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A CONSTRUCTION OF CERTAIN SURFACES OF GENERAL TYPE

Abstract. For each integer $a = 2,3,4$, we construct a family of minimal complex algebraic surfaces with torsion group $\mathbb{Z}/2$ such that $c_1^2 = 2\chi(\mathcal{O}) - 1$ and $\chi(\mathcal{O}) = a$, where c_1 and $\chi(\mathcal{O})$ are the first Chern class and the Euler characteristic of the structure sheaf.

1. Introduction

As one can see in Bombieri's paper [2] on pluricanonical maps, it is sometimes effective to employ the torsion parts of the Picard groups in studying regular surfaces of general type. This is true in particular for the case of surfaces with small invariants, especially for those with vanishing geometric genus.

One reason for this is that such surfaces generally tend to assume several topological types for single values of numerical invariants. Since the torsion group for a regular surface is isomorphic to the first homology group with integral coefficients, one can use it to distinguish the topological types. Another reason for the importance of the torsion groups is that they sometimes have effect on the nonbirationality of pluricanonical maps. In fact in [5], Ciliberto and Mendes Lopes classified minimal surfaces with $c_1^2 = 2\chi(\mathcal{O}) - 2$ having 2-torsion by showing that such surfaces have non-birational bicanonical maps (and also quoting results from [1], [3], and [4]).

The aim of this short note is to give an explicit construction of a family of minimal algebraic surfaces each member X of which has $c_1^2 = 2\chi(\mathcal{O}) - 1$ and torsion group $\mathrm{Tors}(X) \simeq \mathbb{Z}/2$. Note that the minimal surfaces with the numerical invariants on this line have vanishing irregularity, hence, in particular, that the case $\chi(\mathcal{O}) = 1$ on this line is that of the Godeaux surfaces. By a result given in the previous paper [11], the order of the torsion group for a surface with the invariants on this line is at most 3 if $\chi(\mathcal{O}) = 2$, is at most 2 if $3 \leq \chi(\mathcal{O}) \leq 6$, and is 1 if $7 \leq \chi(\mathcal{O})$. The author has already given a complete description for the case $\chi(\mathcal{O}) = 2$ and $\mathrm{Tors}(X) \simeq \mathbb{Z}/3$ ([10]). Meanwhile, we have U. Perrson's results on the geography of Chern numbers ([12]), and the possible values of the torsion groups are completely understood for the case $\chi(\mathcal{O}) = 1$ (that of Godeaux surfaces, see [8] and [13]). Thus our next game, in view of the sharpness of the bound, is the case of $\mathrm{Tors}(X) \simeq \mathbb{Z}/2$ and $2 \leq \chi(\mathcal{O}) \leq 6$. We shall construct examples of the case $\mathrm{Tors}(X) \simeq \mathbb{Z}/2$ and $\chi(\mathcal{O}) = a$ for each integer $2 \leq a \leq 4$ in the present paper.

For the case $\chi(\mathcal{O}) = 3,4$, surfaces with these invariants already appear in a paper by Du Val [7] in which he gave, with some assumptions, a list for the minimal regular

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surfaces of non-standard case for the non-birationality of the bicanonical maps (see also [4, 6]). In fact one can easily show that our surfaces for the case $\chi(\mathcal{O}) = 3, 4$ are exactly those in Du Val's list (double covers of rational surfaces) but described in a different way. But since imposing singularities on the branch divisor in general makes uncertain the existence of such a branch divisor and also the minimality of the obtained double cover model, it is not clear at least to the author whether the existence follows easily from Du Val's list (especially for the case of small $\chi(\mathcal{O})$). In any case we construct the surfaces also for the case $\chi(\mathcal{O}) = 2$, and even for the case $\chi(\mathcal{O}) = 3, 4$.

Our construction here is in a sense – although not exactly – a combination of two classical methods known as the Campedelli construction and the Godeaux construction. Namely, we first take the minimal resolution Y of the double cover of the Hirzebruch surface $\Sigma_0 \simeq \mathbb{P}^1 \times \mathbb{P}^1$ branched along a singular curve B, and then take the quotient $X = Y/G$ of Y by a certain free action by $G \simeq \mathbb{Z}/2$. Here B is a member of the quadruple anticanonical system $|-4K_{\Sigma_0}|$ having two $[3,3]$-points and $2k$ ordinary quadruple points ($\chi(\mathcal{O}) = 4-k$, $0 \le k \le 2$). This construction comes from an effort to obtain a concrete description for such surfaces. In a different paper (of which the first part is [9]), the author shall give, together with exclusion of the case $\mathrm{Tors}(X) \simeq \mathbb{Z}/2$ and $5 \le \chi(\mathcal{O}) \le 6$, a complete description for the surfaces of the case $\mathrm{Tors}(X) \simeq \mathbb{Z}/2$ and $\chi(\mathcal{O}) = 4$, in which our construction for $\chi(\mathcal{O}) = 4$ appears as a general case of one of the two types. For the case of $\mathrm{Tors}(X) \simeq \mathbb{Z}/2$ and $2 \le \chi(\mathcal{O}) \le 3$, however, a complete classification seems much more difficult. So we first give here examples of such surfaces.

Throughout this paper, we work over the complex number field $\mathbb{C}$.

