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SIMPLE FINITE GROUP SCHEMES AND THEIR INFINITESIMAL DEFORMATIONS

Abstract. We show that the classification of simple finite group schemes over an algebraically closed field reduces to the classification of abstract simple finite groups and of simple restricted Lie algebras in positive characteristic. Both these simple objects have been classified. We review this classification. Finally, we address the problem of determining the infinitesimal deformations of simple finite group schemes.

1. Introduction

In the first part of this paper, we show that a *simple finite group scheme* over an algebraically closed field can be of two types: either the constant group scheme associated to a simple (abstract) finite group or the group scheme of height one associated to a simple restricted Lie algebra. The classification of these two kinds of simple objects (the simple Lie algebras have been classified only for $p \neq 2,3$) was certainly among the greatest achievements of the mathematics of the last century.

Simple finite groups were classified during the years 1955–1985 thanks to the contribution of many mathematicians (see [26, 27, 2] for a nice historical account).

Simple Lie algebras over an algebraically closed field F of characteristic $p \neq 2,3$ have recently been classified by Block–Wilson–Premet–Strade (see [5, 29, 28]). The classification says that for $p \geq 7$ the simple Lie algebras can be of two types: of classical type and of generalized Cartan type. The algebras of *classical type* are obtained by considering the simple Lie algebras in characteristic zero, by taking a model over the integers and then reducing modulo the prime p. The algebras of *generalized Cartan type* are the finite-dimensional analogues of the four classes of infinite-dimensional complex simple Lie algebras, which occurred in Cartan's classification of Lie pseudogroups. In characteristic $p = 5$, apart from the above two types of algebras, there is one more family of simple Lie algebras called Melikian algebras. In characteristic $p = 2,3$, there are many exceptional simple restricted Lie algebras and the classification seems still far away.

After passing in review these two classification results, in the last part of this paper we address the problem of determining the *infinitesimal deformations* of such simple finite group schemes. The group schemes associated to simple finite (abstract) groups and to simple Lie algebras of classical type are known to be rigid apart from some bad characteristic of the base field. We show that this is not the case for the group schemes associated to simple Lie algebras of Cartan type. In particular we determine the infinitesimal deformations of the group schemes of height one associated to the

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restricted simple Lie algebras of Cartan type. We do this by computing the second restricted cohomology group of these algebras with values in the adjoint representation. It remains an open problem to extend the above results to the other simple restricted Lie algebras.

2. Finite group schemes

Let k be a field of characteristic $p \geq 0$. There are three equivalent ways to define a finite group scheme over k.

DEFINITION 1. *A finite group scheme G over k can be defined equivalently as*

(i) a finite scheme G which is a group object in the category of schemes;

(ii) a scheme of the form $G = \mathrm{Spec}(A)$, where A is a finite dimensional commutative k-Hopf algebra;

(iii) a scheme G whose functor of points F_G takes values in finite groups, that is

$$\begin{aligned} F_G : \{k-\text{schemes}\} &\to \{\text{finite groups}\} \\ S &\mapsto G(S) := \mathrm{Hom}_k(S,G). \end{aligned}$$

The following result collects the basic structure of finite group schemes.

THEOREM 1. *Let G be a finite group scheme over the field k.*

(1) If $\mathrm{char}(k) = 0$ then G is étale.

(2) If $\mathrm{char}(k) = p > 0$ then there is a unique extension

$$1 \to G^0 \to G \to G^{et} \to 1$$

such that G^0 is connected and G^{et} is étale. If k is perfect then the above exact sequence splits, that is $G = G^0 \rtimes G^{et}$.

(3) If $\mathrm{char}(k) = p > 0$ and G is connected, there is a minimal natural number $n \geq 1$ such that $G = \mathrm{Ker}(F^n)$ (it is called the height of G), where $F^n : G \to G^{(p^n)}$ is the the n-th iteration of the Frobenius map. In the filtration of normal closed subgroups:

$$1 \lhd \mathrm{Ker}(F) \lhd \mathrm{Ker}(F^2) \lhd \cdots \lhd \mathrm{Ker}(F^n) = G^0,$$

each factor $\mathrm{Ker}(F^{i+1})/\mathrm{Ker}(F^i)$ has height one (i.e. it has vanishing Frobenius).

