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Università e Politecnico di Torino

In Memoriam Aristide Sanini

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Preface

Aristide Sanini was born in 1938 in San Secondo Parmense (a village close to Parma) and studied at the University of Parma. He spent almost all of his academic career at the Politecnico in Turin, having moved there in 1962, when he became assistant to the chair of Carmelo Longo. Aristide Sanini retired in 2000, to "leave room for young people" as he used to say. He died in the summer of 2007 after a long illness.

Before 1962 he had been a primary school teacher in Giuseppe Verdi's home village, Busseto (also near Parma). Therefore he used to say jokingly that he was "il maestro di Busseto".

His main field of research was Differential Geometry, and at the same time he gained the reputation of an extremely active and successful teacher of Geometry at the Politecnico. In addition, he devoted much of his energy to administrative duties: he was on the management board of Collegio Einaudi (a residence for students of the University and Politecnico) for twelve years, and served as both Vice-Rector and Head of the Department of Mathematics.

He was straightforward in his speech, and never at a loss for an answer. All his colleagues will miss him greatly.

This special issue of the Rendiconti incorporates the three lectures given by Sergio Console, Emilio Musso and John Wood on the afternoon of 27 June 2008, during the *Giornata di Geometria in memoria di Aristide Sanini*, supported by both the Istituto Nazionale di Alta Matematica (through its group GNSAGA) and the Politecnico di Torino.

The editors wish to thank the following persons, all of whom contributed to the Giornata and to its organization: Claudio Canuto, Pier Paolo Civalleri, Guido Fiegna, Donato Firrao, Franco Profumo, Fulvio Ricci, Anita Tabacco, Edoardo Vesentini, Rodolfo Zich. They are also grateful to Georgi Mihaylov for translating and transcribing Aristide Sanini's article that concludes this volume.

Sergio Console, Paolo Valabrega
23 December 2009

A. Sanini

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In Memoriam Aristide Sanini

S. Console

SOME RESEARCH TOPICS OF ARISTIDE SANINI*

I am sure that Aristide would make some ironic comment at the thought of me writing a note about his papers. He would probably also object to me not writing in Italian, which I remember he would say with some pride was his second language, his first being the dialect of his village, San Secondo Parmense, near Parma. But, in my opinion, the fact that most of his papers are in Italian (and in local journals) has unfairly limited the diffusion of his work.

I hope not to act against his will trying to focus on some of his research topics. Inevitably, I will put more stress on those that I understand better. These are topics that were developed when, or shortly before, I was his student, and some I learned directly from him during his frequent visits to the library in the Mathematics Department in the University of Turin.

I shall begin with a few biographical notes.

Sanini studied in Parma, under Professor Carmelo Longo, who belonged to Enrico Bompiani's school. He moved from Parma to the Politecnico di Torino to take up a post as Longo's assistant in the 1960's. At this beginning stage of his career, his research was mainly in projective differential geometry. This is apparent by looking at his early papers [S2, S3, S1, S6, S5, S7, S9], as well as the paper based on a talk he gave in Bologna in 1990 on Bompiani's contributions to Riemannian geometry [S35].

In the 1970's, Sanini began to focus on Finsler geometry, with special emphasis on Finsler connections on the tangent bundle of a manifold. For a reference to some of his results on this topic, I refer readers to [12], and in particular page 152.

He began his collaboration with Franco Tricerri later in this decade. This is witnessed by the paper [S17] and the monograph [S20]. Despite the fact that these are their only works in common, their friendship and mathematical relationship endured until the tragic death of Tricerri and his family in 1994. This event left a deep wound, above all in Aristide's personal life, but also in his mathematical career.

Another long lasting collaboration was with Renzo Caddeo and later Paola Piu, both from Cagliari [S31, S42]. Their joint work began in the 1980's when Aristide started to turn his attention to harmonic maps, and later submanifold geometry.

Before I start describing Aristide's research topics, it is worth saying a little about his attitude and way of working, as far as I could understand and learn from him. He read thoroughly, and did not confine his attention to papers strictly related to his current research interests. For example, I remember a long conversation with him about a book on mechanics he was reading not long before his death. When he began a new research topic, he would read quite deeply all around the subject. Then

*An extended version of a talk given at the *Giornata di Geometria in memoria di Aristide Sanini* held at the Politecnico di Torino on 27 June 2008.

he would start computing using his own methods (for instance, he could handle very long calculations with moving frames). At this stage, he began writing research notes on which he would base a series of seminars. (The image on page 379 gives two examples.) Writing one or more papers on the chosen topic was the final step.

I am personally very fond of the technical report “Fibrati di Grassmann e applicazioni armoniche” [S33], which has been translated and published in this volume. I vividly remember when he gave a series of seminars in 1988 based on these notes, and soon afterwards I started my research under his supervision based on problems he discussed there. This report is both a summary of his research on harmonic maps and an example of his approach to the subject, and contains some original results.

I would divide the research of Aristide Sanini roughly as follows:

1960’s: projective-differential geometry;

1970’s: Finsler spaces;

1970-80’s: geometry of foliations;

1980-90’s: harmonic maps, Gauss maps;

1990’s: submanifolds of Lie groups.

This paper is devoted to the description of the last two periods (Sections 1, 2).

An appendix at the end of this survey reproduces part of a paper Sanini and I wrote together in 1998 [S43]. I remember that Aristide was very happy with this, but it remained unpublished since some time later we found out that Cecília Ferreira had obtained a similar result [9]. But I think that our proof is simpler and since, to be frank, most of the ideas were Aristide’s, this extract gives a concrete example of his mathematics.

The main result can be described as follows. Let M be an oriented surface of Euclidean space E^3 with no umbilical point and let $\varphi : M \to SO(3)$ be the function mapping each point x of M to the orthogonal matrix determined by the orthonormal frame $\{e_1, e_2, e_3\}$, where e_1 and e_2 are the unit vectors of the principal directions at x. Of course, this map can be locally identified with the Gauss map of M into the flag manifold of triples of orthogonal one-dimensional vector subspaces of $\mathbb{R}^3$ studied in [9]. It is shown that φ *is harmonic if and only if M is a surface of revolution for which the product of the radius of a parallel and the curvature of a given meridian is constant* (Theorem 3).

Anno Accademico 1969-70

Aristide SANINI

"CONNESSIONI SU SPAZI DI FINSLER"

Istituto Matematico del Politecnico di Torino

SERIE DI PUBBLICAZIONI
DELL'ISTITUTO MATEMATICO DEL
POLITECNICO DI TORINO

-1981-

SERIE III : SEMINARI
N° 15

Aristide SANINI
(Istituto Matematico del Politecnico di Torino)

OPERATORI DIFFERENZIALI SU VARIETA' RIEMANNIANE
E STRUTTURE DI EINSTEIN

Pervenuto il 6 marzo 1981

1. Harmonic maps, Gauss maps

1.1. Harmonic maps and deformations of metrics

I shall begin with an account of Sanini's contributions to harmonic maps in the 1980's. Recall that a map of Riemannian manifolds $\phi : (M,g) \to (N,g')$ is *harmonic* if it is a critical point of the *energy functional*

$$E(\phi) := \frac{1}{2}\int_M \mathrm{tr}_g \phi^* g' dv_g\,,$$

where dv_g denotes the volume element of M with respect to the metric g. The real number $e(\phi) := \frac{1}{2}\mathrm{tr}_g \phi^* g'$ is called *energy density*. (See also the paper of J. Wood in this volume [18].)

If ϕ_t is a one-parameter variation of $\phi = \phi_0$ and

$$v = \frac{d\phi_t}{dt}_{|t=0} \in \phi^{-1}TN$$

is the corresponding *variation vector field of* (ϕ_t) then

$$\frac{dE(\phi_t)}{dt}_{|t=0} = -\int_M (\tau(\phi), v) dv_g = -\langle \tau(\phi), v\rangle\,,$$

where $\tau(\phi) := \mathrm{tr}_g Dd\phi$ is the *tension field* of ϕ.

Hence $\tau(\phi) = 0$ is the Euler–Lagrange equation for the energy functional $E(\phi)$, and ϕ is harmonic if and only if $\tau(\phi) = 0$.

The energy functional

$$E(\phi) := \frac{1}{2}\int_M \mathrm{tr}_g \phi^* g' dv_g\,,$$

depends in an essential way on the metric.

A smooth map $\phi : (M,g) \to (N,g')$ is called *weakly conformal* if

$$\phi^* h = \lambda^2 g \tag{1}$$

for some function $\lambda : M \to [0,\infty)$ (cf. [18]).

In [S26], Sanini carried out an investigation of the conditions for the energy to be stationary with respect to a deformation of the metric. More precisely,

1. arbitrary deformations: *E is critical if and only if* $\dim M = 2$ *and ϕ is weakly conformal or* $\dim M > 2$ *and ϕ is constant*.

2. isovolumetric deformations: *E is critical if and only if* $\dim M = 2$ *and ϕ is weakly conformal or* $\dim M > 2$ *and ϕ is either a homothetic immersion or constant.*

(cf. Theorem 1 in the paper of Wood in this volume [18]).

The *stress energy tensor* of a map $\phi : (M,g) \to (N,g')$ is the tensor field $S(\phi) = e(\phi)g - \phi^* g'$. Its divergence is $\operatorname{div} S(\phi) = -\langle \tau(\phi), d\phi \rangle$ and, in particular, if ϕ is harmonic, $S(\phi)$ is conservative (i.e., its divergence is identically zero).

Actually, (cf. a proof of Theorem 1 in [18]), the Euler–Lagrange operator for the variation of energy is precisely the stress energy tensor.

More generally, given $\phi : (M,g) \to (N,g')$ with energy density $e(\phi) := \frac{1}{2}\mathrm{tr}_g \phi^* g'$, Uhlenbeck [16] introduced the *m-energy* functional

$$E^m(\phi) := \frac{1}{2}\int_M \left(\frac{2}{m} e_\phi\right)^{m/2} dv_g\,, \qquad m = \dim M,$$

which agrees with the energy for $m = 2$ and depends only on the conformal structure of M. Then

$$\frac{dE(\phi, g_t)}{dt}_{|t=0} = \frac{1}{2}\langle S^m(\phi), h\rangle\,, \qquad h = \frac{dg_t}{dt}_{|t=0}\,,$$

where $S^m(\phi) = \left(\frac{2}{m}e_\phi\right)^{m/2-1}\left(\frac{2}{m}e_\phi g - \phi^* g'\right)$ is the analog of the stress-energy tensor.

In [S34] the following results are proved:

1. *The m-energy functional $E^m(\phi)$ is critical with respect to deformations of g if and only if ϕ is weakly conformal.*
2. *If ϕ is weakly conformal, then ϕ is a local minimum of $E^m(\phi)$.*

The second part was obtained by computing the second derivative of $E^m(\phi)$.

The tangent bundle TM of a Riemannian manifold (M,g) can be endowed with a Riemannian metric, the so-called Sasaki metric, which makes the submersion $\pi : TM \to M$ Riemannian. This metric can be described as follows.

Elements of TM are pairs $(x,\dot{x})$, with $\dot{x} \in T_x M$. A local coordinate system (x^i) on M determines a local coordinate system $(x^i, \dot{x}^i)$ on TM, which associates to the vector $\dot{x}$ its components with respect to the natural basis $\partial_i = \frac{\partial}{\partial x^i}$ at the point x.