Notation. Let S be a compact complex manifold of dimension 2. We denote by $c_1(S)$, $p_g(S)$ and $q(S)$, the first Chern class, the geometric genus and the irregularity of S, respectively. The torsion group $\mathrm{Tors}(S) = \mathrm{TorPic}(S)$ is the torsion part of the Picard group of S. If V is a complex manifold, K_V is a canonical divisor of V. For a coherent sheaf $\mathcal{F}$ on V, we denote by $h^i(\mathcal{F})$ and $\chi(\mathcal{F})$, the dimension of the i-th cohomology group and the Euler characteristic of $\mathcal{F}$, respectively. Let $f : V \to W$ be a morphism to a complex manifold W, and D a divisor on W. We denote by $f^*(D)$ and $f_*^{-1}(D)$ the total transform and the strict transform of D, respectively. The symbol $\sim$ means a linear equivalence of divisors. We denote by $\Sigma_d \to \mathbb{P}^1$ the Hirzebruch surface of degree d. The divisors Δ_0 and Γ are its minimal section and its fiber, respectively. Throughout this paper, X is a minimal algebraic surface of general type with $c_1^2 = 2\chi(\mathcal{O}_X) - 1$.

2. A lemma

Let us give a lemma which we shall use in the construction of the families of surfaces stated in Section 1. Let W be a compact connected complex manifold, and G a group acting on W. Let B be an effective reduced divisor on W such that $B \sim nF$ for a non-trivial divisor F and an integer $n \ge 2$. Then we have a Galois cover $V \to W$ of mapping degree n with branch locus B and with Galois group $\mathbb{Z}/n$. The variety V is a subvariety of the total space of the line bundle F. We assume that the divisors B and F are stable under the action of G. We say that an action of G on V is a lifting of the one on W, if the action of G on V and that on W are compatible with the projection $V \to W$. Let us give a criterion for the existence of a lifting. Let h be a meromorphic function on W corresponding to the principal divisor $B - nF$. Then $c_g = (g^*h)/h$ is a non-zero constant function for any $g \in G$, and $g \mapsto c_g$ gives a character c of G. Let $\mathrm{Char}(G)$ be the character group of G, and Ψ the endomorphism of $\mathrm{Char}(G)$ given by $\chi \mapsto n\chi$. We denote by $\mathrm{Im}(\Psi)$ the image of the morphism $\Psi : \mathrm{Char}(G) \to \mathrm{Char}(G)$.

LEMMA 1. *The action of G on W lifts to one on V if and only if $c \in \mathrm{Im}(\Psi)$. If $c \in \mathrm{Im}(\Psi)$, then there exist exactly $\sharp\mathrm{ker}(\Psi)$ liftings of the action of G, where $\mathrm{ker}(\Psi)$ is the kernel of the morphism Ψ.*

Proof. See the Appendix. □

3. A family of surfaces with $\mathrm{Tors}(X) \simeq \mathbb{Z}/2$

Let us start the construction of a family of minimal surfaces X with $c_1^2 = 2\chi(\mathcal{O}) - 1$, $\chi(\mathcal{O}) = 4 - k$, and $\mathrm{Tors}(X) \simeq \mathbb{Z}/2$ for each integer $0 \le k \le 2$. Let $W = \mathbb{P}^1 \times \mathbb{P}^1$ be the Hirzebruch surface of degree 0, and $(X_0 : X_1)$ and $(Y_0 : Y_1)$ homogeneous coordinates of $\mathbb{P}^1$. We define an involution ι_0 of W by

$$\iota_0 : ((X_0 : X_1), (Y_0 : Y_1)) \mapsto ((X_1 : X_0), (Y_1 : Y_0)).$$

We put $x = X_1/X_0$ and $y = Y_1/Y_0$. Let G be a group of automorphisms of W generated by ι_0. Then $G \simeq \mathbb{Z}/2$ acts naturally on W, and W has exactly 4 fixed points of ι_0, namely $p_1 : (x,y) = (1,1)$, $p_2 : (x,y) = (1,-1)$, $p_3 : (x,y) = (-1,1)$ and $p_4 : (x,y) = (-1,-1)$. Let $q : W_0 \to W$ be the blowing-up of W at $2k+2$ points $w_1, \dots, w_{2k+2}$, where $\{w_{2j+1}\}_{0 \le j \le k}$ is a set of distinct $k+1$ points on $W \setminus \{p_1, \dots, p_4\}$, and $w_{2j+2} = \iota_0(w_{2j+1})$ for each integer $0 \le j \le k$. The action of G on W lifts to one on W_0. We denote by $E_i^0 = q^{-1}(w_i)$ the exceptional curve of the first kind lying over w_i for $1 \le i \le 2k+2$. Let $q' : W_2 \to W_0$ be the blowing-up of W_0 at two points w'_1 and w'_2, where $w'_1 \in E_1^0$ and $w'_2 = \iota_0(w'_1) \in E_2^0$. We denote by $E_i^\vee = q'^{-1}(w'_i)$ the exceptional curve of the first kind lying over w'_i for $i = 1, 2$. We use the same symbol E_i^0 for the total transform on W_2 of the divisor E_i^0. We put $\overline{q} = q \circ q' : W_2 \to W$. Note that the action of G on W lifts to one on W_2.

LEMMA 2. *Assume that the configuration of the $k+1$ points w_{2j+1}'s $(0 \le j \le k)$ and that of w'_1 are sufficiently general. Then there exists a reduced curve B'_2 on W_2 satisfying the following five conditions* :

1) $B'_2 \in |\overline{q}^*(8\Delta_0 + 8\Gamma) - \sum_{i=1,2} 3(E_i^0 + E_i^\vee) - \sum_{3 \le i \le 2k+2} 4E_i^0|$,
2) $B'_2 \cap q'^{-1}_*(E_i^0) = \emptyset$ *for* $i = 1, 2$,
3) $B'_2 \cap \overline{q}^{-1}(\{p_1, \dots, p_4\}) = \emptyset$,
4) B'_2 *has at most negligible singularities,*
5) B'_2 *is stable under the action of* G *on* W_2.

