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# ON SOME COMPLETE METRIC SPACES OF STRONGLY SUMMABLE SEQUENCES OF FUZZY NUMBERS

**Abstract.** In this article we introduce and study the notions of $\Delta_{(r)}$-lacunary strongly summable, $\Delta_{(r)}$-Cesàro strongly summable, $\Delta_{(r)}$-statistically convergent and $\Delta_{(r)}$-lacunary statistically convergent sequences of fuzzy numbers. Consequently we construct the spaces of these four types of difference sequences and investigate the relationship among these spaces. Further we show that the spaces of $\Delta_{(r)}$-lacunary strongly summable and $\Delta_{(r)}$-Cesàro strongly summable sequences are complete metric spaces.

## 1. Introduction

The concept of fuzzy sets and fuzzy set operations was first introduced by Zadeh [15] and subsequently several authors have studied various aspects of the theory and applications of fuzzy sets. Bounded and convergent sequences of fuzzy numbers were introduced by Matloka [8] where it was shown that every convergent sequence is bounded. Nanda [9] studied the spaces of bounded and convergent sequence of fuzzy numbers and showed that they are complete metric spaces.

Functional analytic studies of the space of strongly Cesàro summable sequences of complex terms and other closely related spaces of strongly summable sequences can be found in [3].

The notion of difference sequence of complex terms was introduced by Kizmaz [5]. In this article, we define the difference operator $\Delta_{(r)}$, which we describe in the next section. With the help of this new operator, we then extend the notion of difference sequence to sequences of fuzzy numbers.

The idea of the statistical convergence of a sequence was introduced by Fast [2] and Schoenberg [14] independently, in order to extend the notion of convergence of sequences. It is also found in Zygmund [16]. Later on it was linked with summability by Fridy and Orhan [4], Maddox [7], Rath and Tripathy [11] and many others. In [10] Nuray and Savaş extended the idea to sequences of fuzzy numbers and discussed the concept of statistically Cauchy sequences of fuzzy numbers.

Among the various types of fuzzy sets, those that are defined on the set $\mathbb{R}$ of real numbers have special significance. Membership functions of these sets, which have the form

$$A : \mathbb{R} \longrightarrow [0,1]$$

clearly have a quantitative meaning and under certain conditions can be viewed as fuzzy numbers or fuzzy intervals. To view them in this way, they should capture our intuitive conception of approximate numbers or intervals, such as “numbers that are close to a given real number” or “numbers that are around a given interval of real numbers”. Such concepts are essential for characterizing states of fuzzy variables and consequently, play an important role in many applications, including fuzzy control, decision making, approximate reasoning, optimization and statistics with imprecise probabilities.

## 2. Definitions and preliminaries

Let $D$ denote the set of all closed bounded intervals $A=[A_1,A_2]$ on the real line $\mathbb{R}$. For $A,B\in D$ define

$$A\leq B \quad \text{iff} \quad A_1\leq B_1 \text{ and } A_2\leq B_2,$$
$$h(A,B)=\max(|A_1-B_1|,|A_2-B_2|).$$

Then $(D,h)$ is a complete metric space. Also $\leq$ is a partial order relation in $D$.

A fuzzy number is a fuzzy subset of the real line $\mathbb{R}$ which is bounded, convex and normal. Let $L(\mathbb{R})$ denote the set of all fuzzy numbers which are upper semi-continuous and have compact support. In other words, if $X\in L(\mathbb{R})$ then for any $\alpha\in[0,1]$, $X^\alpha$ is compact where

$$X^\alpha=\begin{cases}\{t\in\mathbb{R}:X(t)\geq\alpha\} & \text{if } \alpha\in(0,1],\\ \{t\in\mathbb{R}:X(t)>0\} & \text{if } \alpha=0.\end{cases}$$

Define a map $d:L(\mathbb{R})\times L(\mathbb{R})\longrightarrow\mathbb{R}$ by

$$d(X,Y)=\sup_{0\leq\alpha\leq1}h(X^\alpha,Y^\alpha).$$

It is straightforward to see that $d$ is a metric on $L(\mathbb{R})$. In fact $(L(\mathbb{R}),d)$ is a complete metric space.