For any vector field $X \in \mathfrak{X}(M)$, its horizontal lift X^H and its vertical lift X^V are uniquely determined. This lift operation extends also to Finsler fields on M, i.e., to vector fields depending also on the directional variable $\dot{x}$. Thus any vector field $\widetilde{Z}$ tangent to TM can be written uniquely as

$$\widetilde{Z} = \widetilde{Z}^H + \widetilde{Z}^V\,,$$

where $\widetilde{Z}^H$ and $\widetilde{Z}^V$ are the horizontal and vertical component of $\widetilde{Z}$ respectively.

The Sasaki metric $\overline{g}$ on TM is then uniquely determined by the conditions

$$\overline{g}(X^H, Y^H) = \overline{g}(X^V, Y^V) = g(X,Y)\,, \qquad \overline{g}(X^H, Y^V) = 0\,, \quad \forall X, Y \in \mathfrak{X}(M)\,.$$

Let TM be the tangent bundle of (M,g), endowed with the Sasaki metric $\overline{g}$.

A vector field ξ on M may be considered as a map

$$\varphi_\xi : (M,g) \to (TM,\overline{g}).$$

A joint work Caddeo–Sanini [S31] studies conditions under which the induced metric $\varphi_\xi^*\overline{g}$ on M is harmonic with respect to g.

This is equivalent to the requirement that the identity map $\mathrm{id} : (M,g) \to (M,\varphi_\xi^*\overline{g})$ is harmonic. It turns out that this happens when:

1. ξ *is a conformal vector field and M is a surface,*
2. ξ *is a Killing vector field and M is locally flat,*
3. ξ *is a Killing vector field with constant length and M has constant curvature,* $\dim M > 2$.

Given a map $\phi : (M,g) \to (N,g')$, its differential Φ is a map between the Riemannian manifolds TM and TN, endowed with their respective Sasaki metrics.

Recall that the map ϕ is *totally geodesic* if $Dd\phi = 0$.

Computing $\overline{D}d\Phi$, where $\overline{D}$ is the metric connection on $\Phi^*(T(TN)$ induced by the Sasaki metric on TN, it is shown in [S25] that ϕ *is totally geodesic if and only if* Φ *is totally geodesic*.

Moreover, the tension field of Φ at $(x,\dot{x}) \in TM$, $\tau(\Phi) = \mathrm{tr}(Dd\Phi)$, is related to that of ϕ by

$$\tau(\Phi) = \left\{\tau(\phi) + \sum_i R^N(Dd\phi(\dot{x},e_i)d\phi\dot{x})d\phi e_i\right\}^H + \{\mathrm{div}Dd\phi)(\dot{x})\}^V , \tag{2}$$

where e_i is an orthonormal basis at x and ^{H} (respectively ^{V}) denotes the horizontal (respectively vertical) projection.

Thus ([S25, Proposition 3]) *if* ϕ *is harmonic, then* Φ *is harmonic if and only if*

1. $\sum_i R^N(Dd\phi(\dot{x},e_i)d\phi\dot{x})d\phi e_i = 0$*, for any* $X \in \mathfrak{X}(M)$,
2. $\mathrm{div}Dd\phi = 0$.

The Laplacian of the energy density $e(\phi)$ can be expressed as follows (cf. [6]):

$$\Delta e(\phi) = |Dd\phi|^2 + \sum_i \langle d\phi(\mathrm{Ric}^M e_i), d\phi e_i\rangle - \sum_{i,j}\langle R^N(d\phi e_i, d\phi e_j)d\phi e_i, d\phi e_j\rangle , \tag{3}$$

where Ric is the (1,1) Ricci tensor. Moreover, the energy density of a totally geodesic map is constant. Hence, by (2), (3) and the above result [S25, Proposition 3] one gets that *if* Φ *is harmonic and M is compact then* ϕ *is totally geodesic.*

In a subsequent paper, [S29], Sanini considered an isometric immersion $f\colon M \to N$. The differential of f then defines an immersion $F_1\colon T_1M \to T_1N$ between the bundles of unit tangent vectors. This is related to the Gauss map of an m-dimensional submanifold of N as a map of M into the Grassmannian of m-planes in N (associating to each point x of M its tangent space at x) [13].

We add a few comments concerning the metric on the unit tangent bundles T_1M. The unit tangent bundle is the hypersurface of TM given by elements $(x,\dot{x})$ such that $|\dot{x}| = 1$. It can therefore be endowed with the induced metric. In this case, T_1M has constant mean curvature ([S29, Proposition 2]).

Actually, as Sanini learned later from [11] and [10], there is a one-parameter family of metrics on T_1M making the submersion $T_1M \to M$ Riemannian. We will call these metrics "Sasaki-like metrics". The Sasaki metric corresponds to setting the parameter equal to 1. This is at the origin of what he later described as "a strange result", that I shall now discuss. He studied the harmonicity of F_1 when f is not totally geodesic and N *is a space of constant curvature* c. Using similar computations as in his previous paper [S25], he showed that in this case F_1 *is harmonic (with respect to the Sasaki metrics on* T_1M *and* T_1N*) if and only if*

1. $c = 0$ *and* $f(M)$ *is a minimal Einstein submanifold,*
2. $c = \dim M$ *and* $f(M)$ *is a totally umbilical submanifold of* N.

Actually, if one modifies the metric of the unit tangent bundle by a constant [10] (i.e., if one considers a "Sasaki-like metric" on T_1M instead – see Subsection 1.2) then this condition becomes a relationship among this constant, the sectional curvature of N and the dimension of M (see [S33, Proposition 4][1]).

A submanifold M of a Riemannian manifold N is called *pseudoumbilical* if the mean curvature vector is an umbilical normal section, i.e., if $A_H = |H|^2\mathrm{id}$, where A denotes the shape operator.

Also in the paper [S29], the following generalization of the Ruh–Vilms Theorem [15] was proved.

THEOREM 1 ([S29, Theorem 3]). *Let* N *be a space of constant curvature* c. *If* $f : M \to N$ *is a pseudoumbilical Einstein submanifold with parallel mean curvature, then the restriction to* T_1M *of the vertical component of the tension field* $\tau(F)$ *of the differential* $F : TM \to TN$ *of* f *is orthogonal to* T_1N.

Conversely, if the vertical component of $\tau(F)_{|T_1M}$ *is orthogonal to* T_1N*, then*

1. *the mean curvature of* M *is parallel,*
2. *the following conditions are equivalent:*
   (a) *M is Einstein,*
   (b) *M is pseudoumbilical,*

[1] Here and later, I refer to the numbering in the translation printed in this volume

3. *the quadratic form*

$$\begin{aligned} Q(X) &= g_N(\mathrm{Ric}^M(X) - mA_H X - c(m-1)X, X) \\ &= -\textstyle\sum_i g_N(\alpha(e_i, X), \alpha(e_i, X)) \end{aligned}$$

is proportional to the metric of M, where $m = \dim M$, α is the second fundamental form and (e_i) is an orthonormal frame of M.

We will see in the next subsection that to say that Q is proportional to the metric is equivalent to the conformality of the Gauss map of M into the Grassmannian of m-planes.

1.2. Gauss maps and harmonic maps

Recall that the "classical" Gauss map γ maps any point x of a orientable surface immersed in $\mathbb{R}^3$ to the unit vector N_x applied at the origin of $\mathbb{R}^3$.

If (M, g) is an m-dimensional Riemannian manifold isometrically immersed in $\mathbb{R}^n$ (or, more generally, a space of constant curvature), then one can define several generalizations of the Gauss map.

• The *Gauss map into the Grassmannian* which maps any $x \in M$ to the subspace of $\mathbb{R}^n$ parallel to T_xM, i.e.,

$$\gamma : M \to G_m(n)$$

with $G_m(n)$ the Grassmannian of m-planes of $\mathbb{R}^n$ endowed with its canonical metric as a symmetric space.

• The *"spherical" Gauss map* (defined by Chern and Lashof) is the mapping

$$\nu : T_1M^\perp \to S^{n-1}$$

sending any unit normal vector to the point of S^{n-1} obtained by its parallel transport to the origin of $\mathbb{R}^n$.

The harmonicity of the Gauss map can be read in term of the submanifold geometry of M.

For example, a classical result by Chern is that an orientable surface $f : M^2 \hookrightarrow \mathbb{R}^n$ is harmonic if and only if the Gauss map $M \to G_2(n) \cong Q_{n-2}$ (complex quadric in $\mathbb{C}P^{n-1}$) is *antiholomorphic*.

Moreover, Ruh and Vilms [15] proved that $\gamma : M \to G_m(n)$ is harmonic if and only if the mean curvature vector is parallel, i.e., $\nabla^\perp H = 0$.

We refer to [S33, Theorem 4] for results of Obata on weak conformality of the Gauss map γ for submanifolds of spaces of constant curvature. The condition that γ is weakly conformal is equivalent to the fact that the quadratic form Q (see Theorem 1) is proportional to the metric of M, i.e., $Q(X) = \ell^2 g(X, X)$, for any X tangent to M, cf. [S33].

More generally, for a submanifold of an arbitrary Riemannian manifold $f : M \hookrightarrow N$ one can define the following generalized Gauss maps (Jensen–Rigoli [10], Wood [17]).

• The *Gauss map into the Grassmann bundle*:

$$\gamma : M \to G_m(TN)$$

sending any point m to $f_*(T_pM)$ regarded as a m-plane in $T_{f(p)}N$, and thus a point in the Grassmann bundle $G_m(TN)$. The latter is endowed with the one-parameter family of metrics (the "Sasaki-like metrics" we mentioned above and which we will describe soon).

• The *"spherical" Gauss map*:

$$\begin{array}{rccc} \nu : & T_1M^\perp & \to & T_1N \\ & (x,\xi) & \mapsto & (f(x),\xi)\,. \end{array}$$

Let $\mathrm{O}(M)$ be the principal bundle on M of orthonormal frames of M endowed with the canonical form $\theta = (\theta^i)$, which is a $\mathbb{R}^m$-valued 1-form and the $\mathfrak{o}(m)$-valued connection 1-form $\omega = (\omega^i_j)$ determined by the Levi-Civita connection on M. The Grassmann bundle $G_p(TM)$ is the associated bundle to $\mathrm{O}(M)$ with typical fiber the Grassmannian of p-planes in $\mathbb{R}^m$

$$G_p(m) = \frac{O(m)}{O(p) \times O(m-p)}\,.$$

Let us consider the quadratic form on $\mathrm{O}(M)$

$$W = \sum (\theta^i)^2 + \lambda^2 \sum (\omega^a_r)^2\,,$$

where $r = 1,\dots,p,\;\; a = p+1,\dots,m$ and λ is a positive constant.

Since W is $O(p)\times O(m-p)$-invariant and vanishes on the fibers of the submersion $\mathrm{O}(M) \to G_p(TM)$, it induces a family of positive definite quadratic forms ds_λ on $G_p(TM)$, the "Sasaki-like metrics". The Sasaki metric on T_1M corresponds to $p = 1$ and $\lambda = 1$.

In the technical report [S33], the tension field of these generalized Gauss maps is computed. Thus, some known results (both of Sanini and other authors) are computed with a unified method.

We give an example, which I remember well, since it is related to the first paper I wrote following the advice and suggestions of Sanini [3], and a later joint paper [S37].