Note that $\sum_{3\le i\le 2k+2} 4E_i^0 = 0$ if $k=0$. We shall give a proof of the lemma above at the end of this section. We define a reduced curve B_2 on W_2 by

$$B_2 = B_2' + \sum_{i=1,2} q'^{-1}_{*}(E_i^0).$$

Then B_2 is stable under the action of G, and singularities of B_2 are at most negligible ones. Moreover we have $B_2 \sim 2F_2$, where

$$F_2 \sim \overline{q}^*(4\Delta_0 + 4\Gamma) - \sum_{i=1,2} (E_i^0 + 2E_i^\vee) - \sum_{3\le i\le 2k+2} 2E_i^0.$$

Let $f_2 : Y_2 \to W_2$ be the double cover of W_2 with branch locus B_2, and $\tilde{Y} \to Y_2$ the minimal desingularization of Y_2. Then we obtain a surjective morphism $f : \tilde{Y} \to W_2$ of mapping degree 2 with branch locus B_2. We have $f^*(q'^{-1}_{*}(E_i^0)) = 2E_i$ for a (-1)-curve E_i on $\tilde{Y}$ for each $i = 1,2$. We denote by $p : \tilde{Y} \to Y$ the blowing-down of the two (-1)-curves E_1 and E_2. Then we see easily that

$$K_Y^2 = 2(2(4-k)-1), \qquad \chi(\mathcal{O}_Y) = 2(4-k). \tag{1}$$

LEMMA 3. *Assume that the configuration of the $k+1$ points w_{2j+1}'s $(0\le j\le k)$ and that of w_1' are sufficiently general. Then the fixed part of the canonical system $|K_{\tilde{Y}}|$ is $\sum_{i=1,2} 2E_i$, and the variable part of $|K_{\tilde{Y}}|$ is free from base points. In particular, Y is minimal.*

Proof. Since W is a rational surface, we have $|K_{\tilde{Y}}| = f^*|K_{W_2} + F_2|$, where

$$F_2 + K_{W_2} \sim \overline{q}^*(2\Delta_0 + 2\Gamma) - \sum_{i=1,2} E_i^\vee - \sum_{3\le i\le 2k+2} E_i^0.$$

We study the linear system $|K_{W_2} + F_2|$. We denote by L_{w_i} the unique member of $|\Gamma|$ passing through w_i, and by M_{w_i} the unique member of $|\Delta_0|$ passing through w_i, where $1 \le i \le 2k+2$.

First, we give a proof for the case $k=0$ or 1. Assume that $k=0$ or 1. The linear system $|\overline{q}^*(\Delta_0+\Gamma) - \sum_{i=1,2} E_i^\vee| + |\overline{q}^*(\Delta_0+\Gamma) - \sum_{3\le i\le 2k+2} E_i^0|$ is a subsystem of $|F_2 + K_{W_2}|$. Note that both $L_{w_1} + M_{w_2}$ and $L_{w_2} + M_{w_1}$ are members of $|\Delta_0 + \Gamma|$ passing through w_1 and w_2. Thus the fixed part of $|\overline{q}^*(\Delta_0+\Gamma) - \sum_{i=1,2} E_i^\vee|$ is $\sum_{i=1,2} q'^{-1}_{*}(E_i^0)$, and the variable part of $|\overline{q}^*(\Delta_0+\Gamma) - \sum_{i=1,2} E_i^\vee|$ is free from base points. Moreover $|\overline{q}^*(\Delta_0+\Gamma) - \sum_{3\le i\le 2k+2} E_i^0|$ is free from base points. Thus the assertion follows for the case $k=0$ or 1.

Next we give a proof for the case $k=2$. Take a member C_1 of $|2\Delta_0 + \Gamma|$ passing through the 5 points w_1, w_3, w_4, w_5, w_6. This is possible, since $\dim|2\Delta_0+\Gamma| = 5$. Let C_2 be a member of $|2\Delta_0 + \Gamma|$ passing through the 5 points w_1, w_2, w_3, w_5 and w_6. Then the 4 members

$$C_1 + L_{w_2}, \quad C_2 + L_{w_4}, \quad \iota_0^*(C_1) + L_{w_1} = \iota_0^*(C_1 + L_{w_2}), \quad \iota_0^*(C_2) + L_{w_3} = \iota_0^*(C_2 + L_{w_4})$$

of $|2\Delta_0 + 2\Gamma|$ pass through the 6 points $w_1, \dots, w_6$, hence they are corresponding to members of $|K_{W_2} + F_2|$. We use these 4 divisors to study the canonical system $|K_{\tilde{Y}}|$.

Let C_1', C_1'' and D be effective divisors on W satisfying $C_1 = C_1' + D$ and $\iota_0^*(C_1) = C_1'' + D$, where C_1' and C_1'' have no common irreducible components. Then we have $\iota_0^*(D) = D$ and $C_1'' = \iota_0^*(C_1')$. Let us show that $D = 0$, namely, that C_1 and $\iota_0^*(C_1)$ have no common irreducible components, on the assumption that the configurations of w_{2j+1}'s $(0 \le j \le 2)$ are sufficiently general. We see easily that if the configuration of the 3 points w_{2j+1}'s $(0 \le j \le 2)$ is sufficiently general, then the following five conditions are satisfied:

i) no members of $|2\Delta_0 + \Gamma|$ stable under ι_0 pass through the 3 points w_1, w_3, w_5,
ii) each member of $|\Delta_0|$ contains at most one out of the 6 points $w_1, \ldots, w_6$,
iii) each member of $|\Gamma|$ contains at most one out of the 6 points $w_1, \ldots, w_6$,
iv) no members of $|\Delta_0 + \Gamma|$ stable under ι_0 pass through the 2 points w_3, w_5,
v) no members of $|2\Delta_0 + \Gamma|$ passing through the 4 points $w_3, \ldots, w_6$ are tangent to L_{w_1} at w_1.

