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E. Ballico*

GRAPH CURVES WITH GONALITY ONE

Abstract. A graph curve is a nodal curve C such that each irreducible component of C is smooth, rational and meets exactly 3 others (Bayer and Eisenbud). Here we classify the graph curves with gonality 1 with respect to balanced line bundles in the sense of L. Caporaso.

1. Introduction

L. Caporaso defined, constructed and studied a universal Picard variety over $\overline{\mathcal{M}}_g$ ([2]). In her construction she introduced the following notion. Let X be a genus g semistable curve. A degree d line bundle L on X satisfies the Basic Inequality if and only if

$$\left| \deg(L|C) - \frac{d(2p_a(C) - 2 + \sharp(C \cap \overline{X\backslash C}))}{(2g-2)} \right| \leq \sharp(C \cap \overline{X\backslash C})/2 \tag{1}$$

for all proper subcurves C of X. Let X be a semistable curve. For any integer d let $A(X,d)$ denote the set of all line bundles on X with degree d and satisfying the Basic Inequality ([2], p. 611) (or semibalanced in the sense of [7], Definition 1.1). For any integer $r \geq 0$ set $W^r_d(X) := \{L \in A(X,d) : h^0(X,L) \geq r+1\}$. We say that a depth 1 sheaf F on X has pure rank 1 if its restriction to X_{reg} is a pure rank 1 vector bundle. For the elementary properties of depth 1 coherent sheaves on reduced curves, see [8], parts VII and VIII. Let F be a sheaf on X with depth 1 and pure rank 1. Set

$$\mathrm{Sing}(F) := \{P \in X : F \text{ is not locally free at } P\}.$$

Hence $\mathrm{Sing}(F) \subseteq \mathrm{Sing}(X)$. The degree $\deg(F)$ of F may be defined by the Riemann–Roch formula $\chi(F) = \deg(F) + \chi(\mathscr{O}_X)$. Now assume that X is stable. Set $S := \mathrm{Sing}(F)$. Let $u_S : X_S \to X$ be the quasi-stable curve obtained by "blowing-up" S, i.e. X_S is semistable and connected,

$$u_S|(X_S\backslash u^{-1}(S)) : X_S\backslash u^{-1}(S) \to X\backslash S$$

is an isomorphism and for every $P \in S$ the scheme $E_P := u_S^{-1}(P)$ is a smooth rational curve intersecting the other components of X_S at two points. Let $v : C \to X$ be the partial normalization of X in which we only normalize the points of S. Set $M := v^*(F)/\mathrm{Tors}(v^*(F))$; then M is a line bundle on C and $\deg(M) = \deg(F) - \sharp(S)$ (see Corollary 1). See C as a subcurve of X_S (the complementary of the union of all exceptional divisors E_P, $P \in S$).

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Let L be any line bundle on X_S such that $L|C \cong M$ and $\deg(L|E_P) = 1$ for all $P \in S$. We have $\deg(L) = \deg(F)$ (Remark 2). The sheaf F on X is said to satisfy the Basic Inequality if the line bundle L on X_S satisfies the Basic Inequality.

L. Caporaso began the study of the Brill–Noether theory of stable curves from the point of view of rank 1 sheaves (or line bundles) satisfying the Basic Inequality ([3], [4]). Here we consider a very particular class of stable curves: the graph curves considered in [1]. A graph curve is a nodal curve X such that each irreducible component of X is smooth, rational and intersects exactly 3 other components, each of them only at one point. Hence $p_a(X) \ge 3$ and X has $2p_a(X)-2$ irreducible components. See [1] for many properties of graph curves and the introduction of [1] for several reasons to study them.

Let X be a nodal projective curve defined over an algebraically closed field $\mathbb{K}$ with $\mathrm{char}(\mathbb{K}) = 0$. Let $\mathscr{B}(X)$ denote the set of the irreducible components of X. The *dual graph* $\|X\|$ of X is the following non-oriented graph with multiple edges and loops. There is a bijection between the set of all vertices of $\|X\|$ and $\mathscr{B}(X)$. If $v \ne w$ are vertices of $\|X\|$ and T_v, T_w are the associated irreducible components of X, then v and w are connected by $\sharp(T_v \cap T_w)$ edges. Each vertex of $\|X\|$, say associated to $T \in \mathscr{B}(X)$, has exactly $\sharp(\mathrm{Sing}(T))$ loops. The graph of a graph curve is trivalent and the converse holds if we restrict to graphs without loops and multiple edges and to nodal curves with smooth irreducible components, any two of them intersecting at most at one point. If $g \ge 9$ there are many graph curves with $W^1_1(X) \ne \emptyset$, i.e. with gonality 1 with respect to line bundles satisfying the Basic Inequality. To describe them we introduce the following definition.

DEFINITION 1. *Let X be a graph curve of genus g. Let V_X denote the set of all $T \in \mathscr{B}(X)$ such that $X\backslash T$ has 3 connected components. An element of V_X is called a totally disconnecting component of X. Fix $T \in V_X$. Let C_1, C_2, C_3 be the connected components of $\overline{X\backslash T}$, with the convention $p_a(C_1) \ge p_a(C_2) \ge p_a(C_3)$. Set $g_i := p_a(C_i)$. Hence $g = g_1+g_2+g_3$. We will say that T is allowable (and write $T \in V'_X$) if $2g_1 \le g$, i.e. if $g_1 \le g_2+g_3$.*

THEOREM 1. *Let X be a graph curve of genus $g \ge 3$. There is a bijection between $W^1_1(X)$ and V'_X. Every $L \in W^1_1(X)$ is spanned and $h^0(X,L) = 2$.*

Let Y be any nodal and connected projective curve. The singular point P of Y is called a *disconnecting node* of Y if $Y\backslash\{P\}$ is not connected. Since Y is nodal, $Y\backslash\{P\}$ has exactly 2 connected components for any disconnecting node P of Y. Let $v : C \to Y$ be the partial normalization of Y in which we only normalize the disconnecting node P. Since P is a disconnecting node of Y, C has two connected components. Hence $h^0(C, \mathscr{O}_C) = 2$. Set $F_{[P]} := v_*(\mathscr{O}_C)$. The coherent sheaf $F_{[P]}$ has depth 1, pure rank 1, $\deg(F_{[P]}) = 1$, $\mathrm{Sing}(F_{[P]}) = \{P\}$, $h^0(Y, F_{[P]}) = 2$ and $F_{[P]}$ is spanned (Lemmas 1 and 2).