THEOREM 2. [10] *The spherical Gauss map $\nu : T_1M^\perp \to T_1N$ of a submanifold M of a space of constant curvature N and with* $\operatorname{codim} M \geq 2$ *is harmonic if and only if the following conditions hold*

1. *f is minimal,*
2. *the second fundamental form is* conformal, *i.e.,*

   $$\operatorname{tr}(A_\xi A_\eta) = \lambda\langle \xi,\eta\rangle\,,$$

   for any normal vectors ξ,η, where λ is a function on M.

Surfaces with conformal second fundamental form are in fact homogeneous (even symmetric) as soon as one assumes their mean curvature vector field H is parallel in the normal bundle. Indeed, *assume $f: M \to N^n(c)$ is an isometric immersion of a compact connected surface in an n-dimensional real space-form of constant curvature c (with $n = 4$ or 5). If the second fundamental form of M is conformal and nonzero and the mean curvature vector of M is parallel in the normal bundle, then either*

1. *$n = 4$ and M is a Veronese surface in a 4-sphere, or*
2. *$n = 4$ and M is a Clifford torus in a Euclidean 4-space, or*
3. *$n = 5$ and f is an immersion of a real projective plane into a 5-sphere, which is factored through a Veronese surface in a suitable 4-sphere in the 5-sphere [3].*

Submanifolds with conformal second fundamental form are related with a widely studied class of immersed submanifolds, the *isotropic immersions*. An immersion $f : M \hookrightarrow \overline{M}$ is said to be isotropic if for any $x \in M$, we have $\|\alpha(v,v)\| = \lambda(x)\|v\|^2$ for any $v \in T_xM$, where λ is a positive smooth function on M (the isotropy function).

The main link between the above classes of submanifolds is the following: *if $f : M \hookrightarrow \overline{M}$ has conformal second fundamental form and assuming that the codimension is $p = \frac{1}{2}m(m+1)$, then f is isotropic and the isotropy function coincides with the conformality function* [S37].

I am also in some way linked personally to the next paper Sanini wrote on Gauss maps, since I helped him to write it in English [S38]. In this paper, he studies the Gauss map γ: $(M,g) \to (G_m(n),\Gamma)$ and considers submanifolds satisfying the weaker property that the tension field of the Gauss map γ is orthogonal to its image, i.e., $\tau_\gamma \perp \mathrm{im}(\gamma)$. This is equivalent to the stress energy tensor of γ having zero divergence (cf. [1]) and is characterized by the condition

$$\sum_i \alpha(e_i, X) \cdot \nabla^\perp_{e_i} H = 0\,,$$

where H is the mean curvature vector field and (e_i) is an orthonormal frame of M.

In particular, *if M compact and orientable then $\|H\|$ constant*.

A detailed study is carried out for surfaces in $\mathbb{R}^n$ or more generally in spaces of constant curvature with $\tau_\gamma \perp \mathrm{im}(\gamma)$. For example it is shown that *the surfaces $M^2 \subseteq N^3(c)$ satisfying $\sum_i \alpha(e_i,X) \cdot \nabla^\perp_{e_i} H = 0$ with $\nabla^\perp H \neq 0$ are ruled by geodesics intersecting orthogonally a plane curve L of constant curvature in $N^3(c)$. For $c = 0$ they are round cones*.

2. Submanifolds of Lie groups

In the second half of the 1990's, Sanini started to turn his attention to submanifold geometry in Lie groups.

In particular, he considered the Heisenberg group

$$H_3 = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{R} \right\}$$

endowed with the left invariant Riemannian metric

$$ds^2 = dx^2 + dy^2 + (dz - x\,dy)^2 .$$

The Heisenberg group (H_3, ds^2), although diffeomorphic to $\mathbb{R}^3$, has a very different behavior from the point of view of its Riemannian (sub)manifold geometry. Indeed H_3 is a nilpotent Lie group admitting large classes of both minimal and constant mean curvature surfaces.

A remarkable property, explicitly proved in [S41], is that, however, H_3 *does not admit totally umbilical surfaces.*

The (generalized) Gauss map $\gamma : M \to G_m(TH_3)$ of a surface M of the Heisenberg group H_3 was examined in [S41].

Using the above property, it is proved that *the Gauss map γ is conformal if and only if M is minimal*. Moreover, a characterization of a surface M with constant mean curvature having vertically harmonic Gauss map is given. Namely, in case M is minimal, it is a surface having the same analytical representation in $\mathbb{R}^3$ as a plane parallel to the axis of revolution of H_3. In case M has nonvanishing constant mean curvature, M is a "round cylinder" (in the above sense) with rulings parallel to the axis of revolution of H_3. Vertically harmonic means that the vertical component of the tension field with respect to the submersion $G_2(TH_3) \to M$ vanishes.

In a joint paper with Piu [S42] they consider surfaces in the Heisenberg group (H_3, ds^2) of the form $S = \exp uX \exp vY$, $(u, v) \in \mathbb{R}^2$, where

$$X = \begin{pmatrix} 0 & a & c \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad Y = \begin{pmatrix} 0 & \alpha & \gamma \\ 0 & 0 & \beta \\ 0 & 0 & 0 \end{pmatrix}$$

are two linearly independent vectors tangent to H_3 at the identity. They prove that

1. *S is a minimal surface with Gauss map γ vertically harmonic if and only if $[X, Y] = 0$ (which is equivalent to $a\beta - \alpha b = 0$).*

2. *S is a minimal surface with γ harmonic if and only if $[X, Y] = 0$ and the one-parameter subgroup $\sigma(u) = \exp uX$ either is a geodesic of H_3, or has torsion equal to zero (i.e., $a^2 + b^2 - c^2 = 0$).*

Moreover, if $\sigma(u)$ is not a geodesic and has vanishing torsion, then the ruled surface S_1 generated by principal normal lines is flat along $\sigma(u)$.

Appendix. A surface of revolution of a remarkable type

Let M be an oriented surface of Euclidean space E^3 with no umbilical point. Recall from the introduction that one can consider the map $\varphi : M \to SO(3)$ mapping each point x of M to the orthogonal matrix determined by the orthonormal frame $\{e_1, e_2, e_3\}$, where e_1 and e_2 are the unit vectors of the principal directions at x. This map can be locally identified with the Gauss map of M into the flag manifold of triples of orthogonal one dimensional vector subspaces. The compact Lie group $SO(3)$ is endowed with a biinvariant metric g'.

The following result will be proven.

THEOREM 3. *The map* $\varphi : (M,g) \to (SO(3), g')$ *is harmonic if and only if* M *is a surface of revolution for which the product of the radius of a parallel and the curvature of a given meridian is constant.*

As a first step, the surfaces of revolution satisfying the above condition will be constructed explicitly. One gets a family of surfaces as the general solution of an ordinary differential equation of second order. Observe that, for instance, spheres are not in this family (but round cones and cylinders are).

The next step will be to show that the surfaces constructed in the previuos section are the only surfaces for which φ is harmonic.

Observe that by Pluzhnikov's Theorem [14], a mapping f of a Riemannian manifold (M,g) into a Lie group G, endowed with a biinvariant metric g', is harmonic if and only if the form $f^*\theta$ has null divergence, where $f^*\theta$ is the induced form on M by the Maurer–Cartan form θ on G, cf. also [4].

Surfaces of revolution with ϕ harmonic

Let M be a surface in E^3, generated by revolution of the meridian curve $(x(u), 0, z(u))$, $x(u) > 0$, along the z axis. We assume that the meridian curve is referred to arc length, hence $(x')^2 + (z')^2 = 1$. Thus the surface M is parametrized by

$$P(u,v) = (x(u)\cos v,\ x(u)\sin v,\ z(u))\,.$$

A unit normal vector is $e_3 = (-z'\cos v, -z'\sin v, x')$. The first fundamental form is given by

$$ds^2 = du^2 + x^2 dv^2 \tag{4}$$

and the principal curvatures are

$$\begin{aligned} \alpha_{11} &= \alpha(e_1,e_1) = x'z'' - x''z', &&\text{(curvature of the meridian)},\\ \alpha_{22} &= \alpha(e_2,e_2) = z'/x, \end{aligned}$$

where $e_1 = P_u$, $e_2 = P_v/|P_v|$ is an orthonormal frame of the tangent space and α is the second fundamental form.

Set $X := (e_1, e_2, e_3)$ (where the e_i are thought as column vectors), the induced form by the Maurer–Cartan form θ of $SO(3)$ is given by

$$
\begin{aligned}
\varphi^*\theta &= X^{-1}dX \\
&= \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} \left\{ \left(\frac{de_1}{du} \frac{de_2}{du} \frac{de_3}{du} \right) du + \left(\frac{de_1}{dv} \frac{de_2}{dv} \frac{de_3}{dv} \right) dv \right\} \\
&= \begin{pmatrix} 0 & 0 & x''z' - x'z'' \\ 0 & 0 & 0 \\ x'z'' - x''z' & 0 & 0 \end{pmatrix} du + \begin{pmatrix} 0 & -x' & 0 \\ x' & 0 & -z' \\ 0 & z' & 0 \end{pmatrix} dv.
\end{aligned} \tag{5}
$$

The divergence of a 1-form $\beta = \sum \beta_i dx^i$ is given by

$$\delta\beta = -\sum g^{ij}\nabla_j \beta_i = -\sum g^{ij}\{\partial_j \beta_i - \Gamma^k_{ji}\beta_k\},$$

where g is the Riemannian metric and Γ are the Christoffel symbols.
Using (4), one gets $\Gamma^1_{11} = \Gamma^2_{11} = \Gamma^2_{22} = 0$ and $\Gamma^1_{22} = -xx'$.

The condition that $\varphi^*\theta$ has null divergence can be read off by the only equation

$$\partial_u(x'z'' - x''z') + \frac{x'}{x}(x'z'' - x''z') = 0,$$

which is equivalent to

$$\partial_u\{x(x'z'' - x''z')\} = 0. \tag{6}$$

Hence we have the following

LEMMA 1. *The only surfaces of revolution M for which the map $\varphi : (M, g) \to (SO(3), g')$ is harmonic are the ones for which the product of the radius of a parallel and the curvature of a given meridian is constant.*

If the meridian has equation $y = 0$, $z = f(x)$, the above condition is equivalent to the second order ordinary differential equation

$$\frac{f''}{(1 + f'^2)^{\frac{3}{2}}} = \frac{k}{x} \qquad (k \text{ constant}),$$

whose solutions (depending on the constants k and $c > 0$) are

$$f(x) = \pm \int \frac{\log(cx^k)}{\sqrt{1 - \log^2(cx^k)}}\, dx.$$

Observe that, for $k = 0$ one gets the round cone and, with an obvious change of variables, the round cylinder.

