Rend. Sem. Mat. Univ. Pol. Torino
Vol. 67, 4 (2009), 395 – 406
In Memoriam Aristide Sanini

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CONFORMAL VARIATIONAL PROBLEMS, HARMONIC MAPS AND HARMONIC MORPHISMS

Abstract. We discuss some aspects of harmonic maps and morphisms related to conformality, especially some recent results on smoothness and infinitesimal behaviour of twistor and transform methods for finding harmonic maps, and the dual notion of harmonic morphism.

1. Introduction

Amongst Aristide Sanini's interests were conformal variational problems. He wrote two papers on this subject [24, 25]. In the first of these, he characterized weakly conformal maps from surfaces as maps whose energy is extremal *with respect to variations of the metric*.

On the other hand, harmonic maps extremize the energy *with respect to variations of the map*. The intersection of these classes is the class of minimal branched immersions; in particular, all harmonic maps from the 2-sphere are automatically weakly conformal, and so are minimal branched immersions. There are many twistor and transform methods for the construction of such mappings into various symmetric spaces, starting with harmonic 2-spheres in complex projective space. However, the constructions are algebraic and are not, in general, smooth or even continuous. After reminding the reader of these ideas, in Section 6, we discuss some recent results on the smoothness of the Gauss transform.

An infinitesimal variation of a harmonic map is called a *Jacobi field*; if a Jacobi field comes from a genuine variation, it is called *integrable*. We discuss these ideas in Sections 7 and 8, in particular, the integrability of Jacobi fields along harmonic 2-spheres in $\mathbb{C}P^2$.

Then we remind the reader of Uhlenbeck's idea of 'adding a uniton', and we mention some recent developments which allow us to give completely explicit formulae for harmonic 2-spheres in the unitary group and related spaces.

Related to the Gauss transform is the twistor method for finding harmonic 2-spheres in S^4. In Section 10, we study the infinitesimal behaviour of this method, seeing that Jacobi fields are no longer always integrable.

Then, in Section 11, we discuss horizontally weakly conformal maps, characterizing them in a way dual to that of Sanini; this leads to a discussion of harmonic morphisms in Section 12*ff*. where we see how to dualize some of the twistor theory for weakly conformal harmonic maps to give formulae for harmonic morphisms.

*An expanded version of the author's talk at the *Giornata di Geometria in Memoria di Aristide Sanini*, held at the Politecnico di Torino on 27 June 2008. This work was partially supported by the Gulbenkian Foundation, Portugal.

2. Harmonic maps between Riemannian manifolds

Let $\phi:(M,g)\to(N,h)$ be a smooth map between compact smooth Riemannian manifolds. The *energy* or *Dirichlet integral of* ϕ is

$$E(\phi)=\int_M e(\phi)\,\omega_g=\int_M \frac{1}{2}|\mathrm{d}\phi|^2\,\omega_g$$

where ω_g denotes the volume measure induced by the metric g and, for any $p\in M$,

$$\begin{aligned}|\mathrm{d}\phi_p|^2 &= \text{Hilbert–Schmidt square norm of } \mathrm{d}\phi_p\\ &= g^{ij}h_{\alpha\beta}\,\phi_i^\alpha\phi_j^\beta\,.\end{aligned}$$

Here $\phi_i^\alpha=\partial u^\alpha/\partial x^i$ denote the partial derivatives of ϕ with respect to some local coordinates (x^i) on M and (u^α) on N, (g_{ij}) and $(h_{\alpha\beta})$ are the components of the metric tensor g and h, and $(h^{\alpha\beta})$ is the inverse matrix of $(h_{\alpha\beta})$.