DEFINITION 2. *Let X be a graph curve of genus g and P a disconnecting node* *of X. Let C_1 and C_2 denote the closures in X of the 2 connected components of $X\setminus\{P\}$. The disconnecting node P is said to be allowable if g is even and $p_a(C_1) = p_a(C_2) = g/2$.*

THEOREM 2. *Let X be a graph curve of genus $g \geq 3$. Let Φ denote the set of all allowable disconnecting nodes of X. Let Ψ denote the set of all isomorphism classes of depth 1 sheaves F on X with pure rank 1, degree 1, $h^0(X,F) \geq 2$, $Sing(F) \neq \emptyset$, and satisfying the Basic Inequality. The map $P \mapsto F_{[P]}$ induces a bijection $\beta : \Phi \to \Psi$. Every $F \in \Psi$ is spanned and satisfies $h^0(X,F) = 2$.*

If the graph curve X has no disconnecting nodes, then much more can be said (see [4], Proposition 5.2.7, for the case of degree 2 sheaves).

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2. The proofs and related results

Let X be a graph curve of genus g. The inequality (1) for the irreducible components of X gives the following necessary condition for the Basic Inequality for $L \in \mathrm{Pic}^d(X)$:

$$|\deg(L|T) - d/(2g-2)| \leq 3/2 \tag{2}$$

for all $T \in \mathcal{B}(X)$.

EXAMPLE 1. Fix an integer $g \geq 9$. Here we show the existence of many graph curves X of genus g with $V_X' \neq \emptyset$ or $V_X\setminus V_X' \neq \emptyset$. Fix integers $g_i \geq 3$, $1 \leq i \leq 3$, such that $g_1 + g_2 + g_3 = g$, and let X_i be any graph curve of genus g_i. Fix $P_i \in \mathrm{Sing}(X_i)$ and let $f_{i,P_i} : Y_i \to X_i$ be the quasi-stable curve obtained from X_i by "blowing-up" the point P_i. Hence $p_a(Y_i) = g_i$ and there is a unique component $E_{i,P_i} := f_i^{-1}(P_i)$ of Y_i intersecting the other components of Y_i at two points. Fix $Q_i \in E_{i,P_i}$, $1 \leq i \leq 3$, and 3 distinct points Q_1', Q_2', Q_3' of $D := \mathbb{P}^1$. Take as X the curve obtained from $Y_1 \sqcup Y_2 \sqcup Y_3 \sqcup D$ by gluing together the point Q_i and the point Q_i' for all $i = 1,2,3$. X is a genus g graph curve and $D \in V_X$. $D \in V_X'$ if and only if $\max\{g_1, g_2, g_3\} \leq g/2$.

EXAMPLE 2. Fix an even integer $g \geq 6$. Here we show the existence of many genus g graph curves with an allowable disconnecting node. Fix graph curves X_i, $i = 1,2$, of genus $g/2$ and $P_i \in \mathrm{Sing}(X_i)$. Let $u_i : Y_i \to X_i$ be the "blowing-up" of P_i. Hence each Y_i is a quasi-stable curve of genus $g/2$ with a unique exceptional component $E_i := u_i^{-1}(P_i)$. Fix $Q_i \in E_i \cap (Y_i)_{reg}$. Let X be the only nodal curve obtained from $Y_1 \sqcup Y_2$ by gluing Q_1 and Q_2. X has genus g and each irreducible component of X is smooth and rational. Let $u : Y_1 \sqcup Y_2 \to X$ denote the quotient map. Let T be an irreducible component of X, say coming from an irreducible component D of Y_1. If $D \neq E_1$, then $Q_1 \notin D$ and hence $\sharp(T \cap \overline{X\setminus T}) = \sharp(D \cap \overline{Y_1\setminus D}) = 3$. If $T = E_1$, then T intersects $u(E_2)$ and the images in X of the two irreducible components of Y_1 associated to the two irreducible components of X_1 containing P_1. Hence X is a graph curve. Set $P := u(Q_1) = u(Q_2)$. The point P is an allowable disconnecting node of X.

REMARK 1. Let X be a reduced and quasi-projective curve, $P \in X$, and F a sheaf on X with pure rank 1 and depth 1. The germ F_P of F at P is a torsion free $\mathcal{O}_{X,P}$-module with pure rank 1. Hence there exists an inclusion $j : F_P \hookrightarrow M$ with M a free $\mathcal{O}_{X,P}$-module with rank 1. The minimal integer $\dim_{\mathbb{K}}(M/F_P)$ for all such pairs (j,M) is an important invariant of the germ F_P. Call $\ell(F,P)$ this integer. We have $\ell(F,P) \geq 0$ and $\ell(F,P) = 0$ if and only if F_P is a free $\mathcal{O}_{X,P}$-module. This invariant may be computed on the formal completion of $\mathcal{O}_{X,P}$. Let $m_{X,P}$ be the maximal ideal of the local ring $\mathcal{O}_{X,P}$. Notice that $m_{X,P}$ is a free $\mathcal{O}_{X,P}$-module if and only if $P \in X_{reg}$. Hence if $P \in \mathrm{Sing}(X)$ and $F_P \cong m_{X,P}$, then $\ell(F,P) = 1$. Now assume that X is projective. Fix a finite set $S \subseteq \mathrm{Sing}(X)$ and let $f : C \to X$ be the partial normalization of X in which we only normalize the points of S. The support of the torsion of $f^*(F)$ is contained in the finite set $f^{-1}(S)$. Set $G := f^*(F)/\mathrm{Tors}(f^*(F))$. G is a coherent sheaf on C with depth 1 and pure rank 1. Since X and C are projective, the integers $\deg(F)$ and $\deg(G)$ are well-defined and satisfy the Riemann–Roch formulas $\chi(F) = \deg(F) + \chi(\mathcal{O}_X)$, $\chi(G) = \deg(G) + \chi(\mathcal{O}_C)$ even if X or C are not connected. We have

$$\deg(G) = \deg(F) - \sum_{P \in S} \ell(F,P). \tag{3}$$

We need this formula only when each point of S is either an ordinary node of X or an ordinary cusp of X. In this case we may decompose f into $\sharp(S)$ partial normalizations of a singular point which is either an ordinary node or an ordinary cusp. Hence for nodes or ordinary cusps it is sufficient to prove it when $\sharp(S) = 1$, say $S = \{P\}$. In this case (3) is obviously true if F_P is free. If F is not locally free at P and P is either an ordinary node or an ordinary cusp, then the germ of F at P is formally isomorphic to the maximal ideal of the local ring $\mathcal{O}_{X,P}$ ([6] or, for nodes, [8], pp. 163–166). Hence it is sufficient to check (3) when $F = \mathcal{I}_P$. In this case (3) holds, because $\ell(\mathcal{I}_P, P) = 1$ and G is the ideal sheaf of the length two scheme $f^{-1}(P)$.