The general case of surfaces with ϕ harmonic

Let M be a surface of E^3, with no umbilical point. Using the same notations as in [2], at any point $x \in M$ one has an orthonormal frame $\{e_1, e_2, e_3\}$, where e_3 is the unit normal vector and e_1, e_2 are the unit vectors of the principal directions. In terms of differential forms (if x is the position vector field, ω^i the dual forms to e_i and ω^i_j are the connections forms, with $\omega^i_j + \omega^j_i = 0$) one has

$$\begin{aligned} dx &= \omega^1 e_1 + \omega^2 e_2, \\ de_1 &= \omega^2_1 e_2 + \omega^3_1 e_3, \\ de_2 &= \omega^1_2 e_1 + \omega^3_2 e_3, \\ de_3 &= \omega^1_3 e_1 + \omega^2_3 e_2. \end{aligned}$$

In particular, one has the structure equations

$$d\omega^1 = -\omega^1_2 \wedge \omega^2, \quad d\omega^2 = -\omega^2_1 \wedge \omega^1. \tag{7}$$

We set

$$\omega^2_1 = h\omega^1 + k\omega^2, \quad \omega^3_1 = a\omega^1, \quad \omega^3_2 = c\omega^2, \tag{8}$$

where a, c (with $a > c$) are the principal curvatures of M at x and

$$H = \frac{1}{2}(a + c), \qquad K = ac$$

are the mean curvature and the Gaussian curvature, respectively. The Gauss and Codazzi equations read

$$K = ac = h_2 - k_1 - h^2 - k^2, \tag{9}$$

$$c_1 = (a - c)k, \qquad a_2 = (a - c)h, \tag{10}$$

where, here and in the sequel, $h_2 = e_2(h)$ and so on. Further, one has

$$[e_1, e_2] = \nabla_{e_1} e_2 - \nabla_{e_2} e_1 = -he_1 - ke_2. \tag{11}$$

If φ denotes the map from M to $SO(3)$ given by the orthonormal frame $\{e_1, e_2, e_3\}$, then the 1-form $\varphi^*\theta$ with values in $\mathfrak{so}(3)$, induced by the Maurer–Cartan form on $SO(3)$ is given by

$$\varphi^*\theta = \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} (de_1\, de_2\, de_3) = \omega^\alpha_\beta(e_1)\omega^1 + \omega^\alpha_\beta(e_2)\omega^2. \tag{12}$$

Using the Hodge $*$ operator, we get

$$*\varphi^*\theta = -\omega^\alpha_\beta(e_2)\omega^1 + \omega^\alpha_\beta(e_1)\omega^2. \tag{13}$$

Thus φ is harmonic if and only if $\delta\varphi^*\theta d * \varphi^*\theta = 0$. Explicitly,

$$(14) \quad -d\left(\omega^\alpha_\beta(e_2)\right) \wedge \omega^1 + \omega^\alpha_\beta(e_2)\omega^1_2 \wedge \omega^2 + d\left(\omega^\alpha_\beta(e_1)\right) \wedge \omega^2 - \omega^\alpha_\beta(e_1)\omega^2_1 \wedge \omega^1 = 0.$$

Setting

$$\alpha = 1,\ \beta = 2; \quad \alpha = 1,\ \beta = 3; \quad \alpha = 2,\ \beta = 3,$$

respectively, and using (8), we get the following conditions expressing the harmonicity of φ:

$$(15) \qquad h_1 + k_2 = 0,$$

$$(16) \qquad a_1 + ak = 0,$$

$$(17) \qquad c_2 - ch = 0.$$

Note that (15) is equivalent to the fact that the codifferential of the connection form ω^2_1 vanishes. Using the above, equations (10), (11) and

$$[e_1, e_2](a) = -ha_1 - ka_2, \qquad [e_1, e_2](c) = -hc_1 - kc_2,$$

we get the equations

$$(18) \qquad ch_1 = -(a+c)hk, \qquad ak_2 = (a+c)hk.$$

When multiplied by a and c respectively, and added, using (15), these imply

$$(19) \qquad (c^2 - a^2)hk = 0.$$

Since M has no umbilical point, one cannot have $a = c$. If $c = -a$ (i.e., M is minimal), by (18) one would have $h_1 = k_2 = 0$ and hence, by (10), (16) and (17), $h = k = 0$, thus $a = c = 0$, by (9).

Hence in order that (18) hold, we must have $hk = 0$. We consider the case $h = 0$ (the other is similar and actually equivalent). For $h = 0$ the integral curves of the field e_1 are geodesics in M. Thus they are plane curves, since they are curvature lines (cf. for instance [8, page 140] or [5, page 152]). Moreover, by (10), (17) and (18), it follows that $a_2 = c_2 = k_2 = 0$, which means that the integral curves of the field e_2 are circles. Indeed, if $\overline{\nabla}$ denotes the Levi-Civita connection of E^3, we have

$$\overline{\nabla}_{e_2} e_2 = -ke_1 + ce_3, \qquad \overline{\nabla}_{e_2}\overline{\nabla}_{e_2} e_2 = -(k^2 + c^2)e_2,$$

which implies that the curvature lines tangent to e_2 are plane curves and have constant curvature $\sqrt{k^2 + c^2}$.

Thus the surfaces for which φ is harmonic are of revolution. To end the proof of the theorem, we show that the product of the principal curvature a and the radius $1/\sqrt{k^2 + c^2}$ of the circle is constant.

We already remarked that

$$e_2\left(\frac{a}{\sqrt{k^2+c^2}}\right)=0.$$

Moreover, by (16), (10) and (9), we have

$$\begin{aligned}
e_1\left(\frac{a}{\sqrt{k^2+c^2}}\right) &= a_1(k^2+c^2)^{-1/2}-a(k^2+c^2)^{-3/2}(kk_1+cc_1)\\
&=(k^2+c^2)^{-3/2}\left(-ak(k^2+c^2)-ak(-ac-k^2)-ac(a-c)k\right)\\
&=0,
\end{aligned}$$

proving the constancy.

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In Memoriam Aristide Sanini

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CONFORMAL VARIATIONAL PROBLEMS, HARMONIC MAPS AND HARMONIC MORPHISMS

Abstract. We discuss some aspects of harmonic maps and morphisms related to conformality, especially some recent results on smoothness and infinitesimal behaviour of twistor and transform methods for finding harmonic maps, and the dual notion of harmonic morphism.

1. Introduction

Amongst Aristide Sanini's interests were conformal variational problems. He wrote two papers on this subject [24, 25]. In the first of these, he characterized weakly conformal maps from surfaces as maps whose energy is extremal *with respect to variations of the metric*.

On the other hand, harmonic maps extremize the energy *with respect to variations of the map*. The intersection of these classes is the class of minimal branched immersions; in particular, all harmonic maps from the 2-sphere are automatically weakly conformal, and so are minimal branched immersions. There are many twistor and transform methods for the construction of such mappings into various symmetric spaces, starting with harmonic 2-spheres in complex projective space. However, the constructions are algebraic and are not, in general, smooth or even continuous. After reminding the reader of these ideas, in Section 6, we discuss some recent results on the smoothness of the Gauss transform.

An infinitesimal variation of a harmonic map is called a *Jacobi field*; if a Jacobi field comes from a genuine variation, it is called *integrable*. We discuss these ideas in Sections 7 and 8, in particular, the integrability of Jacobi fields along harmonic 2-spheres in $\mathbb{C}P^2$.

Then we remind the reader of Uhlenbeck's idea of 'adding a uniton', and we mention some recent developments which allow us to give completely explicit formulae for harmonic 2-spheres in the unitary group and related spaces.

Related to the Gauss transform is the twistor method for finding harmonic 2-spheres in S^4. In Section 10, we study the infinitesimal behaviour of this method, seeing that Jacobi fields are no longer always integrable.

Then, in Section 11, we discuss horizontally weakly conformal maps, characterizing them in a way dual to that of Sanini; this leads to a discussion of harmonic morphisms in Section 12*ff*. where we see how to dualize some of the twistor theory for weakly conformal harmonic maps to give formulae for harmonic morphisms.

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2. Harmonic maps between Riemannian manifolds

Let $\phi : (M,g) \to (N,h)$ be a smooth map between compact smooth Riemannian manifolds. The *energy* or *Dirichlet integral of* ϕ is

$$E(\phi) = \int_M e(\phi)\,\omega_g = \int_M \frac{1}{2}|\mathrm{d}\phi|^2\,\omega_g$$

where ω_g denotes the volume measure induced by the metric g and, for any $p \in M$,

$$\begin{aligned}|\mathrm{d}\phi_p|^2 &= \text{Hilbert–Schmidt square norm of } \mathrm{d}\phi_p\\ &= g^{ij}h_{\alpha\beta}\,\phi_i^\alpha\phi_j^\beta\,.\end{aligned}$$

Here $\phi_i^\alpha = \partial u^\alpha/\partial x^i$ denote the partial derivatives of ϕ with respect to some local coordinates (x^i) on M and (u^α) on N, (g_{ij}) and $(h_{\alpha\beta})$ are the components of the metric tensor g and h, and $(h^{\alpha\beta})$ is the inverse matrix of $(h_{\alpha\beta})$.

The map ϕ is called *harmonic* if the first variation of E for variations ϕ_t *of the map* ϕ vanishes at ϕ, i.e., $\frac{\mathrm{d}}{\mathrm{d}t}E(\phi_t)\big|_{t=0} = 0$. We compute:

$$\frac{\mathrm{d}}{\mathrm{d}t}E(\phi_t)\Big|_{t=0} = -\int_M \big\langle \tau(\phi), v\big\rangle\,\omega_g \tag{1}$$

where $v = \partial\phi_t/\partial t|_{t=0}$ is the *variation vector field of* (ϕ_t), and $\tau(\phi) = \nabla\mathrm{d}\phi$ is the *tension field of* ϕ given by

$$\begin{aligned}\tau(\phi) &= \nabla\,\mathrm{d}\phi = \mathrm{Tr}\,\nabla\mathrm{d}\phi = \sum_{i=1}^m \nabla\mathrm{d}\phi(e_i,e_i)\\ &= \sum_{i=1}^m \big\{\nabla^\phi_{e_i}\big(\mathrm{d}\phi(e_i)\big) - \mathrm{d}\phi(\nabla^M_{e_i}e_i)\big\}\end{aligned}$$

for any orthonormal frame $\{e_i\}$. In local coordinates, this reads

$$\begin{aligned}\tau(\phi)^\gamma &= g^{ij}\left(\frac{\partial^2\phi^\gamma}{\partial x^i\partial x^j} - \Gamma^k_{ij}\frac{\partial\phi^\gamma}{\partial x^k} + L^\gamma_{\alpha\beta}\frac{\partial\phi^\alpha}{\partial x^i}\frac{\partial\phi^\beta}{\partial x^j}\right)\\ &= \Delta^M\phi^\gamma + g(\mathrm{grad}\,\phi^\alpha, \mathrm{grad}\,\phi^\beta)\,L^\gamma_{\alpha\beta}\,.\end{aligned}$$

Here, Γ^k_{ij} (resp. $L^\gamma_{\alpha\beta}$) denotes the Christoffel symbols on (M,g) (resp. (N,h)), and Δ^M denotes the *Laplace–Beltrami operator on functions* $f : M \to \mathbb{R}$ given by

$$\begin{aligned}\Delta^M f &= \nabla\,\mathrm{grad}\,f = \nabla\mathrm{d}f = -\mathrm{d}^*\mathrm{d}f = \mathrm{Tr}\,\nabla\mathrm{d}f\\ &= \sum_{i=1}^m \big\{e_i\big(e_i(f)\big) - \big(\nabla^M_{e_i}e_i\big)f\big\}\\ &= \frac{1}{\sqrt{|g|}}\frac{\partial}{\partial x^i}\left(\sqrt{|g|}\,g^{ij}\frac{\partial f}{\partial x^j}\right) = g^{ij}\left(\frac{\partial^2 f}{\partial x^i\partial x^j} - \Gamma^k_{ij}\frac{\partial f}{\partial x^k}\right).\end{aligned}$$

3. Examples of harmonic maps

From (1), we see that $\phi : M \to N$ is harmonic if and only if it satisfies the *harmonic* or *tension field equation*:

$$\tau(\phi) \equiv \operatorname{Tr} \nabla \mathrm{d}\phi = 0\,. \tag{2}$$

We list some standard examples.