Proof. (1) By a result of Cartier, every k-group scheme is smooth if $\mathrm{char}(k) = 0$ (see [34, Chap. 11.4]). Therefore every finite group scheme in characteristic zero is étale.

(2) The above exact sequence is obtained by taking G^0 the connected component of G containing the identity and G^{et} to be $\mathrm{Spec}(\pi_0(k[G]))$, where $\pi_0(k[G])$ is the maximal separable sub-Hopf algebra of the algebra $k[G]$ of regular functions on G. If k is perfect, then $G^{et} \cong \mathrm{Spec}(k[G]_{red})$, where $k[G]_{red}$ is the maximal reduced quotient of $k[G]$, which gives the splitting (see [34, Chap. 6]).

(3) The kernel of F^n is represented by the Hopf algebra $k[G]/(x^{p^n} \mid x \in I)$, where I is the augmentation ideal of $k[G]$ (that is, the maximal ideal corresponding to the origin of G). The first assertion follows from the fact that if G is connected then the augmentation ideal I is nilpotent. The second assertion is clear.

□

Therefore every finite group scheme can be realized as an extension of étale and height one finite group schemes (these latter occur only if $\mathrm{char}(k) = p > 0$). Now we provide an explicit description of these two building blocks.

THEOREM 2. *If k is algebraically closed the we have a bijection*

$$\{\textit{Étale } k\textit{-group schemes}\} \longleftrightarrow \{\textit{Finite (abstract) groups}\}.$$

Proof. The bijection is realized explicitly as follows: to an étale k-group scheme G one associates the finite group $G(k)$ of its k-points. Conversely, to an abstract finite group Γ one associates the finite group scheme whose k-Hopf algebra is the k-algebra k^Γ of functions from Γ to k with comultiplication given by $\Delta(e_\rho) = \bigoplus_{\rho=\sigma\cdot\tau} e_\sigma \otimes e_\tau$, where e_ρ is the function sending $\rho \in \Gamma$ into 1 and the other elements of Γ to 0. One can check the above maps are one the inverse of the other (see [34, Chapter 6.4]). □

In order to describe the finite group schemes of height one, we need to recall the definition of the restricted Lie algebras (sometimes called p-Lie algebras) over a field k of positive characteristic.

DEFINITION 2 ([17]). *A Lie algebra L over a field k of characteristic $p > 0$ is said to be* restricted *(or a p-Lie algebra) if it is endowed with a map (called p-map) $[p] : L \to L$, $x \mapsto x^{[p]}$, which satisfies the following conditions:*

(i) $\mathrm{ad}(x^{[p]}) = \mathrm{ad}(x)^{[p]}$ *for every $x \in L$.*

(ii) $(\alpha x)^{[p]} = \alpha^p x^{[p]}$ *for every $x \in L$ and every $\alpha \in k$.*

(iii) $(x_0 + x_1)^{[p]} = x_0^{[p]} + x_1^{[p]} + \sum_{i=1}^{p-1} s_i(x_0, x_1)$ *for every $x_0, x_1 \in L$, where the element $s_i(x_0, x_1) \in L$ is defined by*

$$s_i(x_0,x_1) = -\frac{1}{i}\sum_u \mathrm{ad}x_{u(1)} \circ \mathrm{ad}x_{u(2)} \circ \cdots \circ \mathrm{ad}x_{u(p-1)}(x_1),$$

the summation being over all the maps $u : [1, \cdots, p-1] \to \{0,1\}$ taking i times the value 0.

The last two conditions in the above definition can be rephrased by saying that the map $a \mapsto a^p - a^{[p]}$ from L into the universal enveloping algebra $\mathfrak{U}_L$ is p-semilinear, where a^p denotes the p-th self product of a in $\mathfrak{U}_L$. We give examples of restricted Lie algebras.

EXAMPLES 1.

(i) Let A an associative k-algebra, where $\mathrm{char}(k) = p > 0$. Then the Lie algebra $\mathrm{Der}_F A$ of k-derivations of A into itself is a restricted Lie algebra with bracket $[D_1, D_2] = D_1 \circ D_2 - D_2 \circ D_1$ and p-map $D \mapsto D^p := D \circ \cdots \circ D$.