The map ϕ is called *harmonic* if the first variation of E for variations ϕ_t *of the map* ϕ vanishes at ϕ, i.e., $\frac{\mathrm{d}}{\mathrm{d}t}E(\phi_t)\big|_{t=0}=0$. We compute:

$$\frac{\mathrm{d}}{\mathrm{d}t}E(\phi_t)\Big|_{t=0}=-\int_M\langle\tau(\phi),v\rangle\,\omega_g \tag{1}$$

where $v=\partial\phi_t/\partial t|_{t=0}$ is the *variation vector field of* (ϕ_t), and $\tau(\phi)=\nabla\mathrm{d}\phi$ is the *tension field of* ϕ given by

$$\begin{aligned}\tau(\phi) &= \nabla\,\mathrm{d}\phi=\mathrm{Tr}\,\nabla\mathrm{d}\phi=\sum_{i=1}^m\nabla\mathrm{d}\phi(e_i,e_i)\\ &= \sum_{i=1}^m\big\{\nabla^\phi_{e_i}\big(\mathrm{d}\phi(e_i)\big)-\mathrm{d}\phi(\nabla^M_{e_i}e_i)\big\}\end{aligned}$$

for any orthonormal frame $\{e_i\}$. In local coordinates, this reads

$$\begin{aligned}\tau(\phi)^\gamma &= g^{ij}\left(\frac{\partial^2\phi^\gamma}{\partial x^i\partial x^j}-\Gamma^k_{ij}\frac{\partial\phi^\gamma}{\partial x^k}+L^\gamma_{\alpha\beta}\frac{\partial\phi^\alpha}{\partial x^i}\frac{\partial\phi^\beta}{\partial x^j}\right)\\ &= \Delta^M\phi^\gamma+g(\mathrm{grad}\,\phi^\alpha,\,\mathrm{grad}\,\phi^\beta)\,L^\gamma_{\alpha\beta}\,.\end{aligned}$$

Here, Γ^k_{ij} (resp. $L^\gamma_{\alpha\beta}$) denotes the Christoffel symbols on (M,g) (resp. (N,h)), and Δ^M denotes the *Laplace–Beltrami operator on functions* $f:M\to\mathbb{R}$ given by

$$\begin{aligned}\Delta^M f &= \nabla\,\mathrm{grad}\,f=\nabla\mathrm{d}f=-\mathrm{d}^*\mathrm{d}f=\mathrm{Tr}\,\nabla\mathrm{d}f\\ &= \sum_{i=1}^m\big\{e_i\big(e_i(f)\big)-\big(\nabla^M_{e_i}e_i\big)f\big\}\\ &= \frac{1}{\sqrt{|g|}}\frac{\partial}{\partial x^i}\left(\sqrt{|g|}\,g^{ij}\frac{\partial f}{\partial x^j}\right)=g^{ij}\left(\frac{\partial^2 f}{\partial x^i\partial x^j}-\Gamma^k_{ij}\frac{\partial f}{\partial x^k}\right).\end{aligned}$$

3. Examples of harmonic maps

From (1), we see that $\phi : M \to N$ is harmonic if and only if it satisfies the *harmonic* or *tension field equation*:

(2) $$\tau(\phi) \equiv \operatorname{Tr} \nabla \mathrm{d}\phi = 0\,.$$

We list some standard examples.

1. A smooth map $\phi : \mathbb{R}^m \supseteq U \to \mathbb{R}^n$ is harmonic if and only if $\Delta\phi = 0$ where Δ is the usual Laplacian on $\mathbb{R}^m$.

2. A smooth map $\phi : (M,g) \to \mathbb{R}^n$ is harmonic if and only if $\Delta^M \phi = 0$ where Δ^M is the Laplace–Beltrami operator on (M,g).

Note that both the above equations are *linear*.

3. A smooth map from an interval of $\mathbb{R}$ or from S^1 to N is harmonic if and only if it defines a *geodesic* of N parametrized linearly.

4. *Holomorphic* and *antiholomorphic* maps between Kähler manifolds are harmonic; in fact they minimize energy in their homotopy class.

5. *Harmonic morphisms*, i.e., maps which preserve Laplace's equation, are harmonic maps, see Section 12.

