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In Memoriam Aristide Sanini

S. Console

SOME RESEARCH TOPICS OF ARISTIDE SANINI*

I am sure that Aristide would make some ironic comment at the thought of me writing a note about his papers. He would probably also object to me not writing in Italian, which I remember he would say with some pride was his second language, his first being the dialect of his village, San Secondo Parmense, near Parma. But, in my opinion, the fact that most of his papers are in Italian (and in local journals) has unfairly limited the diffusion of his work.

I hope not to act against his will trying to focus on some of his research topics. Inevitably, I will put more stress on those that I understand better. These are topics that were developed when, or shortly before, I was his student, and some I learned directly from him during his frequent visits to the library in the Mathematics Department in the University of Turin.

I shall begin with a few biographical notes.

Sanini studied in Parma, under Professor Carmelo Longo, who belonged to Enrico Bompiani's school. He moved from Parma to the Politecnico di Torino to take up a post as Longo's assistant in the 1960's. At this beginning stage of his career, his research was mainly in projective differential geometry. This is apparent by looking at his early papers [S2, S3, S1, S6, S5, S7, S9], as well as the paper based on a talk he gave in Bologna in 1990 on Bompiani's contributions to Riemannian geometry [S35].

In the 1970's, Sanini began to focus on Finsler geometry, with special emphasis on Finsler connections on the tangent bundle of a manifold. For a reference to some of his results on this topic, I refer readers to [12], and in particular page 152.

He began his collaboration with Franco Tricerri later in this decade. This is witnessed by the paper [S17] and the monograph [S20]. Despite the fact that these are their only works in common, their friendship and mathematical relationship endured until the tragic death of Tricerri and his family in 1994. This event left a deep wound, above all in Aristide's personal life, but also in his mathematical career.

Another long lasting collaboration was with Renzo Caddeo and later Paola Piu, both from Cagliari [S31, S42]. Their joint work began in the 1980's when Aristide started to turn his attention to harmonic maps, and later submanifold geometry.

Before I start describing Aristide's research topics, it is worth saying a little about his attitude and way of working, as far as I could understand and learn from him. He read thoroughly, and did not confine his attention to papers strictly related to his current research interests. For example, I remember a long conversation with him about a book on mechanics he was reading not long before his death. When he began a new research topic, he would read quite deeply all around the subject. Then

*An extended version of a talk given at the *Giornata di Geometria in memoria di Aristide Sanini* held at the Politecnico di Torino on 27 June 2008.

he would start computing using his own methods (for instance, he could handle very long calculations with moving frames). At this stage, he began writing research notes on which he would base a series of seminars. (The image on page 379 gives two examples.) Writing one or more papers on the chosen topic was the final step.

I am personally very fond of the technical report "Fibrati di Grassmann e applicazioni armoniche" [S33], which has been translated and published in this volume. I vividly remember when he gave a series of seminars in 1988 based on these notes, and soon afterwards I started my research under his supervision based on problems he discussed there. This report is both a summary of his research on harmonic maps and an example of his approach to the subject, and contains some original results.

I would divide the research of Aristide Sanini roughly as follows:

1960's: projective-differential geometry;

1970's: Finsler spaces;

1970-80's: geometry of foliations;

1980-90's: harmonic maps, Gauss maps;

1990's: submanifolds of Lie groups.

This paper is devoted to the description of the last two periods (Sections 1, 2).

An appendix at the end of this survey reproduces part of a paper Sanini and I wrote together in 1998 [S43]. I remember that Aristide was very happy with this, but it remained unpublished since some time later we found out that Cecília Ferreira had obtained a similar result [9]. But I think that our proof is simpler and since, to be frank, most of the ideas were Aristide's, this extract gives a concrete example of his mathematics.

The main result can be described as follows. Let M be an oriented surface of Euclidean space E^3 with no umbilical point and let $\varphi : M \to SO(3)$ be the function mapping each point x of M to the orthogonal matrix determined by the orthonormal frame $\{e_1, e_2, e_3\}$, where e_1 and e_2 are the unit vectors of the principal directions at x. Of course, this map can be locally identified with the Gauss map of M into the flag manifold of triples of orthogonal one-dimensional vector subspaces of $\mathbb{R}^3$ studied in [9]. It is shown that φ *is harmonic if and only if M is a surface of revolution for which the product of the radius of a parallel and the curvature of a given meridian is constant* (Theorem 3).

