γ_C^2 is surjective. From this and with the help of some examples that we found in [10], we deduce that for the general curve of genus $g > 152$, γ_C^2 is surjective.

Very recently, Calabri, Ciliberto and Miranda in [5] have shown that γ_C^2 has maximal rank for the general curve of any genus g, namely it is injective for $g \leq 17$ and surjective for $g \geq 18$. Their proof is achieved by degeneration to a stable binary curve, i.e. a the union of two rational curves meeting at $g+1$ points.

Finally in Section 7 we report on results on the first and the second Gaussian map for a curve on an abelian surface obtained in [13].

The main result of [13] roughly says that if a curve C is contained in an abelian surface, the corank of γ_C^2 is at least 2, hence it is never surjective.

Recall that by an important theorem of Wahl ([27]), we know that if a curve is a hyperplane section of a K3 surface, the first Gaussian (or Wahl) map is not surjective (cf. also [3]).

In [13] we also proved that for a "sufficiently ample" curve C contained in an abelian surface the first Wahl map is surjective (see Theorem 7.1 for a precise statement).

Finally recall Ciliberto-Harris-Miranda's theorem ([6], see also Voisin's proof in [24]), stating that the first Wahl map of the generic curve of genus g is surjective as soon as this is numerically possible, i.e. for $g \geq 10$, with the exception of $g = 11$. For $g < 10$ and $g = 11$ it is known that the generic curve lies on a K3 surface ([20]).

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2. Definition of Gaussian maps

Let Y be a smooth complex projective variety and let $\Delta_Y \subset Y \times Y$ be the diagonal. Let L and M be line bundles on Y. For a non-negative integer k, the *k-th Gaussian map* associated to these data is the restriction to diagonal map

$$\gamma_{L,M}^k : H^0(Y \times Y, I_{\Delta_Y}^k \otimes L \boxtimes M) \to H^0(Y, I_{\Delta_Y|\Delta_Y}^k \otimes L \otimes M) \cong H^0(Y, S^k\Omega_Y^1 \otimes L \otimes M). \tag{1}$$

Usually *first* Gaussian maps are simply referred to as *Gaussian maps*. The exact sequence

$$0 \to I_{\Delta_Y}^{k+1} \to I_{\Delta_Y}^k \to S^k\Omega_Y^1 \to 0 \tag{2}$$

(where $S^k\Omega^1_Y$ is identified to its image via the diagonal map), twisted by $L\boxtimes M$, shows that the domain of the k-th Gaussian map is the kernel of the previous one:

$$\gamma^k_{L,M} : \ker\gamma^{k-1}_{L,M} \to H^0(S^k\Omega^1_Y \otimes L \otimes M).$$

In what follows, we will deal with Gaussian maps of order one and two, assuming also that the two line bundles L and M coincide. The map γ^0_L is the multiplication map of global sections

$$H^0(X,L)\otimes H^0(X,L) \to H^0(X,L^2) \tag{3}$$

which obviously vanishes identically on $\wedge^2 H^0(L)$. Consequently,

$$H^0(Y\times Y, I_{\Delta_Y}\otimes L\boxtimes L)$$

decomposes as $\wedge^2 H^0(L)\oplus I_2(L)$, where $I_2(L)$ is the kernel of $S^2H^0(X,L)\to H^0(X,L^2)$. Since γ^1_L vanishes on symmetric tensors, one writes

$$\gamma^1_L : \wedge^2 H^0(L) \to H^0(\Omega^1_Y\otimes L^2). \tag{4}$$

Again, $H^0(Y\times Y, I^2_{\Delta_Y}\otimes L\boxtimes L)$ decomposes as the sum of $I_2(L)$ and the kernel of (4). Since γ^2_L vanishes identically on skew-symmetric tensors, one usually writes

$$\gamma^2_L : I_2(L) \to H^0(S^2\Omega^1_Y\otimes L^2). \tag{5}$$

(In general, Gaussian maps of even, respectively odd, order vanish identically on skew-symmetric, respectively symmetric, tensors.) The second Gaussian map γ^2_C is the second Gaussian map of the canonical line bundle. In what follows we will also deal with the first Wahl map, γ^1_C, which is the first Gaussian map of the canonical bundle. We have:

$$\gamma^1_C : \wedge^2 H^0(K_C) \to H^0(3K_C)$$

$$\gamma^2_C : I_2(K_C) \to H^0(4K_C).$$

Let us now also give the expressions of γ^1_C and γ^2_C in local coordinates. Fix a basis $\{\omega_i\}$ of $H^0(K_C)$. In local coordinates, assume that $\omega_i = f_i(z)dz$. Then we have

$$\gamma^1_C(\omega_i\wedge\omega_j) = (f_i' f_j - f_j' f_i)(dz)^3.$$

Let $Q\in I_2(K_C)$, $Q=\sum_{i,j} a_{ij}\omega_i\otimes\omega_j$, recall that $\sum_{i,j} a_{ij} f_i f_j \equiv 0$, and since $a_{i,j}$ are symmetric, we also have $\sum_{i,j} a_{ij} f_i' f_j \equiv 0$. The local expression of $\gamma^2_C(Q)$ is

$$\gamma^2_C(Q) = \sum_{i,j} a_{ij} f_i'' f_j (dz)^4 = -\sum_{i,j} a_{ij} f_i' f_j' (dz)^4. \tag{6}$$

3. The Siegel metric

Let M_g, respectively $M_g^{(n)}$, be the moduli space of smooth genus g curves, respectively of smooth genus g curves with a fixed n-level structure.

Let A_g, respectively $A_g^{(n)}$, be the moduli space of g-dimensional principally polarized Abelian varieties, respectively of g-dimensional principally polarized Abelian varieties with a n-level structure.

Denote by

$$H_g := \{Z \in M(g,\mathbb{C}) \mid Z = {}^tZ,\ \mathrm{Im}Z > 0\}$$

the Siegel space so that A_g is the quotient of H_g by the action of $Sp(2g,\mathbb{Z})$ and $A_g^{(n)}$ is the quotient of H_g by $\ker(Sp(2g,\mathbb{Z}) \to Sp(2g,\mathbb{Z}/n\mathbb{Z}))$. Denote by $j : M_g \to A_g$, and $j^{(n)} : M_g^{(n)} \to A_g^{(n)}$ the period maps which send a curve to its Jacobian.

