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WEAKLY COMPATIBLE MAPS IN CONE METRIC SPACES

Abstract. The object of this paper is to establish a theorem for a unique common fixed point of four self mappings, weakly compatibile in pairs and satisfying a generalized contractive condition in a cone metric space. Our result generalizes and synthesizes the results of Abbas–Jungck [1], Arshad et al. [2], Huang–Zhang [3] and Vetro [8].

1. Introduction

There has been a number of generalizations of metric space. One such generalization is that of a cone metric space initiated by Huang and Zhang [3]. In this space they replaced the set of real numbers of a metric space by an ordered Banach space and gave some fundamental results for a self map satisfying a contractive condition. In [1] Abbas and Jungck, generalized the result of [3] for two self maps through weak compatibility in a normal cone metric space. Along the same lines, Vetro [8] proved some fixed point theorems for two self maps satisfying a contractive condition through weak compatibility. In [5] the authors introduce the concept of compatibility in cone metric space. Recently, Rezapour and Hamlbarani [6] were able to omit the assumption of normality in a cone metric space, which is a milestone in developing fixed point theory. Also in [2] Arshad et al. proved a fixed point theorem for three self map adopting the contractive condition of [7] through weak compatibility.

In this paper, we establish a theorem postulating a unique common fixed point for four self maps through weak compatibility satisfying a more generalized contractive condition than the one adopted in [1, 2, 3, 8] in a non-normal cone metric space. Our results generalize, extend and unify several well-known fixed point results in cone metric spaces. An example illustrates the main result of this paper.

2. Preliminaries

DEFINITION 1 ([3]). *Let E be a real Banach space and P be a subset of E. Then P is called a* cone *if*

(i) *P is closed, nonempty and $P \neq \{0\}$;*
(ii) *$a,b \in \mathbb{R}$, $a,b \geq 0$, and $x,y \in P$ imply $ax+by \in P$;*
(iii) *$x \in P$ and $-x \in P$ imply $x = 0$.*

Given a cone $P \subseteq E$, we define a partial ordering "$\leq$" in E by $x \leq y$ if $y-x \in P$. We write $x < y$ to denote $x \leq y$ and $x \neq y$, and we write $x \ll y$ to denote $y-x \in P^0$, where P^0 stands for the interior of P.

Note 1. For $a \geq 0$ and $x \in P$, taking $b = 0$ in (ii) we have $ax \in P$.
2. taking $a = b = 0$ in (ii) we have $0 \in P$.

PROPOSITION 1 ([4]). *Let P be a cone in a real Banach space E. If $a \in P$ and $a \leq ka$ for some $k \in [0,1)$, then $a = 0$.*

Proof. For $a \in P, k \in [0,1)$ and $a \leq ka$ gives $(k-1)a \in P$. As $0 \leq k < 1$ we have $1-k > 0$ which gives $1/(1-k) > 0$. So by Note 1, $\frac{1}{1-k}(k-1)a \in P$ implies $-a \in P$. Now $a \in P$ and $-a \in P$, which implies $a = 0$, by Definition 1, (iii). □

PROPOSITION 2 ([4]). *Let P be a cone in a real Banach space E. If $a \in E$ and $a \ll c$ for all $c \in P^0$, then $a = 0$.*

REMARK 1. (See [6].) We have $\lambda P^0 \subseteq P^0$ for $\lambda > 0$, and $P^0 + P^0 \subseteq P^0$.

DEFINITION 2 ([3]). *Let X be a nonempty set. Suppose that we are given a mapping $d\colon X \times X \to E$ that satisfies:*

(a) $0 \leq d(x,y)$ *for all $x,y \in X$, and $d(x,y) = 0$ if and only if $x = y$;*
(b) $d(x,y) = d(y,x)$ *for all $x,y \in X$;*
(c) $d(x,y) \leq d(x,z) + d(z,y)$ *for all $x,y,z \in X$.*

Then d is called a cone metric *on X, and (X,d) is called a* cone metric space.

For examples of cone metric spaces we refer to Huang–Zhang [3].