(ii) Let G a group scheme over k, where $\mathrm{char}(k) = p > 0$. Then the Lie algebra $\mathrm{Lie}(G)$ associated to G is a restricted Lie algebra with respect to the p-map given by the differential of the homomorphism $G \to G$, $x \mapsto x^p := x \circ \cdots \circ x$.

The Lie algebra $\mathrm{Lie}(G)$ associated to a k-group scheme G in positive characteristic p (as in the above example) depends only on the first height truncation $\mathrm{Ker}(F) \lhd G$ of G. Indeed, we have the following

THEOREM 3. *Let k be a field of characteristic* $p > 0$. *Then there is a bijection*

$$\{\text{Restricted } k\text{-Lie algebras}\} \longleftrightarrow \{\text{Finite } k\text{-group schemes of height one}\}.$$

Proof. The bijection is realized explicitly as follows: to a finite group scheme G of height 1, one associates the restricted Lie algebra $\mathrm{Lie}(G)$ with p-map given by the differential of the map $x \mapsto x^p$. Conversely, to a restricted Lie algebra $(L, [p])$, one associates the finite group scheme corresponding to the dual of the restricted enveloping Hopf algebra $\mathfrak{U}^{[p]}(L) := \mathfrak{U}(L)/(x^p - x^{[p]})$. One can check that the above map are inverse one of the other (see [11, Chapter 2.7]). $\square$

As a consequence of the structure theorems for finite group schemes discussed in this section, we get

COROLLARY 1. *A simple finite group scheme over an algebraically closed field k is either the étale group scheme associated to a simple (abstract) finite group or (if* $\mathrm{char}(k) = p > 0$*) the height one group scheme associated to a simple restricted Lie algebra.*

Both these simple objects have been classified (the last ones assuming that $p \neq 2, 3$)!

3. Classification of simple finite groups

The simple finite groups are divided in four families:

- Cyclic groups $\mathbb{Z}/p\mathbb{Z}$ with prime order;
- Alternating groups A_n, $n \geq 5$;
- Simple groups of Lie type, including

– Classical groups of Lie type:

(1) $A_n(q)$: the Projective Special Linear Group $\mathrm{PSL}_{n+1}(\mathbb{F}_q)$.

(2) ${}^2A_n(q^2)$: the Projective Special Unitary Group $\mathrm{PSU}_{n+1}(\mathbb{F}_{q^2})$, with respect to the Hermitian form on $\mathbb{F}_{q^2}^{n+1}$

$$\Psi(w,v) = \sum_{i=1}^{n+1} w_i^q v_i,$$

where $w = (w_1, \cdots, w_{n+1})$ and $v = (v_1, \cdots, v_{n+1})$.

(3) $B_n(q)$: the subgroup $O_{2n+1}(\mathbb{F}_q)$ of the special orthogonal group in dimension $2n+1$ formed by the elements having spinor norm 1.

(4) $C_n(q)$: the Projective Symplectic group $\mathrm{PSp}_{2n}(\mathbb{F}_q)$.

(5) $D_n(q)$: the subgroup $O_{2n}^{+}(\mathbb{F}_q)$ of the projective special split orthogonal group in dimension $2n$ formed by the elements having unit spinor norm.

(6) ${}^2D_n(q)$: the subgroup $O_{2n}^{-}(\mathbb{F}_{q^2})$ of elements of spinor norm 1 in the projective special non-split orthogonal group in dimension $2n$.

– Exceptional and twisted groups of Lie type, obtained via the Steinberg construction, starting with an automorphism of a Dynkin diagram and an automorphism of a finite field. The resulting groups are denoted: ${}^2B_2(2^{2n+1})$, ${}^3D_4(q^3)$, $E_6(q)$, ${}^2E_6(q^2)$, $E_7(q)$, $E_8(q)$, $F_4(q)$, ${}^2F_4(2^{2n+1})$, $G_2(q)$, ${}^2G_2(3^{2n+1})$ (see [7]).

- 26 sporadic groups (see [35]).

4. Classification of simple restricted Lie algebras

First of all, we show that there is a bijection between simple restricted Lie algebras, that is restricted Lie algebras without restricted ideals (i.e. ideals closed under the p-map), and simple Lie algebras (not necessarily restricted).