For $X,Y\in L(\mathbb{R})$, we define

$$X\leq Y \quad \text{iff} \quad X^\alpha\leq Y^\alpha \text{ for any } \alpha\in[0,1].$$

A subset $E$ of $L(\mathbb{R})$ is said to be bounded above if there exists a fuzzy number $M$, called an upper bound of $E$, such that $X\leq M$ for every $X\in E$. $M$ is called the least upper bound or supremum of $E$ if $M$ is an upper bound and $M$ is the smallest of all upper bounds. A lower bound and the greatest lower bound or infimum are defined similarly. $E$ is said to be bounded if it is both bounded above and bounded below.

Let $r\in\mathbb{R}$. Then $\overline{r}\in L(\mathbb{R})$ is defined by

$$\overline{r}(t)=\begin{cases}1 & \text{if } t=r\\ 0 & \text{if } t\neq r.\end{cases}$$

For $X,Y\in L(\mathbb{R})$, the arithmetic operations are defined as follows:

$$\begin{aligned}(X\oplus Y)(t) &= \sup_{s\in\mathbb{R}}\{X(s)\wedge Y(t-s)\},\\ (X-Y)(t) &= \sup_{s\in\mathbb{R}}\{X(s)\wedge Y(s-t)\},\\ (X\otimes Y)(t) &= \sup_{s\in\mathbb{R}\setminus\{0\}}\{X(s)\wedge Y(t/s)\},\\ (X\div Y)(t) &= \sup_{s\in\mathbb{R}}\{X(st)\wedge Y(s)\},\end{aligned}$$

where $t \in \mathbb{R}$ and $X(s) \wedge Y(t) = \min(X(s), Y(t))$. Furthermore, if $X^\alpha = [a_1^\alpha, b_1^\alpha]$ and $Y^\alpha = [a_2^\alpha, b_2^\alpha]$, then we have

$$\begin{aligned}
(X \oplus Y)^\alpha &= [a_1^\alpha + a_2^\alpha,\ b_1^\alpha + b_2^\alpha], \\
(X - Y)^\alpha &= [a_1^\alpha - b_2^\alpha,\ b_1^\alpha - a_2^\alpha], \\
(X \otimes Y)^\alpha &= \left[\min_{i,j\in\{1,2\}} a_i^\alpha b_j^\alpha,\ \max_{i,j\in\{1,2\}} a_i^\alpha b_j^\alpha\right], \\
(Y^{-1})^\alpha &= \left[\frac{1}{b_2^\alpha}, \frac{1}{a_2^\alpha}\right], \quad a_2^\alpha > 0.
\end{aligned}$$

A sequence $X = (X_k)$ of fuzzy numbers is a function $X$ from the set $\mathbb{N}$ of all positive integers into $L(\mathbb{R})$. The fuzzy number $X_k$ denotes the value of the function at $k \in \mathbb{N}$ and is called the $k$-th term or general term of the sequence. A set $E^F$ of sequences $(X_n)$ of fuzzy numbers is said to be a sequence space of fuzzy numbers if, for $(X_n), (Y_n) \in E^F$, we have

$$(X_n) + (Y_n) = (X_n \oplus Y_n) \in E^F \quad \text{and} \quad r(X_n) = (rX_n) \in E^F,$$

where

$$rX_n(t) = \begin{cases} X_n(r^{-1}t) & \text{if } r \neq 0, \\ \bar{0} & \text{if } r = 0. \end{cases}$$

We can now state the following definitions (see [7, 9]):

A sequence $X = (X_k)$ of fuzzy numbers is said to be convergent to the fuzzy number $X_0$, written as $\lim_k X_k = X_0$, if for every $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$d(X_k, X_0) < \varepsilon \quad \text{for } k > n_0.$$

The set of convergent sequences is denoted by $c^F$. A sequence $X = (X_k)$ of fuzzy numbers is said to be Cauchy if for every $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$d(X_k, X_l) < \varepsilon \quad \text{for } k, l > n_0.$$

A sequence $X = (X_k)$ of fuzzy numbers is said to be bounded if the set $\{X_k : k \in \mathbb{N}\}$ of fuzzy numbers is bounded. The set of such bounded sequences is denoted by $\ell_\infty^F$.