The Torelli theorem states that j is injective, while $j^{(n)}$ is two-to-one on the image and ramified over the hyperelliptic locus. In fact multiplication by -1 in $H^1(C,\mathbb{Z})$ $= H^1(JC,\mathbb{Z})$, where JC is the Jacobian of the curve C, is induced by an automorphism of abelian varieties but not by an automorphism of non-hyperelliptic curves. The local Torelli theorem says that outside the hyperelliptic locus and restricted to the hyperelliptic locus the period map is an immersion (cf. [21]). From now on we shall work on $M_g^{(n)}$ and $A_g^{(n)}$, with $n \geq 3$, since they are smooth, everything works in the same way on M_g and A_g but in the orbifold context.

We will now define the Siegel metric.

The Siegel space H_g is a homogeneous space and it can be seen as the quotient $Sp(2g,\mathbb{R})/U(g)$. We call the unique (up to scalar) invariant metric Siegel metric.

Let F be the homogeneous vector bundle on H_g associated to the standard g-dimensional representation of $U(g,\mathbb{C})$. The Hodge metric h on F is the only (up to multiplication by scalars) invariant metric on the homogeneous bundle F. Moreover through the identification

$$\Omega^1_{H_g} \simeq S^2F$$

the Hodge metric on F defines the Siegel metric on H_g .

The Siegel metric on H_g defines a metric on $A_g^{(n)}$ and A_g and, through the period map, an induced metric on $M_g^{(n)}$ and M_g outside the hyperelliptic locus, and on the hyperelliptic locus itself. We call all these metrics Siegel metrics.

These metrics can be described in terms of polarized variation of Hodge structures. More precisely, on $A_g^{(n)}$ we have the universal family $\phi : \mathcal{A} \to A_g^{(n)}$, and the polarized variation of Hodge structures associated to the local system $R^1\phi_*\mathbb{Z}$. The associated Hodge bundle $\mathcal{F}^1$ can be identified with $\phi_*(\Omega^1_{\mathcal{A}|A_g^{(n)}})$, where $\Omega^1_{\mathcal{A}|A_g^{(n)}}$ is the sheaf of relative holomorphic one forms. The polarization induces a Hermitian metric on $R^1\phi_*\mathbb{C}$ and on $\mathcal{F}^1$, which we call the Hodge metric. In fact the pullback of $\mathcal{F}^1$ on H_g is the bundle F and the pullback of the metric is the Hodge metric on F. Hence the Siegel metric is induced by the Hodge metric through the identification $S^2\mathcal{F}^1 \cong \Omega^1_{A_g^{(n)}}$.

On $M_g^{(n)}$ we have the universal family $\psi : \mathcal{C} \to M_g^{(n)}$ with induced relative dualizing sheaf $K_{\mathcal{C}|M_g^{(n)}}$. The local system $R^1\psi_*\mathbb{Z}$ coincides with the pullback of $R^1\phi_*\mathbb{Z}$ through the period map: at a point $[C] \in M_g^{(n)}$, we have $H^1(C,\mathbb{Z}) \cong H^1(JC,\mathbb{Z})$. The non-degenerate Hermitian product on $H^1(C,\mathbb{C})$, defined by the polarization is the following: for any $[\eta],[\xi] \in H^1(C,\mathbb{C})$, we have

$$\langle[\eta],[\xi]\rangle = i\int_C \eta \wedge \overline{\xi}.$$

The Hodge bundle can be identified with $\psi_*(K_{\mathcal{C}|M_g^{(n)}})$, and the corresponding Hodge metric yields a metric on $S^2\mathcal{F}^1 \cong j^{(n)*}\Omega^1_{A_g^{(n)}}$, hence on $j^{(n)*}\mathcal{T}_{A_g^{(n)}}$, and by restriction the Siegel metric on $\mathcal{T}_{M_g^{(n)}}$.

We finally observe that for the sake of simplicity we defined the Siegel metric on the fine moduli space $M_g^{(n)}$, but we also have a Siegel metric on M_g viewed as an orbifold.

Recall that outside the hyperelliptic locus we have the sequence of tangent bundles:

$$0 \to \mathcal{T}_{M_g^{(n)}} \to j^{(n)*}\mathcal{T}_{A_g^{(n)}} \xrightarrow{\pi} \mathcal{N} \to 0, \tag{7}$$

whose dual, under the identifications

$$j^{(n)*}\Omega^1{}_{A_g^{(n)}} \cong S^2(\psi_*K_{\mathcal{C}|M_g^{(n)}}), \qquad \Omega^1{}_{M_g^{(n)}} \cong \psi_*(K^2{}_{\mathcal{C}|M_g^{(n)}}),$$

is

$$0 \to \mathcal{I}_2 \to S^2(\psi_*K_{\mathcal{C}|M_g^{(n)}}) \xrightarrow{m} \psi_*(K^2{}_{\mathcal{C}|M_g^{(n)}}) \to 0, \tag{8}$$

where $\mathcal{I}_2 := \mathcal{N}^*$ and m is the multiplication map.

The Hermitian connection of the variation of Hodge structures $\mathcal{R}^1\psi_*\mathbb{C}$, the Gauss-Manin connection, defines a Hermitian connection on $\mathcal{F}^1 = \psi_*K_{\mathcal{C}|M_g^{(n)}}$, thus on $\mathcal{F}^{1*}$, as well as $S^2\mathcal{F}^1$ and $S^2\mathcal{F}^{1*} \simeq j^{(n)*}\mathcal{T}_{A_g^{(n)}}$, which we denote by ∇.

The exact sequence (7) defines a second fundamental form,

$$\sigma \in \mathrm{Hom}(\mathcal{T}_{M_g^{(n)}}, \mathcal{N} \otimes \Omega^1_{M_g^{(n)}}),\ \sigma : s \mapsto \pi(\nabla(s)).$$

Similarly the exact sequence (8) defines the second fundamental form

$$\rho \in \mathrm{Hom}\left(\mathcal{I}_2, \psi_*(K^2_{\mathcal{C}|M_g^{(n)}}) \otimes \Omega^1_{M_g^{(n)}}\right).$$

Clearly σ yields a linear map $\tilde{\sigma} : \mathcal{T}_{M_g^{(n)}} \otimes \mathcal{T}_{M_g^{(n)}} \to \mathcal{N}$, and $\rho : \mathcal{I}_2 \to \Omega^1_{M_g^{(n)}} \otimes \Omega^1_{M_g^{(n)}}$ is the dual of $\tilde{\sigma}$.

In [12] we computed the curvature form R of $\mathcal{T}_{M_g^{(n)}}$ in terms of the curvature form $\tilde{R}$ of $j^{(n)*}(\mathcal{T}_{A_g^{(n)}})$ and the second fundamental form σ. In fact we have

$$\langle R(s),t\rangle = \langle \tilde{R}(s),t\rangle - \langle \sigma(s),\sigma(t)\rangle, \tag{9}$$

where s,t are local sections of $\mathcal{T}_{M_g^{(n)}}$.