Remark 1 immediately gives the following result.

COROLLARY 1. *Let X be a reduced projective curve and F a coherent sheaf on X with depth 1 and pure rank 1. Fix $S \subseteq Sing(F)$. Assume that each point of S is an ordinary node or an ordinary cusp of X. Let $v : C \to X$ be the partial normalization of X in which we only normalize the points of S. Set $M := v^*(F)/Tors(v^*(F))$. Then $Sing(M) = v^{-1}(Sing(F)\backslash S)$ and $\deg(M) = \deg(F) - \sharp(S)$.*

LEMMA 1. *Let X be a nodal projective curve. Fix $S \subseteq Sing(X)$ and let F be a coherent sheaf on X with depth 1, pure rank 1 and $S \subseteq Sing(F)$. Let $v : C \to X$ be the partial normalization of X in which we only normalize the points of S. Set $M := v^*(F)/Tors(v^*(F))$. Then M has depth 1, pure rank 1, $\sharp(Sing(M)) = \sharp(Sing(F)) - \sharp(S)$, $\deg(M) = \deg(F) - \sharp(S)$, $h^0(X,F) = h^0(C,M)$ and $F \cong v_*(M)$. If F is spanned, then M is spanned.*

Proof. By Corollary 1 it only remains to check that $h^0(X,F) = h^0(C,M)$, $F \cong v_*(M)$ and that M is spanned if F is spanned. There is a natural map $\tau : F \to v_*(M)$ which is an isomorphism outside S. The map τ is an isomorphism at each point $P \in S$ because the germ of F at P is isomorphic to the germ of the ideal sheaf $\mathcal{I}_P$, while M is free in a neighborhood of the two points $v^{-1}(P)$. Since $F \cong v_*(M)$, $h^0(X,F) = h^0(C,M)$. Now assume that F is spanned. Since the tensor product is a right exact functor, $v^*(F)$ is spanned. Hence any quotient of $v^*(F)$ is spanned. $\square$

REMARK 2. Let X be a stable curve of genus g and F a depth 1 sheaf on X with pure rank 1. Assume $\mathrm{Sing}(F) \neq \emptyset$. Let $v : C \to X$ be the partial normalization of X in which we normalize only the points of $\mathrm{Sing}(F)$. Set $M := v^*(F)/\mathrm{Tors}(v^*(F))$. Remark 1 shows that M is a line bundle on C and that $\deg(M) = \deg(F) - \sharp(\mathrm{Sing}(F))$. We have $\chi(\mathscr{O}_C) = 1 - g + \sharp(\mathrm{Sing}(F))$, but C may be disconnected. Let D be the only quasi-stable curve with X as its stable model and such that $\{u^{-1}(P)\}_{P\in \mathrm{Sing}(F)}$ is the set of the exceptional components of D, where $u : D \to X$ is the stable reduction. The curve D is connected and $p_a(D) = g$. There is an inclusion $j : C \hookrightarrow D$ such that $v = u \circ j$ and $\overline{D \setminus j(C)}$ is the union of the exceptional components of D. Any line bundle L on D such that $j^*(L) \cong M$ and $\deg(L|E) = 1$ for every exceptional component E of D will be said to be *associated* to F or to M. The existence of L is trivial for the following reason. Let W be a reduced projective curve and W_1 a union of some of the irreducible components of W. Assume $\emptyset \neq W_1 \neq W$ and set $W_2 := \overline{W \setminus W_1}$. Then the restriction maps $\mathrm{Pic}(W) \to \mathrm{Pic}(W_i)$, $i = 1,2$, induce a surjection $\mathrm{Pic}(W) \to \mathrm{Pic}(W_1) \times \mathrm{Pic}(W_2)$. Since D has $\sharp(\mathrm{Sing}(F))$ exceptional components and $\overline{D \setminus j(C)}$ is the union of the exceptional components of D, we have $\deg(L) = \deg(M) + \sharp(\mathrm{Sing}(F)) = \deg(F)$. We only need the set of integers $\{L|T\}_{T\in\mathscr{B}(D)}$ to check the Basic Inequality. We use M to study the cohomological properties (e.g. the spannedness) of F.

LEMMA 2. *Fix a nodal projective curve X and $P \in \mathrm{Sing}(X)$. Let $v : C \to X$ be the partial normalization of X in which we only normalize the point P. Let $u : D \to X$ be the blowing-up of P and $E_P := u^{-1}(P)$ the exceptional component of D. Set $F := v_*(\mathscr{O}_C)$. Then F is a coherent sheaf on X with depth 1, pure rank 1 and degree 1. Let L be any line bundle on D associated to F (Remark 2). If P is a disconnecting node of X, then $h^0(X,F) = h^0(D,L) = 2$ and F and L are spanned. If P is not a disconnecting node of X, then $h^0(X,F) = h^0(D,L) = 1$.*