The natural density of a set $K$ of positive integers is denoted by $\delta(K)$ and defined by

$$\delta(K) = \lim_n \frac{1}{n} \operatorname{card}\{k \leq n : k \in K\}.$$

A sequence $X = (X_k)$ of fuzzy numbers is said to be statistically convergent to a fuzzy number $X_0$ if for every $\varepsilon > 0$, we have

$$\lim_n \frac{1}{n} \operatorname{card}\{k \leq n : d(X_k, X_0) \geq \varepsilon\} = 0.$$

We write st-$\lim X_k = X_0$.

Throughout the article we denote by $w^F$, the set of all sequences $X = (X_k)$ of fuzzy numbers.

By a lacunary sequence $\theta = (k_p)$ where $p = 1, 2, 3, \dots$ and $k_0 = 0$, we mean an increasing sequence of non-negative integers with $h_p = (k_p - k_{p-1}) \to \infty$ as $p \to \infty$. We set $I_p = (k_{p-1}, k_p]$ for $p = 1, 2, 3 \dots$.

Let $r$ be a non-negative integer. Then for a lacunary sequence $\theta$ we define:

$$L_\theta^F(\Delta_{(r)}) = \{X \in w^F : \lim_{p\to\infty} \frac{1}{h_p} \sum_{k \in I_p} d(\Delta_{(r)} X_k, X_0) = 0 \text{ for some } X_0\},$$

where $(\Delta_{(r)} X_k) = (X_k - X_{k-r})$ and $\Delta_{(0)} X_k = X_k$ for all $k \in \mathbb{N}$. In this expansion it is important to note that we take $X_{k-r} = 0$ for non-positive values of $k - r$.

Kizmaz [5] introduced the difference operator $\Delta$, to define the spaces $c(\Delta)$, $c_0(\Delta)$ and $\ell_\infty(\Delta)$ of complex sequences. If we apply this operator to sequence of fuzzy numbers, we have $(\Delta X_k) = (X_k - X_{k+1})$ and $\Delta^0 X_k = X_k$ for all $k \in \mathbb{N}$. If we take $r = 1$ in the new difference operator, we get the operator $\Delta_{(1)}$. It is obvious that for a space $Z$, we have $X = (X_k) \in Z(\Delta_{(1)})$ if and only if $X = (X_k) \in Z(\Delta)$.

If $X \in L_\theta^F(\Delta_{(r)})$, then we say that $X$ is $\Delta_{(r)}$-lacunary strongly summable sequence of fuzzy numbers.

A sequence $X = (X_k) \in w^F$ is said to be $\Delta_{(r)}$-Cesàro strongly summable if $X$ belongs to the set

$$C^F(\Delta_{(r)}) = \left\{X \in w^F \ : \ \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} d(\Delta_{(r)} X_k, X_0) = 0 \text{ for some } X_0\right\}.$$

In particular for $\theta = (2^p)$ with $p = 1, 2, 3, \dots$, we have $L_\theta^F(\Delta_{(r)}) = C^F(\Delta_{(r)})$.