The exact sequence (7) at $[C] \in M_g^{(n)}$ is

$$0 \to H^1(T_C) \to S^2(H^0(K_C))^* \to I_2(C)^* \to 0, \tag{10}$$

thus σ yields a homomorphism

$$\sigma : H^1(T_C) \to \mathrm{Hom}(I_2(K_C), H^0(2K_C)). \tag{11}$$

Analogously, at $[C] \in M_g^{(n)}$ the exact sequence (8) is:

$$0 \to I_2(K_C) \to S^2(H^0(K_C)) \xrightarrow{m} H^0(2K_C) \to 0, \tag{12}$$

hence the second fundamental form ρ gives a homomorphism

$$\rho : I_2(K_C) \to \mathrm{Hom}(H^1(T_C), H^0(2K_C)) \tag{13}$$

and for every $v \in H^1(T_C)$, and for every $Q \in I_2(C)$, we have

$$\sigma(v)(Q) = \rho(Q)(v).$$

In [12], to compute the curvature form of $\mathcal{T}_{M_g^{(n)}}$ at $[C] \in M_g^{(n)}$, we used some results of [14]. Namely in [14] it is proven that the map ρ, whose image is in $S^2H^0(2K_C)$, is a lifting of the second Gaussian map γ_C^2, as in the following diagram

$$\begin{tikzcd} I_2(K_C) \arrow[r, "\rho"] \arrow[dr, "\gamma_C^2"'] & S^2H^0(2K_C) \arrow[d, "m"] \\ & H^0(4K_C) \end{tikzcd} \tag{14}$$

where the map m is given by multiplication.

Moreover in [14] an explicit computation of the image of ρ at the tangent direction given by a Schiffer variation at a point P of C is given in terms of the second Gaussian map, namely we have:

$$\xi_P(\rho(Q)(\xi_P)) = 2\pi i \gamma_C^2(Q)(P), \tag{15}$$

for every quadric $Q \in I_2$.

Let us briefly recall the definition of ξ_P. Consider the exact sequence

$$0 \to T_C \to T_C(P) \to T_C(P)_{|P} \to 0.$$

Notice that $H^0(T_C(P)_{|P}) \cong \mathbb{C}$. If we denote the coboundary map by

$$\delta : H^0(T_C(P)_{|P}) \to H^1(T_C),$$

we have $\dim(\mathrm{Im}(\delta)) = 1$. Any non-zero element ξ_P in $\mathrm{Im}(\delta)$ is called a Schiffer variation. Let us a choose a local coordinate z in a neighborhood of P. Under the Dolbeault isomorphism $H^1(T_C) \cong H^{0,1}(T_C)$, it is represented by the form $\theta_P = \frac{1}{z}\overline{\partial} b_P \otimes \frac{\partial}{\partial z}$, where b_P is a bump function around P. Notice that if we choose b_P to be one in a neighborhood of P for this choice of local coordinate z, ξ_P depends only on the choice of z and in fact also formula (15) depends on z.

Using these results, in [12] we obtained a closed expression for the holomorphic sectional curvature of $\mathcal{T}_{M_g^{(n)}}$ at $[C] \in M_g^{(n)}$ along the tangent directions given by the Schiffer variations ξ_P. More precisely, the following holds.

THEOREM 1. *([12]) The holomorphic sectional curvature of $\mathcal{T}_{M_g^{(n)}}$ at $[C] \in M_g^{(n)}$ computed at the tangent vector ξ_P is given by*

$$\begin{aligned} H(\xi_P) &= \frac{1}{\langle \xi_P, \xi_P \rangle \langle \xi_P, \xi_P \rangle} \langle R(\xi_P), \xi_P \rangle (\xi_P, \overline{\xi_P}) \\ &= -1 - \frac{1}{64\pi^2 (\sum_j |f_j(P)|^2)^4} \sum_i |\gamma_C^2(Q_i)(P)|^2, \end{aligned}$$

where $\{Q_i\}$ is an orthonormal basis of I_2, $\{\omega_j\}$ is an orthonormal basis of $H^0(K_C)$ and in a local coordinate z around P, we have $\omega_j = f_j(z)dz$.

In the above formula, -1 is the value of the holomorphic sectional curvature of $A_g^{(n)}$ calculated along the tangent directions at $[C] \in M_g^{(n)}$ given by ξ_P, for all $P \in C$, while the term $-\frac{1}{64\pi^2(\sum_j |f_j(P)|^2)^4} \sum_i |\gamma_C^2(Q_i)(P)|^2$ represents the contribution given by the second fundamental form.

4. Second Gaussian map and curvature

We first recall some results on the second Gaussian map obtained in [10].

Assume that C is either a hyperelliptic curve of genus $g \geq 3$, or a trigonal curve of genus $g \geq 4$. Let $|F|$ denote the g_2^1 in the hyperelliptic case, the g_3^1 in the trigonal case. Let $\phi_F : C \to \mathbb{P}^1$ be the induced morphism and $\nu : \mathbb{P}^1 \hookrightarrow \mathbb{P}^{g-1}$ be the Veronese embedding, so that in the hyperelliptic case $\phi_K = \nu \circ \phi_F$, where ϕ_K is the canonical map. Observe that in the hyperelliptic case the hyperelliptic involution τ acts as $-Id$ on $H^0(K_C)$, so we have an exact sequence

$$0 \to I_2(K_C) \to S^2(H^0(K_C)) \to H^0(2K_C)^+ \to 0,$$

where $H^0(2K_C)^+$ denotes the τ-invariant part of $H^0(2K_C)$ whose dimension is $(2g-1)$ and $I_2(K_C)$ is the vector space of the quadrics containing the rational normal curve.

Set $L := K_C - F$, and fix a basis $\{x,y\}$ of $H^0(F)$, and a basis $\{t_1,\dots,t_r\}$ of $H^0(L)$ both in the hyperelliptic and in the trigonal case. We have a linear map

$$\psi : \Lambda^2(H^0(L)) \to I_2, \quad t_i \wedge t_j \mapsto Q_{ij} = xt_i \odot yt_j - xt_j \odot yt_i.$$

In both cases the linear map $\psi : \Lambda^2(H^0(L)) \to I_2$ is an isomorphism as can be easily checked or found in [1].