DEFINITION 3 ([3]). *Let (X,d) be a cone metric space. Let $\{x_n\}$ be a sequence in X and $x \in X$. If for every $c \in E$ with $0 \ll c$ there is a positive integer N_c such that for all $n > N_c$ we have $d(x_n,x) \ll c$, then the sequence $\{x_n\}$ is said to* converge *to x, and x is called the* limit *of $\{x_n\}$. We write $\lim_{n\to\infty} x_n = x$ or $x_n \to x$, as $n \to \infty$.*

DEFINITION 4 ([3]). *Let (X,d) be a cone metric space. Let $\{x_n\}$ be a sequence in X. If for any $c \in E$ with $0 \ll c$ there is a N such that for all $n,m > N$ we have $d(x_n,x_m) \ll c$, then the sequence $\{x_n\}$ is said to be a* Cauchy sequence *in X.*

REMARK 2. It follows from the above definitions that if $\{x_{2n}\}$ is a subsequence of a Cauchy sequence $\{x_n\}$ in a cone metric space (X,d) and $x_{2n} \to z$ then $x_n \to z$.

Proof. As $x_{2n} \to z$ and $\{x_n\}$ is Cauchy sequence, for any $c \in E$ with $0 \ll c$ there is a N such that for all $n > N$,

$$d(x_n,x_{2n}) \ll c/2 \quad \text{and} \quad d(x_{2n},z) \ll c/2.$$

Now,

$$d(x_n,z) \leq d(x_n,x_{2n}) + d(x_{2n},z) \ll c/2 + c/2 \;=\; c.$$

Therefore $d(x_n,z) \ll c$, for all $n > N$. Thus $x_n \to z$.

DEFINITION 5 ([3]). *Let (X,d) be a cone metric space. If every Cauchy sequence in X is convergent in X, then X is called a* complete *cone metric space.*

PROPOSITION 3. *Let (X,d) be a cone metric space and P be a cone in a real Banach space E. If $u \leq v$ and $v \ll w$ then $u \ll w$.*

DEFINITION 6 ([1]). *Let f and g be self maps of a set X. If $w = fx = gx$, for some $x \in X$, then w is called a* point of coincidence *of f and g.*

DEFINITION 7 ([6]). *Let X be any set. A pair of self maps (f,g) is said to be* weakly compatible *if $u \in X$ and $fu = gu$ imply $gfu = fgu$.*

PROPOSITION 4 ([1]). *Let (f,g) be a pair of weakly compatible self maps of a set X. If f and g have a unique point of coincidence $w = fx = gx$, then w is the unique common fixed point of f and g.*

LEMMA 1. *Let (X,d) be a cone metric space with respect to a cone P in a real Banach space E, and take $k_1, k_2, k > 0$. Suppose that $x_n \to x$ and $y_n \to y$, in X and*

$$ka \leq k_1 d(x_n,x) + k_2 d(y_n,y). \tag{1}$$

Then $a = 0$.

Proof. As $x_n \to x$ and $y_n \to y$, there exists a positive integer N_c such that

$$\frac{c}{(k_1+k_2)} - d(x_n,x), \ \ \frac{c}{(k_1+k_2)} - d(y_n,y) \ \in \ P^0, \qquad \forall n > N_c.$$

Therefore, by Remark 1, we have

$$\frac{k_1 c}{(k_1+k_2)} - k_1 d(x_n,x), \ \ \frac{k_2 c}{(k_1+k_2)} - k_2 d(y_n,y) \ \in \ P^0, \qquad \forall n > N_c.$$

Again by adding and Remark 1, we have $c - k_1 d(x_n,x) - k_2 d(y_n,y) \in P^0$ for all $n > N_c$. From (1) and Proposition 3 we have $c - ka \in P^0$, i.e. $ka \ll c$ for each $c \in P^0$. Finally, by Proposition 2, we have $a = 0$ since $k > 0$. □

3. Main results

THEOREM 1. *Let (X,d) be a complete cone metric space with respect to a cone P contained in a real Banach space E. Suppose that A, B, S and T are self mappings $X \to X$ satisfying:*

3.1. $A(X) \subseteq T(X), B(X) \subseteq S(X)$;

3.2. the pairs (A,S) and the (B,T) are weakly compatible;

3.3. one of $A(X), S(X), B(X), T(X)$ is complete;

3.4. for some $\lambda, \mu, \delta, \gamma \in [0,1)$ with $\lambda + \mu + \delta + 2\gamma < 1$ we have

$$d(Ax,By) \leq \lambda d(Ax,Sx) + \mu d(By,Ty) + \delta d(Sx,Ty) + \gamma[d(Ax,Ty) + d(Sx,By)]$$

for all $x, y \in X$.