1. A smooth map $\phi : \mathbb{R}^m \supseteq U \to \mathbb{R}^n$ is harmonic if and only if $\Delta\phi = 0$ where Δ is the usual Laplacian on $\mathbb{R}^m$.

2. A smooth map $\phi : (M,g) \to \mathbb{R}^n$ is harmonic if and only if $\Delta^M \phi = 0$ where Δ^M is the Laplace–Beltrami operator on (M,g).

Note that both the above equations are *linear*.

3. A smooth map from an interval of $\mathbb{R}$ or from S^1 to N is harmonic if and only if it defines a *geodesic* of N parametrized linearly.

4. *Holomorphic* and *antiholomorphic* maps between Kähler manifolds are harmonic; in fact they minimize energy in their homotopy class.

5. *Harmonic morphisms*, i.e., maps which preserve Laplace's equation, are harmonic maps, see Section 12.

4. Weakly conformal maps

A smooth map $\phi : (M,g) \to (N,h)$ is called *weakly conformal* if

$$\phi^* h = \lambda^2 g \tag{3}$$

for some function $\lambda : M \to [0,\infty)$; explicitly, for all $p \in M$,

$$h\big(\mathrm{d}\phi_p(X), \mathrm{d}\phi_p(Y)\big) = \lambda(p)^2 g(X,Y) \qquad (X, Y \in T_pM)\,;$$

equivalently,

$$\mathrm{d}\phi_p^* \circ \mathrm{d}\phi_p = \lambda(p)^2 \operatorname{Id}_{T_pM}\,.$$

In local coordinates, equation (3) reads

$$h_{\alpha\beta}\phi_i^\alpha \phi_j^\beta = \lambda^2 g_{ij}\,.$$

A. Sanini characterized weak conformality as follows.

THEOREM 1 ([24]). *A non-constant map ϕ is a critical point of the energy with respect to variations* of the metric *if and only if* $\dim M = 2$ *and ϕ is weakly conformal.*

Proof. The Euler–Lagrange operator for such variations is the *stress-energy tensor* $S(\phi) = e(\phi)g - \phi^* h$. If this is zero, taking the trace shows that $\dim M = 2$, then comparing with equation (3) shows that ϕ is weakly conformal with $\lambda^2 = e(\phi)$. □

5. Harmonic maps and minimal branched immersions

Let M^2 be a *surface*, i.e., a Riemannian manifold of dimension two. Then the energy integral is unchanged under conformal changes of the metric, so that the concept of harmonic map from a surface depends only on its conformal structure; in particular, if M^2 is orientable, we can (and will) take it to be a *Riemann surface*, i.e., one-dimensional complex manifold; then methods of complex analysis may be used.

Let $\phi : M^2 \to N$ be a weakly conformal map from a surface. Then, an easy calculation shows that, away from points where $d\phi$ is zero, the mean curvature is $2\lambda^2$ times the tension field, hence, *a weakly conformal map from a surface is harmonic if and only if it is minimal away from points where its differential is zero*. Such a map is called a *minimal branched immersion*; the points where $d\phi$ is zero are called *branch points*, and are described in [17].

The following fact was established by the author [30] and many others.

LEMMA 1. *Any harmonic map from the* 2*-sphere* S^2 *is weakly conformal and so is a minimal branched immersion.*

Proof. The $(2,0)$-part of the stress energy tensor is a holomorphic section of $T^*_{2,0}S^2$. Such a section must vanish since this bundle has negative degree. □

6. Smoothness of transforms

Let $\pi : \mathbb{C}^{n+1} \setminus \{\vec{0}\} \to \mathbb{C}P^n$ be the canonical projection. For any smooth map $\phi : M^2 \to \mathbb{C}P^n$, write $\phi = [\Phi]$ to mean that $\Phi : U \to \mathbb{C}^{n+1}$ is a smooth map on an open subset of M^2 with $\phi = \pi \circ \Phi$ away from zeros; thus Φ represents ϕ in homogeneous coordinates. We denote orthogonal projection onto ϕ (resp. $\phi^\perp$) by π_ϕ (resp. $\pi_\phi^\perp$). Note that the linear map $\Phi \mapsto \pi_\phi^\perp(\partial\Phi/\partial z)$ (resp. $\Phi \mapsto \pi_\phi^\perp(\partial\Phi/\partial\bar{z})$) represents the partial derivative $\partial\phi/\partial z$ (resp. $\partial\phi/\partial\bar{z}$).

A smooth map $\phi = [\Phi] : M^2 \to \mathbb{C}P^n$ has two *Gauss transforms*, a ∂'-transform:

$$G'(\phi) = \left[\pi_\phi^\perp \frac{\partial\Phi}{\partial z}\right],$$

defined at points where $\partial\phi/\partial z$ is non-zero, and a ∂''-transform:

$$G''(\phi) = \left[\pi_\phi^\perp \frac{\partial\Phi}{\partial\bar{z}}\right],$$

defined at points where $\partial\phi/\partial\bar{z}$ is non-zero. These are both independent of the choice of Φ.

For simplicity, assume now that M^2 is oriented. If ϕ is harmonic and not antiholomorphic (resp. holomorphic), then $G'(\phi)$ (resp. $G''(\phi)$) extends over the zeros of $\partial\phi/\partial z$ (resp. $\partial\phi/\partial\bar{z}$) to give a harmonic map. Then, following work of other authors (see [14]), the next result was established by J. Eells and the author [14]; we give the formulation in [9].

THEOREM 2. *All harmonic maps from S^2 to CP^n are obtained from holomorphic maps by applying the ∂'-Gauss transform up to n times.*

To use this to study the space of harmonic maps, we need to answer the question, *Is the Gauss transform smooth, or even continuous?* In general, it is not; however, for integers k, d and E with k and E non-negative, set

$$\begin{aligned}\mathrm{Hol}^*_k(S^2,\mathbb{C}P^2) &= \text{the space of full holomorphic maps of degree } k;\\ \mathrm{Harm}_{d,E}(S^2,\mathbb{C}P^2) &= \text{the space of harmonic maps of degree } d \text{ and energy } 4\pi E.\end{aligned}$$

Then the following was established by L. Lemaire and the author.

THEOREM 3 ([21]). *The Gauss transform*

$$G' : \mathrm{Hol}^*_k(S^2,\mathbb{C}P^2) \to \mathrm{Harm}(S^2,\mathbb{C}P^2)$$

is smooth if restricted to the subspace $\mathrm{Hol}^*_{k,r}(S^2,\mathbb{C}P^2)$ *of holomorphic maps of fixed total ramification index r. In fact, it gives a diffeomorphism*

$$G' : \mathrm{Hol}^*_{k,r}(S^2,\mathbb{C}P^2) \to \mathrm{Harm}_{k-r-2,\,3k-r-2}(S^2,\mathbb{C}P^2)\,.$$

7. Infinitesimal deformations and transformations

Let $(\phi_{t,s})$ be a 2-parameter variation of ϕ; write $v = \dfrac{\partial\phi_{t,s}}{\partial t}\Big|_{(0,0)}$ and $w = \dfrac{\partial\phi_{t,s}}{\partial s}\Big|_{(0,0)}$ for the corresponding variation vector fields. Set

$$H_\phi(v,w) = \frac{\partial^2 E}{\partial t\partial s}(\phi_{t,s})\Big|_{(0,0)} = \int_M \langle J_\phi v, w\rangle\,\omega_g$$

where

$$J_\phi v = \Delta^\phi v - \mathrm{Tr}\,R^N(\mathrm{d}\phi, v)\mathrm{d}\phi\,.$$

Here $\Delta^\phi = -\mathrm{Tr}\,\nabla^2$ is the Laplacian on $\phi^{-1}TN$. The linear operator J_ϕ is called the *Jacobi operator* along ϕ.

The following is easy to establish [22].

LEMMA 2. (i) *If (ϕ_t) is a one-parameter family of maps with $\phi_0 = \phi$ and $\partial\phi_t/\partial t\big|_{t=0} = v$, then*

$$J_\phi(v) = -\frac{\partial}{\partial t}\tau(\phi_t)\Big|_{t=0}.$$

(ii) *If ϕ_t is a one-parameter family of* harmonic *maps with $\phi_0 = \phi$, then $v = \partial\phi_t/\partial t\big|_{t=0}$ is a* Jacobi field *along ϕ, i.e., $J_\phi(v) = 0$.*

So J_ϕ is the *linearization* of the tension field τ, up to a sign convention.

8. Integrability of Jacobi fields

DEFINITION 1. *A Jacobi field v along a harmonic map ϕ is said to be* integrable *if there is a one-parameter family (ϕ_t) of harmonic maps with $\phi_0 = \phi$ and* $\left.\dfrac{\partial \phi_t}{\partial t}\right|_{t=0} = v$.

We ask the question, *For what manifolds are all Jacobi fields integrable?*

One reason that this is important is the following result of D. Adams and L. Simon.

THEOREM 4 ([1]). *Let (M,g) and (N,h) be real-analytic Riemannian manifolds. If all Jacobi fields along harmonic maps from M to N are integrable, then* $\mathrm{Harm}(M,N)$ *is a real-analytic manifold* with tangent spaces given by the Jacobi fields.

Since we can construct all harmonic maps from S^2 to $\mathbb{C}P^n$ explicitly as above, it is natural to ask what is known in this case. In the case $n = 1$, R. Gulliver and B. White [18] showed that all Jacobi fields along harmonic maps are integrable. For the case $n = 2$, L. Lemaire and the author showed the following.

THEOREM 5 ([22]). *All Jacobi fields along harmonic maps from S^2 to $\mathbb{C}P^2$ are integrable.*

The idea is that the Gauss transform and its inverse are smooth away from branch points, so if a harmonic map $\phi : S^2 \to \mathbb{C}P^2$ is the Gauss transform $G'(f)$ of a holomorphic map $f : S^2 \to \mathbb{C}P^2$, then the inverse of G' maps a Jacobi field along ϕ into one along f. We then show that this Jacobi field is actually holomorphic. The key step is to show that it extends across the branch points, then a GAGA principle tells us that it's actually given by rational functions and so explicitly integrable. It follows that the original harmonic map is integrable. The methods make essential use of the low dimensions, and so are unlikely to generalize to higher n.

9. Factorization into unitons

The Gauss transform is an example of K. Uhlenbeck's operation of 'adding a uniton' which transforms harmonic maps $M^2 \to \mathrm{U}(n)$ from a surface to the unitary group into other harmonic maps $M^2 \to \mathrm{U}(n)$ as follows. Any harmonic map ϕ defines a connection $A^\phi = \frac{1}{2}\phi^{-1}\mathrm{d}\phi$ on the trivial bundle $\underline{\mathbb{C}}^n = M^2 \times \mathbb{C}^n$ and thus a covariant derivative $D^\phi = \mathrm{d} + A^\phi$; then a *uniton* or *flag factor* for ϕ is a subbundle β of the trivial bundle which is (i) *holomorphic with respect to $D^\phi_{\bar{z}}$* (i.e. the sections of β are closed under $D^\phi_{\bar{z}}$), and (ii) *closed under A^ϕ_z*. Uhlenbeck showed the following.