A sequence $X = (X_k) \in w^F$ is said to be $\Delta_{(r)}$-statistically convergent if $X$ belongs to the set

$$S^F(\Delta_{(r)}) = \left\{X \in w^F \ : \ \lim_n \frac{1}{n} \operatorname{card}\{k \le n : d(\Delta_{(r)} X_k, X_0) \ge \varepsilon\} = 0 \right. \\ \left. \text{for every } \varepsilon > 0 \text{ and some } X_0\right\}.$$

A sequence $X = (X_k) \in w^F$ is said to be $\Delta_{(r)}$-lacunary statistically convergent if $X$ belongs to the set

$$S_\theta^F(\Delta_{(r)}) = \left\{X \in w^F \ : \ \lim_p \frac{1}{h_p} \operatorname{card}\{k \in I_p : d(\Delta_{(r)} X_k, X_0) \ge \varepsilon\} = 0 \right. \\ \left. \text{for every } \varepsilon > 0 \text{ and some } X_0\right\}.$$

## 3. Main results

THEOREM 1. *Let $\theta$ be a lacunary sequence. Then if a sequence $X = (X_k)$ is $\Delta_{(r)}$-lacunary strongly summable then it is $\Delta_{(r)}$-lacunary statistically convergent.*

*Proof.* Suppose $X=(X_k)$ is $\Delta_{(r)}$-lacunary strongly summable to $X_0$. Then

$$\lim_{p\to\infty}\frac{1}{h_p}\sum_{k\in I_p} d(\Delta_{(r)}X_k,X_0)=0.$$

The result follows from the inequality:

$$\sum_{k\in I_p} d(\Delta_{(r)}X_k,X_0)\geq \varepsilon\ \mathrm{card}\{k\in I_p : d(\Delta_{(r)}X_k,X_0)\geq\varepsilon\},$$

and the proof is complete. □

THEOREM 2. *If a sequence $X=(X_k)$ is $\Delta_{(r)}$-bounded and $\Delta_{(r)}$-statistically convergent, then it is $\Delta_{(r)}$-Cesàro strongly summable.*

*Proof.* Suppose $X=(X_k)$ is $\Delta_{(r)}$-bounded and $\Delta_{(r)}$-statistically convergent to $X_0$. Since $X=(X_k)$ is $\Delta_{(r)}$-bounded, we can find a fuzzy number $M$ such that

$$d(\Delta_{(r)}X_k,X_0)\leq M \text{ for all } k\in\mathbb{N}.$$

Again, since $X=(X_k)$ is $\Delta_{(r)}$-statistically convergent to $X_0$, for every $\varepsilon>0$

$$\lim_n\frac{1}{n}\ \mathrm{card}\{k\leq n : d(\Delta_{(r)}X_k,X_0)\geq\varepsilon\}=0.$$

Now the result follows from the following inequality:

$$\begin{aligned}\frac{1}{n}\sum_{1\leq k\leq n} d(\Delta_{(r)}X_k,X_0) &= \frac{1}{n}\sum_{\substack{1\leq k\leq n\\ d(\Delta_{(r)}X_k,X_0)\geq\varepsilon}} d(\Delta_{(r)}X_k,X_0)+\frac{1}{n}\sum_{\substack{1\leq k\leq n\\ d(\Delta_{(r)}X_k,X_0)<\varepsilon}} d(\Delta_{(r)}X_k,X_0)\\ &\leq \frac{M}{n}\ \mathrm{card}\{k\leq n : d(\Delta_{(r)}X_k,X_0)\geq\varepsilon\}+\varepsilon.\end{aligned}$$

The proof is complete. □

THEOREM 3. *Let $\theta$ be a lacunary sequence. Then if a sequence $X=(X_k)$ is $\Delta_{(r)}$-bounded and $\Delta_{(r)}$-lacunary statistically convergent, then it is $\Delta_{(r)}$-lacunary strongly summable.*

*Proof.* This follows by similar arguments as applied to prove Theorem 2. □

THEOREM 4. *Let $\theta$ be a lacunary sequence and $X=(X_k)$ is $\Delta_{(r)}$-bounded. Then $X$ is $\Delta_{(r)}$-lacunary statistically convergent if and only if it is $\Delta_{(r)}$-lacunary strongly summable.*

*Proof.* The proof follows by combining Theorem 1 and Theorem 3. □

THEOREM 5. *If a sequence $X=(X_k)$ is $\Delta_{(r)}$-statistically convergent and $\liminf_p\left(\frac{h_p}{p}\right)>0$ then it is $\Delta_{(r)}$-lacunary statistically convergent.*