Then A, B, S and T have a unique common fixed point in X.

Proof. Let $x_0 \in X$ be any point in X. Using *3.4* we construct sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\begin{cases} Ax_{2n} = Tx_{2n+1} = y_{2n}, \\ Bx_{2n+1} = Sx_{2n+2} = y_{2n+1}, \end{cases} \tag{2}$$

for all n. Our first aim is to show that $\{y_n\}$ is a Cauchy sequence in X.

STEP 1. Taking $x = x_{2n}$ and $y = x_{2n+1}$ in *3.4* we get

$$\begin{aligned} d(Ax_{2n}, Bx_{2n+1}) \quad &\leq \quad \lambda d(Ax_{2n}, Sx_{2n}) + \mu d(Bx_{2n+1}, Tx_{2n+1}) + \delta d(Sx_{2n}, Tx_{2n+1}) \\ &\qquad + \gamma[d(Ax_{2n}, Tx_{2n+1}) + d(Sx_{2n}, Bx_{2n+1})]. \end{aligned}$$

Using (2) we get

$$\begin{aligned} d(y_{2n}, y_{2n+1}) \quad &\leq \quad \lambda d(y_{2n}, y_{2n-1}) + \mu d(y_{2n+1}, y_{2n}) + \delta d(y_{2n-1}, y_{2n}) \\ &\qquad + \gamma[d(y_{2n}, y_{2n}) + d(y_{2n-1}, y_{2n+1})] \\ &\leq \quad \lambda d(y_{2n}, y_{2n-1}) + \mu d(y_{2n+1}, y_{2n}) + \delta d(y_{2n-1}, y_{2n}) \\ &\qquad + \gamma[d(y_{2n-1}, y_{2n}) + d(y_{2n}, y_{2n+1})]. \end{aligned}$$

By writing $d(y_n, y_{n+1}) = d_n$, we have

$$d_{2n} \leq \lambda d_{2n-1} + \mu d_{2n} + \delta d_{2n-1} + \gamma[d_{2n} + d_{2n-1}],$$

i.e. $(1 - \mu - \gamma)d_{2n} \leq (\lambda + \delta + \gamma)d_{2n-1}$, which implies

$$d_{2n} \leq h d_{2n-1}, \tag{3}$$

where $h = (\lambda + \delta + \gamma)/(1 - \mu - \gamma)$. In view of *3.4*, we deduce that $h < 1$.

Taking $x = x_{2n+2}$ and $y = x_{2n+1}$ in *3.4*, we get

$$\begin{aligned} &d(Ax_{2n+2}, Bx_{2n+1}) \\ &\quad \leq \lambda d(Ax_{2n+2}, Sx_{2n+2}) + \mu d(Bx_{2n+1}, Tx_{2n+1}) + \delta d(Sx_{2n+2}, Tx_{2n+1}) \\ &\qquad + \gamma[d(Ax_{2n+2}, Tx_{2n+1}) + d(Sx_{2n+2}, Bx_{2n+1})]. \end{aligned}$$

Using (2) we get

$$\begin{aligned} d(y_{2n+2}, y_{2n+1}) \quad &\leq \quad \lambda d(y_{2n+2}, y_{2n+1}) + \mu d(y_{2n+1}, y_{2n}) + \delta d(y_{2n+1}, y_{2n}) \\ &\qquad + \gamma[d(y_{2n+2}, y_{2n}) + d(y_{2n+1}, y_{2n+1})] \\ &\leq \quad \lambda d(y_{2n+2}, y_{2n+1}) + \mu d(y_{2n+1}, y_{2n}) + \delta d(y_{2n+1}, y_{2n}) \\ &\qquad + \gamma[d(y_{2n+2}, y_{2n+1}) + d(y_{2n+1}, y_{2n})]. \end{aligned}$$

We therefore have

$$d_{2n+1} \leq \lambda d_{2n+1} + \mu d_{2n} + \delta d_{2n} + \gamma[d_{2n+1} + d_{2n}],$$

i.e. $(1-\lambda-\gamma)d_{2n+1} \leq (\mu+\delta+\gamma)d_{2n}$, which implies

$$d_{2n+1} \leq kd_{2n}, \tag{4}$$

where $k = (\mu+\delta+\gamma)/(1-\lambda-\gamma)$. By condition *3.4*, we have $k < 1$.