THEOREM 4. *There is a bijection*

$$\{\textit{Simple restricted Lie algebras}\} \longleftrightarrow \{\textit{Simple Lie algebras}\}.$$

Explicitly, to a simple restricted Lie algebra $(L,[p])$ we associate its derived algebra $[L,L]$. Conversely, to a simple Lie algebra M we associate the restricted subalgebra $M^{[p]}$ of $\mathrm{Der}_F(M)$ generated by $\mathrm{ad}(M)$ (which is called the universal p-envelope of M).

Proof. We have to prove that the above maps are well-defined and are inverse one of the other.

• Consider a simple restricted Lie algebra $(L,[p])$. The derived subalgebra $[L,L] \lhd L$ is a non-zero ideal (since L is not abelian) and therefore $[L,L]_p = L$, where $[L,L]_p$ denotes the p-closure of $[L,L]$ inside L.

Take a non-zero ideal $0 \neq I \lhd [L,L]$. Since $[L,L]_p = L$, we deduce from [13, Chapter 2, Prop. 1.3] that I is also an ideal of L and therefore $I_p = L$ by the restricted simplicity of $(L,[p])$. From loc. cit., it follows also that $[L,L] = [I_p, I_p] = [I,I] \subset I$ from which we deduce that $I = L$. Therefore $[L,L]$ is simple.

Since $\mathrm{ad} : L \to \mathrm{Der}_F(L)$ is injective and $[L,L]_p = L$, it follows by loc. cit. that $\mathrm{ad} : L \to \mathrm{Der}_F([L,L])$ is injective. Therefore we have that $[L,L] \subset L \subset \mathrm{Der}_F([L,L])$ and hence $[L,L]^{[p]} = [L,L]_p = L$.

• Conversely, start with a simple Lie algebra M and consider its universal p-envelop $M < M^{[p]} < \mathrm{Der}_F(M)$.

Take any restricted ideal $I \lhd_p M^{[p]}$. By loc. cit., we deduce $[I, M^{[p]}] \subset I \cap [M^{[p]}, M^{[p]}] = I \cap [M,M] = I \cap M \lhd M$. Therefore, by the simplicity of M, either $I \cap M = M$ or $I \cap M = 0$. In the first case, we have that $M \subset I$ and therefore $M^{[p]} = I$. In the second case, we have that $[I, M^{[p]}] = 0$ and therefore $I = 0$ because $M^{[p]}$ has trivial center. We conclude that $M^{[p]}$ is simple restricted.

Moreover, by loc. cit., we have that $[M^{[p]}, M^{[p]}] = [M,M] = M$. □

Note that the intersection of the two types of Lie algebras appearing in the above correspondence is the set of restricted simple Lie algebras, namely those restricted Lie algebra which does not have any proper ideal.

Simple Lie algebras (and their minimal p-envelopes) over $k = \overline{k}$ of $\mathrm{char}(k) = p > 3$ have been classified by Block–Wilson–Premet–Strade (1984–2005), answering to a conjecture of Kostrikin–Shafarevich (1966). They are divided into two types:

- Lie algebras of *classical type*;
- Lie algebras of *(generalized) Cartan type*.

4.1. Lie algebras of classical type

The Lie algebras of classical type are reduction of simple Lie algebras in characteristic zero. Simple Lie algebras over an algebraically closed field of *characteristic zero* were classified at the beginning of the 19th century by Killing and Cartan. The classification proceeds as follows: first the non-degeneracy of the Killing form is used to establish a correspondence between simple Lie algebras and irreducible root systems and then the irreducible root systems are classified by mean of their associated Dynkin diagrams. Explicitly:

DYNKIN DIAGRAMS $\longleftrightarrow$	SIMPLE LIE ALGEBRAS
$A_n\,(n \geq 1)$	$\mathfrak{sl}(n+1)$
$B_n\,(n \geq 2)$	$\mathfrak{so}(2n+1)$
$C_n\,(n \geq 3)$	$\mathfrak{sp}(2n)$
$D_n\,(n \geq 4)$	$\mathfrak{so}(2n)$
E_6, E_7, E_8, F_4, G_2	Exceptional Lie algebras

where $\mathfrak{sl}(n+1)$ is the special linear algebra, $\mathfrak{so}(2n+1)$ is the special orthogonal algebra of odd rank, $\mathfrak{sp}(2n)$ is the symplectic algebra and $\mathfrak{so}(2n)$ is the special orthogonal algebra of even rank. For the simple Lie algebras corresponding to the exceptional Dynkin diagrams, see the book [19] or the nice account in [3].