THEOREM 6 ([28]). (i) *The map $\widetilde{\phi} : M^2 \to \mathrm{U}(n)$ given by $\widetilde{\phi} = \phi(\pi_\beta - \pi_\beta^\perp)$ is harmonic. We say that $\widetilde{\phi}$ is obtained from ϕ by* adding the uniton β *or* by the flag transform with flag factor β.

(ii) *Any harmonic map* $\phi : S^2 \to \mathrm{U}(n)$ *can written as a finite product:*

$$\phi = \text{const.}\, (\pi_{\beta_1} - \pi^{\perp}_{\beta_1}) \cdot \ \ldots \ \cdot (\pi_{\beta_r} - \pi^{\perp}_{\beta_r})\,.$$

Such a product is called a *uniton factorization* of the harmonic map ϕ, and the minimum number of flag factors required is called the *uniton number*. Given an arbitrary harmonic map ϕ, one method of factorization is to add the uniton $\alpha^0 = \ker A^{\phi}_z$ giving a new harmonic map ϕ^1, then repeat the process. After a finite number r of steps one reaches a constant map; then setting $\beta_i = (\alpha^{r-i})^{\perp}$ gives a factorization, called the *factorization by* A_z*-kernels*. Dually, we may use A_z-images [32]. However, neither of these is the most efficient way in the sense of minimizing the number of steps; that is provided by using the kernel of the bottom coefficient of the extended solution as proposed by Uhlenbeck [28]; dually, we may use the image of the adjoint of the top coefficient.

Conversely, to build all possible harmonic maps, we do successive flag transforms starting with the constant map, giving a sequence of harmonic maps $\phi_0 = \text{const.}$, $\phi_1, \ldots, \phi_r = \phi$. To do this, we must know all the possible flag factors (unitons) at each stage.

However, there are two problems:

(i) to find unitons, we must find a holomorphic (or, at least, meromorphic) basis for the trivial bundle $\underline{\mathbb{C}}^n$ with respect to $D^{\phi_i}_{\bar z}$ for each i; to find this we must, in general, solve $\bar\partial$-problems;

(ii) like the Gauss transform, adding a uniton may not depend smoothly, or even continuously, on the data.

M. J. Ferreira, B. A. Simões and the author solved the first problem as follows.

THEOREM 7 ([15]). *For the (dual of) Uhlenbeck's factorization, all the possible unitons at each stage can be found explicitly in terms of projections of holomorphic functions, without solving* $\bar\partial$*-problems, giving explicit formulae for all harmonic maps from the* 2*-sphere to* $\mathrm{U}(n)$.

By thinking of them as stationary Ward solitons, B. Dai and C.-L. Terng [13] also obtained explicit formulae for the unitons of the Uhlenbeck factorization.

The author and M. Svensson [27] developed the ideas in [15] to show how to find explicit formulae for the harmonic maps corresponding to *any* factorization of $\mathrm{U}(n)$ including those in [13] and [15]. Thus we obtain explicit algebraic parametrizations of all harmonic maps $S^2 \to \mathrm{U}(n)$ by meromorphic functions, which can be used to study continuity.

10. Smoothness of twistor methods

With a history going back to Weierstrass, *twistor methods* have been successful in constructing harmonic maps, especially from the 2-sphere to symmetric spaces. The first case of this was the following.

Thinking of $\mathbb{C}P^3$ as the set of complex lines through the origin, let $\pi : \mathbb{C}P^3 \to \mathbb{H}P^1 = S^4$ be the *Calabi–Penrose twistor map* given by sending a complex line to the quaternionic line containing it. E. Calabi showed the following.

THEOREM 8 ([10, 11]). *Every harmonic map* $\phi : S^2 \to S^4$ *is* $\pm$ *the projection* $\pi \circ f$ *of a horizontal holomorphic map* $f : S^2 \to \mathbb{C}P^3$.

Here 'horizontal' means that the image of the differential $\mathrm{d}f$ at each point is orthogonal to the kernel of $\mathrm{d}\pi$.

L. Lemaire and the author used this to study Jacobi fields along harmonic maps from S^2 to S^4 and to S^3, obtaining the following result [23].

THEOREM 9. *(i) For each* $d = 1, 2, \dots$ *, the map* $f \mapsto \phi = \pi \circ f$ *is a diffeomorphism of the space of holomorphic horizontal maps* $f : S^2 \to \mathbb{C}P^3$ *of degree* d *onto the space of harmonic maps* $\phi : S^2 \to S^4$ *of energy* $4\pi d$.

(ii) If ϕ (equivalently, f) is full, the Jacobi fields along ϕ correspond to infinitesimal deformations of the horizontal holomorphic map f.

(iii) There are some non-full harmonic maps $\phi : S^2 \to S^4$ which have non-integrable Jacobi fields.

(iv) There are some non-full harmonic maps $\phi : S^2 \to S^3$ which have non-integrable Jacobi fields.

11. The dual problem: horizontal weak conformality

The following definition can be regarded as the dual of that of weak conformality given in Section 4.

DEFINITION 2. $\phi : (M,g) \to (N,h)$ *is called* horizontally weakly conformal (HWC) $($*or* semiconformal$)$ *if, for each* $p \in M$, either

(i) $\mathrm{d}\phi_p = 0$, *in which case we call* p *a* critical point, or

(ii) $\mathrm{d}\phi_p$ *maps the horizontal space* $\mathcal{H}_p = \{\ker(\mathrm{d}\phi_p)\}^\perp$ *conformally onto* $T_{\phi(p)}N$, *i.e.*, $\mathrm{d}\phi_p$ *is surjective and there exists a number* $\lambda(p) \neq 0$ *such that*

$$h\big(\mathrm{d}\phi_p(X), \mathrm{d}\phi_p(Y)\big) = \lambda(p)^2 g(X,Y) \quad (X, Y \in \mathcal{H}_p),$$

in which case we call p *a* regular point.

Equivalently, ϕ is HWC if and only if, for each $p \in M$,

$$\mathrm{d}\phi_p \circ \mathrm{d}\phi_p^* = \lambda(p)^2 \, \mathrm{Id}_{T_{\phi(p)}N}$$

for some $\lambda(p) \in [0, \infty)$. In local coordinates, this reads

$$g^{ij}\phi_i^\alpha \phi_j^\beta = \lambda^2 h^{\alpha\beta}.$$

These should be compared with the formulae in Section 4. The function $\lambda : M \to [0,\infty)$ is called the *dilation* of ϕ; it is smooth away from the critical points; on setting it equal to zero at the critical points, it becomes continuous on M with λ^2 smooth.

Note that, whereas a non-constant weakly conformal map ϕ is an immersion away from the points where $\mathrm{d}\phi$ vanishes, a non-constant horizontally weakly conformal map is a submersion away from those points.

We then have the following dual of Sanini's result above, see [6].

THEOREM 10. *A non-constant map ϕ is a critical point of the energy with respect to horizontal variations of the metric if and only if* $\dim N = 2$ *and ϕ is horizontally weakly conformal.*

Proof. The map is a critical point if and only if its stress-energy tensor is zero on horizontal vectors. It is easily seen that this holds if and only if $\dim N = 2$ and ϕ is HWC. □

12. Harmonic morphisms

We can now study a type of map which in many ways, is dual to that of harmonic maps. A smooth map $\phi : (M,g) \to (N,h)$ is called a *harmonic morphism* if, for every harmonic function $f : V \to \mathbb{R}$ defined on an open subset V of N with $\phi^{-1}(V)$ non-empty, the composition $f \circ \phi$ is harmonic on $\phi^{-1}(V)$. We have the following *characterization* due independently to B. Fuglede and T. Ishihara.

THEOREM 11 ([16, 19]). *A smooth map $\phi : M \to N$ between Riemannian manifolds is a harmonic morphism if and only if it is both* harmonic *and* horizontally weakly conformal.

Proof. The 'if' part is a simple application of the chain rule for a function of a function. The converse direction requires the local existence of enough harmonic functions, obvious in the real-analytic case, but more delicate in the smooth case. □

We list some properties of harmonic morphisms.

1. The *composition* of two harmonic morphisms is a harmonic morphism.

2. Harmonic morphisms *preserve harmonicity* of maps, i.e., the composition $f \circ \phi : M \to P$ of a harmonic map $f : N \to P$ with a harmonic morphism $\phi : M \to N$ is a harmonic map.

3. *If* $\dim N = 1$, then *the harmonic morphisms are precisely the harmonic maps*; in particular, *if $N = \mathbb{R}$, then the harmonic morphisms are precisely the harmonic functions*.

4. *A map $\phi : N \to P$ between surfaces* is a harmonic morphism if and only if it is weakly conformal.

5. The concept of *harmonic morphism to a surface* depends only on the conformal structure of the surface. Hence the notion of *harmonic morphism to a Riemann*

surface is well-defined.

We next list some examples of harmonic morphisms.

1. For any $m \in \{1, 2, \dots\}$, *radial projection*

$$\mathbb{R}^m \setminus \{\vec{0}\} \to S^{m-1}, \quad \vec{x} \mapsto \vec{x}/|\vec{x}|$$

is a harmonic morphism with dilation $\lambda(\vec{x}) = 1/|\vec{x}|$. More generally, a *horizontally conformal submersion with* gradλ *tangent to the fibres is a harmonic morphism if and only if it has minimal fibres*.

2. The *Hopf maps* $S^3 \to S^2$, $S^7 \to S^4$, $S^{15} \to S^8$, $S^{2n+1} \to \mathbb{C}P^n$, $S^{4n+3} \to \mathbb{H}P^n$ are harmonic morphisms with constant dilation. More generally, *a Riemannian submersion is a harmonic morphism if and only if its fibres are minimal*.

3. (J.Y. Chen [12]) *Stable harmonic maps* from a compact Riemannian manifold to S^2 are harmonic morphisms.

13. Twistor theory for harmonic morphisms

The following was proved by the author [31] for submersions and extended to maps with critical points by M. Ville [29].

THEOREM 12. *Given a non-constant harmonic morphism $\phi : M^4 \to N^2$ from an orientable Einstein 4-manifold to a Riemann surface, there is a Hermitian structure J on M^4 such that ϕ is holomorphic with respect to J, and J is parallel along the fibres of ϕ.*

Conversely, the author showed that, if M^4 is also anti-self-dual, a Hermitian structure gave rise to local harmonic morphisms, away from points where it is Kähler. This was generalized as follows.

Hermitian structures correspond to holomorphic sections of the twistor space Z^6 of M^4. V. Apostolov and P. Gauduchon [2] showed that local existence of harmonic morphisms is equivalent to local existence of Hermitian structures, and this is equivalent to the self-dual part W_+ of the Weyl tensor being degenerate. When W_+ is identically zero, i.e., M^4 is anti-self-dual, the twistor space has an *integrable* complex structure so that there are lots of Hermitian structures, and so lots of harmonic morphisms.

Twistor methods have been extended to give various classes of holomorphic harmonic maps and morphisms from higher-dimensional spaces to surfaces, see [26] and [6, Chapters 8 and 9].

14. Explicit formulae for harmonic morphisms

Starting from Theorem 12, explicit formulae can be given for harmonic morphisms to surfaces from 4-dimensional real or complex space-forms. By dimension reduction, we obtain the following *mini-twistor* formulae in 3-dimensional space forms given by

P. Baird and the author [3] for $\mathbb{R}^3$ and [4] for S^3 and $\mathbb{H}^3$; we give here the version for $\mathbb{R}^3$. In fact, the first part of this result is essentially due to C. G. J. Jacobi [20]; the converse (ii) was established in [3].