In view of (3) and (4) we have

$$d_{2n+1} \leq kd_{2n} \leq hkd_{2n-1} \leq k^2hd_{2n-2} \leq \ldots \leq k^{n+1}h^n d_0$$

where $d_0 = d(y_0, y_1)$, and

$$d_{2n} \leq hd_{2n-1} \leq hkd_{2n-2} \leq h^2kd_{2n-3} \leq \ldots \leq h^n k^n d_0.$$

Therefore, $d_{2n+1} \leq k^{n+1}h^n d_0$ and $d_{2n} \leq h^n k^n d_0$. Also,

$$d(y_n, y_{n+p}) \leq d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + \cdots + d(y_{n+p}, y_{n+p-1}),$$

i.e.

$$d(y_n, y_{n+p}) \leq d_n + d_{n+1} + \cdots + d_{n+p-1}. \tag{5}$$

We next suppose that $n = 2m$ is even. By (5) we have

$$\begin{aligned} d(y_{2m}, y_{2m+p}) &\leq [h^m k^m + h^m k^{m+1} + h^{m+1}k^{m+1} + h^{m+1}k^{m+2} + \cdots] d_0 \\ &= h^m k^m [1 + k + hk + hk^2 + h^2k^2 + \cdots] d_0 \\ &= h^m k^m [(1 + hk + h^2k^2 + \cdots) + (k + hk^2 + h^2k^3 + \cdots)] d_0 \\ &= h^m k^m (1+k)[1 + hk + h^2k^2 + h^3k^3 \cdots] d_0. \end{aligned}$$

Since $hk < 1$ and P is closed, we conclude that

$$d(y_{2m+p}, y_{2m}) \leq (hk)^m \frac{1+k}{1-hk} d_0. \tag{6}$$

Now for $c \in P^0$, there exists $r > 0$ such that $c - y \in P^0$ if $\|y\| < r$. Choose a positive integer N_c such that $\|(hk)^m(1+k)d_0/(1-hk)\| < r$ for all $m \geq N_c$, which implies that

$$c - (hk)^m \frac{1+k}{1-hk} d_0 \in P^0.$$

On the other hand, (6) means that

$$(hk)^m \frac{1+k}{1-hk} d_0 - d(y_{2m+p}, y_{2m}) \in P.$$

So we have $c - d(y_{2m+p}, y_{2m}) \in P^0$ for all $m > N_c$ and for all p by Proposition 3.

The same argument applies if $n = 2m+1$ is odd. Thus, $d(y_{n+p}, y_n) \ll c$, for all p and for all $n \geq N_c$. Hence $\{y_n\}$ is a Cauchy sequence in X.

Case 1: $S(X)$ is complete. Since $\{y_n\}$ is a Cauchy sequence in X, it follows that $\{y_{2n+1} = Sx_{2n+2}\}$ is a Cauchy sequence in $S(X)$, which is complete. So $y_{2n+1} \to z = Su$ for some $u \in X$. Now,

$$\begin{aligned} d(Au, Su) &\leq d(Au, Bx_{2n+1}) + d(Bx_{2n+1}, Su) \\ &= d(y_{2n+1}, Su) + d(Au, Bx_{2n+1}). \end{aligned}$$