Anno Accademico 1969-70

Aristide SANINI

"CONNESSIONI SU SPAZI DI FINSLER"

Istituto Matematico del Politecnico di Torino

SERIE DI PUBBLICAZIONI
DELL'ISTITUTO MATEMATICO DEL
POLITECNICO DI TORINO

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SERIE III : SEMINARI

N° 15

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1. Harmonic maps, Gauss maps

1.1. Harmonic maps and deformations of metrics

I shall begin with an account of Sanini's contributions to harmonic maps in the 1980's. Recall that a map of Riemannian manifolds $\phi : (M,g) \to (N,g')$ is *harmonic* if it is a critical point of the *energy functional*

$$E(\phi) := \frac{1}{2}\int_M \mathrm{tr}_g \phi^* g' dv_g \,,$$

where dv_g denotes the volume element of M with respect to the metric g. The real number $e(\phi) := \frac{1}{2}\mathrm{tr}_g \phi^* g'$ is called *energy density*. (See also the paper of J. Wood in this volume [18].)

If ϕ_t is a one-parameter variation of $\phi = \phi_0$ and

$$v = \frac{d\phi_t}{dt}_{|t=0} \in \phi^{-1}TN$$

is the corresponding *variation vector field of* (ϕ_t) then

$$\frac{dE(\phi_t)}{dt}_{|t=0} = -\int_M (\tau(\phi), v) dv_g = -\langle \tau(\phi), v\rangle \,,$$

where $\tau(\phi) := \mathrm{tr}_g Dd\phi$ is the *tension field* of ϕ.

Hence $\tau(\phi) = 0$ is the Euler–Lagrange equation for the energy functional $E(\phi)$, and ϕ is harmonic if and only if $\tau(\phi) = 0$.

The energy functional

$$E(\phi) := \frac{1}{2}\int_M \mathrm{tr}_g \phi^* g' dv_g \,,$$

depends in an essential way on the metric.

A smooth map $\phi : (M,g) \to (N,g')$ is called *weakly conformal* if

(1) $$\phi^* h = \lambda^2 g$$

for some function $\lambda : M \to [0,\infty)$ (cf. [18]).

In [S26], Sanini carried out an investigation of the conditions for the energy to be stationary with respect to a deformation of the metric. More precisely,

1. arbitrary deformations: *E is critical if and only if* $\dim M = 2$ *and ϕ is weakly conformal or* $\dim M > 2$ *and ϕ is constant.*

2. isovolumetric deformations: *E is critical if and only if* $\dim M = 2$ *and ϕ is weakly conformal or* $\dim M > 2$ *and ϕ is either a homothetic immersion or constant.*

(cf. Theorem 1 in the paper of Wood in this volume [18]).

The *stress energy tensor* of a map $\phi : (M,g) \to (N,g')$ is the tensor field $S(\phi) = e(\phi)g - \phi^* g'$. Its divergence is $\operatorname{div} S(\phi) = -\langle \tau(\phi), d\phi \rangle$ and, in particular, if ϕ is harmonic, $S(\phi)$ is conservative (i.e., its divergence is identically zero).

Actually, (cf. a proof of Theorem 1 in [18]), the Euler–Lagrange operator for the variation of energy is precisely the stress energy tensor.

More generally, given $\phi : (M,g) \to (N,g')$ with energy density $e(\phi) := \frac{1}{2}\mathrm{tr}_g \phi^* g'$, Uhlenbeck [16] introduced the *m-energy* functional

$$E^m(\phi) := \frac{1}{2}\int_M \left(\frac{2}{m} e_\phi\right)^{m/2} dv_g\,, \qquad m = \dim M,$$

which agrees with the energy for $m = 2$ and depends only on the conformal structure of M. Then

$$\frac{dE(\phi, g_t)}{dt}_{|t=0} = \frac{1}{2}\langle S^m(\phi), h\rangle\,, \qquad h = \frac{dg_t}{dt}_{|t=0}\,,$$

where $S^m(\phi) = \left(\frac{2}{m} e_\phi\right)^{m/2-1}\left(\frac{2}{m} e_\phi g - \phi^* g'\right)$ is the analog of the stress-energy tensor.

In [S34] the following results are proved:

1. *The m-energy functional $E^m(\phi)$ is critical with respect to deformations of g if and only if ϕ is weakly conformal.*

2. *If ϕ is weakly conformal, then ϕ is a local minimum of $E^m(\phi)$.*

The second part was obtained by computing the second derivative of $E^m(\phi)$.

The tangent bundle TM of a Riemannian manifold (M,g) can be endowed with a Riemannian metric, the so-called Sasaki metric, which makes the submersion $\pi : TM \to M$ Riemannian. This metric can be described as follows.

Elements of TM are pairs $(x,\dot{x})$, with $\dot{x} \in T_x M$. A local coordinate system (x^i) on M determines a local coordinate system $(x^i, \dot{x}^i)$ on TM, which associates to the vector $\dot{x}$ its components with respect to the natural basis $\partial_i = \frac{\partial}{\partial x^i}$ at the point x.