These simple Lie algebras admit a model over the integers via the (so-called) Chevalley bases. Thus, via reduction modulo a prime p, one obtains a restricted Lie algebra over $\mathbb{F}_p$, which is simple up to a quotient by a small ideal. For example $\mathfrak{sl}(n)$ is not simple if p divides n, but its quotient $\mathfrak{psl}(n) = \mathfrak{sl}(n)/(I_n)$ by the unit matrix I_n becomes simple. There are similar phenomena occurring only for $p = 2,3$ for the other Lie algebras (see [28, Page 209]). The restricted simple algebras obtained in this way are called algebras of *classical type*. Their Killing form is non-degenerate except at a finite number of primes. Moreover, they can be characterized as those restricted simple Lie algebras admitting a projective representation with nondegenerate trace form (see [4]).

4.2. Lie algebras of Cartan type

However, there are restricted simple Lie algebras which have no analogous in characteristic zero and therefore are called non-classical. The first example of a non-classical restricted simple Lie algebra is due to E. Witt, who in 1937 realized that the derivation algebra $W(1) := \mathrm{Der}_k(k[X]/(X^p))$ over a field k of characteristic $p > 3$ is simple with a degenerate Killing form. In the succeeding three decades, many more non-classical restricted simple Lie algebras have been found (see [18, 14, 1, 15]). The first comprehensive conceptual approach to constructing these non-classical restricted simple Lie algebras was proposed by Kostrikin–Shafarevich and Kac (see [22, 23, 21]). They showed that all the known examples can be constructed as finite-dimensional analogues of the four classes of infinite-dimensional complex simple Lie algebras, which occurred in Cartan's classification of Lie pseudogroups (see [6]).

Denote by $\mathcal{O}(m)$ the divided power k-algebra in m-variables. It is the commutative and associative algebra with unit defined by the generators x^α for $\alpha \in \mathbb{N}^m$ satisfying the relations

$$x^\alpha \cdot x^\beta = \binom{\alpha+\beta}{\alpha} x^{\alpha+\beta} := \prod_{i=1}^{m} \binom{\alpha_i+\beta_i}{\alpha_i} x^{\alpha+\beta}.$$

For any m-tuple $\underline{n} \in \mathbb{N}^m$, we define the truncated subalgebra of $\mathcal{O}(m)$

$$\mathcal{O}(m;\underline{n}) = \mathrm{span}\langle x^\alpha \,|\, 0 \le \alpha_i < p^{n_i} \rangle.$$

The simple Lie algebras of Cartan type (over an algebraically closed field k of characteristic $p > 3$) are divided in four families, called Witt–Jacobson, Special, Hamiltonian and Contact algebras, plus an exceptional family of Lie algebras in characteristic $p = 5$, called Melikian algebras. We list the simple *graded* Lie algebras of Cartan type. The general simple Lie algebras of Cartan type are filtered deformations of these graded Lie algebras (see [28, Chap. 6]).

(1) WITT–JACOBSON: The Witt-Jacobson Lie algebra $W(m;\underline{n})$ is the subalgebra of $\mathrm{Der}_k\mathcal{O}(m;\underline{n})$ of special derivations:

$$W(m;\underline{n}) = \{D \in \mathrm{Der}_k\mathcal{O}(m;\underline{n}) \, : D(x^{(a)}) = \sum_{i=1}^{m} x^{(a-\varepsilon_i)} D(x_i)\}.$$

The Witt-Jacobson Lie algebra is a free $\mathcal{O}(m,\underline{n})$ generated by the special derivations ∂_i defined by $\partial_i(x^{(a)}) = x^{(a-\varepsilon_i)}$.