Using *3.4* with $x = u$ and $y = x_{2n+1}$, we have

$$\begin{aligned} d(Au, Su) &\leq d(y_{2n+1}, Su) + \lambda d(Au, Su) + \mu d(Bx_{2n+1}, Tx_{2n+1}) + \delta d(Su, Tx_{2n+1}) \\ &\quad + \gamma[d(Au, Tx_{2n+1}) + d(Bx_{2n+1}, Su)] \\ &= d(y_{2n+1}, Su) + \lambda d(Au, Su) + \mu d(y_{2n+1}, y_{2n}) + \delta d(Su, y_{2n}) \\ &\quad + \gamma[d(Au, y_{2n}) + d(y_{2n+1}, Su)] \\ &\leq d(y_{2n+1}, Su) + \lambda d(Au, Su) + \mu[d(y_{2n+1}, Su) + d(Su, y_{2n})] + \delta d(Su, y_{2n}) \\ &\quad + \gamma[d(Au, Su) + d(Su, y_{2n}) + d(y_{2n+1}, Su)]. \end{aligned}$$

Thus

$$(1 - \lambda - \gamma) d(Au, Su) \leq (\mu + \delta + \gamma) d(y_{2n}, Su) + (1 + \mu + \gamma) d(y_{2n+1}, Su).$$

As $y_{2n} \to Su, y_{2n+1} \to Su$ and $1 - \lambda - \gamma > 0$, using Lemma 1, we have $d(Au, Su) = 0$, and we get $Au = Su$. Thus $Au = Su = z$. Therefore z is a point of coincidence of the pair (A, S). Since (A, S) is weakly compatible, $Az = Sz$.

STEP 2. As $A(X) \subseteq T(X)$, there exists $v \in X$ such that $z = Au = Tv$. So

$$z = Au = Su = Tv. \tag{7}$$

Taking $x = u$ and $y = v$ in *3.4* we have

$$d(Au, Bv) \leq \lambda d(Au, Su) + \mu d(Bv, Tv) + \delta d(Su, Tv) + \gamma[d(Au, Tv) + d(Bv, Su)].$$

Using (7) we have

$$d(z, Bv) \leq (\mu + \gamma)\, d(z, Bv).$$

As $\mu + \gamma < 1$, using Proposition 1, it follows that $d(Bv, z) = 0$ and we get $Bv = z$. As the pair (B, T) is weak compatible we get $Bz = Tz$. Taking $x = z, y = z$ in *3.4* and using $Az = Sz$, $Bz = Tz$ we get

$$d(Az, Bz) \leq (\delta + 2\gamma)\, d(Az, Bz).$$

As $\delta + 2\gamma \in [0, 1)$ we get $Az = Bz$, by Proposition 1 and we have $Az = Sz = Bz = Tz$. Thus z is a point of coincidence of the four self maps A, B, S, T.

Case 2: $T(X)$ is complete. The proof of this case is similar to Case 1.

Case 3: $A(X)$ is complete. $\{y_n\}$ is a Cauchy sequence in X. Hence $\{y_{2n} = Ax_{2n}\}$ is a Cauchy sequence in $A(X)$, which is complete. Hence $y_{2n} \to z = Aw$ for some $w \in X$. As $A(X) \subseteq T(X)$ there exists $p \in X$ such that $z = Aw = Tp$. It follows from Case 2 that $Az = Bz = Sz = Tz$. Thus, also in this case, the maps A, B, S, T have a common point of coincidence.

Case 4: $B(X)$ is complete. The proof of this case is similar to Case 3.

STEP 3. We have $z = Bz = Sz$. Let $Au = Su$ be another point of coincidence of the pair (A,S). Now

$$\begin{aligned} d(z,Au) &\leq d(z,Bx_{2n+1}) + d(Bx_{2n+1},Au) \\ &= d(z,y_{2n+1}) + d(Au,Bx_{2n+1}). \end{aligned}$$

Taking $x = u$ and $y = x_{2n+1}$ in *3.4* we get

$$\begin{aligned} d(z,Au)) &\leq d(z,y_{2n+1}) + \lambda d(Au,Su) + \mu d(Bx_{2n+1},Tx_{2n+1}) + \delta d(Su,Tx_{2n+1}) \\ &\quad +\gamma[d(Au,Tx_{2n+1}) + d(Bx_{2n+1},Su)] \\ &= d(z,y_{2n+1}) + \mu d(y_{2n+1},y_{2n}) + \delta d(Au,y_{2n}) + \gamma[d(Au,y_{2n}) \\ &\quad +d(y_{2n+1},Au)] \\ &\leq d(z,y_{2n+1}) + \mu[d(y_{2n+1},z) + d(z,y_{2n})] + \delta[d(Au,z) + d(z,y_{2n})] \\ &\quad +\gamma[d(Au,z) + d(z,y_{2n}) + d(y_{2n+1},z) + d(z,Au)]. \end{aligned}$$