For any vector field $X \in \mathfrak{X}(M)$, its horizontal lift X^H and its vertical lift X^V are uniquely determined. This lift operation extends also to Finsler fields on M, i.e., to vector fields depending also on the directional variable $\dot{x}$. Thus any vector field $\widetilde{Z}$ tangent to TM can be written uniquely as

$$\widetilde{Z} = \widetilde{Z}^H + \widetilde{Z}^V\,,$$

where $\widetilde{Z}^H$ and $\widetilde{Z}^V$ are the horizontal and vertical component of $\widetilde{Z}$ respectively.

The Sasaki metric $\overline{g}$ on TM is then uniquely determined by the conditions

$$\overline{g}(X^H, Y^H) = \overline{g}(X^V, Y^V) = g(X,Y)\,, \qquad \overline{g}(X^H, Y^V) = 0\,, \quad \forall X, Y \in \mathfrak{X}(M)\,.$$

Let TM be the tangent bundle of (M,g), endowed with the Sasaki metric $\overline{g}$.

A vector field ξ on M may be considered as a map

$$\varphi_\xi : (M,g) \to (TM,\overline{g}).$$

A joint work Caddeo–Sanini [S31] studies conditions under which the induced metric $\varphi_\xi^*\overline{g}$ on M is harmonic with respect to g.

This is equivalent to the requirement that the identity map $\mathrm{id} : (M,g) \to (M,\varphi_\xi^*\overline{g})$ is harmonic. It turns out that this happens when:

1. ξ *is a conformal vector field and M is a surface,*

2. ξ *is a Killing vector field and M is locally flat,*

3. ξ *is a Killing vector field with constant length and M has constant curvature,* $\dim M > 2$.

Given a map $\phi : (M,g) \to (N,g')$, its differential Φ is a map between the Riemannian manifolds TM and TN, endowed with their respective Sasaki metrics.

Recall that the map ϕ is *totally geodesic* if $Dd\phi = 0$.

Computing $\overline{D}d\Phi$, where $\overline{D}$ is the metric connection on $\Phi^*(T(TN)$ induced by the Sasaki metric on TN, it is shown in [S25] that ϕ *is totally geodesic if and only if* Φ *is totally geodesic*.

Moreover, the tension field of Φ at $(x,\dot{x}) \in TM$, $\tau(\Phi) = \mathrm{tr}(Dd\Phi)$, is related to that of ϕ by

$$\tau(\Phi) = \left\{ \tau(\phi) + \sum_i R^N(Dd\phi(\dot{x},e_i)d\phi\dot{x})d\phi e_i \right\}^H + \{\mathrm{div} Dd\phi)(\dot{x})\}^V , \tag{2}$$

where e_i is an orthonormal basis at x and ^{H} (respectively ^{V}) denotes the horizontal (respectively vertical) projection.

Thus ([S25, Proposition 3]) *if* ϕ *is harmonic, then* Φ *is harmonic if and only if*

1. $\sum_i R^N(Dd\phi(\dot{x},e_i)d\phi\dot{x})d\phi e_i = 0$*, for any* $X \in \mathfrak{X}(M)$,

2. $\mathrm{div} Dd\phi = 0$.

The Laplacian of the energy density $e(\phi)$ can be expressed as follows (cf. [6]):

$$\Delta e(\phi) = |Dd\phi|^2 + \sum_i \langle d\phi(\mathrm{Ric}^M e_i), d\phi e_i\rangle - \sum_{i,j} \langle R^N(d\phi e_i, d\phi e_j)d\phi e_i, d\phi e_j\rangle , \tag{3}$$

where Ric is the (1,1) Ricci tensor. Moreover, the energy density of a totally geodesic map is constant. Hence, by (2), (3) and the above result [S25, Proposition 3] one gets that *if* Φ *is harmonic and M is compact then* ϕ *is totally geodesic.*

In a subsequent paper, [S29], Sanini considered an isometric immersion $f\colon M \to N$. The differential of f then defines an immersion $F_1\colon T_1M \to T_1N$ between the bundles of unit tangent vectors. This is related to the Gauss map of an m-dimensional submanifold of N as a map of M into the Grassmannian of m-planes in N (associating to each point x of M its tangent space at x) [13].

We add a few comments concerning the metric on the unit tangent bundles T_1M. The unit tangent bundle is the hypersurface of TM given by elements $(x,\dot{x})$ such that $|\dot{x}| = 1$. It can therefore be endowed with the induced metric. In this case, T_1M has constant mean curvature ([S29, Proposition 2]).