Assume that $D \in |2\Delta_0 + \Gamma|$. Then $D \in |2\Delta_0 + \Gamma|$ is stable under ι_0, and passes through the 3 points w_1, w_3, w_5, which contradicts the condition i). Thus we have $D \notin |2\Delta_0 + \Gamma|$.

Assume that $D \in |2\Delta_0|$. Then we have $C_1 \in |\Delta_0| + |\Delta_0| + |\Gamma|$, which contradicts the conditions ii) and iii). Thus we have $D \notin |2\Delta_0|$.

Assume that $D \in |\Delta_0 + \Gamma|$. Then $C_1' \in |\Delta_0|$ contains at most one out of the 5 points w_1, $w_3, \ldots, w_6$ by the condition ii). Thus, since $\iota_0^*(D) = D$, the divisor D passes through the 4 points w_3, w_4, w_5 and w_6. This contradicts the condition iv). Thus we have $D \notin |\Delta_0 + \Gamma|$.

Assume that $D \in |\Delta_0|$. Then D is a member of $|\Delta_0|$ stable under ι_0. Note that $w_1, \ldots, w_6 \in W \setminus \{p_1, \ldots, p_4\}$, where $\{p_1, \ldots, p_4\}$ is the set of all fixed points of ι_0 on W. Thus by the condition ii), the divisor D contains none of the 6 points $w_1, \ldots, w_6$. It follows that both C_1' and $C_1'' = \iota_0^*(C_1')$ contain the 4 points w_3, w_4, w_5 and w_6, which contradicts $C_1' \cdot C_1'' = 2$. Thus we have $D \notin |\Delta_0|$.

Assume that $D \in |\Gamma|$. Then we have $C_1 \in |\Delta_0| + |\Delta_0| + |\Gamma|$, which contradicts the conditions ii) and iii). Hence we have $D \notin |\Gamma|$.

Thus, by the argument above, the divisors C_1 and $\iota_0^*(C_1)$ have no common irreducible components. Moreover C_1 and L_{w_1} have no common irreducible components by the conditions ii) and iii). By the condition v), we have $C_1 \cap L_{w_1} = w_1 + w_7$ for a certain point $w_7 \neq w_1$ on W. It follows that

$$(C_1 + L_{w_2}) \cap (\iota_0^*(C_1) + L_{w_1}) = w_7 + w_8 + \sum_{1 \le i \le 6} w_i,$$

where $w_8 = \iota_0(w_7)$. From this we can deduce that the fixed part of $|K_{W_2} + F_2|$ equals $\sum_{i=1,2} q'^{-1}_*(E_i^0)$, and that the base locus of the variable part of $|K_{W_2} + F_2|$ is at most $\bar{q}^{-1}(\{w_7, w_8\})$ on the assumption that the configuration of the 4 points w_1, w_3, w_5 and w_1' are sufficiently general.

By the same method as in the case of C_1, we see that if the configuration of w_1,

w_3, w_5 and w'_1 are sufficiently general, then

$$(C_2 + L_{w_4}) \cap (\iota_0^*(C_2) + L_{w_3}) = w'_7 + w'_8 + \sum_{1 \le i \le 6} w_i,$$

where $w'_7 \in L_{w_3}$ and $w'_8 \in L_{w_4}$ are certain points on W. It follows that the base locus of the variable part of $|K_{W_2} + F_2|$ is at most $\bar{q}^{-1}(\{w'_7, w'_8\})$. Thus the assertion follows for the case $k = 2$, since we have $\{w_7, w_8\} \cap \{w'_7, w'_8\} = \emptyset$. □

In what follows, we assume that the configuration of the $k+1$ points w_{2j+1}'s $(0 \le j \le k)$ and that of w'_1 are sufficiently general as in Lemma 3, hence that Y is minimal. We put

$$F_2 = \bar{q}^*(\sum_{i=1,2} 2(L_{w_i} + M_{w_i})) - \sum_{i=1,2} (E_i^0 + 2E_i^\vee) - \sum_{3 \le i \le 2k+2} 2E_i^0,$$

where L_{w_i} and M_{w_i} are the divisors as in the proof of Lemma 3. Then the divisors B_2 and F_2 are stable under the action of G. Let h be a meromorphic function on W_2 corresponding to the principal divisor $B_2 - 2F_2$. Then $c_{\iota_0} = (\iota_0^* h)/h$ is a non-zero constant. We use the same symbol p_i for the point on W_2 lying over the fixed point $p_i \in W$ of ι_0. Since $\{p_1, \ldots, p_4\} \cap \operatorname{supp}(B_2 - 2F_2) = \emptyset$, we infer $h(p_1) \ne 0$, hence $c_{\iota_0} = 1$. Thus by Lemma 1, there exist exactly two liftings to Y_2 of the action of G on W_2.