An easy computation in local coordinates shows that we have

$$\gamma_C^2(Q_{ij}) = \gamma_F^1(x \wedge y)\gamma_L^1(t_i \wedge t_j),$$

hence, for any quadric Q of rank four, we have

$$\gamma_C^2(Q) = \gamma_F^1(x \wedge y)\gamma_L^1(\psi^{-1}(Q)).$$

So, if we denote by $q_1,\dots,q_l$ the ramification points of either the g_2^1, or of the g_3^1, we see that the image of γ_C^2 is contained in $H^0(4K_C - (q_1 + \cdots + q_l))$ and $\mathrm{rank}(\gamma_C^2) = \mathrm{rank}(\gamma_L^1)$. In fact,

$$\mathrm{div}(\gamma_C^2(Q)) = \mathrm{div}(\gamma_F^1(x \wedge y)) + \mathrm{div}(\gamma_L^1(\psi^{-1}(Q))) = q_1 + \cdots + q_l + \mathrm{div}(\gamma_L^1(\psi^{-1}(Q)).$$

Therefore $\gamma_C^2(Q)(q_i) = 0$ for all $i = 1,\dots,l$.

Using the above observations and the computation of the rank of the first Gaussian map of the canonical linear series for hyperelliptic curves done in [9], in [10] we proved the following

PROPOSITION 1. *([10, Lem. 4.1, Prop. 4.2]) Let C be a hyperelliptic curve of genus $g \geq 3$. Then the rank of γ_C^2 is $2g-5$ and its image is contained in*

$$H^0(4K_C - (q_1 + \cdots + q_{2g+2})),$$

where $\{q_1,\dots,q_{2g+2}\}$ are the Weierstrass points.

Assume now that C is non-hyperelliptic trigonal curve of genus $g \geq 4$. Let $|F|$ be the g_3^1 on C, assume $F = p_1 + p_2 + p_3$, $p_i \in C$. Denote by $L = K_C - F = K_C - p_1 - p_2 - p_3$, $\deg(L) = 2g-5$, $h^0(L) = g-2$. Thus $H^0(L) \subset H^0(K_C)$ and $\gamma_L^1 = \gamma^1_{K|\Lambda^2 H^0(K_C - p_1 - p_2 - p_3)}$. In [9] it is proven that for the general trigonal curve of genus $g \geq 4$, $\dim(\mathrm{coker}(\gamma_K^1)) = g+5$, moreover specific examples of trigonal curves (whose genera are all equal to 1 modulo 3) such that the corank of γ_K^1 is $g+5$ are exhibited. Using results of [15], in [4] Brawner proved that $\dim(\mathrm{coker}(\gamma_K^1)) = g+5$ for any trigonal curve of genus $g \geq 4$.

In [10] we determined the rank of γ_C^2 for trigonal curves, computing the rank of γ_L^1 and generalizing the computation done in [15] and [4] for γ_K^1.

THEOREM 2. *([10]) For any trigonal non-hyperelliptic curve C of genus $g \geq 4$, the image of γ_C^2 is contained in $H^0(4K_C - (q_1 + \cdots + q_{2g+4}))$, where $q_1 + \cdots + q_{2g+4}$ is the ramification divisor of the g_3^1.*

If $g \geq 8$, the rank of γ_C^2 is $4g-18$.

We also recall

THEOREM 3. *([10]) Assume that C is smooth curve of genus $g \geq 5$, which is non-hyperelliptic and non-trigonal. Then for any $P \in C$ there exists a quadric $Q \in I_2$ such that $\gamma_C^2(P) \neq 0$. Equivalently, for all $P \in C$, $\mathrm{Im}(\gamma_C^2) \not\subset H^0(4K_C - P)$.*

Assume $[C] \in M_g^{(n)}$, with $g \geq 4$, C non-hyperelliptic. Then Theorem 1 allows us to define a function $F : C \to \mathbb{R}$, given by the holomorphic sectional curvature evaluated along the tangent vectors given by the Schiffer variations:

$$F(P) = H(\xi_P) = -1 - \frac{1}{64\pi^2(\sum_j |f_j(P)|^2)^4} \sum_i |\gamma_C^2(Q_i)(P)|^2 \leq -1,$$

where $\{Q_i\}$ is an orthonormal basis of $I_2(K_C)$, $\{\omega_j\}$ is an orthonormal basis of $H^0(K_C)$, and $\omega_j = f_j(z)dz$ is a local expression around P.

PROPOSITION 2. *([12])*

- *If $g = 4$, the set of points $P \in C$ such that $F(P) = -1$ is finite, which implies that F is non-constant.*
- *If $g \geq 5$ and C is neither hyperelliptic nor trigonal, then $F(P) < -1$ for all $P \in C$.*
- *If C is a trigonal curve of genus ≥ 4, $F(P) = H(\xi_P) = -1$ for every $P \in C$ which is a ramification point of the g_3^1, while there exist points $x \in C$ such that $F(x) < -1$, hence F is not constant.*

Proof. Assume C has genus 4, then the dimension of I_2 is one and I_2 can be generated by a quadric Q of rank 4 which has norm 1. So $F(P) = -1 - \frac{1}{64\pi^2(\sum_j |f_j(P)|^2)^4} |\gamma_C^2(Q)(P)|^2$ for all $P \in C$, hence there is a finite number of points P such that $\gamma_C^2(Q)(P) = 0$, so in these points we have $F(P) = -1$, while $F(P) < -1$ elsewhere .

As regards the second statement, we observe that $F(P) = -1$ if and only if $\gamma_C^2(Q_i)(P) = 0$ for all i, where $\{Q_i\}$ is an orthonormal basis of I_2. But then we must have $\gamma_C^2(Q)(P) = 0$ for all $Q \in I_2$. So the proof follows by Theorem 3.

The last statement follows from Theorem 2 and from the observation that if x is a point in C such that $\gamma_C^2(Q_1)(x) \neq 0$, we have $F(x) < -1$. □

REMARK 1. The previous statements imply that for any curve $C \in M_g^{(n)}$, not hyperelliptic, nor trigonal, for every point $P \in C$ the holomorphic sectional curvature of $M_g^{(n)}$, at C along the tangent directions given by ξ_P is strictly smaller than the holomorphic sectional curvature of $A_g^{(n)}$.

On the other hand, in the trigonal case, along the Schiffer variations at the ramification points of the g_3^1, (which are a basis of the tangent space to the trigonal locus) the holomorphic sectional curvature of $M_g^{(n)}$, coincides with the holomorphic sectional curvature of $A_g^{(n)}$.