Actually, as Sanini learned later from [11] and [10], there is a one-parameter family of metrics on T_1M making the submersion $T_1M \to M$ Riemannian. We will call these metrics "Sasaki-like metrics". The Sasaki metric corresponds to setting the parameter equal to 1. This is at the origin of what he later described as "a strange result", that I shall now discuss. He studied the harmonicity of F_1 when f is not totally geodesic and N *is a space of constant curvature* c. Using similar computations as in his previous paper [S25], he showed that in this case F_1 *is harmonic (with respect to the Sasaki metrics on* T_1M *and* T_1N*) if and only if*

1. $c = 0$ *and* $f(M)$ *is a minimal Einstein submanifold,*
2. $c = \dim M$ *and* $f(M)$ *is a totally umbilical submanifold of* N.

Actually, if one modifies the metric of the unit tangent bundle by a constant [10] (i.e., if one considers a "Sasaki-like metric" on T_1M instead – see Subsection 1.2) then this condition becomes a relationship among this constant, the sectional curvature of N and the dimension of M (see [S33, Proposition 4][1]).

A submanifold M of a Riemannian manifold N is called *pseudoumbilical* if the mean curvature vector is an umbilical normal section, i.e., if $A_H = |H|^2\mathrm{id}$, where A denotes the shape operator.

Also in the paper [S29], the following generalization of the Ruh–Vilms Theorem [15] was proved.

THEOREM 1 ([S29, Theorem 3]). *Let* N *be a space of constant curvature* c. *If* $f : M \to N$ *is a pseudoumbilical Einstein submanifold with parallel mean curvature, then the restriction to* T_1M *of the vertical component of the tension field* $\tau(F)$ *of the differential* $F : TM \to TN$ *of* f *is orthogonal to* T_1N.

Conversely, if the vertical component of $\tau(F)_{|T_1M}$ *is orthogonal to* T_1N*, then*

1. *the mean curvature of* M *is parallel,*
2. *the following conditions are equivalent:*
   (a) *M is Einstein,*
   (b) *M is pseudoumbilical,*

[1] Here and later, I refer to the numbering in the translation printed in this volume

3. *the quadratic form*

$$\begin{aligned} Q(X) &= g_N(\mathrm{Ric}^M(X) - mA_H X - c(m-1)X, X) \\ &= -\textstyle\sum_i g_N(\alpha(e_i, X), \alpha(e_i, X)) \end{aligned}$$

is proportional to the metric of M, where $m = \dim M$, α is the second fundamental form and (e_i) is an orthonormal frame of M.

We will see in the next subsection that to say that Q is proportional to the metric is equivalent to the conformality of the Gauss map of M into the Grassmannian of m-planes.

1.2. Gauss maps and harmonic maps

Recall that the "classical" Gauss map γ maps any point x of a orientable surface immersed in $\mathbb{R}^3$ to the unit vector N_x applied at the origin of $\mathbb{R}^3$.

If (M, g) is an m-dimensional Riemannian manifold isometrically immersed in $\mathbb{R}^n$ (or, more generally, a space of constant curvature), then one can define several generalizations of the Gauss map.

• The *Gauss map into the Grassmannian* which maps any $x \in M$ to the subspace of $\mathbb{R}^n$ parallel to $T_x M$, i.e.,

$$\gamma : M \to G_m(n)$$

with $G_m(n)$ the Grassmannian of m-planes of $\mathbb{R}^n$ endowed with its canonical metric as a symmetric space.

• The *"spherical" Gauss map* (defined by Chern and Lashof) is the mapping

$$\nu : T_1 M^{\perp} \to S^{n-1}$$

sending any unit normal vector to the point of S^{n-1} obtained by its parallel transport to the origin of $\mathbb{R}^n$.

The harmonicity of the Gauss map can be read in term of the submanifold geometry of M.

For example, a classical result by Chern is that an orientable surface $f : M^2 \hookrightarrow \mathbb{R}^n$ is harmonic if and only if the Gauss map $M \to G_2(n) \cong Q_{n-2}$ (complex quadric in $\mathbb{C}P^{n-1}$) is *antiholomorphic*.

Moreover, Ruh and Vilms [15] proved that $\gamma : M \to G_m(n)$ is harmonic if and only if the mean curvature vector is parallel, i.e., $\nabla^{\perp} H = 0$.

We refer to [S33, Theorem 4] for results of Obata on weak conformality of the Gauss map γ for submanifolds of spaces of constant curvature. The condition that γ is weakly conformal is equivalent to the fact that the quadratic form Q (see Theorem 1) is proportional to the metric of M, i.e., $Q(X) = \ell^2 g(X, X)$, for any X tangent to M, cf. [S33].

More generally, for a submanifold of an arbitrary Riemannian manifold $f : M \hookrightarrow N$ one can define the following generalized Gauss maps (Jensen–Rigoli [10], Wood [17]).

• The *Gauss map into the Grassmann bundle*:

$$\gamma : M \to G_m(TN)$$

sending any point m to $f_*(T_pM)$ regarded as a m-plane in $T_{f(p)}N$, and thus a point in the Grassmann bundle $G_m(TN)$. The latter is endowed with the one-parameter family of metrics (the "Sasaki-like metrics" we mentioned above and which we will describe soon).