Proof. Since $F = v_*(\mathscr{O}_C)$, $h^0(X,F) = h^0(C,\mathscr{O}_C)$. The latter integer is the number of connected components of C. Hence this integer is either 1 (if P is not a disconnecting node) or 2 (if P is a disconnecting node). In both cases we easily see that $h^0(C,\mathscr{O}_C) = h^0(D,L)$ (use that C may be identified with a subcurve of D (the complementary of the exceptional divisor E_P), that $M \cong L|C$, that $L|E_P$ has degree 1 and the Mayer–Vietoris exact sequence (6) below with E_P instead of D). Now assume that P is a disconnecting node. Let G be the subsheaf of F spanned by $H^0(X,F)$. Hence $h^0(X,G) = h^0(X,F) = 2$. Since G is a subsheaf of a depth 1 sheaf, it has depth 1. Fix a general $\sigma \in H^0(X,F)$. The section σ is associated to a general section η of $H^0(C,\mathscr{O}_C)$. Hence η has no zero. Hence σ has no zero outside P. Hence G has pure rank 1 and either $G = F$ or F/G is supported by P. Assume $G \neq F$. Since P is an ordinary node and $P \in \mathrm{Sing}(F)$, there is $H \in \mathrm{Pic}(X)$ such that $G \subseteq H \subset F$ and F/H is the skyscraper sheaf $\mathbb{K}_P$ supported by P and with $h^0(X,\mathbb{K}_P) = 1$. We have $h^0(X,H) = 2$. Since $\deg(L|T) = 0$ for every

$T \in \mathcal{B}(D)\setminus\{E_P\}$, we have $\deg(H|T) \le 0$ for every $T \in \mathcal{B}(X)$. Since X is connected and $h^0(X,H) \ge 2$, we obtained a contradiction. □

PROPOSITION 1. *Let X be a genus g graph curve. Then $\mathcal{O}_X$ is the only $L \in A(X,0)$ such that $h^0(X,L) > 0$.*

Proof. Fix $L \in A(X,0)$ such that $h^0(X,L) > 0$ and assume $L \ne \mathcal{O}_X$. Setting $d = 0$ in (2) we get $\deg(L|T) \in \{-1,0,1\}$ for all $T \in \mathcal{B}(X)$. Set $S_i := \{T \in \mathcal{B}(X) : \deg(L|T) = i\}$. Since $\deg(L) = 0$, $\sharp(S_1) = \sharp(S_{-1})$. Fix $\sigma \in H^0(X,L)\setminus\{0\}$ and set $Z := \{P \in X : \sigma(P) = 0\}$. Since $L \ne \mathcal{O}_X$, $Z \ne \emptyset$. If $T \in S_{-1}$, then $T \subseteq Z$. If $T \in S_0$, then either $T \subseteq Z$ or $T \cap Z = \emptyset$. If $T \in S_1$, then either $T \subseteq Z$ or $\sharp(Z \cap T) \le 1$. Since $\emptyset \ne Z \ne X$, we get $S_1 \ne \emptyset$. Set $W := \{T \in \mathcal{B}(X) : T \subseteq Z\}$ and $Y := \cup_{T \in W} T$. Since $Z \ne X$, $Y \ne X$. Since $S_{-1} \ne \emptyset$ and $S_{-1} \subseteq W$, $Y \ne \emptyset$. Since X is connected, $m := \sharp(Y \cap \overline{X\setminus Y}) > 0$. Set $A := \{T \in \mathcal{B}(\overline{X\setminus Y}) : T \cap Y \ne \emptyset\}$. We just saw that $A \subseteq S_1$ and that any $T \in A$ intersects Y at a unique point. Thus $\deg(L|\overline{X\setminus Y}) \ge m$. By taking $C := \overline{X\setminus Y}$ and $d = 0$ in the Basic Inequality (1) we get $|m| \le m/2$, contradiction. □

PROPOSITION 2. *Fix an integer $d < 0$ and a genus g graph curve X. Then $h^0(X,L) = 0$ for every $L \in A(X,d)$.*

Proof. Assume the existence of $L \in A(X,d)$ such that $h^0(X,L) > 0$. The inequality (2) gives $\deg(L|T) \le 1$ for all $T \in \mathcal{B}(X)$. Set $S_i := \{T \in \mathcal{B}(X) : \deg(L|T) = i\}$ and $S_- := \cup_{i<0} S_i$. Since $d < 0$, $S_- \ne \emptyset$. Fix $\sigma \in H^0(X,L)\setminus\{0\}$ and set $Z := \{P \in X : \sigma(P) = 0\}$. Since $d < 0$, $Z \ne \emptyset$. If $T \in S_-$, then $T \subseteq Z$. If $T \in S_0$, then either $T \subseteq Z$ or $T \cap Z = \emptyset$. If $T \in S_1$, then either $T \subseteq Z$ or $\sharp(Z \cap T) \le 1$. Since $\emptyset \ne Z \ne X$, we get $S_1 \ne \emptyset$. Set $W := \{T \in \mathcal{B}(X) : T \subseteq Z\}$ and $Y := \cup_{T \in W} T$. Since $Z \ne X$, $Y \ne X$. Since $S_- \subseteq W$ and $S_- \ne \emptyset$, $Y \ne \emptyset$. Let C be a connected component of $\overline{X\setminus Y}$. Since X is connected, $m := \sharp(C \cap \overline{X\setminus C}) > 0$. Set $A := \{T \in \mathcal{B}(\overline{X\setminus Y}) : T \cap Y \ne \emptyset\}$. We just saw that $A \subseteq S_1$ and that any $T \in A$ intersects Y at a unique point. Thus $\deg(L|C) \ge m$. Since X is semistable, either $m \ge 2$ or $p_a(C) = 1$. Since $d(2p_a(C) - 2 + m) \le 0$, the Basic Inequality (1) gives $|m| \le m/2$, contradiction. □

Proof of Theorem 1. Fix $L \in W^1_1(X)$. Since $g \ge 3$, $1/(2g-2) < 1/2$. Hence the inequality (2) gives $\deg(L|T) \in \{-1,0,1\}$ for all $T \in \mathcal{B}(X)$. Set $S_i := \{T \in \mathcal{B}(X) : \deg(L|T) = i\}$ and $Y_i := \cup_{T \in S_i} T$. Let B' be the base locus of L and B the union of the irreducible components of X contained in B'. We have $Y_{-1} \subseteq B$. If $T \in S_0$ then either $T \subseteq B$ or $T \cap B' = \emptyset$. If $T \in S_1$, then either $T \subseteq B$ or $\sharp(T \cap B') \le 1$.