LEMMA 4. *There exists a unique free action of G on $\tilde{Y}$ which is obtained by lifting the action on W_2. This action on $\tilde{Y}$ induces one on Y free from fixed points.*

Proof. The fiber $f_2^{-1}(p_i)$ is a set of 2 points for each $1 \le i \le 4$. We take the unique lifting to Y_2 of the action of G such that the induced action of G on $f_2^{-1}(p_1)$ is free from fixed points. We obtain an action of G on $f_2^{-1}(p_i)$ by restricting this lifting. Since $\{p_1, \ldots, p_4\}$ is the set of all fixed points of the action of G on W_2, we only need to show that the action of G on $f_2^{-1}(p_i)$ is free for any $2 \le i \le 4$.

Let L_s be a member of $|\Gamma|$ given by $x - s = 0$, and M_s a member of $|\Delta_0|$ given by $y - s = 0$ for each $s \in \mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$. Then we have $\bar{q}^{-1}(L_1) \simeq \mathbb{P}^1$ and $\mathcal{O}_{\bar{q}^{-1}(L_1)}(F_2) \simeq \mathcal{O}_{\mathbb{P}^1}(4)$. Putting $U_i = \bar{q}^{-1}(L_1) \setminus \{p_i\}$ $(i = 1, 2)$, we have $\bar{q}^{-1}(L_1) = \bigcup_{i=1,2} U_i$. For each $i = 1, 2$, we take a coordinate z_i on U_i such that $\iota_0 : z_i \mapsto -z_i$ on U_i and $z_1 z_2 = 1$ on $U_1 \cap U_2$ hold. Note that the fixed point $p_1 \in U_2$ is given by $z_2 = 0$, and that the fixed point $p_2 \in U_1$ is given by $z_1 = 0$. Let $\bigcup_{i=1,2} U_i \times \mathbb{C}$ be the total space of the line bundle $\mathcal{O}_{\bar{q}^{-1}(L_1)}(F_2)$. We take a fiber coordinate ζ_i on $U_i \times \mathbb{C}$ such that

$$\zeta_1 = \frac{\zeta_2}{z_2^4}\,. \tag{2}$$

Let $g_i = 0$ be a defining equation of $B_2|_{\bar{q}^{-1}(L_1)}$ on U_i such that $g_1 = g_2/z_2^8$. Then $f_2^{-1}(\bar{q}^{-1}(L_1))$ is a subvariety of $\bigcup_{i=1,2} U_i \times \mathbb{C}$ locally defined by $\zeta_i^2 - g_i = 0$. Since B_2 is stable under the action of G, the function $\iota_0^* g_1/g_1 = \iota_0^* g_2/g_2$ is holomorphic on $\bar{q}^{-1}(L_1)$, hence a constant. From this together with $g_2(p_1) \ne 0$, we infer $\iota_0^* g_i = g_i$ for

$i = 1,2$. Thus, since the action of G on $f_2^{-1}(p_1)$ is non-trivial, the automorphism of $f_2^{-1}(\overline{q}^{-1}(L_1))$ corresponding to $\iota_0 \in G$ is given by $(z_2, \zeta_2) \mapsto (-z_2, -\zeta_2)$ on $U_2 \times \mathbb{C}$. By this together with (2), we see that this automorphism is given by $(z_1, \zeta_1) \mapsto (-z_1, -\zeta_1)$ on $U_1 \times \mathbb{C}$. Thus the action on $f_2^{-1}(p_2)$ is free from fixed points.

Note that we have $p_1, p_3 \in \overline{q}^{-1}(M_1)$ and $p_3, p_4 \in \overline{q}^{-1}(L_{-1})$. Using M_1 and L_{-1} in place of L_1, we see that the action of G on $f_2^{-1}(p_i)$ is free for $i = 3,4$ in the same way. Thus the assertion follows. □

PROPOSITION 1. *Let X be a quotient of Y by the free action of G given in Lemma 4. Then X is a minimal algebraic surface of general type with $c_1^2 = 2\chi(\mathcal{O}) - 1$, $\chi(\mathcal{O}) = 4 - k$ and* $\mathrm{Tors}(X) \simeq \mathbb{Z}/2$.

Proof. Since the projection $\pi : Y \to X$ is an unramified Galois cover of degree 2, we infer from (1) and Lemma 3 that X is a minimal surface such that $c_1^2 = 2\chi(\mathcal{O}) - 1$, $\chi(\mathcal{O}) = 4 - k$ and $\mathbb{Z}/2 \subset \mathrm{Tors}(X)$. The isomorphism $\mathrm{Tors}(X) \simeq \mathbb{Z}/2$ then follows from [11, Theorem 1]. □

4. Proof of Lemma 2

Finally, we prove Lemma 2. Take homogeneous coordinates $(X_0 : X_1)$ and $(Y_0 : Y_1)$ as in the beginning of the previous section such that w_1 is given by $(x,y) = (0,0)$. Let C_3 be the unique member of $|\Delta_0 + \Gamma|$ whose strict transform on W_0 passes through w_1'. Then C_3 is defined by $\mu x + \nu y = 0$ for certain constants μ and $\nu \in \mathbb{C}$. The point w_{2j+1} is given by $(x,y) = (\alpha_j, \beta_j)$ for each integer $1 \le j \le k$, where α_j and $\beta_j \in \mathbb{C}$ are certain constants.