Thus

$$(1-\delta-2\gamma)d(z,Au) \leq (1+\mu+\gamma)d(z,y_{2n+1}) + (\mu+\delta+\gamma)d(z,y_{2n}).$$

Since $y_{2n+1} \to z$ and $y_{2n} \to z$, and we have $1-\delta-2\gamma > 0$, using Lemma 1 we obtain $d(z,Au) = 0$ and so $Au = z$. Hence the point of coincidence of (A,S) is unique. As the pair (A,S) is weakly compatible by Proposition 4, z is the unique common fixed point of A and S. Hence $z = Az = Bz = Sz = Tz$ is the unique fixed point of A,B,S,T. ☐

Taking and $T = S$ in Theorem 1 we have the following corollary for three self mappings:

COROLLARY 1. *Let (X,d) be a complete cone metric space with respect to a cone P contained in a real Banach space E. Let A,B and S be self mappings on X satisfying:*

1. $A(X) \cup B(X) \subseteq S(X)$*;*
2. *the pairs (A,S) and (B,S) are weakly compatible;*
3. *one of $S(X)$ or $A(X) \cup B(X)$ is complete;*
4. *for some $\lambda,\mu,\delta,\gamma \in [0,1)$ with $\lambda+\mu+\delta+2\gamma < 1$, we have*

$$d(Ax,By) \leq \lambda d(Ax,Sx) + \mu d(By,Sy) + \delta d(Sx,Sy) + \gamma[d(Ax,Sy) + d(By,Sy)],$$

for all $x,y \in X$.

Then A,B and S have a unique common fixed point in X.

Proof. (The case in which $A(X) \cup B(X)$ is complete.) This follows from the cases in which $A(X)$ or $B(X)$ is complete. For these we have $y_{2n} = Ax_{2n} \to z \in A(X)$ and $y_{2n} = Ax_{2n} \to z \in B(X)$ as discussed above in Cases 3 and 4 respectively. ☐

In [2] Arshad et al. established the following result:

THEOREM 2 ([2]). *Let (X,d) be a cone metric space and P be an order cone. Let $S,T,f : X \to X$ be such that $S(X) \cup T(X) \subseteq f(X)$. Assume that the following conditions hold:*

(i) $d(Sx,Ty) \leq \alpha d(fx,Sx) + \beta d(fy,Ty) + \gamma d(fx,fy)$ *for all $x,y \in X$ with $x \neq y$, where α,β,γ are non-negative real numbers with $\alpha+\beta+\gamma < 1$;*

(ii) $d(Sx,Tx) < d(fx,Sx) + \beta d(fx,Tx)$ *for all $x \in X$ with $Sx \neq Tx$.*

If $f(X)$ or $S(X) \cup T(X)$ is a complete subspace of X, then S,T and f have a unique point of coincidence. Moreover, if (S,f) and (T,f) are weakly compatible, then S,T and f have a unique common fixed point.

REMARK 3. Corollary 1 is a more complete result. Its contractive condition is more general than the one adopted in the above theorem. Moreover, it does not require assumption (ii) at all.

Taking $B = A$ and $T = S$ in Theorem 1, we obtain

COROLLARY 2. *Let (X,d) be a complete cone metric space with respect to a cone P contained in a real Banach space E. Let A and S be self mappings on X satisfying:*

1. *$A(X) \subseteq S(X)$;*
2. *the pair (A,S) is weakly compatible;*
3. *one of $A(X)$ or $S(X)$ is complete;*
4. *for some $\lambda,\mu,\delta,\gamma \in [0,1)$ with $\lambda+\mu+\delta+2\gamma < 1$ we have*

$$d(Ax,Ay) \leq \lambda d(Ax,Sx) + \mu d(Ay,Sy) + \delta(Sx,Sy) + \gamma[d(Ax,Sy) + d(Sx,Ay)]$$

for all $x,y \in X$.