5. The hyperelliptic locus

We will now explain some results obtained in [10] and [12] on the hyperelliptic locus $HE_g \subset M_g^{(n)}$. Recall that by local Torelli, the restriction of the period map to HE_g is an injective immersion (cf. [21]). Therefore we have the exact sequence

$$0 \to \mathcal{T}_{HE_g} \to \mathcal{T}_{A_g^{(n)}|HE_g} \to \mathcal{N}_{HE_g|A_g^{(n)}} \to 0,$$

and we denote by

$$\sigma_{HE} : \mathcal{T}_{HE_g} \to \mathrm{Hom}(\mathcal{T}_{HE_g}, \mathcal{N}_{HE_g|A_g^{(n)}})$$

the associated second fundamental form and by ρ_{HE} the second fundamental form of the dual exact sequence. At the point $[C] \in HE_g$ the dual exact sequence is

$$0 \to I_2 \to S^2(H^0(K_C)) \to H^0(2K_C)^+ \to 0,$$

where $H^0(2K_C)^+$ is the invariant part of $H^0(2K_C)$ under the hyperelliptic involution and I_2 is the vector space of the quadrics containing the rational normal curve, so that

$$\rho_{HE} : I_2 \to \mathrm{Hom}(\mathcal{T}_{HE_g,[C]}, H^0(2K_C)^+).$$

We recall that the set of Schiffer variations at the Weierstrass points P_i generates $\mathcal{T}_{HE_g,[C]}$.

In [12], we observed that we have a formula which is similar to (15) at a Weierstrass point $P \in C$:

$$\xi_P(\rho_{HE}(Q)(\xi_P)) = \gamma_C^2(Q)(P)$$

Let us denote by H_{HE} the holomorphic sectional curvature of $\mathcal{T}_{HE_g}$, if $[C] \in HE_g$ and $P \in C$ is a Weiestrass point, we have the same expression for $H_{HE}(\xi_P)$ as in Theorem 1, namely

$$H_{HE}(\xi_P) = -1 - \frac{1}{64\pi^2(\sum_j |f_j(P)|^2)^4} \sum_i |\gamma_C^2(Q_i)(P)|^2 \tag{16}$$

where $\{Q_i\}$ is an orthonormal basis of I_2 and $\{\omega_j =_{loc.} f_j(z)dz\}$ is an orthonormal basis of $H^0(K_C)$.

So, using Proposition 1, we have the following

COROLLARY 1. *([12]) Let $[C] \in HE_g$, then $H_{HE}(\xi_P) = -1$, for any Weierstrass point $P \in C$.*

Proof. The proof immediately follows from (16) and from Proposition 1. $\square$

6. Curves on K3 surfaces and results for the general curve

In this section we will explain some results of [11] on the second Gaussian map for curves on K3 surfaces, from which we have also deduced surjectivity of the 2nd Gaussian map for the general curve of sufficiently high genus. Then we will also discuss a theorem of [5] in which, with different methods, they prove that the second Gaussian map for the general curve C of any genus has maximal rank.

First of all let us recall that Wahl ([27]) has given a deformation theoretic interpretation of the first Gaussian map, showing that if a canonical curve can be extended in projective space as a hyperplane section of a surface which is not a cone, then the first Gaussian map is not surjective.

In particular in [27] it is proven that if a curve lies on a K3 surface, the first Gaussian map cannot be surjective (see also [3]).

The obstruction to the surjectivity of the first Gaussian map for a curve in a K3 surface is given by the extension class of the cotangent sequence

$$0 \to K_C^{-1} \to \Omega^1_{X|C} \to K_C \to 0, \tag{17}$$

which is a non-trivial element in the kernel of the dual of the first Gaussian map (see [3]).

To study the second Gaussian map γ_C^2 for a curve in a K3 surface X it is natural to consider the “symmetric square” of the cotangent extension

$$0 \to \Omega^1_{X|C} \otimes K_C^{-1} \to S^2\Omega^1_{X|C} \to K_C^2 \to 0. \tag{18}$$

This does not give any obstruction to the surjectivity of γ_C^2 for the general curve in a general K3 surface, while it gives an obstruction if C is any curve in an abelian surface, as it is proven in [13]. In fact in the next section we will discuss a result obtained in [13], that asserts that if C is a curve in an abelian surface X, then the corank of γ_C^2 is at least two.

The main result that we obtained for curves in K3 surfaces is the following

THEOREM 4. *([11]) If X is a general polarized K3 surface of degree $2g-2$ with $g > 280$ and C is a general hyperplane section of X, then γ_C^2 is surjective.*

Let us explain the strategy of the proof of Theorem 4. We have the following commutative diagram

$$\begin{tikzcd}
I_2(\mathcal{O}_X(C)) \arrow[r, "\gamma^2_{\mathcal{O}_X(C)}"] \arrow[dd, "r"] & H^0(S^2\Omega^1_X \otimes \mathcal{O}_X(2C)) \arrow[dr, "p_1"] & \\
& & H^0(S^2\Omega^1_{X|C} \otimes K_C^2) \arrow[dl, "p_2"] \\
I_2(K_C) \arrow[r, "\gamma_C^2"'] & H^0(K_C^4) &
\end{tikzcd} \tag{19}$$

where r and p_1 are restriction maps, and p_2 comes from the conormal extension. More precisely, consider the second symmetric square of the cotangent exact sequence (18) tensored by K_C^2:

$$0 \to \Omega^1_{X|C} \otimes K_C \to S^2\Omega^1_{X|C} \otimes K_C^2 \to K_C^4 \to 0, \tag{20}$$

then we have

$$H^0(S^2\Omega^1_{X|C} \otimes K_C^2) \xrightarrow{p_2} H^0(K_C^4) \to H^1(\Omega^1_{X|C} \otimes K_C) \cong H^0(T_{X|C})^*,$$

hence p_2 is surjective by the following lemma.

LEMMA 1. *If X is a general K3 surface and C a general curve of genus at least* 13 *in the very ample linear system $|\mathcal{O}_X(C)|$ then $H^0(T_{X|C}) = 0$.*

Proof. By the exact sequence given by restriction of T_X to C, $H^0(T_{X|C})$ injects in $H^1(T_X(-C))$, which vanishes by lemma (2.3) of [8]. □

The theorem follows if we prove that also the maps $\gamma^2_{\mathcal{O}_X(C)}$ and p_1 are surjective. In fact in [11] we exhibited examples of pairs (X,C) where X is a K3 and C is a very ample curve in X of any genus g sufficiently high ($g \geq 281$) for which $\gamma^2_{\mathcal{O}_X(C)}$ and p_1 are surjective.