Put $\eta^{\iota_0}(X_0, X_1; Y_0, Y_1) = \eta(X_1, X_0; Y_1, Y_0)$ for each homogeneous polynomial $\eta(X_0, X_1; Y_0, Y_1) \in H^0(\mathcal{O}_W(l\Delta_0 + m\Gamma))$ of bidegree (l,m). Then $\eta \mapsto \eta^{\iota_0}$ gives an involution of $H^0(\mathcal{O}_W(l\Delta_0 + m\Gamma))$, and this involution induces an action of $G = \langle \iota_0 \rangle \simeq \mathbb{Z}/2$ on $H^0(\mathcal{O}_W(l\Delta_0 + m\Gamma))$. Let $V^+_{(l,m)}$ be the space consisting of all elements in $H^0(\mathcal{O}_W(l\Delta_0 + m\Gamma))$ stable under this action. We denote by $\Lambda^+_{(l,m)} = \mathbb{P}(V^+_{(l,m)})$ the subsystem of $|l\Delta_0 + m\Gamma|$ corresponding to the subspace $V^+_{(l,m)}$. If D is an effective divisor on W_2, we denote by $\Lambda^+_{(l,m)}(D)$ the space consisting of all members C's of $\Lambda^+_{(l,m)}$ such that $\overline{q}^*C - D$ is effective. We put $\tilde{\Lambda}^+_{(l,m)}(D) = \overline{q}^*(\Lambda^+_{(l,m)}(D)) - D$. Moreover we put $\Lambda^+ = \Lambda^+_{(8,8)}(\sum_{i=1,2} 3(E_i^0 + E_i^\vee) + \sum_{3 \le i \le 2k+2} 4E_i^0)$ and $\tilde{\Lambda}^+ = \tilde{\Lambda}^+_{(8,8)}(\sum_{i=1,2} 3(E_i^0 + E_i^\vee) + \sum_{3 \le i \le 2k+2} 4E_i^0)$.

Proof of Lemma 2. First, we give a proof for the case $k = 1$. In what follows, we assume that α_1, β_1, μ and ν are sufficiently general. Then we have

$$\dim \Lambda^+_{(2,2)}(\sum_{i=1,2} (E_i^0 + E_i^\vee) + \sum_{i=3,4} E_i^0) = 1.$$

It is easily verified that the base locus of this linear pencil is $\{w_i\}_{1 \le i \le 4} \cup \{w_9, w_{10}\}$, where the point w_9 is given by

$$x = \frac{\beta_1(\mu\beta_1 + \nu\alpha_1)}{\mu\alpha_1 + \nu\beta_1} \quad \text{and} \quad y = \frac{\alpha_1(\mu\beta_1 + \nu\alpha_1)}{\mu\alpha_1 + \nu\beta_1},$$

and $w_{10} = \iota_0(w_9)$. We use the same symbol w_i for the point on W_2 lying over $w_i \in W$, where $i = 9, 10$. It is also easily verified that $\tilde{\Lambda}^+_{(2,2)}(\sum_{i=3,4} E_i^0)$ is free from base points. Thus from

$$3\tilde{\Lambda}^+_{(2,2)}(\sum_{i=1,2}(E_i^0 + E_i^\vee) + \sum_{i=3,4} E_i^0) + \tilde{\Lambda}^+_{(2,2)}(\sum_{i=3,4} E_i^0) \subset \tilde{\Lambda}^+,$$

we infer that the base locus of $\tilde{\Lambda}^+$ is at most $\{w_9, w_{10}\}$. Meanwhile, since $\iota_0^*(C_3)$ passes through w_1, we have

$$2(C_3 + \iota_0^*(C_3) + L_{\alpha_1} + L_{1/\alpha_1} + M_{\beta_1} + M_{1/\beta_1}) \in \Lambda^+,$$

where L_s and M_s are the divisors as in the proof of Lemma 4 for each $s \in \mathbb{C} \cup \{\infty\}$. Thus, since $C_3 + \iota_0^*(C_3) + L_{\alpha_1} + L_{1/\alpha_1} + M_{\beta_1} + M_{1/\beta_1}$ passes through neither w_9 nor w_{10}, we infer that the linear system $\tilde{\Lambda}^+$ is free from base points. By Bertini's theorem, any general member B_2' of $\tilde{\Lambda}^+$ satisfies all the conditions given in Lemma 2.

Next, we give a proof for the case $k = 0$. In what follows, we assume that μ and ν are sufficiently general. Then we have $\dim \Lambda^+_{(2,2)}(\sum_{i=1,2}(E_i^0 + E_i^\vee)) = 2$. It is easily verified that $\tilde{\Lambda}^+_{(2,2)}(\sum_{i=1,2}(E_i^0 + E_i^\vee))$ is free from base points. We therefore infer, since we have

$$3\tilde{\Lambda}^+_{(2,2)}\Big(\sum_{i=1,2}(E_i^0 + E_i^\vee)\Big) + \overline{q}^*\Lambda^+_{(2,2)} \subset \tilde{\Lambda}^+,$$

that $\tilde{\Lambda}^+$ is free from base points. Thus any general member B_2' of $\tilde{\Lambda}^+$ satisfies all the conditions given in Lemma 2.

Finally, we give a proof for the case $k = 2$. In what follows, we assume that $\alpha_1, \alpha_2, \beta_1, \beta_2, \mu, \nu$ are sufficiently general. We see easily that $\dim \Lambda^+_{(2,2)}(\sum_{1 \le i \le 6} E_i^0) = 1$, and that the base locus of this linear system is $\{w_i\}_{1 \le i \le 6} \cup \{w_{11}, w_{12}\}$, where the point w_{11} is given by

$$x = \frac{(\beta_1\beta_2 - 1)(\alpha_1\beta_2 - \alpha_2\beta_1)}{(\beta_1 - \beta_2)(\alpha_1\alpha_2 - \beta_1\beta_2)} \quad \text{and} \quad y = \frac{(\alpha_1\alpha_2 - 1)(\alpha_1\beta_2 - \alpha_2\beta_1)}{(\alpha_1 - \alpha_2)(\alpha_1\alpha_2 - \beta_1\beta_2)},$$