4. Weakly conformal maps

A smooth map $\phi : (M,g) \to (N,h)$ is called *weakly conformal* if

(3) $$\phi^* h = \lambda^2 g$$

for some function $\lambda : M \to [0,\infty)$; explicitly, for all $p \in M$,

$$h\big(\mathrm{d}\phi_p(X), \mathrm{d}\phi_p(Y)\big) = \lambda(p)^2 g(X,Y) \qquad (X, Y \in T_p M)\,;$$

equivalently,

$$\mathrm{d}\phi_p^* \circ \mathrm{d}\phi_p = \lambda(p)^2 \operatorname{Id}_{T_p M}\,.$$

In local coordinates, equation (3) reads

$$h_{\alpha\beta}\phi_i^\alpha \phi_j^\beta = \lambda^2 g_{ij}\,.$$

A. Sanini characterized weak conformality as follows.

THEOREM 1 ([24]). *A non-constant map ϕ is a critical point of the energy with respect to variations* of the metric *if and only if* $\dim M = 2$ *and ϕ is weakly conformal.*

Proof. The Euler–Lagrange operator for such variations is the *stress-energy tensor* $S(\phi) = e(\phi)g - \phi^* h$. If this is zero, taking the trace shows that $\dim M = 2$, then comparing with equation (3) shows that ϕ is weakly conformal with $\lambda^2 = e(\phi)$. □

5. Harmonic maps and minimal branched immersions

Let M^2 be a *surface*, i.e., a Riemannian manifold of dimension two. Then the energy integral is unchanged under conformal changes of the metric, so that the concept of harmonic map from a surface depends only on its conformal structure; in particular, if M^2 is orientable, we can (and will) take it to be a *Riemann surface*, i.e., one-dimensional complex manifold; then methods of complex analysis may be used.

Let $\phi : M^2 \to N$ be a weakly conformal map from a surface. Then, an easy calculation shows that, away from points where $d\phi$ is zero, the mean curvature is $2\lambda^2$ times the tension field, hence, *a weakly conformal map from a surface is harmonic if and only if it is minimal away from points where its differential is zero*. Such a map is called a *minimal branched immersion*; the points where $d\phi$ is zero are called *branch points*, and are described in [17].

The following fact was established by the author [30] and many others.

LEMMA 1. *Any harmonic map from the* 2*-sphere* S^2 *is weakly conformal and so is a minimal branched immersion.*

Proof. The $(2,0)$-part of the stress energy tensor is a holomorphic section of $T^*_{2,0}S^2$. Such a section must vanish since this bundle has negative degree. □

6. Smoothness of transforms

Let $\pi : \mathbb{C}^{n+1} \setminus \{\vec{0}\} \to \mathbb{C}P^n$ be the canonical projection. For any smooth map $\phi : M^2 \to \mathbb{C}P^n$, write $\phi = [\Phi]$ to mean that $\Phi : U \to \mathbb{C}^{n+1}$ is a smooth map on an open subset of M^2 with $\phi = \pi \circ \Phi$ away from zeros; thus Φ represents ϕ in homogeneous coordinates. We denote orthogonal projection onto ϕ (resp. $\phi^\perp$) by π_ϕ (resp. $\pi_\phi^\perp$). Note that the linear map $\Phi \mapsto \pi_\phi^\perp(\partial\Phi/\partial z)$ (resp. $\Phi \mapsto \pi_\phi^\perp(\partial\Phi/\partial\bar{z})$) represents the partial derivative $\partial\phi/\partial z$ (resp. $\partial\phi/\partial\bar{z}$).

A smooth map $\phi = [\Phi] : M^2 \to \mathbb{C}P^n$ has two *Gauss transforms*, a ∂'-transform:

$$G'(\phi) = \left[\pi_\phi^\perp \frac{\partial\Phi}{\partial z}\right],$$

defined at points where $\partial\phi/\partial z$ is non-zero, and a ∂''-transform:

$$G''(\phi) = \left[\pi_\phi^\perp \frac{\partial\Phi}{\partial\bar{z}}\right],$$

defined at points where $\partial\phi/\partial\bar{z}$ is non-zero. These are both independent of the choice of Φ.