(2) SPECIAL: The Special Lie algebra is the subalgebra of $W(m,\underline{n})$, $m \geq 3$, of derivations preserving the special form $\omega_S = dx_1 \wedge \cdots \wedge dx_m$:

$$S(m;\underline{n})^{(1)} = \{D \in W(m;\underline{n}) \,|\, D(\omega_S) = 0\}^{(1)},$$

where (1) denotes the derived algebra.

(3) HAMILTONIAN: The Hamiltonian algebra is the subalgebra of $W(2r,\underline{n})$ of derivations preserving the Hamiltonian form $\omega_H = dx_1 \wedge dx_{r+1} + \cdots + dx_r \wedge dx_{2r}$:

$$H(2r;\underline{n})^{(2)} = \{D \in W(2r;\underline{n}) \,|\, D(\omega_H) = 0\}^{(2)},$$

where (2) denotes the double derived algebra.

(4) CONTACT: The Contact algebra is the subalgebra of $W(2r+1,\underline{n})$ of derivations preserving the contact form $\omega_K = \sum_{i=1}^{r}(x_i dx_{i+r} - x_{i+r}dx_i) + dx_{2r+1}$ up to multiples:

$$K(2r+1;\underline{n})^{(1)} = \{D \in W(2r+1;\underline{n}) \,|\, D(\omega_K) \in \mathcal{O}(2r+1;\underline{n})\omega_K\}^{(1)}.$$

(5) MELIKIAN (only if $\mathrm{char}(k) = 5$): The Melikian algebra (introduced in [24]) is defined as

$$M(n_1,n_2) = \mathcal{O}(2,(n_1,n_2)) \oplus W(2,(n_1,n_2)) \oplus W(\widetilde{2,(n_1,n_2)}),$$

where $W(\widetilde{2,(n_1,n_2)})$ is a copy of $W(2,(n_1,n_2))$ and the Lie bracket is defined by the following rules (for all $D,E \in W(2,(n_1,n_2))$ and $f,g \in \mathcal{O}(2,(n_1,n_2))$):

$$\begin{cases} [D,E] := [D;E], \\ [D,\widetilde{E}] := \widetilde{[D,E]} + 2\,\mathrm{div}(D)\widetilde{E}, \\ [D,f] := D(f) - 2\,\mathrm{div}(D) f, \\ [f_1\widetilde{D_1} + f_2\widetilde{D_2}, g_1\widetilde{D_1} + g_2\widetilde{D_2}] := f_1g_2 - f_2g_1, \\ [f,\widetilde{E}] := fE, \\ [f,g] := 2\,(gD_2(f) - fD_2(g))\widetilde{D_1} + 2\,(fD_1(g) - gD_1(f))\widetilde{D_2}, \end{cases}$$

where $\mathrm{div}(f_1D_1 + f_2D_2) := D_1(f_1) + D_2(f_2) \in \mathcal{O}(2,(n_1,n_2))$.

The above algebras are restricted if and only if $\underline{n} = \underline{1}$. In general, denoting with $X(m,\underline{n})$ one of the above simple Lie algebras, its minimal p-envelope $X(m,\underline{n})_{[p]}$ inside its derivation algebra $\mathrm{Der}_k(X(m,\underline{n}))$ is equal to (see [28, Theo. 7.2.2]):

$$X(m;\underline{n})_{[p]} = X(m;\underline{n}) + \sum_{i=1}^{m} \sum_{0<j_i<n_i} k\cdot \partial_i^{p^{j_i}}.$$

In this way we obtain all the simple restricted Lie algebras, up to filtered deformations. We observe that in each of the above simple Lie algebras of Cartan type, the Killing form is always degenerate.

5. Deformations of simple finite group schemes

Having at our hand a complete classification of the simple finite group schemes, it is natural to study their properties. Here we are interested in their deformations. According to Grothendieck's philosophy, one should first understand the infinitesimal deformations. It is a classical result (of difficult attribution) that the infinitesimal deformations of the finite group scheme G are parametrized by $H^2(G,\mathfrak{g})$, the second cohomology group of G with values in the adjoint representation $\mathfrak{g} = \mathrm{Lie}(G)$.

5.1. Deformations of simple (abstract) groups

From Maschke's theorem, it follows that every constant group scheme is rigid over a field of characteristic not dividing the order of the group. In particular this applies to the finite group schemes associated to simple finite groups. We do not know what happens if the characteristic of k divides the order of the group.