THEOREM 13. (i) *Let g and h be holomorphic functions on an open subset of $\mathbb{C}$. Then any smooth local solution $\phi : \mathbb{R}^3 \supseteq U \to \mathbb{C}$, $z = \phi(x_1, x_2, x_3)$ to the equation*

$$-2g(z)x_1 + (1 - g(z)^2)x_2 + \mathrm{i}(1 + g(z)^2)x_3 = 2h(z) \tag{4}$$

is a harmonic morphism.

(ii) *Every harmonic morphism is given this way locally, up to composition with isometries on the domain and weakly conformal maps on the codomain.*

When M^4 is of Minkowski signature, Hermitian structures become *shear-free ray congruences*; on complexifying, they both become *holomorphic foliations by null planes*, see [5]. To find harmonic morphisms into Lorentzian surfaces, we replace the complex analytic functions g and h in (4) by functions analytic with respect to the hyperbolic numbers [7]. All cases can be unified by using the *bicomplex numbers*, see [8].

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AN EXPERIMENTAL STUDY OF GOLDSTEIN–PETRICH CURVES

Abstract. Numerical methods are used to prove the existence of closed embedded curves in $\mathbb{R}^2$ and $\mathbb{H}^2$ whose shapes are invariant under the Goldstein–Petrich flow. Numerical routines are used to compute all closed spherical free elasticæ. The evidence of the experiments shows the existence of a unique, up to rotations, embedded closed free elastica of S^2 with symmetry group of order h, for every positive integer $h \geq 2$. The order of the symmetry groups and the number of self-intersection points of any closed free elastica in S^2 are determined.

1. Introduction

The interplay between integrable evolution equations and the motion of curves has been the focus of intense research in the past decades, both in geometry and mathematical physics (see for instance [6, 7, 8, 4, 11, 12, 14, 15, 16, 20, 21, 23] and the literature therein). In the seminal paper [11], Goldstein and Petrich related the mKdV hierarchy to the motions of curves in the plane. Later, this approach has been generalized to other 2-dimensional Klein geometries [6, 7, 8] or to higher-dimensional homogeneous spaces [20, 21]. The second Goldstein–Petrich flow for curves in a 2-dimensional Riemannian space form $\mathcal{S}^2_\varepsilon$ of constant curvature $\varepsilon = 0, 1, -1$, is defined by the *modified Korteweg–de Vries* equation

$$\kappa_t + \kappa_{sss} + \frac{3}{2}\kappa^2\kappa_s = 0, \tag{1}$$

where $\kappa(s,t)$ denotes the (geodesic) curvature. Closed curves whose shape is invariant under the flow defined by (1) are referred to as *Goldstein–Petrich* curves. In this case, $\kappa(s,t)$ must be a periodic solution of (1) in the form of a traveling wave, so $\kappa = \kappa(s + (\varepsilon - \lambda)t)$ and

$$\kappa_{ss} + \frac{\kappa^3}{2} + (\varepsilon - \lambda)\kappa = \mu, \tag{2}$$

where μ is a constant of integration. On physical and geometrical grounds it is natural to demand that the curves are closed and without self intersections. Generic periodic solutions of (2) need not correspond to closed curves [23]. And, even if κ corresponded to a closed curve, in general there would be points of self-intersection. It is known that, in the spherical case, there exists a countable family of simple Goldstein–Petrich curves, with $\mu = 0$ [2, 17]. These particular solutions are known as *elastic curves* and they constitute a classical topic in mathematical physics and geometry. However, in the

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Euclidean or hyperbolic case, closed elasticæ always have self intersections [2, 17, 23]. It is then a natural problem to look for numerical procedures to recognize whether or not there exist simple closed Goldstein–Petrich planar or hyperbolic curves. Another natural problem is to develop effective computational methods to find all simple closed elasticæ in the 2-sphere.

On the base of our numerical approach, we exhibit explicit examples of simple Goldstein–Petrich curves in the Euclidean plane and in the Poincaré disk. Experimental evidences confirm the existence of 1-parameter families of simple closed Goldstein–Petrich curves with symmetry groups $\mathbb{Z}_n$, for every $n \in \mathbb{N}$. In the second part of the paper we examine free elasticæ in S^2. The curvature can be expressed in term of the Jacobi elliptic function $\mathrm{cn}(-,\sqrt{\lambda})$, $\lambda \in (0,1/2)$. We compute the monodromy of the spin Frenet system and we construct a map $\phi : \mathbb{Q} \cap (0,1/2) \to (0,1/2)$ such that $\lambda = \phi(p/q)$ gives a closed elastica. Every closed spherical elastica arises in this fashion, for some $p/q \in \mathbb{Q} \cap (0,1/2)$. Our tests suggest that if $\lambda = \phi(p/2q)$ then the corresponding elastica has symmetry group $\mathbb{Z}_q$ and possesses $(p-1)q$ points of self intersection. In particular, if $p = 1$ we obtain a simple closed elastica with symmetry group $\mathbb{Z}_q$, for every $q \geq 2$. If $\lambda = \phi(p/(2q+1))$ then the curve has symmetry group $\mathbb{Z}_{2q+1}$ and possesses $(2p-1)(2q+1)$ points of self intersection. As an application we compute and visualize embedded Pinkall's tori in $\mathbb{R}^3$ with non trivial symmetry groups $\mathbb{Z}_n$, for every $n \in \mathbb{N}$ [26].

Acknowledgments. Numerical computations and visualization have been performed with the software *Mathematica 6*.

2. Goldstein–Petrich curves

2.1. Frames

We let $\mathbb{R}^3_\varepsilon$, $\varepsilon = -1,0,1$, be the real 3-dimensional space with coordinates (x^1,x^2,x^3) endowed with the bilinear form

$$\langle \mathbf{x},\mathbf{y}\rangle_\varepsilon = \varepsilon x^1y^1 + x^2y^2 + x^3y^3 =^t \mathbf{x}g^\varepsilon\mathbf{x}$$

and with the orientation $dx^1 \wedge dx^2 \wedge dx^3 > 0$. If $\varepsilon = -1$ we also fix a time-orientation by saying that a null-vector $\mathbf{x}$ is future-directed if $x^1 > 0$. We denote by $\mathcal{S}^2_\varepsilon \subset \mathbb{R}^3_\varepsilon$ the space-like 2-dimensional submanifolds

$$\begin{cases} \mathcal{S}^2_1 &= \{\mathbf{x} \in \mathbb{R}^3_1 : \|\mathbf{x}\|^2_1 = 1\} \cong S^2, \\ \mathcal{S}^2_{-1} &= \{\mathbf{x} \in \mathbb{R}^3_{-1} : \|\mathbf{x}\|^2_{-1} = -1, x^1 \geq 1\} \cong \mathbb{H}^2, \\ \mathcal{S}^2_0 &= \{\mathbf{x} \in \mathbb{R}^3_0 : x^1 = 1\} \cong \mathbb{R}^2. \end{cases}$$

We indicate by $\mathrm{SO}_\varepsilon(3)$ the connected component of the identity of the automorphism group of $\mathbb{R}^3_\varepsilon$. It is convenient to think of $\mathrm{SO}_\varepsilon(3)$ as the frame manifold of all oriented basis $\mathbf{a} = (\mathbf{a}_1,\mathbf{a}_1,\mathbf{a}_3)$ of $\mathbb{R}^3_\varepsilon$ such that $\langle \mathbf{a}_i,\mathbf{a}_j\rangle = g^\varepsilon_{ij}$. If $\varepsilon = -1$ we also require that $\mathbf{a}_1$ is future-directed. We then let $\mathfrak{so}_\varepsilon(3)$ be the Lie algebra of $\mathrm{SO}_\varepsilon(3)$, whose elements are

3×3 matrices of the form

$$\begin{pmatrix} 0 & -\varepsilon y^1 & -\varepsilon y^2 \\ y^1 & 0 & -y^3 \\ y^2 & y^3 & 0 \end{pmatrix}.$$

REMARK 1. The hyperbolic plane $\mathcal{S}^2_{-1}$ is identified with the unit disk Δ endowed with its standard Poincaré metric

$$\frac{4}{(1-x^2-y^2)^2}(dx^2+dy^2)$$

by means of

$$\mathbf{x}\in\mathcal{S}^2_{-1}\to\left(\frac{x^2}{x^1},\frac{x^3}{x^1}\right)\in\Delta.$$

Let $\gamma : I\subset\mathbb{R}\to\mathcal{S}^2_\varepsilon$ be a regular curve parameterized by the arclength. For each $s\in I$ we set $\mathbf{t}(s)=\gamma'(s)$ and we denote by $\mathbf{n}(s)\in\mathbb{R}^3_\varepsilon$ the unique unit vector such that $(\gamma(s),\mathbf{t}(s),\mathbf{n}(s))\in \mathrm{SO}_\varepsilon(3)$. The map

$$\mathrm{F} : s\in I\to(\gamma(s),\mathbf{t}(s),\mathbf{n}(s))\in\mathrm{SO}_\varepsilon(3)$$

is the *Frenet frame field* of γ. It satisfies the *Frenet–Serret equations*

$$\gamma'=\mathbf{t},\quad \mathbf{t}'=-\varepsilon\gamma+\kappa\mathbf{n},\quad \mathbf{n}'=-\kappa\mathbf{t}$$

where $\kappa : I\to\mathbb{R}$ is the *curvature* of γ. The Serret–Frenet equations can be conveniently written in the form

$$\mathrm{F}'=\mathrm{F}\mathcal{K}(\kappa) \tag{3}$$

where

$$\mathcal{K}(\kappa)=\begin{pmatrix} 0 & -\varepsilon & 0 \\ 1 & 0 & -\kappa \\ 0 & \kappa & 0 \end{pmatrix}. \tag{4}$$

REMARK 2. By the existence and uniqueness of solutions of linear systems of o.d.e. it follows that for every smooth function $\kappa : I\to\mathbb{R}$ and every $\mathbf{a}\in\mathrm{SO}_\varepsilon(3)$ there exists a unique $F : I\to\mathrm{SO}_\varepsilon(3)$ with initial condition $F(0)=\mathbf{a}$ and satisfying (3). For this reason we will simply say that F is a *Frenet frame with curvature* κ.