To do this we followed the strategy used in [7] to study the first Wahl map. More precisely, from the exact sequence

$$\begin{aligned} 0 \to I^3_{\Delta_X} \otimes p^*(\mathcal{O}_X(C)) \otimes q^*(\mathcal{O}_X(C)) &\to I^2_{\Delta_X} \otimes p^*(\mathcal{O}_X(C)) \otimes q^*(\mathcal{O}_X(C)) \\ &\to I^2_{\Delta_X}/I^3_{\Delta_X} \otimes p^*(\mathcal{O}_X(C)) \otimes q^*(\mathcal{O}_X(C)) \to 0 \end{aligned} \tag{21}$$

and taking global sections, we see that $\gamma^2_{\mathcal{O}_X(C)}$ is surjective if $H^1(I^3_{\Delta_X} \otimes p^*(\mathcal{O}_X(C)) \otimes q^*(\mathcal{O}_X(C))) = 0$.

The idea used in [7] is to consider the blow-up Y of $X \times X$ along the diagonal Δ_X and to use Kawamata–Viehweg vanishing theorem ([18, 23]). Let E be the exceptional divisor and denote by $\pi : Y \to X \times X$ the natural morphism and by $f := p \circ \pi$, $g := q \circ \pi$. Then

$$\begin{aligned} H^1(I^3_{\Delta_X} \otimes p^*(\mathcal{O}_X(C)) \otimes q^*(\mathcal{O}_X(C))) &\cong H^1(Y, f^*(\mathcal{O}_X(C)) \otimes g^*(\mathcal{O}_X(C))(-3E)) \\ &\cong H^1(Y, f^*(\mathcal{O}_X(C)) \otimes g^*(\mathcal{O}_X(C)) \otimes K_Y(-4E)), \end{aligned}$$

since $K_Y = \mathcal{O}_Y(E)$. So by the Kawamata–Viehweg vanishing theorem, it suffices to prove that $f^*(\mathcal{O}_X(C)) \otimes g^*(\mathcal{O}_X(C))(-4E)$ is big and nef.

Now notice that if one decomposes $\mathcal{O}_X(C)$ as $\otimes_{i=1}^4 A_i$, where A_i are line bundles on X, then $L = \otimes_{i=1}^4 (f^*(A_i) \otimes g^*(A_i)(-E))$. To obtain that L is big and nef, we asked suitable conditions on the line bundles A_i, and we studied the sublinear system of $|f^*(A_i) \otimes g^*(A_i)(-E)|$ given by $\mathbb{P}(\Lambda^2(H^0(A_i)))$ (cf. lemma 3.3 of [11]).

Consider now the map

$$p_1 : H^0(S^2\Omega^1_X \otimes \mathcal{O}_X(2C)) \to H^0(S^2\Omega^1_{X|C} \otimes K_C^2).$$

Clearly p_1 is surjective if $H^1(S^2\Omega^1_X \otimes \mathcal{O}_X(C)) = 0$.

To prove this vanishing we observed that, given a decomposition of $\mathcal{O}_X(C)$ as $\mathcal{O}_X(D) \otimes \mathcal{O}_X(D')$, we have

$$H^1(S^2\Omega^1_X \otimes \mathcal{O}_X(C)) = H^1(X \times X, I^2_{\Delta_X}/I^3_{\Delta_X} \otimes p^*(\mathcal{O}_X(D)) \otimes q^*(\mathcal{O}_X(D'))),$$

hence its vanishing is implied by that of $H^1(X \times X, I^2_{\Delta_X} \otimes p^*(\mathcal{O}_X(D)) \otimes q^*(\mathcal{O}_X(D')))$ and of $H^2(X \times X, I^3_{\Delta_X} \otimes p^*(\mathcal{O}_X(D)) \otimes q^*(\mathcal{O}_X(D')))$.

So, with the same argument as above, it suffices to show that $f^*(\mathcal{O}_X(D)) \otimes g^*(\mathcal{O}_X(D'))(-4E)$ is big and nef. The strategy is now to choose $\mathcal{O}_X(D) = \otimes_{i=1}^4 A_i$ and $D' = D + B$ with B nef and effective, and take $C \in |2D + B|$.

The above decompositions are shown on concrete examples of K3 surfaces X and of curves C in X, which are explicitely constructed via their Picard lattices (cf. proposition 3.4 of [11]).

Using this result and some examples given in [10], we deduce the following

COROLLARY 2. *([11]) For the general curve of genus greater than* 152*, the second Gaussian map* γ^2_C *is surjective.*

Proof. By Theorem 4 and the semicontinuity of the corank of γ^2_C, for a general curve of genus greater than 280, γ^2_C is surjective. Surjectivity for the general curve of genus $153 \leq g \leq 280$ can be proved exhibiting examples of curves of genus g with surjective second Gaussian map, which are either hyperplane sections of a polarized K3 surface as in the proof of Theorem 4 given in [11], or lying in the product of two curves as in [10, theorem 3.1].

More precisely let C_1, C_2 be two smooth curves of respective genera g_1, g_2, choose divisors D_i on C_i of degree d_i, $i = 1,2$. Set $X = C_1 \times C_2$, let $C \in |{p_1}^*(D_1) \otimes {p_2}^*(D_2)|$ be a smooth curve, where p_i is the projection from $C_1 \times C_2$ on C_i, then $g(C) = 1 + (g_2 - 1)d_1 + (g_1 - 1)d_2 + d_1 d_2$.

In [10] we proved that if either $g_1, g_2 \geq 2$, $d_i \geq 2g_i + 5$, $i = 1,2$, or $g_1 \geq 2$, $g_2 = 1$, $d_1 \geq 2g_1 + 5$, $d_2 \geq 7$, or $g_2 = 0$, $d_2 \geq 7$, $d_2(g_1 - 1) > 2d_1 \geq 4g_1 + 10$, then γ^2_C is surjective.

Then one has to check directly that these values of $g(C)$ cover all the remaining integers between 153 and 280. □

Note that, for dimensional reasons, surjectivity can be expected for a general curve of genus at least 18, and in fact recently in [5] Calabri, Ciliberto and Miranda showed that for the general curve of genus $g \geq 18$, the second Gaussian map is surjective. More precisely they proved the following

THEOREM 5. *([5], Theorem 1). The second Gaussian map* $\gamma^2_C : I_2(K_C) \to$

$H^0(C,4K_C)$ for C a general curve of any genus g has maximal rank, namely it is injective for $g \leq 17$ and surjective for $g \geq 18$.