Then A and S have a unique common fixed point in X.

REMARK 4. Taking $\lambda = \mu = \gamma = 0$ and $\delta = k$ in Theorem 1, we get Theorem 2.1 of Abbas–Jungck [1] even in a non-normal cone metric space when one of $f(X)$ or $g(X)$ is complete.

REMARK 5. Taking $\lambda = \mu = k$ and $\delta = \gamma = 0$ in Theorem 1, then $k \in [0,1/2)$ we get Theorem 2.4 of Abbas–Jungck [1] even in a non-normal cone metric space.

REMARK 6. Taking $\lambda = \mu = \delta = 0$ and $\gamma = k$ in Theorem 1, then $k \in [0,1/2)$ and we get Theorem 2.4 of Abbas–Jungck [1] even in a non-normal cone metric space.

Thus Theorem 1 of this paper synthesizes and generalizes almost all the results of Abbas–Jungck [1].

The following definition and theorem appear in Vetro [8]:

DEFINITION 8 ([8]). *Let (X,d) be a cone metric space and P be a normal cone with normal constant C. Given mappings $f,g\colon X\to X$, we say f is a g-*weak contraction *if*

$$d(fx,fy)\le \alpha d(fx,gx)+\beta d(fy,gy)+\gamma d(gx,gy)$$

for all $x,y\in X$, where $\alpha,\beta,\gamma\in[0,1)$ and $\alpha+\beta+\gamma<1$.

THEOREM 3 ([8]). *Let (X,d) be a cone metric space and P be a normal cone with normal constant C. Let $f,g\colon X\to X$ be such that $f(X)\subseteq g(X)$. Suppose that f is a g-weak contraction such that*

- *$f(g(x))=g(g(x))$ if and only if $f(x)=g(x)$*

If $f(X)$ or $g(X)$ is a complete subspace of X, then the mappings f and g have a unique common fixed point in X. Moreover, for any $x_0\in X$, the $f-g$ sequences $\{fx_n\}$ of the initial point x_0 converge to the fixed point.

REMARK 7. Corollary 2 is a more general result than Theorem 3. Here the contractive condition is more general and the normality of the cone metric space has not been assumed.

Taking $S=I$, the identity map on X, in above corollary we get

COROLLARY 3. *Let (X,d) be a complete cone metric space. Let A be self mapping on X satisfying:*

- *for some $\lambda,\mu,\delta,\gamma\in[0,1)$ with $\lambda+\mu+\delta+2\gamma<1$ we have*

$$d(Ax,Ay)\le \lambda d(Ax,x)+\mu d(Ay,y)+\delta d(x,y)+\gamma[d(Ax,y)+d(Ay,x)],$$

for all $x,y\in X$.

Then the map A has the unique fixed point in X and for any $x\in X$, the iterative sequence $\{A^n x\}$ converges to the fixed point.

Proof. Existence and uniqueness of the fixed point follows from Corollary 2, by taking $S=I$ there. Taking $T=S=I$, $B=A$ and $x_0=x$ in Theorem 1 we have $y_0=Ax$, $y_1=A^2x$, $\ldots$, $y_{n+1}=A^n x$, etc. Thus for each x, the sequence $\{A^n x\}$ converges to the fixed point z. □

Taking $\gamma=0$ in corollary 3 we have

COROLLARY 4. *Let (X,d) be a complete cone metric space. Let A be self mapping on X satisfying:*

- *for some $\lambda,\mu,\delta\in[0,1)$ with $\lambda+\mu+\delta<1$, we have*

$$d(Ax,Ay)\le \lambda d(Ax,x)+\mu d(Ay,y)+\delta d(x,y),$$

for all $x,y\in X$.

Then the map A has the unique fixed point in X and for any $x \in X$, the iterative sequence $\{A^n x\}$ converges to the fixed point.

REMARK 8. Taking $\lambda = k$ and $\mu = \delta = 0$ in Corollary 4, we get Theorem 1 of Huang–Zhang [3] even for a non-normal cone metric space.