(a) First assume $B \ne \emptyset$. Since $h^0(X,L) > 0$, $B \ne X$. Since X is connected, $B \cap \overline{X\setminus B} \ne \emptyset$. Let A_i, $1 \le i \le z$, be the connected components of $\overline{X\setminus B}$. Since A_i is a connected component of $\overline{X\setminus B}$, $A_i \cap \overline{X\setminus A_i} = A_i \cap B$. Since X is connected, we get $m_i := \sharp(A_i \cap B) > 0$. We just saw that $\deg(L|T) = 1$ and $\sharp(T \cap B) = 1$ for every $T \subseteq A_i$ such that $T \cap B \ne \emptyset$. Hence $a_i := \deg(L|A_i) \ge m_i$. The Basic Inequality (1) applied to A_i gives $p_a(A_i) > 0$ and

$$a_i(2g-2) - m_i(g-1) \le 2p_a(A_i) - 2 + m_i. \qquad (4)$$

Since $A_i \neq X$, $p_a(A_i) \le g-2$. Hence

$$(2a_i - m_i)(g-1) \le 2g - 6 + m_i \tag{5}$$

with $a_i \ge m_i > 0$. The inequality (5) is not satisfied for all (a_i, m_i, g) such that $a_i \ge m_i \ge 1$, $g \ge 3$ and $a_i \ge 2$. Thus if $B \neq \emptyset$ we get a contradiction, unless $a_i = m_i = 1$ for all connected components A_i of $\overline{X \setminus B}$. Assume $a_i = m_i = 1$ for all i. Since X is connected and each connected component of $\overline{X \setminus B}$ intersects B at a unique point, B must be connected. Since $\deg(L|A_i) = 1$, we also see that the point $A_i \cap B$ is the only base point of L contained in A_i, that $L|A_i \cong \mathscr{O}_{A_i}(A_i \cap B)$ and that the restriction map $\rho_i : H^0(X, L) \to H^0(A_i, L|A_i)$ has one-dimensional image. Since $h^0(X, L) \ge 2$, we get $z \ge 2$. From (4) with $a_i = m_i = 1$ we get $p_a(A_i) \ge g/2$ for all i. Since B is connected, $p_a(B) \ge 0$. Since $z \ge 2$ and $g \ge p_a(B) + \sum_{i=1}^{z} p_a(A_i)$, we obtain $z = 2$, g even, $p_a(A_1) = p_a(A_2) = g/2$ and $p_a(B) = 0$. Since $\sharp(B \cap \overline{X \setminus B}) = z = 2 < 3$ and $p_a(B) = 0$, ω_X is not ample, contradiction. The contradiction gives $B = \emptyset$.

(b) Since $B = \emptyset$, $S_{-1} = \emptyset$. Hence there is $D \in \mathscr{B}(X)$ such that $\deg(L|D) = 1$ and $\deg(L|T) = 0$ for all $T \in \mathscr{B}(X) \setminus D$. Hence $B' \subset D \cap X_{reg}$. Since $B' \cap \overline{X \setminus D} = \emptyset$ and $\deg(L|T) = 0$ for all $T \in \overline{X \setminus D}$, we get $L|\overline{X \setminus D} \cong \mathscr{O}_{\overline{X \setminus D}}$ and the existence of $\sigma \in H^0(X, L)$ with no zero in a neighborhood of $\overline{X \setminus D}$. Consider the Mayer–Vietoris exact sequence

$$0 \to L \to L|D \oplus L|\overline{X \setminus D} \to L|D \cap \overline{X \setminus D} \to 0. \tag{6}$$

Since $L|\overline{X \setminus D}$ is trivial, $h^0(\overline{X \setminus D}, L|\overline{X \setminus D})$ is the number, s, of the connected components of $\overline{X \setminus D}$. Since $\|X\|$ is 3-valent, $1 \le s \le 3$. Call M_i, $1 \le i \le s$, the connected components of $\overline{X \setminus D}$. Since $L|\overline{X \setminus D} \cong \mathscr{O}_{\overline{X \setminus D}}$ and every connected component of $\overline{X \setminus D}$ intersects D, the restriction map $H^0(\overline{X \setminus D}, L|\overline{X \setminus D}) \to H^0(D \cap \overline{X \setminus D}, L|D \cap \overline{X \setminus D})$ is injective. Hence (6) gives the injectivity of the restriction map $\rho : H^0(X, L) \to H^0(D, L|D)$. Since $\deg(L|D) = 1$, $h^0(D, L|D) = 2$. Since $h^0(X, L) \ge 2$, the linear map ρ is an isomorphism. Hence $h^0(X, L) = 2$ and $B' \cap D = \emptyset$. Since $B' \subset D \cap X_{reg}$, we get $B' = \emptyset$, i.e. L is spanned.

Let $\phi_L : X \to \mathbb{P}^1$ be the morphism induced by $|L|$. Since $L|M_i \cong \mathscr{O}_{M_i}$ and M_i is connected, $\phi_L(M_i)$ is a point $Q_i \in \mathbb{P}^1$. Since ρ is bijective and $\deg(L|D) = 1$, $\phi_L|D$ is an isomorphism. Since $\phi_L|M_i$ is constant, we get $\sharp(M_i \cap D) \le 1$. Since each M_i is a connected component of $\overline{X \setminus D}$, we have $M_i \cap D \neq \emptyset$. Thus $\sharp(M_i \cap D) = 1$. Since X is a graph curve, $\sharp(D \cap \overline{X \setminus D}) = 3$. Thus $s = 3$. Since $s = 3$, $D \in V_X$. Conversely, take $D \in V_X$ and call C_i, $1 \le i \le 3$, the connected components of $\overline{X \setminus D}$. Fix an isomorphism $u : D \to \mathbb{P}^1$. Let $f : X \to \mathbb{P}^1$ be unique morphism such that $f|D = u$ and $f(C_i) = u(C_i \cap D)$ for all i. Set $L := f^*(\mathscr{O}_{\mathbb{P}^1}(1))$. Hence L is a degree 1 spanned line bundle. Obviously, $\deg(L|C) = 1$ for any subcurve C of X containing D, while $\deg(L|C) = 0$ for any subcurve C of X not containing D. It is clear that the second construction is the inverse of the first one. Hence to conclude the proof of Theorem 1 it is sufficient to check that L satisfies the Basic Inequality if and only if $D \in V'_X$.