The proof they gave in [5] relies on the study of the limit of the second Gaussian map when the general curve of genus g degenerates to a general stable binary curve, i.e. the union of two rational curves meeting at $g+1$ points. The theorem then follows by upper semicontinuity. For such a stable binary curve C they explicitly write down the ideal $I_2(K_C)$ and they first describe the 2nd Gaussian map for C modulo torsion, then they deal with the torsion part. By direct computations performed with Maple, they verify the injectivity for a general binary curve of genus $g \leq 17$ and the surjectivity for $g = 18$. Finally they complete the argument by induction on g, for $g \geq 19$.

We have observed in the proof of Corollary 2 that examples of curves whose second Gaussian map is surjective were already given in [10] (for curves in the product of two curves) and we recall that other examples were given in [2] (for complete intersections). Notice that using complete intersections it is not possible to deduce surjectivity for the general curve of any sufficiently high genus, due to restrictions on the genus.

On the other hand, Theorem 4 shows that general curves on K3 surfaces of sufficiently high genus behave as general curves in the moduli space, with respect to the second Gaussian map.

Finally, we also observe that with the method used in [11] to prove Theorem 4 it is impossible to reach the optimal lower bound for the genus of the curve C, in fact the conditions that we gave on the line bundles A_i and the decomposition $\mathcal{O}_X(C) = \mathcal{O}_X(2D+B)$ force the genus of C to be high.

Moreover, the vanishing of $H^1(S^2\Omega^1_X \otimes \mathcal{O}_X(C))$ itself, already implies that the curve C must be of genus at least 31, as one can check looking at the restriction of $\Omega^1_X \otimes \Omega^1_X(C)$ to C and the induced cohomology exact sequence.

7. Curves on abelian surfaces

In this section we discuss some results obtained in collaboration with E. Colombo and G. Pareschi in [13] on the first and the second Gaussian map for curves on abelian surfaces.

The first result says that the first Gaussian map for a "sufficiently ample" curve in an abelian surface is surjective. In fact we have the following

THEOREM 6 ([13]). *Let C be a smooth irreducible curve contained in an abelian surface X. Assume that the first Gaussian map of the line bundle $\mathcal{O}_X(C)$ on the surface X is surjective and that the multiplication map*

$$\gamma^0_{X,C} : H^0(X,\mathcal{O}_X(C)) \otimes H^0(C,K_C) \to H^0(K_C^2)$$

is surjective (for example, both conditions hold if $\mathcal{O}_X(C)$ is at least a 5-th power of an ample line bundle on X, see [22]). Then the first Gaussian map of C is surjective.

REMARK 2. Recently in [19] it is shown that for a curve C of genus $g > 145$ sitting on a very general abelian surface X, the first Gaussian map of the line bundle $\mathcal{O}_X(C)$ on the surface X is surjective. Hence by Theorem 6 we obtain the surjectivity of the first Gaussian map of C, and therefore a new proof of the surjectivity of the map γ^1_C for the general curve of genus > 145.

The main theorem of [13] asserts that if a curve C is contained in an abelian surface, the second Gaussian map γ^2_C is not surjective. More precisely, let us introduce the following notation. Given a subspace $W \subset H^0(K_C)$, we will denote

$$S^2W \cdot H^0(K_C^2)$$

the image of $S^2W \otimes H^0(K_C^2)$ in $H^0(K_C^4)$ via the natural multiplication map. If W is 2-dimensional and base point free, the base-point-free pencil trick implies that $S^2W \cdot H^0(K_C^2)$ has codimension 2 in $H^0(K_C^4)$. If C is embedded in abelian surface, then $H^0(\Omega^1_X)$ is naturally a (base-point-free) 2-dimensional subspace of $H^0(K_C)$. We have the following

THEOREM 7. *([13]) Let C be a curve contained in abelian surface X. Then the image of the second Gaussian map γ^2_C is contained in $S^2H^0(\Omega^1_X) \cdot H^0(K_C^2)$ (notation as above). Therefore the corank of γ^2_C is at least* 2. *Moreover, if the second Gaussian map of the surface X is surjective, then the image of the map γ^2_C concides with $S^2H^0(\Omega^1_X) \cdot H^0(K_C^2)$.*

The above theorem can also be stated as follows. Given a subspace $V \subset H^1(\mathcal{O}_C)$, let $\overline{V} \subset H^0(K_C)$ be its conjugate.

COROLLARY 3. *([13]) Let $C \subset \mathbb{P}^{g-3}$ be a canonically embedded curve of genus g, obtained from the complete canonical embedding $C \subset \mathbb{P}H^1(\mathcal{O}_C) = \mathbb{P}^{g-1}$ by projection from a line $\mathbb{P}V \subset \mathbb{P}H^1(\mathcal{O}_C)$, $\dim V = 2$. If C is a hyperplane section of an abelian surface $X \subset \mathbb{P}^{g-2}$ then*

$$\mathrm{Im}(\gamma^2_C) \subseteq S^2\overline{V} \cdot H^0(K_C^2).$$

The proofs of the above results rely on cohomological methods, which are similar to the ones used in [3].

Let us sketch the proof of Theorem 7.

First of all let us see the Gaussian maps defined as the H^0 of maps of coherent sheaves on the variety Y. This is achieved as follows: let p and q be the two projections of $Y \times Y$. Applying p_* to the exact sequences (2) tensored by a line bundle M one gets the exact sequences

$$0 \to p_*(I^{k+1}_{\Delta_Y} \otimes q^*M) \to p_*(I^k_{\Delta_Y} \otimes q^*M) \xrightarrow{\varphi^k} S^k\Omega^1_Y \otimes M. \tag{22}$$

The Gaussian maps $\gamma^k_{L,M}$ of (1) are obtained by tensoring with L and taking $H^0(L \otimes \varphi^k)$.

Let us spell out how the Gaussian maps γ^1_C look like in this setting. Let R_C be the kernel of the evaluation map of K_C:

$$0 \to R_C \xrightarrow{f} H^0(K_C) \otimes \mathcal{O}_C \to K_C \to 0 \tag{23}$$

(i.e. sequence (22) for $Y = C$, $M = K_C$, $k = 0$). By (22) (same setting) for $k = 1$ we have the natural map

$$R_C \xrightarrow{g} K_C^2. \tag{24}$$

Tensoring with K_C and taking H^0 one obtains the first Wahl map

$$\gamma_C^1 : H^0(R_C \otimes K_C) \to H^0(K_C^3). \tag{25}$$

One has the exact sequence

$$0 \to I_{\Delta_C}^2 \to \mathcal{O}_{C\times C} \to \mathcal{O}_{\Delta_C^2} \to 0 \tag{26}$$

where Δ_C^2 denotes the first infinitesimal neighborhood.