• The *"spherical" Gauss map*:

$$\begin{array}{rccc} \nu : & T_1M^{\perp} & \to & T_1N \\ & (x,\xi) & \mapsto & (f(x),\xi)\,. \end{array}$$

Let $\mathrm{O}(M)$ be the principal bundle on M of orthonormal frames of M endowed with the canonical form $\theta = (\theta^i)$, which is a $\mathbb{R}^m$-valued 1-form and the $\mathfrak{o}(m)$-valued connection 1-form $\omega = (\omega^i_j)$ determined by the Levi-Civita connection on M. The Grassmann bundle $G_p(TM)$ is the associated bundle to $\mathrm{O}(M)$ with typical fiber the Grassmannian of p-planes in $\mathbb{R}^m$

$$G_p(m) = \frac{O(m)}{O(p)\times O(m-p)}\,.$$

Let us consider the quadratic form on $\mathrm{O}(M)$

$$W = \sum(\theta^i)^2 + \lambda^2\sum(\omega^a_r)^2\,,$$

where $r = 1,\dots,p,\ \ a = p+1,\dots,m$ and λ is a positive constant.

Since W is $O(p)\times O(m-p)$-invariant and vanishes on the fibers of the submersion $\mathrm{O}(M)\to G_p(TM)$, it induces a family of positive definite quadratic forms ds_λ on $G_p(TM)$, the "Sasaki-like metrics". The Sasaki metric on T_1M corresponds to $p = 1$ and $\lambda = 1$.

In the technical report [S33], the tension field of these generalized Gauss maps is computed. Thus, some known results (both of Sanini and other authors) are computed with a unified method.

We give an example, which I remember well, since it is related to the first paper I wrote following the advice and suggestions of Sanini [3], and a later joint paper [S37].

THEOREM 2. [10] *The spherical Gauss map* $\nu : T_1M^{\perp}\to T_1N$ *of a submanifold* M *of a space of constant curvature* N *and with* $\operatorname{codim} M \geq 2$ *is harmonic if and only if the following conditions hold*

1. f *is minimal,*
2. *the second fundamental form is* conformal, *i.e.,*

$$\operatorname{tr}(A_\xi A_\eta) = \lambda\langle\xi,\eta\rangle\,,$$

for any normal vectors ξ,η, *where* λ *is a function on* M.

Surfaces with conformal second fundamental form are in fact homogeneous (even symmetric) as soon as one assumes their mean curvature vector field H is parallel in the normal bundle. Indeed, *assume* $f\colon M \to N^n(c)$ *is an isometric immersion of a compact connected surface in an n-dimensional real space-form of constant curvature c (with* $n = 4$ *or 5). If the second fundamental form of M is conformal and nonzero and the mean curvature vector of M is parallel in the normal bundle, then either*

1. $n = 4$ *and M is a Veronese surface in a 4-sphere, or*

2. $n = 4$ *and M is a Clifford torus in a Euclidean 4-space, or*

3. $n = 5$ *and* f *is an immersion of a real projective plane into a 5-sphere, which is factored through a Veronese surface in a suitable 4-sphere in the 5-sphere [3].*

Submanifolds with conformal second fundamental form are related with a widely studied class of immersed submanifolds, the *isotropic immersions*. An immersion $f : M \hookrightarrow \overline{M}$ is said to be isotropic if for any $x \in M$, we have $\|\alpha(v,v)\| = \lambda(x)\|v\|^2$ for any $v \in T_xM$, where λ is a positive smooth function on M (the isotropy function).

The main link between the above classes of submanifolds is the following: *if* $f : M \hookrightarrow \overline{M}$ *has conformal second fundamental form and assuming that the codimension is* $p = \frac{1}{2}m(m+1)$, *then* f *is isotropic and the isotropy function coincides with the conformality function* [S37].

I am also in some way linked personally to the next paper Sanini wrote on Gauss maps, since I helped him to write it in English [S38]. In this paper, he studies the Gauss map $\gamma\colon (M,g) \to (G_m(n),\Gamma)$ and considers submanifolds satisfying the weaker property that the tension field of the Gauss map γ is orthogonal to its image, i.e., $\tau_\gamma \perp \mathrm{im}(\gamma)$. This is equivalent to the stress energy tensor of γ having zero divergence (cf. [1]) and is characterized by the condition

$$\sum_i \alpha(e_i,X) \cdot \nabla^{\perp}_{e_i} H = 0\,,$$

where H is the mean curvature vector field and (e_i) is an orthonormal frame of M.

In particular, *if M compact and orientable then* $\|H\|$ *constant*.