For simplicity, assume now that M^2 is oriented. If ϕ is harmonic and not antiholomorphic (resp. holomorphic), then $G'(\phi)$ (resp. $G''(\phi)$) extends over the zeros of $\partial\phi/\partial z$ (resp. $\partial\phi/\partial\bar{z}$) to give a harmonic map. Then, following work of other authors (see [14]), the next result was established by J. Eells and the author [14]; we give the formulation in [9].

THEOREM 2. *All harmonic maps from S^2 to CP^n are obtained from holomorphic maps by applying the ∂'-Gauss transform up to n times.*

To use this to study the space of harmonic maps, we need to answer the question, *Is the Gauss transform smooth, or even continuous?* In general, it is not; however, for integers k, d and E with k and E non-negative, set

$$\mathrm{Hol}_k^*(S^2,\mathbb{C}P^2) = \text{the space of full holomorphic maps of degree } k;$$
$$\mathrm{Harm}_{d,E}(S^2,\mathbb{C}P^2) = \text{the space of harmonic maps of degree } d \text{ and energy } 4\pi E.$$

Then the following was established by L. Lemaire and the author.

THEOREM 3 ([21]). *The Gauss transform*

$$G' : \mathrm{Hol}_k^*(S^2,\mathbb{C}P^2) \to \mathrm{Harm}(S^2,\mathbb{C}P^2)$$

is smooth if restricted to the subspace $\mathrm{Hol}_{k,r}^*(S^2,\mathbb{C}P^2)$ *of holomorphic maps of fixed total ramification index r. In fact, it gives a diffeomorphism*

$$G' : \mathrm{Hol}_{k,r}^*(S^2,\mathbb{C}P^2) \to \mathrm{Harm}_{k-r-2,\,3k-r-2}(S^2,\mathbb{C}P^2)\,.$$

7. Infinitesimal deformations and transformations

Let $(\phi_{t,s})$ be a 2-parameter variation of ϕ; write $v = \dfrac{\partial \phi_{t,s}}{\partial t}\Big|_{(0,0)}$ and $w = \dfrac{\partial \phi_{t,s}}{\partial s}\Big|_{(0,0)}$ for the corresponding variation vector fields. Set

$$H_\phi(v,w) = \frac{\partial^2 E}{\partial t \partial s}(\phi_{t,s})\Big|_{(0,0)} = \int_M \langle J_\phi v, w\rangle\, \omega_g$$

where

$$J_\phi v = \Delta^\phi v - \mathrm{Tr} R^N(\mathrm{d}\phi, v)\mathrm{d}\phi\,.$$

Here $\Delta^\phi = -\mathrm{Tr}\nabla^2$ is the Laplacian on $\phi^{-1}TN$. The linear operator J_ϕ is called the *Jacobi operator* along ϕ.

The following is easy to establish [22].

LEMMA 2. (i) *If (ϕ_t) is a one-parameter family of maps with $\phi_0 = \phi$ and $\partial\phi_t/\partial t\big|_{t=0} = v$, then*

$$J_\phi(v) = -\frac{\partial}{\partial t}\tau(\phi_t)\Big|_{t=0}\,.$$

(ii) *If ϕ_t is a one-parameter family of* harmonic *maps with $\phi_0 = \phi$, then $v = \partial\phi_t/\partial t\big|_{t=0}$ is a* Jacobi field along ϕ, *i.e.*, $J_\phi(v) = 0$.

So J_ϕ is the *linearization* of the tension field τ, up to a sign convention.

8. Integrability of Jacobi fields

DEFINITION 1. *A Jacobi field v along a harmonic map ϕ is said to be* integrable *if there is a one-parameter family (ϕ_t) of harmonic maps with $\phi_0 = \phi$ and* $\left.\dfrac{\partial\phi_t}{\partial t}\right|_{t=0} = v$.