5.2. Deformations of Lie algebras of classical type

If G is a finite group scheme of height one associated to a restricted Lie algebra $(\mathfrak{g},[p])$, the cohomology $H^i(G;\mathfrak{g})$ of G with values in the adjoint representation is equal to the restricted cohomology $H^i_*(\mathfrak{g},\mathfrak{g})$ of $\mathfrak{g}$ with values in the adjoint representation (see [16]).

The restricted cohomology is related to the ordinary cohomology by two spectral sequences (see [20, 12]):

$$\begin{cases} E_1^{p,q} = \mathrm{Hom}_{\mathrm{Fr}}(S^p\mathfrak{g}, H^{q-p}(\mathfrak{g},M)) \Rightarrow H_*^{p+q}(\mathfrak{g},M) \text{ if } p\neq 2,\\ E_2^{p,q} = \mathrm{Hom}_{\mathrm{Fr}}(\Lambda^q\mathfrak{g}, H_*^{p}(\mathfrak{g},M)) \Rightarrow H^{p+q}(\mathfrak{g},M),\end{cases}$$

where $S^p\mathfrak{g}$ and $\Lambda^q\mathfrak{g}$ denote, respectively, the p-th symmetric power and the q-th alternating power, and Hom_{Fr} denotes the homomorphisms which are Frobenius linear. In the case of the adjoint representation and for centerless Lie algebras, the above spectral sequences give rise to the following relations for the low cohomology groups:

$$\begin{cases} H^1_*(\mathfrak{g},\mathfrak{g}) = H^1(\mathfrak{g},\mathfrak{g}),\\ 0\to H^2_*(\mathfrak{g},\mathfrak{g}) \to H^2(\mathfrak{g},\mathfrak{g}) \to \mathrm{Hom}_{\mathrm{Fr}}(\mathfrak{g}, H^1(\mathfrak{g},\mathfrak{g})).\end{cases}$$

It is a classical fact that simple Lie algebras are rigid in characteristic zero.

THEOREM 5 (Whitehead). *If* $\mathfrak{g}$ *is a simple Lie algebra over a field k of characteristic* 0*, then* $H^*(\mathfrak{g},\mathfrak{g}) = 0$*. In particular* $\mathfrak{g}$ *is rigid.*

The proof uses the non-degeneracy of the Killing form. If $\mathrm{char}(k) = p$ does not divide the discriminant of the Killing form, then the same proof gives the rigidity of simple Lie algebras of classical type in characteristic p. Indeed Rudakov ([25]) has shown the following

THEOREM 6 (Rudakov). *If* k *is a field of* $\mathrm{char}(k) = p \geq 5$ *and* $\mathfrak{g}$ *is a simple Lie algebra of classical type, then* $\mathfrak{g}$ *is rigid.*

Note, however, that the preceding result is false if $p = 2,3$ (see [10, 9, 8]).

5.3. Deformations of Lie algebras of Cartan type

The Lie algebras of Cartan type do have infinitesimal deformations, unlike the Lie algebras of classical type. We computed in [31] (building upon [30, 33, 32]) the infinitesimal deformations of the simple finite group schemes associated to simple and restricted Lie algebras of Cartan type.

THEOREM 7 ([31]).

$$\begin{cases} h^2_*(W(m;\underline{1}),W(m;\underline{1})) = m, \\ h^2_*(S(m;\underline{1}),S(m;\underline{1})) = m, \\ h^2_*(H(2r;\underline{1}),H(2r;\underline{1})) = 2r+1, \\ h^2_*(K(2r+1;\underline{1}),K(2r+1;\underline{1})) = 2r+1, \\ h^2_*(M(1,1),M(1,1)) = 5. \end{cases}$$

Moreover, in each of the above cases, we get explicit generators for the above second cohomology groups. It would be interesting to extend the above computation to the others simple restricted Lie algebras of Cartan type (i.e. the ones with $\underline{n} \neq \underline{1}$).

5.4. Further developments

We hope that, using similar techniques to the ones used in [31], it could be possible to answer to the following questions (for all simple finite group schemes G):

- what is the space of *obstructions* $H^3(G,\mathfrak{g})$?
- what is the *semi-versal deformation space* of G?

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