A detailed study is carried out for surfaces in $\mathbb{R}^n$ or more generally in spaces of constant curvature with $\tau_\gamma \perp \mathrm{im}(\gamma)$. For example it is shown that *the surfaces* $M^2 \subseteq N^3(c)$ *satisfying* $\sum_i \alpha(e_i,X) \cdot \nabla^{\perp}_{e_i} H = 0$ *with* $\nabla^{\perp} H \neq 0$ *are ruled by geodesics intersecting orthogonally a plane curve L of constant curvature in* $N^3(c)$. *For* $c = 0$ *they are round cones*.

2. Submanifolds of Lie groups

In the second half of the 1990's, Sanini started to turn his attention to submanifold geometry in Lie groups.

In particular, he considered the Heisenberg group

$$H_3 = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{R} \right\}$$

endowed with the left invariant Riemannian metric

$$ds^2 = dx^2 + dy^2 + (dz - x\,dy)^2 .$$

The Heisenberg group (H_3, ds^2), although diffeomorphic to $\mathbb{R}^3$, has a very different behavior from the point of view of its Riemannian (sub)manifold geometry. Indeed H_3 is a nilpotent Lie group admitting large classes of both minimal and constant mean curvature surfaces.

A remarkable property, explicitly proved in [S41], is that, however, H_3 *does not admit totally umbilical surfaces.*

The (generalized) Gauss map $\gamma : M \to G_m(TH_3)$ of a surface M of the Heisenberg group H_3 was examined in [S41].

Using the above property, it is proved that *the Gauss map* γ *is conformal if and only if* M *is minimal*. Moreover, a characterization of a surface M with constant mean curvature having vertically harmonic Gauss map is given. Namely, in case M is minimal, it is a surface having the same analytical representation in $\mathbb{R}^3$ as a plane parallel to the axis of revolution of H_3. In case M has nonvanishing constant mean curvature, M is a "round cylinder" (in the above sense) with rulings parallel to the axis of revolution of H_3. Vertically harmonic means that the vertical component of the tension field with respect to the submersion $G_2(TH_3) \to M$ vanishes.

In a joint paper with Piu [S42] they consider surfaces in the Heisenberg group (H_3, ds^2) of the form $S = \exp uX \exp vY$, $(u, v) \in \mathbb{R}^2$, where

$$X = \begin{pmatrix} 0 & a & c \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad Y = \begin{pmatrix} 0 & \alpha & \gamma \\ 0 & 0 & \beta \\ 0 & 0 & 0 \end{pmatrix}$$

are two linearly independent vectors tangent to H_3 at the identity. They prove that

1. *S is a minimal surface with Gauss map* γ *vertically harmonic if and only if* $[X, Y] = 0$ *(which is equivalent to* $a\beta - \alpha b = 0$*).*

2. *S is a minimal surface with* γ *harmonic if and only if* $[X, Y] = 0$ *and the one-parameter subgroup* $\sigma(u) = \exp uX$ *either is a geodesic of* H_3*, or has torsion equal to zero (i.e.,* $a^2 + b^2 - c^2 = 0$*).*

Moreover, if $\sigma(u)$ is not a geodesic and has vanishing torsion, then the ruled surface S_1 generated by principal normal lines is flat along $\sigma(u)$.

Appendix. A surface of revolution of a remarkable type

Let M be an oriented surface of Euclidean space E^3 with no umbilical point. Recall from the introduction that one can consider the map $\varphi : M \to SO(3)$ mapping each point x of M to the orthogonal matrix determined by the orthonormal frame $\{e_1, e_2, e_3\}$, where e_1 and e_2 are the unit vectors of the principal directions at x. This map can be locally identified with the Gauss map of M into the flag manifold of triples of orthogonal one dimensional vector subspaces. The compact Lie group $SO(3)$ is endowed with a biinvariant metric g'.

The following result will be proven.

THEOREM 3. *The map $\varphi : (M, g) \to (SO(3), g')$ is harmonic if and only if M is a surface of revolution for which the product of the radius of a parallel and the curvature of a given meridian is constant.*

As a first step, the surfaces of revolution satisfying the above condition will be constructed explicitly. One gets a family of surfaces as the general solution of an ordinary differential equation of second order. Observe that, for instance, spheres are not in this family (but round cones and cylinders are).

The next step will be to show that the surfaces constructed in the previuos section are the only surfaces for which φ is harmonic.

Observe that by Pluzhnikov's Theorem [14], a mapping f of a Riemannian manifold (M, g) into a Lie group G, endowed with a biinvariant metric g', is harmonic if and only if the form $f^*\theta$ has null divergence, where $f^*\theta$ is the induced form on M by the Maurer–Cartan form θ on G, cf. also [4].