We ask the question, *For what manifolds are all Jacobi fields integrable?*

One reason that this is important is the following result of D. Adams and L. Simon.

THEOREM 4 ([1]). *Let (M,g) and (N,h) be real-analytic Riemannian manifolds. If all Jacobi fields along harmonic maps from M to N are integrable, then* $\mathrm{Harm}(M,N)$ *is a real-analytic manifold* with tangent spaces given by the Jacobi fields.

Since we can construct all harmonic maps from S^2 to $\mathbb{C}P^n$ explicitly as above, it is natural to ask what is known in this case. In the case $n = 1$, R. Gulliver and B. White [18] showed that all Jacobi fields along harmonic maps are integrable. For the case $n = 2$, L. Lemaire and the author showed the following.

THEOREM 5 ([22]). *All Jacobi fields along harmonic maps from S^2 to $\mathbb{C}P^2$ are integrable.*

The idea is that the Gauss transform and its inverse are smooth away from branch points, so if a harmonic map $\phi : S^2 \to \mathbb{C}P^2$ is the Gauss transform $G'(f)$ of a holomorphic map $f : S^2 \to \mathbb{C}P^2$, then the inverse of G' maps a Jacobi field along ϕ into one along f. We then show that this Jacobi field is actually holomorphic. The key step is to show that it extends across the branch points, then a GAGA principle tells us that it's actually given by rational functions and so explicitly integrable. It follows that the original harmonic map is integrable. The methods make essential use of the low dimensions, and so are unlikely to generalize to higher n.

9. Factorization into unitons

The Gauss transform is an example of K. Uhlenbeck's operation of 'adding a uniton' which transforms harmonic maps $M^2 \to \mathrm{U}(n)$ from a surface to the unitary group into other harmonic maps $M^2 \to \mathrm{U}(n)$ as follows. Any harmonic map ϕ defines a connection $A^\phi = \frac{1}{2}\phi^{-1}\mathrm{d}\phi$ on the trivial bundle $\underline{\mathbb{C}}^n = M^2 \times \mathbb{C}^n$ and thus a covariant derivative $D^\phi = \mathrm{d} + A^\phi$; then a *uniton* or *flag factor* for ϕ is a subbundle β of the trivial bundle which is (i) *holomorphic with respect to $D^\phi_{\bar z}$* (i.e. the sections of β are closed under $D^\phi_{\bar z}$), and (ii) *closed under A^ϕ_z*. Uhlenbeck showed the following.

THEOREM 6 ([28]). (i) *The map $\widetilde{\phi} : M^2 \to \mathrm{U}(n)$ given by $\widetilde{\phi} = \phi(\pi_\beta - \pi_\beta^\perp)$ is harmonic. We say that $\widetilde{\phi}$ is obtained from ϕ by* adding the uniton β *or* by the flag transform with flag factor β.

(ii) *Any harmonic map* $\phi : S^2 \to \mathrm{U}(n)$ *can written as a finite product:*

$$\phi = \text{const.}\,(\pi_{\beta_1} - \pi_{\beta_1}^{\perp}) \cdot \ \ldots \ \cdot (\pi_{\beta_r} - \pi_{\beta_r}^{\perp})\,.$$

Such a product is called a *uniton factorization* of the harmonic map ϕ, and the minimum number of flag factors required is called the *uniton number*. Given an arbitrary harmonic map ϕ, one method of factorization is to add the uniton $\alpha^0 = \ker A_z^{\phi}$ giving a new harmonic map ϕ^1, then repeat the process. After a finite number r of steps one reaches a constant map; then setting $\beta_i = (\alpha^{r-i})^{\perp}$ gives a factorization, called the *factorization by* A_z*-kernels*. Dually, we may use A_z-images [32]. However, neither of these is the most efficient way in the sense of minimizing the number of steps; that is provided by using the kernel of the bottom coefficient of the extended solution as proposed by Uhlenbeck [28]; dually, we may use the image of the adjoint of the top coefficient.