*Proof.* Assume the given conditions. For a given $\varepsilon > 0$, we have

$$\{k \in I_p : d(\Delta_{(r)}X_k, X_0) \geq \varepsilon\} \subseteq \{k \leq n : d(\Delta_{(r)}X_k, X_0) \geq \varepsilon\}.$$

Hence the proof follows from the inequality:

$$\frac{1}{p}\,\mathrm{card}\{k \leq p : d(\Delta_{(r)}X_k, X_0) \geq \varepsilon\} \geq \frac{1}{p}\,\mathrm{card}\{k \in I_p : d(\Delta_{(r)}X_k, X_0) \geq \varepsilon\},$$

and it suffices to replace $1/p$ on the right by $(h_p/p)(1/h_p)$. □

THEOREM 6. *$L_\theta^F(\Delta_{(r)})$ is a complete metric space with metric $g$ defined by*

$$g(X,Y) = \sup_p \left[\frac{1}{h_p}\sum_{k \in I_p} d(\Delta_{(r)}X_k, \Delta_{(r)}Y_k)\right].$$

*Proof.* It is easy to see that $d$ is a metric on $L_\theta^F(\Delta_{(r)})$. To prove completeness, let $(X^i)$ be a Cauchy sequence in $L_\theta^F(\Delta_{(r)})$, where $X^i = (X_k^i) = (X_1^i, X_2^i, \ldots)$ for each $i \in \mathbb{N}$. Then for a given $\varepsilon > 0$, there exists a positive integer $n_0$ such that

$$g(X^i, X^j) = \sup_p \left[\frac{1}{h_p}\sum_{k \in I_p} d(\Delta_{(r)}X_k^i, \Delta_{(r)}X_k^j)\right] < \varepsilon \ \text{ for all } \ i, j \geq n_0.$$

It follows that

$$\frac{1}{h_p}\sum_{k \in I_p} d(\Delta_{(r)}X_k^i, \Delta_{(r)}X_k^j) < \varepsilon \text{ for all } i, j \geq n_0 \text{ and } p \in \mathbb{N}.$$

Hence

$$d(\Delta_{(r)}X_k^i, \Delta_{(r)}X_k^j) < \varepsilon \text{ for all } i, j \geq n_0 \text{ and for all } k \in \mathbb{N}.$$

This implies that $(\Delta_{(r)}X_k^i)$ is a Cauchy sequence in $L(\mathbb{R})$ for all $k \geq 1$. But $L(\mathbb{R})$ is complete and so $(\Delta_{(r)}X_k^i)$ is convergent in $L(\mathbb{R})$ for all $k \geq 1$.

For simplicity, we set $\lim_{i\to\infty} \Delta_{(r)}X_k^i = \lim_{i\to\infty}(X_k^i - X_{k-r}^i) = N_k$, say, for each $k \geq 1$. By considering $k = 1, 2, 3, \ldots, r, \ldots$, we can easily conclude that $\lim_{i\to\infty} X_k^i = X_k$, exists for each $k \geq 1$. It remains to show $X = (X_k) \in L_\theta^F(\Delta_{(r)})$.

Now one can verify that

$$\lim_{j\to\infty} \frac{1}{h_p}\sum_{k \in I_p} d(\Delta_{(r)}X_k^i, \Delta_{(r)}X_k^j) < \varepsilon \ \text{ for all } \ i \geq n_0 \text{ and } p \in \mathbb{N}.$$

Thus

$$\frac{1}{h_p}\sum_{k \in I_p} d(\Delta_{(r)}X_k^i, \Delta_{(r)}X_k) < \varepsilon \ \text{ for all } \ i \geq n_0 \ \text{ and } \ p \in \mathbb{N},$$

which implies that

$$g(X^i, X) < \varepsilon \ \text{ for all } \ i \geq n_0.$$

This shows that $X = (X_k) \in L_\theta^F(\Delta_{(r)})$, and completes the proof. □