Surfaces of revolution with ϕ harmonic

Let M be a surface in E^3, generated by revolution of the meridian curve $(x(u), 0, z(u))$, $x(u) > 0$, along the z axis. We assume that the meridian curve is referred to arc length, hence $(x')^2 + (z')^2 = 1$. Thus the surface M is parametrized by

$$P(u, v) = (x(u)\cos v,\ x(u)\sin v,\ z(u))\,.$$

A unit normal vector is $e_3 = (-z'\cos v, -z'\sin v, x')$. The first fundamental form is given by

$$ds^2 = du^2 + x^2 dv^2 \tag{4}$$

and the principal curvatures are

$$\begin{aligned} \alpha_{11} &= \alpha(e_1, e_1) = x'z'' - x''z', \qquad \text{(curvature of the meridian)},\\ \alpha_{22} &= \alpha(e_2, e_2) = z'/x, \end{aligned}$$

where $e_1 = P_u$, $e_2 = P_v/|P_v|$ is an orthonormal frame of the tangent space and α is the second fundamental form.

Set $X := (e_1, e_2, e_3)$ (where the e_i are thought as column vectors), the induced form by the Maurer–Cartan form θ of $SO(3)$ is given by

$$\varphi^*\theta = X^{-1}dX$$

$$(5)\quad = \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} \left\{ \left(\frac{de_1}{du}\, \frac{de_2}{du}\, \frac{de_3}{du} \right) du + \left(\frac{de_1}{dv}\, \frac{de_2}{dv}\, \frac{de_3}{dv} \right) dv \right\}$$

$$= \begin{pmatrix} 0 & 0 & x''z' - x'z'' \\ 0 & 0 & 0 \\ x'z'' - x''z' & 0 & 0 \end{pmatrix} du + \begin{pmatrix} 0 & -x' & 0 \\ x' & 0 & -z' \\ 0 & z' & 0 \end{pmatrix} dv.$$

The divergence of a 1-form $\beta = \sum \beta_i dx^i$ is given by

$$\delta\beta = -\sum g^{ij}\nabla_j\beta_i = -\sum g^{ij}\{\partial_j\beta_i - \Gamma^k_{ji}\beta_k\},$$

where g is the Riemannian metric and Γ are the Christoffel symbols.
Using (4), one gets $\Gamma^1_{11} = \Gamma^2_{11} = \Gamma^2_{22} = 0$ and $\Gamma^1_{22} = -xx'$.

The condition that $\varphi^*\theta$ has null divergence can be read off by the only equation

$$\partial_u(x'z'' - x''z') + \frac{x'}{x}(x'z'' - x''z') = 0,$$

which is equivalent to

$$(6)\qquad \partial_u\{x(x'z'' - x''z')\} = 0.$$

Hence we have the following

LEMMA 1. *The only surfaces of revolution M for which the map $\varphi : (M,g) \to (SO(3), g')$ is harmonic are the ones for which the product of the radius of a parallel and the curvature of a given meridian is constant.*

If the meridian has equation $y = 0$, $z = f(x)$, the above condition is equivalent to the second order ordinary differential equation

$$\frac{f''}{(1+f'^2)^{\frac{3}{2}}} = \frac{k}{x} \qquad (k \text{ constant}),$$

whose solutions (depending on the constants k and $c > 0$) are

$$f(x) = \pm \int \frac{\log(cx^k)}{\sqrt{1-\log^2(cx^k)}}\, dx.$$

Observe that, for $k = 0$ one gets the round cone and, with an obvious change of variables, the round cylinder.

The general case of surfaces with ϕ harmonic

Let M be a surface of E^3, with no umbilical point. Using the same notations as in [2], at any point $x \in M$ one has an orthonormal frame $\{e_1, e_2, e_3\}$, where e_3 is the unit normal vector and e_1, e_2 are the unit vectors of the principal directions. In terms of differential forms (if x is the position vector field, ω^i the dual forms to e_i and ω^i_j are the connections forms, with $\omega^i_j + \omega^j_i = 0$) one has

$$\begin{aligned} dx &= \omega^1 e_1 + \omega^2 e_2, \\ de_1 &= \omega^2_1 e_2 + \omega^3_1 e_3, \\ de_2 &= \omega^1_2 e_1 + \omega^3_2 e_3, \\ de_3 &= \omega^1_3 e_1 + \omega^2_3 e_2. \end{aligned}$$

In particular, one has the structure equations

$$d\omega^1 = -\omega^1_2 \wedge \omega^2, \quad d\omega^2 = -\omega^2_1 \wedge \omega^1. \tag{7}$$