Conversely, to build all possible harmonic maps, we do successive flag transforms starting with the constant map, giving a sequence of harmonic maps $\phi_0 = $ const., $\phi_1, \ldots, \phi_r = \phi$. To do this, we must know all the possible flag factors (unitons) at each stage.

However, there are two problems:

(i) to find unitons, we must find a holomorphic (or, at least, meromorphic) basis for the trivial bundle $\underline{\mathbb{C}}^n$ with respect to $D_{\bar{z}}^{\phi_i}$ for each i; to find this we must, in general, solve $\bar{\partial}$-problems;

(ii) like the Gauss transform, adding a uniton may not depend smoothly, or even continuously, on the data.

M. J. Ferreira, B. A. Simões and the author solved the first problem as follows.

THEOREM 7 ([15]). *For the (dual of) Uhlenbeck's factorization, all the possible unitons at each stage can be found explicitly in terms of projections of holomorphic functions, without solving* $\bar{\partial}$*-problems, giving explicit formulae for all harmonic maps from the* 2*-sphere to* $\mathrm{U}(n)$.

By thinking of them as stationary Ward solitons, B. Dai and C.-L. Terng [13] also obtained explicit formulae for the unitons of the Uhlenbeck factorization.

The author and M. Svensson [27] developed the ideas in [15] to show how to find explicit formulae for the harmonic maps corresponding to *any* factorization of $\mathrm{U}(n)$ including those in [13] and [15]. Thus we obtain explicit algebraic parametrizations of all harmonic maps $S^2 \to \mathrm{U}(n)$ by meromorphic functions, which can be used to study continuity.

10. Smoothness of twistor methods

With a history going back to Weierstrass, *twistor methods* have been successful in constructing harmonic maps, especially from the 2-sphere to symmetric spaces. The first case of this was the following.

Thinking of $\mathbb{C}P^3$ as the set of complex lines through the origin, let $\pi : \mathbb{C}P^3 \to \mathbb{H}P^1 = S^4$ be the *Calabi–Penrose twistor map* given by sending a complex line to the quaternionic line containing it. E. Calabi showed the following.

THEOREM 8 ([10, 11]). *Every harmonic map $\phi : S^2 \to S^4$ is $\pm$ the projection $\pi \circ f$ of a horizontal holomorphic map $f : S^2 \to \mathbb{C}P^3$.*

Here 'horizontal' means that the image of the differential $\mathrm{d}f$ at each point is orthogonal to the kernel of $\mathrm{d}\pi$.

L. Lemaire and the author used this to study Jacobi fields along harmonic maps from S^2 to S^4 and to S^3, obtaining the following result [23].

THEOREM 9. *(i) For each $d = 1, 2, \dots$, the map $f \mapsto \phi = \pi \circ f$ is a diffeomorphism of the space of holomorphic horizontal maps $f : S^2 \to \mathbb{C}P^3$ of degree d onto the space of harmonic maps $\phi : S^2 \to S^4$ of energy $4\pi d$.*

(ii) If ϕ (equivalently, f) is full, the Jacobi fields along ϕ correspond to infinitesimal deformations of the horizontal holomorphic map f.

(iii) There are some non-full harmonic maps $\phi : S^2 \to S^4$ which have non-integrable Jacobi fields.

(iv) There are some non-full harmonic maps $\phi : S^2 \to S^3$ which have non-integrable Jacobi fields.

11. The dual problem: horizontal weak conformality

The following definition can be regarded as the dual of that of weak conformality given in Section 4.