Let C_1, C_2, C_3 be the connected components of $\overline{X \setminus D}$. Set $g_i := p_a(C_i)$ and assume $g_1 \ge g_2 \ge g_3$. To check the Basic Inequality it is sufficient to check it for all proper connected subcurves of X. Let U be a connected proper subcurve of X. Set $\tau := \sharp(U \cap \overline{X \setminus U})$ and $q := p_a(U)$. First assume that U does not contain D. Hence $\deg(L|U) = 0$. The Basic Inequality is satisfied by U if and only if $|2q-2+\tau| \le \tau(g-1)$. The latter inequality is always satisfied if $\tau \ge 2$, because $0 \le q \le g-1$ and $g \ge 3$. Now assume $\tau = 1$. Since U is connected and does not contain D, there is $i \in \{1,2,3\}$ such that $U \subseteq C_i$. Since C_i is connected, $q \le g_i$. Since $g_i \le g_1$, the pair (L,U) satisfies the Basic Inequality if $2g_1 \le g$, i.e. if $L \in V'_X$. By taking $U := C_1$ we see that if $2g_1 > g$, then L does not satisfy the Basic Inequality. Now assume $D \subseteq U$. Hence $\deg(L|U) = 1$. The Basic Inequality is satisfied if and only if

$$|2g-2q-\tau| \le \tau(g-1). \tag{7}$$

Since $0 \le q \le g-2$, the inequality (7) is satisfied if and only if either $\tau \ge 2$ or $\tau = 1$ and $2q \ge g$. Assume $\tau = 1$. Hence $D \ne U$. Since U is connected, each C_i is connected and $D \subsetneqq U$, we get that U contains at least two of the curves C_1, C_2, C_3, say C_i and C_j. Since U is connected, we get $q \ge g_i + g_j$. Thus if $g_1 \le g_2 + g_3$, then L satisfies the Basic Inequality. $\square$

REMARK 3. Let X be a stable curve of genus $g \ge 4$ such that there is $D \in \mathcal{B}(X)$ with $D \cong \mathbb{P}^1$. Call $C_1, \dots, C_z$ the connected components of $\overline{X \setminus D}$ with the convention $p_a(C_1) \ge \cdots \ge p_a(C_z)$. Set $g_i := p_a(C_i)$. Assume $\sharp(C_i \cap D) = 1$ for all i. Since X is stable, $z \ge 3$. Since $\sharp(D \cap C_i) = 1$ for all i, there is a unique morphism $\phi : X \to \mathbb{P}^1$ such that $\phi|D$ is an isomorphism and $\phi(C_i) = (\phi|D)(C_i \cap D)$ for all i, i.e. each $\phi|C_i$ is constant. Set $L := \phi^*(\mathscr{O}_{\mathbb{P}^1}(1))$. L is a spanned line bundle, $\deg(L|D) = 1$ and $L|C_i \cong \mathscr{O}_{C_i}$ for all i. Following the proof of Theorem 1 we now prove that L satisfies the Basic Inequality if and only if $2g_1 \le g$. Let U be a connected proper subcurve of X. Set $\tau := \sharp(U \cap \overline{X \setminus U})$ and $q := p_a(U)$. First assume $D \nsubseteq U$. Hence $\deg(L|U) = 0$. The Basic Inequality is satisfied by U if and only if $|2q-2+\tau| \le \tau(g-1)$. The latter inequality is always satisfied if $\tau \ge 2$, because $0 \le q \le g-1$ and $g \ge 3$. Now assume $\tau = 1$. Since $U \subseteq C_i$ for some i, and C_i is connected, $q \le g_i$. Hence L satisfies the Basic Inequality with respect to every $U \subseteq C_i$ with $\tau = 1$ if and only if $2g_i \le g$. Taking $U := C_1$ we get that the Basic Inequality is satisfied with respect to all connected subcurves of X not containing D if and only if $2g_1 \le g$. Now assume $D \subseteq U$. In this case the pair (L,U) satisfies the Basis Inequality if and only if (7) is satisfied. Since $0 \le q \le g-1$ and $g \ge 4$, (7) is satisfied if either $\tau \ge 2$ or $\tau = 1$ and $2q \ge g$. Assume $\tau = 1$. Hence $U \ne D$. Since $\tau = 1$, $D \subsetneqq U$ and U is connected, there is a unique index $i \in \{1, \dots, z\}$ such that $C_i \nsubseteq U$. Since U is connected, we get $q \ge g - g_i$. Hence $2q \ge g$ if $2g_i \le g$. Since $g_i \le g_1$, we are done.

REMARK 4. Let X be any graph curve X such that $V'_X \ne \emptyset$. Theorem 1 gives the existence of $L \in A(X,1)$ such that $h^0(X,L) = 2 > \deg(L)/2 + 1$. Hence L does not satisfy Clifford's inequality, contrary to the case of binary curves studied in [3] and to many other cases studied in [4].

PROPOSITION 3. *Let X be a graph curve of genus g and F a sheaf on X with depth* 1 *and pure rank* 1 *satisfying the Basic Inequality. Assume $d := \deg(F) \le 0$ and that F is not locally free. Then $h^0(X,F) = 0$.*