and $w_{12} = \iota_0^*(w_{11})$. We use the same symbol w_i for the point on W_2 lying over $w_i \in W$, where $i = 11, 12$. The linear system $\Lambda^+_{(2,2)}(\sum_{i=1,2}(E_i^0 + E_i^\vee) + \sum_{3 \le i \le 6} E_i^0)$ has a unique member C_4. The divisor C_4 is smooth at $w_1, \ldots, w_6, w_{11}$ and w_{12}, since any distinct 2 members of $\Lambda^+_{(2,2)}(\sum_{1 \le i \le 6} E_i^0)$ intersect each other transversally at these 8 points. We denote by $\overline{C}_4$ the strict transform on W_2 of C_4. Then we have

$$\overline{C}_4 = \overline{q}^*(C_4) - \sum_{i=1,2}(E_i^0 + E_i^\vee) - \sum_{3 \le i \le 6} E_i^0.$$

It is also easily verified that $\dim \tilde{\Lambda}^+_{(2,2)}(\sum_{3\le i\le 6} E_i^0) = 2$, and that this linear system has no base points. Thus from

$$3\Lambda^+_{(2,2)}\Big(\sum_{i=1,2}(E_i^0+E_i^\vee)+\sum_{3\le i\le 6}E_i^0\Big)+\Lambda^+_{(2,2)}\Big(\sum_{3\le i\le 6}E_i^0\Big)\subset\Lambda^+,$$

we infer that the base locus of $\tilde{\Lambda}^+$ is at most $\overline{C}_4$.

The linear system $\Lambda^+_{(4,4)}(\sum_{i=1,2}3E_i^0+\sum_{3\le i\le 6}2E_i^0)$ has a unique member C_5. By the same method as in the proof of Lemma 3, we see that C_4 and C_5 have no common irreducible components. Thus we have

$$C_4\cap C_5 = w_{13}+w_{14}+\sum_{i=1,2}3w_i+\sum_{3\le i\le 6}2w_i, \tag{3}$$

where w_{13} is a point on W and $w_{14}=\iota_0(w_{13})$. If $\{w_{13},w_{14}\}=\{w_3,w_4\}$ holds for general $\alpha_1,\dots,\nu$, then we have $\{w_{13},w_{14}\}=\{w_5,w_6\}$ for general $\alpha_1,\dots,\nu$, which is a contradiction. Thus we have $\{w_{13},w_{14}\}\cap\{w_3,w_4\}=\emptyset$. In the same way, we see that $\{w_{13},w_{14}\}\cap\{w_5,w_6\}=\emptyset$. By the defining equation of C_5, we obtain $\mathrm{mult}_{w_1}C_5=3$. Thus, since the defining equation of C_5 is independent of μ and ν, we infer that $\{w_{13},w_{14}\}\cap\{w_1,w_2\}=\emptyset$ for general μ and ν. Moreover by the defining equation of C_4 and that of C_5, we obtain $C_4\cap\{p_1,\dots,p_4\}=\emptyset$ and $C_5\cap\{w_{11},w_{12}\}=\emptyset$. It follows that

$$\{w_{13},w_{14}\}\cap\{w_1,\dots,w_6,w_{11},w_{12},p_1,\dots,p_4\}=\emptyset. \tag{4}$$

Let us use the same symbol w_i for the point on W_2 lying over $w_i\in W$ for $i=13,14$. Then from (3), (4) and

$$2\Lambda^+_{(4,4)}\Big(\sum_{i=1,2}3E_i^0+\sum_{3\le i\le 6}2E_i^0\Big)\subset\Lambda^+,$$

we infer that the base locus of $\tilde{\Lambda}^+$ is at most $\overline{C}_4\cap\overline{C}_5=\{w_{13},w_{14}\}$, where $\overline{C}_5=\overline{q}^*(C_5)-\sum_{i=1,2}3E_i^0-\sum_{3\le i\le 6}2E_i^0$ is the strict transform on W_2 of C_5.

Now let us show that w_{13} and w_{14} are at most ordinary double points of general members of $\tilde{\Lambda}^+$ using the argument above. Let C_6 be a general member of the linear system $\Lambda^+_{(2,2)}(\sum_{1\le i\le 6}E_i^0)$. We denote by $\overline{C}_6=\overline{q}^*(C_6)-\sum_{1\le i\le 6}E_i^0$ the strict transform on W_2 of C_6. Then since

$$\begin{aligned}\Lambda^+_{(2,2)}\Big(\sum_{1\le i\le 6}E_i^0\Big)+\Lambda^+_{(2,2)}\Big(\sum_{i=1,2}(E_i^0+E_i^\vee)+\sum_{3\le i\le 6}E_i^0\Big)&\\ +\Lambda^+_{(4,4)}\Big(\sum_{i=1,2}3E_i^0+\sum_{3\le i\le 6}2E_i^0\Big)&\subset\Lambda^+,\end{aligned}$$

the divisor $\sum_{4\le i\le 6}\overline{C}_i+\sum_{i=1,2}2q'^{-1}_*(E_i^0)$ is a member of $\tilde{\Lambda}^+$. By $\overline{C}_4\cap\overline{C}_6=\{w_{11},w_{12}\}$ together with (3) and (4), we deduce that both w_{13} and w_{14} are ordinary double points of $\sum_{4\le i\le 6}\overline{C}_i+\sum_{i=1,2}2q'^{-1}_*(E_i^0)$. Thus w_{13} and w_{14} are at most ordinary double points of general members of $\tilde{\Lambda}^+$. Hence any general member B_2' of $\tilde{\Lambda}^+$ satisfies all the conditions given in Lemma 2. □