DEFINITION 2. *$\phi : (M, g) \to (N, h)$ is called* horizontally weakly conformal (HWC) *(or* semiconformal*) if, for each $p \in M$,* either

(i) $\mathrm{d}\phi_p = 0$, *in which case we call p a* critical point, or

(ii) $\mathrm{d}\phi_p$ *maps the horizontal space $\mathcal{H}_p = \{\ker(\mathrm{d}\phi_p)\}^\perp$ conformally onto $T_{\phi(p)}N$, i.e.,* $\mathrm{d}\phi_p$ *is surjective and there exists a number $\lambda(p) \neq 0$ such that*

$$h\big(\mathrm{d}\phi_p(X), \mathrm{d}\phi_p(Y)\big) = \lambda(p)^2 g(X, Y) \quad (X, Y \in \mathcal{H}_p),$$

in which case we call p a regular point.

Equivalently, ϕ is HWC if and only if, for each $p \in M$,

$$\mathrm{d}\phi_p \circ \mathrm{d}\phi_p^* = \lambda(p)^2 \, \mathrm{Id}_{T_{\phi(p)}N}$$

for some $\lambda(p) \in [0, \infty)$. In local coordinates, this reads

$$g^{ij}\phi_i^\alpha \phi_j^\beta = \lambda^2 h^{\alpha\beta}.$$

These should be compared with the formulae in Section 4. The function $\lambda : M \to [0,\infty)$ is called the *dilation* of ϕ; it is smooth away from the critical points; on setting it equal to zero at the critical points, it becomes continuous on M with λ^2 smooth.

Note that, whereas a non-constant weakly conformal map ϕ is an immersion away from the points where $d\phi$ vanishes, a non-constant horizontally weakly conformal map is a submersion away from those points.

We then have the following dual of Sanini's result above, see [6].

THEOREM 10. *A non-constant map ϕ is a critical point of the energy with respect to horizontal variations of the metric if and only if* $\dim N = 2$ *and ϕ is horizontally weakly conformal.*

Proof. The map is a critical point if and only if its stress-energy tensor is zero on horizontal vectors. It is easily seen that this holds if and only if $\dim N = 2$ and ϕ is HWC. □

12. Harmonic morphisms

We can now study a type of map which in many ways, is dual to that of harmonic maps. A smooth map $\phi : (M,g) \to (N,h)$ is called a *harmonic morphism* if, for every harmonic function $f : V \to \mathbb{R}$ defined on an open subset V of N with $\phi^{-1}(V)$ non-empty, the composition $f \circ \phi$ is harmonic on $\phi^{-1}(V)$. We have the following *characterization* due independently to B. Fuglede and T. Ishihara.

THEOREM 11 ([16, 19]). *A smooth map $\phi : M \to N$ between Riemannian manifolds is a harmonic morphism if and only if it is both* harmonic *and* horizontally weakly conformal.

Proof. The 'if' part is a simple application of the chain rule for a function of a function. The converse direction requires the local existence of enough harmonic functions, obvious in the real-analytic case, but more delicate in the smooth case. □

We list some properties of harmonic morphisms.

1. The *composition* of two harmonic morphisms is a harmonic morphism.

2. Harmonic morphisms *preserve harmonicity* of maps, i.e., the composition $f \circ \phi : M \to P$ of a harmonic map $f : N \to P$ with a harmonic morphism $\phi : M \to N$ is a harmonic map.

3. *If* $\dim N = 1$, then *the harmonic morphisms are precisely the harmonic maps*; in particular, *if $N = \mathbb{R}$, then the harmonic morphisms are precisely the harmonic functions*.

4. *A map $\phi : N \to P$ between surfaces* is a harmonic morphism if and only if it is weakly conformal.

5. The concept of *harmonic morphism to a surface* depends only on the conformal structure of the surface. Hence the notion of *harmonic morphism to a Riemann*

surface is well-defined.

We next list some examples of harmonic morphisms.