Proof. By assumption $S := \mathrm{Sing}(F) \neq \emptyset$. Set $s := \sharp(S)$. Let $u_S : X_S \to X$ be the quasi-stable curve obtained by blowing-up S. Let L be any degree d line bundle on X_S associated to F in the sense of Remark 2. By assumption $L \in A(X_S, d)$. Assume $h^0(X,F) > 0$. Since F has no torsion, the natural map $u_S^* : H^0(X,F) \to H^0(X_S,L)$ is injective. Hence $h^0(X_S,L) > 0$. Fix $\sigma \in H^0(X_S,L)\backslash\{0\}$ and set $Z := \{\sigma = 0\}$. Let B be the union of the irreducible components of X_S contained in Z. Fix $T \in \mathcal{B}(X_S)$ which is not contracted by u_S. We have $\sharp(T \cap \overline{X_S\backslash T}) = 3$. Since $p_a(T) = 0$, $p_a(X_S) = g$, $\deg(L) = d$ and L satisfies the Basic Inequality, $|\deg(L|T) - d/(2g-2)| \leq 3/2$. Since $d \leq 0$, we get $\deg(L|T) \leq 1$. Set $S_i := \{T \in \mathcal{B}(X_S) : \deg(L|T) = i\}$, $Y_i := \cup_{T \in S_i} T_S$ and $Y_- := \cup_{i<0} Y_i$. The curve Y_1 contains every irreducible component of X_S contracted by u_S. We have $Y_i \subseteq Z$ for all $i < 0$ and if $T \in S_0$, then either $T \subseteq Z$ or $T \cap Z = \emptyset$. If D is an irreducible component of Y_1, then either $D \subseteq Z$ or $\sharp(D \cap Z) \leq 1$. Since $\deg(L|E_P) = 1$ for every exceptional component E_P, $s > 0$, and $\sum_{T \in \mathcal{B}(X_S)} \deg(L|T) = d \leq 0$, we obtain $Y_- \neq \emptyset$. Hence $B \neq \emptyset$. Since $\sigma \neq 0$, $B \neq X_S$. Let A be any connected component of $\overline{X_S\backslash B}$. Since X_S is connected, $B \cap \overline{X_S\backslash B} \neq \emptyset$. Hence $m := \sharp(A \cap B) > 0$. We saw that $\deg(L|A) \geq m$. Since X_S is semistable, either $p_a(A) \geq 1$ or $m \geq 2$, i.e. $p_a(A) - 1 + m/2 \geq 0$. Since $d \leq 0$, the Basic Inequality gives $\deg(L|A) \leq m/2$, contradiction. □

PROPOSITION 4. *Let X be a graph curve of genus g. There is a bijection between V_X and the set of all spanned line bundles on X with degree* 1.

Proof. Fix a spanned line bundle L on X such that $\deg(L) = 1$. Since L is spanned, it induces a morphism $h_L : X \to \mathbb{P}^r$, $r := h^0(X,L) - 1$. Since L is spanned, $\deg(L|T) \geq 0$ for all $T \in \mathcal{B}(X)$. Hence there is $D \in \mathcal{B}(X)$ such that $\deg(L|D) = 1$ and $\deg(L|T) = 0$ for all $T \in \mathcal{B}(X)\backslash\{D\}$. Let $C_1, \dots, C_s$ be the connected components of $\overline{X\backslash D}$. Notice that h_L contracts each $T \in \mathcal{B}(X)\backslash\{D\}$. Now we repeat the proof of Theorem 1 to get $s = 3$ and that $D \in V_X$. Conversely, fix any $D \in V_X$. In the proof of Theorem 1 we used D to construct a spanned degree 1 line bundle on X. □

Proof of Theorem 2. Let F be a sheaf on X with depth 1, pure rank 1, degree 1, $h^0(X,F) \geq 2$, and satisfying the Basic Inequality. Assume that F is not locally free. Set $S := \mathrm{Sing}(F)$ and $s := \sharp(S) > 0$. Let $u_S : X_S \to X$ be the quasi-stable curve obtained by blowing-up S. Let L be any degree $\deg(F)$ line bundle on X_S associated to F (Remark 2). By assumption $L \in A(X_S, 1)$. As in the previous proofs we have $h^0(X_S,L) \geq h^0(X,F)$ and hence $h^0(X_S,L) \geq 2$. Since X_S is quasi-stable with X as its stable reduction, $2 \leq \sharp(T \cap \overline{X_S\backslash T}) \leq 3$ for every $T \in \mathcal{B}(X_S)$ and $\sharp(T \cap \overline{X_S\backslash T}) = 2$ if and only if T is one of the exceptional components E_P, $P \in S$. Since $2 \leq \sharp(T \cap \overline{X_S\backslash T}) \leq 3$ for every $T \in \mathcal{B}(X_S)$ and $g = p_a(X_S) \geq 3$, the Basic Inequality gives $\deg(L|T) \in \{-1,0,1\}$ for every $T \in \mathcal{B}(X_S)$. Let B' be the base locus of L and B the union of the irreducible components of X_S contained in B'. Set $S_i := \{T \in \mathcal{B}(X_S) : \deg(L|T) = i\}$ and $Y_i := \cup_{T \in S_i} T$. Notice that $E_P \in S_1$ for every $P \in S$. Hence $S_1 \neq \emptyset$. If $T \in S_i$ for some $i < 0$, then $T \subseteq B$. If $T \in S_j$ for some $j \geq 0$, then either $T \subseteq B$ or $\sharp(T \cap B') \leq j$.

(a) Here we assume $B \neq \emptyset$. Since $h^0(X_S,L) > 0$, $B \neq X$. Since X_S is connected, $B \cap \overline{X_S\backslash B} \neq \emptyset$. Let A_i, $1 \leq i \leq z$, be the connected components of $\overline{X_S\backslash B}$. Set $a_i :=$

$\deg(L|A_i)$ and $m_i := \sharp(A_i \cap B) > 0$. We repeat verbatim part (a) of the proof of Theorem 1. Since X_S is not a graph curve, we only use the inequality $p_a(A_i) \le g-1$, which is true for any semistable curve. We first get $a_i \ge m_i$ for all i, and then we get a contradiction unless either $a_i = m_i = 2$ and $p_a(A_i) = g-1$ or $a_i = m_i = 1$ for all i.

(a1) Here we assume $a_i = m_i = 2$ and $p_a(A_i) = g-1$. Since X_S is quasi-stable, every connected subcurve of it with arithmetic genus $g-1$ is the complement of one of the exceptional components. Hence $z = 1$, B is irreducible and B is contracted by u_S. Hence $\deg(L|B) = 1$. Hence $\deg(L) = a_i + \deg(L|B) = 3$, contradiction.