Tensoring sequence (26) with q^*K_C and applying p_*, where p and q are the two projections from $C \times C$, one gets the exact sequence

$$0 \to R_C^2 \xrightarrow{f'} H^0(K_C) \otimes \mathcal{O}_C \xrightarrow{ev} P_C(K_C) \tag{27}$$

where

$$R_C^2 = p_*(q^*(K_C) \otimes I_{\Delta_C}^2), \tag{28}$$

and

$$P_C(K_C) = p_*(q^*(K_C) \otimes \mathcal{O}_{\Delta_C^2})$$

is the bundle of principal parts of K_C.

REMARK 3. We have the commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & R_C^2 \arrow[r, "="] \arrow[d] & R_C^2 \arrow[d] & & \\
0 \arrow[r] & R_C \arrow[r] \arrow[d, "g"] & H^0(K_C) \otimes \mathcal{O}_C \arrow[r] \arrow[d, "ev"] & K_C \arrow[r] \arrow[d, "="] & 0 \\
0 \arrow[r] & K_C^2 \arrow[r] & P_C(K_C) \arrow[r] & K_C \arrow[r] & 0
\end{tikzcd}.$$

Notice that the map g of (24) is surjective if and only if the evaluation map ev is surjective. This is in turn equivalent to the immersivity of the canonical map, which holds if and only if C is non-hyperelliptic.

Sequence (22) for $k=2$, $Y=C$ and $M=K_C$ provides the natural map

$$R_C^2 \xrightarrow{g'} K_C^3. \tag{29}$$

Tensoring with K_C and taking H^0 one obtains the second Wahl map

$$\gamma_C^2 : H^0(R_C^2 \otimes K_C) \to H^0(K_C^4), \tag{30}$$

The coboundary map of the exact sequence (20),

$$f_{e'} : H^0(K_C^4) \to H^1(\Omega_X^1 \otimes K_C) \cong H^1(K_C)^{\oplus 2} \cong \mathbb{C}^{\oplus 2}$$

is identified, by Serre duality, to the extension class $e' \in \mathrm{Ext}^1(K_C^3, \Omega^1_{X|C})$ of sequence (18). The first part of the statement of Theorem 7 is equivalent to:

$$f_{e'} \circ \gamma_C^2 = 0. \tag{31}$$

Let us work in the dual setting.

Applying $\mathrm{Ext}^1(\cdot, \Omega^1_{X|C})$ to the map g' of (29) one gets the map

$$\phi : \mathrm{Ext}^1(K_C^3, \Omega^1_{X|C}) \to \mathrm{Ext}^1(R_C^2, \Omega^1_{X|C})$$

(which is identified to two copies of the dual map of the second Wahl map) and it is easily seen that (31) is equivalent to the fact that

$$\phi(e') = 0. \tag{32}$$

Applying $\mathrm{Ext}^1(\cdot, \Omega^1_{X|C})$ to the map f' of (27) we get the map

$$\psi : \mathrm{Hom}(H^0(K_C), H^1(\Omega^1_{X|C})) = \mathrm{Ext}^1(H^0(K_C) \otimes \mathcal{O}_C, \Omega^1_{X|C}) \to \mathrm{Ext}^1(R_C^2, \Omega^1_{X|C}).$$

Now let us denote by $\tilde{\delta}$ the composition of the coboundary map $H^0(K_C) \to H^1(\mathcal{O}_X)$ of the standard exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(C) \to K_C \to 0,$$

and the map $H^1(d) : H^1(\mathcal{O}_X) \to H^1(\Omega^1_{X|C})$, induced by the derivation $d : \mathcal{O}_X \to \Omega^1_X$. Note that $H^1(d)$ is the zero map since the Hodge–Frölicher spectral sequence degenerates at level of E_1. Hence the map $\tilde{\delta}$ is equal to zero.

To conclude the proof of the first part of Theorem 7, we showed the following result, whose proof can be found in [13].

CLAIM. $\phi(e') = \psi(\tilde{\delta}) = 0$.

The last part of the statement follows from the commutative diagram (constructed as diagram (19))

$$
\begin{tikzcd}
I_2(\mathcal{O}_X(C)) \arrow[r, "\gamma^2_{\mathcal{O}_X(C)}"] \arrow[dd] & H^0(S^2\Omega^1_X \otimes \mathcal{O}_X(2C)) \arrow[dr] & \\
& & H^0(S^2\Omega^1_{X|C} \otimes K_C^2) \arrow[dl] \\
I_2(K_C) \arrow[r, "\gamma^2_C"'] & H^0(K_C^4) &
\end{tikzcd}
\tag{33}
$$

This completes the proof of Theorem 7.

Finally, in [22, Th. 2.2] it is shown that, if $\mathcal{O}_X(C)$ is at least a 7-power of a (necessarily ample) line bundle on X, then the second Gaussian map $\gamma^2_{\mathcal{O}_X(C)}$ is surjective. Hence we have the following

COROLLARY 4. *([13]) Let X be an abelian surface, let $\mathcal{L}$ be an ample line bundle on X and let $k \geq 7$. Then, for every smooth and irreducible curve $C \in |\mathcal{L}^k|$, the image of second Wahl map*

$$\gamma^2_C : I_2(K_C) \to H^0(K_C^4)$$

is the 2-codimensional subspace $S^2H^0(\Omega^1_X) \cdot H^0(K_C^2)$.

REMARK 4.

1. Using Proposition 3.2 of [19] as in Remark 2, one can prove that for a curve sitting on a very general abelian surface of genus $g > 257$ the second Gaussian map $\gamma^2_{\mathcal{O}_X(C)}$ is surjective, hence the image of second Wahl map $\gamma^2_C : I_2(K_C) \to H^0(K_C^4)$ is the 2-codimensional subspace $S^2H^0(\Omega^1_X) \cdot H^0(K_C^2)$.
2. Assume that $X = E_1 \times E_2$, with E_i elliptic curves, let $p_i : X \to E_i$ be the projection maps, D_i divisors of degree d_i on E_i. As in theorem 3.1 of [10], one shows that if $C \in |p_1^*(D_1) \otimes p_2^*(D_2)|$, and $d_i \geq 7$, then $\gamma^2_{\mathcal{O}_X(C)}$ is surjective, hence also in this case the image of γ^2_C is the 2-codimensional subspace $S^2H^0(\Omega^1_X) \cdot H^0(K_C^2)$.

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