1. For any $m \in \{1,2,\dots\}$, *radial projection*

$$\mathbb{R}^m \setminus \{\vec{0}\} \to S^{m-1}, \quad \vec{x} \mapsto \vec{x}/|\vec{x}|$$

is a harmonic morphism with dilation $\lambda(\vec{x}) = 1/|\vec{x}|$. More generally, a *horizontally conformal submersion with* gradλ *tangent to the fibres is a harmonic morphism if and only if it has minimal fibres*.

2. The *Hopf maps* $S^3 \to S^2$, $S^7 \to S^4$, $S^{15} \to S^8$, $S^{2n+1} \to \mathbb{C}P^n$, $S^{4n+3} \to \mathbb{H}P^n$ are harmonic morphisms with constant dilation. More generally, *a Riemannian submersion is a harmonic morphism if and only if its fibres are minimal*.

3. (J.Y. Chen [12]) *Stable harmonic maps* from a compact Riemannian manifold to S^2 are harmonic morphisms.

13. Twistor theory for harmonic morphisms

The following was proved by the author [31] for submersions and extended to maps with critical points by M. Ville [29].

THEOREM 12. *Given a non-constant harmonic morphism $\phi : M^4 \to N^2$ from an orientable Einstein 4-manifold to a Riemann surface, there is a Hermitian structure J on M^4 such that ϕ is holomorphic with respect to J, and J is parallel along the fibres of ϕ.*

Conversely, the author showed that, if M^4 is also anti-self-dual, a Hermitian structure gave rise to local harmonic morphisms, away from points where it is Kähler. This was generalized as follows.

Hermitian structures correspond to holomorphic sections of the twistor space Z^6 of M^4. V. Apostolov and P. Gauduchon [2] showed that local existence of harmonic morphisms is equivalent to local existence of Hermitian structures, and this is equivalent to the self-dual part W_+ of the Weyl tensor being degenerate. When W_+ is identically zero, i.e., M^4 is anti-self-dual, the twistor space has an *integrable* complex structure so that there are lots of Hermitian structures, and so lots of harmonic morphisms.

Twistor methods have been extended to give various classes of holomorphic harmonic maps and morphisms from higher-dimensional spaces to surfaces, see [26] and [6, Chapters 8 and 9].

14. Explicit formulae for harmonic morphisms

Starting from Theorem 12, explicit formulae can be given for harmonic morphisms to surfaces from 4-dimensional real or complex space-forms. By dimension reduction, we obtain the following *mini-twistor* formulae in 3-dimensional space forms given by

P. Baird and the author [3] for $\mathbb{R}^3$ and [4] for S^3 and $\mathbb{H}^3$; we give here the version for $\mathbb{R}^3$. In fact, the first part of this result is essentially due to C. G. J. Jacobi [20]; the converse (ii) was established in [3].

THEOREM 13. (i) *Let g and h be holomorphic functions on an open subset of $\mathbb{C}$. Then any smooth local solution $\phi : \mathbb{R}^3 \supseteq U \to \mathbb{C}$, $z = \phi(x_1, x_2, x_3)$ to the equation*

$$-2g(z)x_1 + (1 - g(z)^2)x_2 + \mathrm{i}(1 + g(z)^2)x_3 = 2h(z) \tag{4}$$

is a harmonic morphism.

(ii) *Every harmonic morphism is given this way locally, up to composition with isometries on the domain and weakly conformal maps on the codomain.*

When M^4 is of Minkowski signature, Hermitian structures become *shear-free ray congruences*; on complexifying, they both become *holomorphic foliations by null planes*, see [5]. To find harmonic morphisms into Lorentzian surfaces, we replace the complex analytic functions g and h in (4) by functions analytic with respect to the hyperbolic numbers [7]. All cases can be unified by using the *bicomplex numbers*, see [8].

Acknowledgments. The author thanks Sergio Console and the organizing committee for the invitation to talk at the *Giornata di Geometria in Memoria di Aristide Sanini*, and Luc Lemaire and Martin Svensson for some comments on a draft of this paper.

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AMS Subject Classification: 53C42 (58E20)

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Lavoro pervenuto in redazione il 27.03.2009