(a2) Here we assume $a_i = m_i = 1$ for all i. Since X_S is connected and each connected component of $\overline{X_S \backslash B}$ intersects B at a unique point, B must be connected. Since $\deg(L|A_i) = 1$, we also see that the point $A_i \cap B$ is the only base point of L contained in A_i, that $L|A_i \cong \mathscr{O}_{A_i}(A_i \cap B)$ and that the restriction map $\rho_i : H^0(X,L) \to H^0(A_i, L|A_i)$ has one-dimensional image. Since $h^0(X,L) \ge 2$, we get $z \ge 2$. From (4) with $a_i = m_i = 1$ we get $p_a(A_i) \ge g/2$ for all i. Since B is connected, $p_a(B) \ge 0$. Since $z \ge 2$ and $g \ge p_a(B) + \sum_{i=1}^{z} p_a(A_i)$, we obtain $z = 2$, g even, $p_a(A_1) = p_a(A_2) = g/2$ and $p_a(B) = 0$. Since ω_{X_S} is semiample and the exceptional components of u_S are the only connected subcurves, J, of X_S such that $\deg(\omega_{X_S}|J) = 0$, there is $P \in S$ such that $B = E_P$. Hence $\deg(L|B) = 1$. Since $z = 2$ and $a_1 = a_2 = 1$, we get $\deg(L) = 3$, contradiction.

(b) Here we assume $B = \emptyset$. Hence $S_{-1} = \emptyset$. Since $\deg(L|E_P) = 1$ for all $P \in \mathrm{Sing}(F)$, we get $s = 1$, $Y_1 = E_P$ (where P is the only point of $\mathrm{Sing}(F)$) and $\deg(L|T) = 0$ for every $T \in \mathscr{B}(X_S)\backslash\{E_P\}$. Thus $Y_0 = \overline{X_S \backslash E_P}$ is isomorphic to the partial normalization of X in which we normalize only the point P. Since $B = \emptyset$, $B' \cap Y_0 = \emptyset$. Hence a general $\sigma \in H^0(X_S, L)$ has no zero in a neighborhood of Y_0. Thus there is an open neighborhood Ω of Y_0 such that $L|\Omega \cong \mathscr{O}_\Omega$ and the trivialization is given by a global section of L. Since each connected component of Y_0 intersects E_P, we obtain the injectivity of the restriction map $\rho : H^0(X_S, L) \to H^0(E_P, L|E_P)$. Since $h^0(E_P, L|E_P) = 2$, and $h^0(X_S, L) \ge 2$, ρ is bijective. Hence $B' \cap E_P = \emptyset$. Hence $B' = \emptyset$, i.e. L is spanned. Let $v : C \to X$ be the partial normalization of X in which we only normalize P. Set $M := v^*(F)/\mathrm{Tors}(v^*(F))$. M is a degree 0 line bundle (Lemma 1). See C as the subcurve Y_0 of X_S. With this identification $M = L|C$ (Remark 2). Hence M is spanned and $\deg(M|T) = 0$ for every $T \in \mathscr{B}(C)$. Hence $M \cong \mathscr{O}_C$. We also get $F \cong v_*(\mathscr{O}_C)$. Any of the inequalities $h^0(X_S, L) \ge 2$ or $h^0(X, F) \ge 2$ gives that P is a disconnecting node of X and that F is spanned (Lemma 2). Conversely, for any disconnecting node Q of X we get in this way a unique spanned sheaf $F_{[Q]}$ with degree 1 and $h^0(X, F_{[Q]}) = 2$ (Lemma 1). It only remains to check that the sheaf $F_{[P]}$ satifies the Basic Inequality if and only if P is allowable.

Let C_i, $i = 1,2$, be the closure in X_S of the 2 connected components of $X_S \backslash \{E_P\}$. Set $g_i := p_a(C_i)$. These components are isomorphic to the closure in X of the 2 connected components of $X \backslash \{P\}$. Notice that $\overline{X_S \backslash C_i} = C_{2-i} \cup E_P$. Since $\deg(L|C_i) = 0$, and $\sharp(C_i \cap (C_{2-i} \cup E_P)) = 1$, C_i satisfies the Basic Inequality for L if and only if $|2p_a(C_i) - 1| \le g-1$. Thus the Basic Inequality holds for the pairs (L, C_1) and (L, C_2) if and only if $2\max\{g_1, g_2\} \le g$, i.e. (since $g = g_1 + g_2$) if and only if g is even and $g_1 = g_2 = g/2$. From now on we assume g even and $g_1 = g_2 = g/2$. Let A be a proper connected subcurve of X_S. Since A is connected, $q := p_a(A) \geq 0$. Set $x := \sharp(A \cap \overline{X_S \setminus A})$. First assume $E_P \nsubseteq A$. Hence $A \subseteq C_i$ for some i. Since A is connected, we get $q \leq p_a(C_i) \leq g/2$. Since $\deg(L|A) = 0$, the Basic Inequality for the pair (L, A) is equivalent to the inequality $|-(2q-2+x)/(2g-2)| \leq x/2$, which is always satisfied, because $x \geq 1$ and $q \leq g/2$. Now assume $E_P \subseteq A$. Hence $\deg(L|A) = 1$. The pair (L, A) satisfies the Basic Inequality if and only if

$$|2g - 2q - x)| \leq x(g-1). \tag{8}$$

Since $0 \leq q < g$, $g \geq 3$ and $x \geq 1$, (8) is satisfied if and only if either $x \geq 2$ or $x = 1$ and $2q \geq g$. Assume $x = 1$. Hence $A \neq E_P$. Since $x = 1$, the connectedness of X and A and the inclusion $E_P \subsetneqq A$ imply the existence of $i \in \{1, 2\}$ such that $C_i \subseteq A$. Thus $q \geq p_a(C_i) \geq g/2$. Hence L satisfies the Basic Inequality. Obviously different disconnecting nodes, say Q_1 and Q_2, give non-isomorphic sheaves, because $\mathrm{Sing}(F_{[Q_1]}) = \{Q_1\} \neq \{Q_2\} = \mathrm{Sing}(F_{[Q_2]})$, giving the injectivity of the map $\beta : \Phi \to \Psi$. The last sentence of the statement of Theorem 2 follows from the surjectivity of β and Lemma 2. $\square$

REMARK 5. Take the set-up of Lemma 2. Assume that X is stable with genus $g \geq 4$ and that P is a disconnecting node of X. Let C_1 and C_2 be the closures in D of the two connected components of $D \setminus E_P$. Thus $g = p_a(C_1) + p_a(C_2)$. The proof of Theorem 2 just given shows that L satisfies the Basic Inequality if and only if g is even and $p_a(C_1) = p_a(C_2)$.

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Edoardo BALLICO,
Dipartimento di Matematica, Università di Trento,
Via Sommarive 14, 38123 Povo (TN), ITALIA
e-mail: ballico@science.unitn.it

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