REMARK 1. Note that if $k=0$ or 1, then the isomorphism class of the quadruple $(W_0, \iota_0|_{W_0}, q'_*(B_2), \sum_{i=1,2} E_i^0)$ depends only on the isomorphism class of X. This is verified as follows. In the construction of X above, the morphism $\pi : Y \to X$ is the unramified double cover corresponding to $\mathrm{Tors}(X)$, and $p : \tilde{Y} \to Y$ is the shortest one among all composites of quadratic transformations such that the variable part of $p^*|K_Y|$ is free from base points. The morphism $\Phi_{-K_{W_0}} \circ q' \circ f$ is the canonical map of $\tilde{Y}$, where $\Phi_{-K_{W_0}} : W_0 \to \mathbb{P}^{6-2k}$ is the anti-canonical map of W_0. We have $\deg \Phi_{-K_{W_0}} = 1$ for $k = 0, 1$ and $\deg \Phi_{-K_{W_0}} = 2$ for $k = 2$. Thus if $k = 0$ or 1, then W_0 is the minimal desingularization of the normalization of the canonical image $Z = \Phi_{K_{\tilde{Y}}}(\tilde{Y}) \subset \mathbb{P}^{6-2k}$, since $\Phi_{-K_{W_0}}$ contracts no (-1)-curves. Now since the divisor $\sum_{i=1,2} E_i^0$ on W_0 is the image by $q' \circ f$ of the fixed part of $p^*|K_Y|$, we infer from the argument above that the isomorphism class of the quadruple $(W_0, \iota_0|_{W_0}, q'_*(B_2), \sum_{i=1,2} E_i^0)$ depends only on the isomorphism class of X. Note also that $q' : W_2 \to W_0$ is the blowing-up of W_0 at all non-negligible singularities of $q'_*(B_2)$.

Appendix

Let us prove Lemma 1. We use the same symbol g for the automorphism of W corresponding to $g \in G$. Let $\{U_i\}_{i \in I}$ be an open covering of W such that the divisor F is given by $f_i = 0$ on U_i, where f_i is a meromorphic function on U_i. We take $\{U_i\}_{i \in I}$ in such a way that there exists a left action of G on I such that $g(U_i) = U_{g \cdot i}$ for any $g \in G$. Let $\cup_{i \in I} U_i \times \mathbb{C}$ be the total space of the line bundle F, such that $(p, \zeta_i) \in U_i \times \mathbb{C}$ and $(p, \zeta_j) \in U_j \times \mathbb{C}$ give the same point on $\cup_{i \in I} U_i \times \mathbb{C}$, if and only if $\zeta_i = (f_i/f_j)(p)\zeta_j$. We denote by $\pi : \cup_{i \in I} U_i \times \mathbb{C} \to W$ the natural projection.

We take a system $(h_i)_{i \in I}$ of defining equations of B such that $h_i = (f_i/f_j)^n h_j$ on $U_i \cap U_j$ hold. Here h_i is a holomorphic function on U_i for each i. Then the variety V is defined by $\zeta_i^n - h_i = 0$ on $U_i \times \mathbb{C}$. Since $h_i/f_i^n = h_j/f_j^n$ gives a meromorphic function on W corresponding to the principal divisor $B - nF$, we have

$$g^* h_{g \cdot i} = c_g (g^* f_{g \cdot i}/f_i)^n h_i \tag{5}$$

on U_i for each $g \in G$, where $c : g \mapsto c_g$ is the character of G given in Lemma 1. Take a constant $c'_g \in \mathbb{C}^\times$ satisfying ${c'_g}^n = c_g$. Then

$$(p, \zeta_i) \mapsto (g(p), \zeta_{g \cdot i}) = (g(p), c'_g (g^* f_{g \cdot i}/f_i)(p)\zeta_i)$$

gives an automorphism of $\cup_{i \in I} U_i \times \mathbb{C}$. This automorphism induces that of V, say ψ_g, since (5) holds.

Now assume that the action of G on W lifts to that on V. We denote by φ_g the automorphism of $\cup_{i \in I} U_i \times \mathbb{C}$ corresponding to $g \in G$. Then from $\varphi_g = (\varphi_g \circ \psi_g^{-1}) \circ \psi_g$ and $\pi \circ (\varphi_g \circ \psi_g^{-1}) = \pi$, we infer that φ_g is given by

$$(p, \zeta_i) \mapsto (g(p), \zeta_{g \cdot i}) = (g(p), \chi_g (g^* f_{g \cdot i}/f_i)(p)\zeta_i), \tag{6}$$

where $\chi_g \in \mathbb{C}^\times$ is a constant such that $\chi_g^n = c_g$. Since $g \mapsto \varphi_g$ is an action of G, we see that $\chi : g \mapsto \chi_g$ is a character of G. Thus we have $c \in \mathrm{Im}(\Psi)$.

Assume conversely that $c \in \mathrm{Im}(\Psi)$. We define an automorphism $\varphi_{\chi,g}$ of V by $(p,\zeta_i) \mapsto (g(p),\zeta_{g\cdot i}) = (g(p),\chi_g(g^* f_{g\cdot i}/f_i)(p)\zeta_i)$ for each $\chi \in \Psi^{-1}(c)$ and $g \in G$. Then it is easily verified that $\varphi_\chi : g \mapsto \varphi_{\chi,g}$ is a lifting of the action of G on W. The set $\{\varphi_\chi\}_{\chi\in\Psi^{-1}(c)}$ is that of all liftings of the action of G on W. $\square$

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