REMARK 9. Taking $\lambda = \mu = k$ and $\delta = 0$ in Corollary 4, $k \in [0, 1/2)$ and we get Theorem 3 of Huang–Zhang [3] even for a non-normal cone metric space.

REMARK 10. Taking $\lambda = \mu = \delta = 0$ and $\gamma = k$ in Corollary 3, $k \in [0, 1/2)$ and we get Theorem 4 of Huang–Zhang [3] even for a non-normal cone metric space.

Thus Theorem 1 of this paper synthesizes and generalizes almost all the results of Huang et. al [3] without assuming normality of the cone metric space.

EXAMPLE 1. (of Theorem 1). Let $X = \mathbb{R}^+$, $E = \mathbb{R}^2$, and consider the cone

$$P = \{(x, y) \in \mathbb{R}^2 : x \geq 0, y \geq 0\} \subseteq E.$$

Fix a real number $\alpha > 0$ and define a cone metric $d : X \times X \to E$ by

$$d(x, y) = |x - y|(1, \alpha).$$

Then (X, d) is a complete cone metric space. Define self maps A, B, S, T on X by

$$A(x) = B(x) = \frac{2}{3}x, \qquad S(x) = T(x) = 2x,$$

for all $x \in \mathbb{R}^+$. Conditions *3.1*, *3.2*, *3.3* of Theorem 1 hold trivially. Condition *3.4* is equivalent to

$$|x - y| \leq 2\lambda x + 2\mu y + 3\delta|x - y| + \gamma[|x - 3y| + |3x - y|]. \tag{8}$$

Consider it in the following four cases:

(a) $y \leq x \leq 3y$, in which (8) becomes

$$(2\lambda + 3\mu + 2\gamma - 1)x + (2\mu - 3\delta + 2\gamma + 1)y \geq 0.$$

(b) $3y \leq x$, in which (8) becomes

$$(2\lambda + 3\delta + 4\gamma - 1)x + (2\mu - 3\delta + 4\gamma + 1)y \geq 0.$$

(c) $3x \leq y$, in which (8) becomes

$$(2\lambda - 3\delta - 4\gamma + 1)x + (2\mu + 3\delta + 4\gamma - 1)y \geq 0.$$

(d) $x \leq y \leq 3x$, in which (8) becomes

$$(2\lambda - 3\delta + 2\gamma + 1)x + (2\mu + 3\delta + 2\gamma - 1)y \geq 0.$$

Thus condition *3.4* also holds good for $\lambda = \frac{1}{4}$, $\mu = \gamma = \frac{1}{5}$ and $\delta = \frac{1}{15}$ and 0 is the unique common fixed point of the maps A, B, S and T.

References

[1] ABBAS M. AND JUNGCK G. Common fixed point results for non commuting mappings without continuity in cone metric spaces. *J. Math. Anal. Appl. 341* (2008), 416–420.

[2] ARSHAD M., AZAM A. AND VETRO P. Some common fixed point results in cone metric spaces. *Fixed Point Theory Appl.* (2009). Art. id 493965.

[3] HUANG L. G. AND ZHANG X. Cone metric spaces and fixed point theorems of contractive mappings. *J. Math. Anal. Appl. 332* (2007), 1468–1476.

[4] ILIĆ D. AND RAKOČEVIĆ V. Quasi-contraction on a cone metric space. *Appl. Math. Lett. 22*, 5 (2009), 728–731.

[5] JAIN S., JAIN S. AND BAHADUR L. Compatibility and weak compatibility for four self maps in a cone metric spaces. *Bull. Math. Analysis Appl. 1* (2009), 1–18.

[6] REZAPOUR S. AND R. HAMLBARANI R. Some notes on the paper 'Cone metric spaces and fixed point theorems of contractive mappings'. *J. Math. Anal. Appl. 345* (2008), 719–724.

[7] SINGH B. AND JAIN S. A fixed point theorem in Menger space through weak compatibility. *J. Math. Anal. Appl. 301* (2005), 439–448.

[8] VETRO P. Common fixed points in cone metric spaces. *Rendiconti Del Circolo Matematico Di Palermo (2) 56* (2007), 464–468.

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