THEOREM 7. *$C^F(\Delta_{(r)})$ is a complete metric space with metric $g'$ defined by*

$$g'(X,Y) = \sup_p \left[ \frac{1}{p} \sum_{1 \le k \le p} d(\Delta_{(r)}X_k, \Delta_{(r)}Y_k) \right].$$

*Proof.* The proof is identical to the proof of Theorem 6. □

## Conclusions

In this paper, with the help of the introduced difference operator $\Delta_{(r)}$ for sequences of fuzzy numbers, we studied lacunary and Cesàro strongly summable difference sequences of fuzzy numbers and investigate their relationship with statistically convergent and lacunary statistically convergent difference sequences of fuzzy numbers. Two further suitable metrics are defined on the spaces of lacunary and Cesàro strongly summable difference sequences of fuzzy numbers under which they become complete metric spaces. The whole paper revolves around difference sequences of fuzzy numbers and there are differences between difference sequences of fuzzy numbers and difference sequences of complex terms. For example, let $(x_k)$ be a sequence of complex terms that converges to $L$. Then $(\Delta x_k)$ converges to 0. But for the fuzzy numbers, when $(X_k)$ converges to $X$ (a fuzzy number), then $(\Delta X_k)$ converges to $Z$ (a fuzzy number), where the area bounded by the curve $Z$ and the real line is twice the area of the curve bounded by $X$ and the real line. Further, the nature of the curve will be symmetric about the membership line, i.e. the line $t = 0$. It may be worthwhile to study further properties of the spaces of this paper for particular values of $r$ in the difference operator $\Delta_{(r)}$.

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### References

[1] DIAMOND P. AND KLOEDEN P. Metric spaces of fuzzy sets. *Fuzzy Sets and Systems 35*, 2 (1990), 241–249.

[2] FAST H. Sur la convergence statistique. *Colloquium Math. 2* (1951), 241–244.

[3] FREEDMAN A. R., SEMBER J. J. AND RAPHAEL M. Some Cesàro-type summability spaces. *Proc. London Math. Soc. (3) 37*, 3 (1978), 508–520.

[4] FRIDY J. A. AND ORHAN C. Statistical limit superior and limit inferior. *Proc. Amer. Math. Soc. 125*, 12 (1997), 3625–3631.

[5] KIZMAZ H. On certain sequence spaces. *Canad. Math. Bull. 24*, 2 (1981), 169–176.

[6] KLIR G. J. AND YUAN B. *Fuzzy sets and fuzzy logic*. Prentice Hall, New Delhi, 2008.

[7] MADDOX I. J. A tauberian condition for statistical convergence. *Math. Proc. Camb. Phil. Soc. 106* (1989), 277–280.

[8] MATLOKA M. Sequences of fuzzy numbers. *BUSEFAL 28* (1986), 28–37.

[9] NANDA S. On sequences of fuzzy numbers. *Fuzzy Sets and Systems 33*, 1 (1989), 123–126.

[10] NURAY F. AND SAVAŞ E. Statistical convergence of sequences of fuzzy numbers. *Math. Slovaca 45*, 3 (1995), 269–273.

[11] RATH D. AND TRIPATHY B. C. Matrix maps on sequence spaces associated with sets of integers. *Indian J. Pure Appl. Math. 27*, 2 (1996), 197–206.

[12] SAVAŞ E. A note on sequence of fuzzy numbers. *Inform. Sci. 124*, 1-4 (2000), 297–300.

[13] SAVAŞ E. On strongly $\lambda$-summable sequences of fuzzy numbers. *Inform. Sci. 125*, 1-4 (2000), 181–186.

[14] SCHOENBERG I. J. The integrability of certain functions and related summability methods. *Amer. Math. Monthly 66* (1959), 361–375.

[15] ZADEH L. A. Fuzzy sets. *Information and Control 8* (1965), 338–353.

[16] ZYGMUND A. *Trigonometric series. Vol. I, II*, third ed. Cambridge Mathematical Library. Cambridge University Press, Cambridge, 2002.

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