We set

$$\omega^2_1 = h\omega^1 + k\omega^2, \quad \omega^3_1 = a\omega^1, \quad \omega^3_2 = c\omega^2, \tag{8}$$

where a, c (with $a > c$) are the principal curvatures of M at x and

$$H = \frac{1}{2}(a+c), \qquad K = ac$$

are the mean curvature and the Gaussian curvature, respectively. The Gauss and Codazzi equations read

$$K = ac = h_2 - k_1 - h^2 - k^2, \tag{9}$$

$$c_1 = (a-c)k, \qquad a_2 = (a-c)h, \tag{10}$$

where, here and in the sequel, $h_2 = e_2(h)$ and so on. Further, one has

$$[e_1, e_2] = \nabla_{e_1} e_2 - \nabla_{e_2} e_1 = -he_1 - ke_2. \tag{11}$$

If φ denotes the map from M to $SO(3)$ given by the orthonormal frame $\{e_1, e_2, e_3\}$, then the 1-form $\varphi^*\theta$ with values in $\mathfrak{so}(3)$, induced by the Maurer–Cartan form on $SO(3)$ is given by

$$\varphi^*\theta = \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} (de_1\, de_2\, de_3) = \omega^\alpha_\beta(e_1)\omega^1 + \omega^\alpha_\beta(e_2)\omega^2. \tag{12}$$

Using the Hodge $*$ operator, we get

$$*\varphi^*\theta = -\omega^\alpha_\beta(e_2)\omega^1 + \omega^\alpha_\beta(e_1)\omega^2. \tag{13}$$

Thus φ is harmonic if and only if $\delta\varphi^*\theta d * \varphi^*\theta = 0$. Explicitly,

$$(14)\quad -d\left(\omega_\beta^\alpha(e_2)\right)\wedge\omega^1 + \omega_\beta^\alpha(e_2)\omega_2^1\wedge\omega^2 + d\left(\omega_\beta^\alpha(e_1)\right)\wedge\omega^2 - \omega_\beta^\alpha(e_1)\omega_1^2\wedge\omega^1 = 0.$$

Setting

$$\alpha = 1,\ \beta = 2;\quad \alpha = 1,\ \beta = 3;\quad \alpha = 2,\ \beta = 3,$$

respectively, and using (8), we get the following conditions expressing the harmonicity of φ:

$$(15)\qquad h_1 + k_2 = 0,$$

$$(16)\qquad a_1 + ak = 0,$$

$$(17)\qquad c_2 - ch = 0.$$

Note that (15) is equivalent to the fact that the codifferential of the connection form ω_1^2 vanishes. Using the above, equations (10), (11) and

$$[e_1, e_2](a) = -ha_1 - ka_2, \qquad [e_1, e_2](c) = -hc_1 - kc_2,$$

we get the equations

$$(18)\qquad ch_1 = -(a+c)hk, \qquad ak_2 = (a+c)hk.$$

When multiplied by a and c respectively, and added, using (15), these imply

$$(19)\qquad (c^2 - a^2)hk = 0.$$

Since M has no umbilical point, one cannot have $a = c$. If $c = -a$ (i.e., M is minimal), by (18) one would have $h_1 = k_2 = 0$ and hence, by (10), (16) and (17), $h = k = 0$, thus $a = c = 0$, by (9).

Hence in order that (18) hold, we must have $hk = 0$. We consider the case $h = 0$ (the other is similar and actually equivalent). For $h = 0$ the integral curves of the field e_1 are geodesics in M. Thus they are plane curves, since they are curvature lines (cf. for instance [8, page 140] or [5, page 152]). Moreover, by (10), (17) and (18), it follows that $a_2 = c_2 = k_2 = 0$, which means that the integral curves of the field e_2 are circles. Indeed, if $\overline{\nabla}$ denotes the Levi-Civita connection of E^3, we have

$$\overline{\nabla}_{e_2}e_2 = -ke_1 + ce_3, \qquad \overline{\nabla}_{e_2}\overline{\nabla}_{e_2}e_2 = -(k^2+c^2)e_2,$$

which implies that the curvature lines tangent to e_2 are plane curves and have constant curvature $\sqrt{k^2+c^2}$.

Thus the surfaces for which φ is harmonic are of revolution. To end the proof of the theorem, we show that the product of the principal curvature a and the radius $1/\sqrt{k^2+c^2}$ of the circle is constant.

We already remarked that

$$e_2\left(\frac{a}{\sqrt{k^2+c^2}}\right)=0.$$

Moreover, by (16), (10) and (9), we have

$$\begin{aligned} e_1\left(\frac{a}{\sqrt{k^2+c^2}}\right) &= a_1(k^2+c^2)^{-1/2}-a(k^2+c^2)^{-3/2}(kk_1+cc_1) \\ &= (k^2+c^2)^{-3/2}\left(-ak(k^2+c^2)-ak(-ac-k^2)-ac(a-c)k\right) \\ &= 0, \end{aligned}$$

proving the constancy.

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