2.2. Jets, differential functions and total derivatives

We now collect some definitions taken from [24]. The space of n-th order jets of functions $u\in C^\infty(\mathbb{R},\mathbb{R})$ is denoted by $J^n(\mathbb{R},\mathbb{R})$. The independent variable is s, the dependent variable and its virtual derivatives up to order n are $u,u_{(1)},\dots,u_{(n)}$. The projective limit of the natural sequence

$$\cdots\to J^n(\mathbb{R},\mathbb{R})\to J^{n-1}(\mathbb{R},\mathbb{R})\to\cdots\to J^1(\mathbb{R},\mathbb{R})\to J^0(\mathbb{R},\mathbb{R})\to\mathbb{R}. \tag{5}$$

is the *infinite jet space*. It is denoted by $\mathbf{J}(\mathbb{R},\mathbb{R})$. A function

$$\mathbf{W}:\mathbf{J}(\mathbb{R},\mathbb{R})\to\mathbb{R}$$

is said to be a *differential function* if there exists a polynomial $W\in\mathbb{R}[x_0,x_1,\dots,x_n]$ such that

$$\mathbf{W}(\mathbf{u})=W(u,u_{(1)},\dots,u_{(n)}),$$

for every $\mathbf{u}=(s,u,u_{(1)},\dots,u_{(n)},\dots)\in\mathbf{J}(\mathbb{R},\mathbb{R})$. The set of the polynomial differential functions will be denoted by $J[\mathbf{u}]$. The total derivative

$$\delta: J[\mathbf{u}]\to J[\mathbf{u}]$$

is defined by

$$\delta(\mathbf{W})(\mathbf{u})=\sum_{j=0}^{\infty}\frac{\partial W}{\partial x_j}|_{\mathbf{u}}u_{(j+1)}.$$

A differential function $\mathbf{W}\in J[\mathbf{u}]$ is *exact* if there exist $\mathcal{P}(\mathbf{W})\in J[\mathbf{u}]$ such that

$$\mathbf{W}=\delta(\mathcal{P}(\mathbf{W})).$$

The *primitive* $\mathcal{P}(\mathbf{W})$ is unique, up to a constant. We use the notation $\int\mathbf{W}ds$ to denote the primitive $\mathcal{P}(\mathbf{W})$ of $\mathbf{W}$ such that $\mathcal{P}(\mathbf{W})|_{\mathbf{0}}=0$. If $\kappa:(s,t)\in\mathbb{R}^2\to\mathbb{R}$ is a smooth function, we put

$$j_s(\kappa):(s,t)\in\mathbb{R}\to(s,\kappa(s,t),\partial_s\kappa|_{(s,t)},\dots.,\partial_s^n\kappa|_{(s,t)}\dots.)\in\mathbf{J}(\mathbb{R},\mathbb{R}).$$

The *mKdV hierarchy* can be described as follows : let $\mathbf{F}_n$ and $\mathbf{G}_n$ be the differential functions defined by

$$\mathbf{F}_1=0,\quad \mathbf{G}_1=-1,\quad \mathbf{F}_2=u_{(1)},\quad \mathbf{G}_2=-\frac{1}{2}u^2$$

and by

$$\mathbf{F}_n=\delta^2(\mathbf{F}_{n-1})+u^2\mathbf{F}_{n-1}+u_{(1)}\int u\mathbf{F}_{n-1}ds,\quad \mathbf{G}_n=-\int u\mathbf{F}_n ds,$$

for every $n\geq 3$. Then, the *n-th member of the mKdV hierarchy* is the evolution equation

$$\kappa_t+\left(\delta^2(\mathbf{F}_n)+u^2\mathbf{F}_n-u_{(1)}\mathbf{G}_n\right)|_{j_s(\kappa)}=0.$$

2.3. The Goldstein–Petrich flows

Following [11], a *local dynamics* of curves is defined by

$$\partial_t\gamma=\mathbf{U}|_{j_s(\kappa)}\mathbf{t}+\mathbf{V}|_{j_s(\kappa)}\mathbf{n}, \tag{6}$$

where $\mathbf{U},\mathbf{V}\in J[\mathbf{u}]$ are two differential functions and where

$$\mathrm{F}(-,t):s\in\mathbb{R}\to\mathrm{F}(s,t)=(\gamma(s,t),\mathbf{t}(s,t),\mathbf{n}(s,t))\in\mathrm{SO}_\varepsilon(3),$$

and

$$\kappa(-,t) : s \in \mathbb{R} \to \kappa(s,t) \in \mathbb{R}$$

denote the Frenet frame and the curvature of the evolving curve, respectively. We also assume that $u\mathbf{V}$ is exact. From (6) and (7) we deduce

$$\mathrm{F}^{-1}d\mathrm{F} = \mathbf{K}|_{j_s(\kappa)}ds + \mathbf{Q}|_{j_s(\kappa)}dt, \tag{7}$$

where

$$\mathbf{K}, \mathbf{Q} : \mathbf{J}(\mathbb{R},\mathbb{R}) \to \mathfrak{so}_\varepsilon(3)$$

are the $\mathfrak{so}_\varepsilon(3)$-valued differential functions

$$\mathbf{K} = \begin{pmatrix} 0 & -\varepsilon & 0 \\ 1 & 0 & -u \\ 0 & u & 0 \end{pmatrix}, \tag{8}$$

and

$$\mathbf{Q} = \begin{pmatrix} 0 & -\varepsilon\mathbf{U} & -\varepsilon\mathbf{V} \\ \mathbf{U} & 0 & -\delta(\mathbf{V}) - u\mathbf{U} \\ \mathbf{V} & \delta(\mathbf{V}) + u\mathbf{U} & 0 \end{pmatrix}, \tag{9}$$

where $\mathbf{U}$ is a primitive of $u\mathbf{V}$ (i.e. $\delta(\mathbf{U}) = u\mathbf{V}$). The compatibility equation of (7) is

$$\partial_t(\mathbf{K}|_{j_s(\kappa)}) - \partial_s(\mathbf{Q})|_{j_s(\kappa)} = [\mathbf{K},\mathbf{Q}]|_{j_s(\kappa)}. \tag{10}$$

An easy inspection shows that (10) is satisfied if and only if

$$\partial_t(\kappa) = \left(\delta^2(\mathbf{V}) + (u^2 + \varepsilon)\mathbf{V} + u_{(1)}\mathbf{U}\right)|_{j_s(\kappa)}. \tag{11}$$

The *first Goldstein–Petrich flow* is defined by the choice

$$\mathbf{V}_{(1)} = 0, \quad \mathbf{U}_{(1)} = -1.$$

The curvature then evolves according to

$$\partial_t(\kappa) + \partial_s(\kappa) = 0.$$

Thus, the first GP-flow is trivial from a geometrical viewpoint (i.e. every curve evolves by rigid motions). The *second Goldstein–Petrich flow* is given by

$$\mathbf{V}_{(2)} = -u_{(1)}, \quad \mathbf{U}_{(2)} = \varepsilon - \frac{1}{2}u^2, \tag{12}$$

which yields the *mKdV* equation

$$\kappa_t + \kappa_{sss} + \frac{3}{2}\kappa^2\kappa_s = 0. \tag{13}$$

The choice

$$\mathbf{V}_{(3)} = \varepsilon u_{(1)} - u_{(3)} - \frac{3}{2}u^2 u_{(1)}, \quad \mathbf{U}_{(3)} = -\varepsilon^2 + \frac{1}{2}\varepsilon u^2 - \frac{3}{8}u^4 + \frac{1}{2}u_{(1)}^2 - uu_{(2)}$$

gives the *third Goldstein–Petrich flow*. The corresponding evolution equation is the third member

$$\kappa_t + \kappa_{sssss} + \frac{15}{8}\kappa^4\kappa_s + \frac{5}{2}\kappa_s^3 + 10\kappa\kappa_s\kappa_{ss} + \frac{5}{2}\kappa^2\kappa_{sss} = 0$$

of the *mKdV* hierarchy.

REMARK 3. More generally, let $\{q_{(h,n)}\}_{1\leq h\leq n}$ be the sequences defined recursively by

$$\begin{cases} q_{(1,2n)} = -2n, & \\ q_{(h,2n)} = (-1)^h(|q_{(h-1,2n-1)}| + |q_{(h,2n-1)}|), & h = 2,...,n, \\ q_{(h,2n)} = q_{(2n-h,2n)}, & h = n+1,\cdots,2n-1, \\ q_{(2n,2n)} = 1, & \end{cases}$$

and by

$$\begin{cases} q_{(1,2n+1)} = -(2n+1), & \\ q_{(h,2n+1)} = (-1)^h(|q_{(h-1,2n)}| + |q_{(h,2n)}|), & h = 2,...,n, \\ q_{(h,2n+1)} = -q_{(2n+1-h,2n)}, & h = n+1,\cdots,2n, \\ q_{(2n+1,2n+1)} = -1. & \end{cases}$$

Then, we set

$$b_{(0,0)} = 1, \quad b_{(h,n)} = \varepsilon^h q_{(h,n)}, \quad h = 1,\cdots n, \quad n \in \mathbb{N}$$

and we consider the differential functions defined recursively by

$$\mathbf{A}_{(1,n)} = 0, \quad \mathbf{A}_{(2,n)} = -u_{(1)}, \quad n \geq 2$$

and by

$$\mathbf{A}_{(h,n)} = \delta^2(\mathbf{A}_{(h-1,n)}) + (u^2+\varepsilon)\mathbf{A}_{(h-1,n)} + u_{(1)}\left(\int u\mathbf{A}_{(h-1,n)}ds - b_{(h-2,n-2)}\right),$$

for each $n \geq 3$ and each $h = 2,\ldots,n$. If we define $\mathbf{V}_{(n)}$ and $\mathbf{U}_{(n)}$ by

$$\mathbf{V}_{(n)} = \mathbf{A}_{(n,n+1)}, \quad \mathbf{U}_{(n)} = \left(\int u\mathbf{V}_{(n)}ds\right) - b_{(n-1,n-1)}$$

we get the *n-th Goldstein–Petrich flow*, whose corresponding evolution equation is the n-th member of the *mKdV* hierarchy.

REMARK 4. If κ is a solution of (11) and if $\mathbf{K}$ and $\mathbf{Q}$ are defined as in (8) and (9) then the 1-form

$$\Theta = \mathbf{K}|_{j_s(\kappa)}ds + \mathbf{Q}|_{j_s(\kappa)}dt$$

fulfils the Maurer-Cartan equation

$$d\Theta + \Theta \wedge \Theta = 0.$$

Thus, by the Cartan-Darboux theorem, there exist $\mathrm{F} : \mathbb{R}\times\mathbb{R} \to \mathrm{SO}_\varepsilon(3)$ such that $\mathrm{F}^{-1}d\mathrm{F} = \Theta$. The map F is unique up to left multiplication by an element of $\mathrm{SO}_\varepsilon(3)$. If we put $\mathrm{F} = (\gamma, \mathbf{t}, \mathbf{n})$, then $\gamma : \mathbb{R}^2 \to \mathbb{R}^3_\varepsilon$ is the analytical parametrization of the dynamics of a curve evolving according to (6).

REMARK 5. The first three basic distinguished functionals of (13) are

$$\mathfrak{L}_0 = \int ds, \quad \mathfrak{L}_1 = \int k(s)ds, \quad \mathfrak{L}_2 = \int \kappa^2(s)ds.$$

2.4. Goldstein–Petrich curves

Closed trajectories whose shapes are invariant under (13) are called *Goldstein–Petrich curves* of $\mathcal{S}^2_\varepsilon$ (GP-curves for brevity). Solutions of (13) which correspond to Goldstein–Petrich curves are periodic solutions in the form of a traveling wave, so $\kappa = \kappa(s+(\varepsilon-\lambda)t)$, for some constant λ and

$$\kappa''' + \left(\frac{3}{2}\kappa^2 + (\varepsilon-\lambda)\right)\kappa' = 0. \tag{14}$$

Integrating (14) we find

$$\kappa'' + \frac{\kappa^3}{2} + (\varepsilon-\lambda)\kappa = -\mu, \tag{15}$$

where μ is a constant. Another integration yields

$$(\kappa')^2 + \frac{1}{4}\kappa^4 + (\varepsilon-\lambda)\kappa^2 + \mu\kappa = -\nu, \tag{16}$$

where ν is another constant of integration. Thus, the *signature* [5, 22, 25] of a GP-curve is one of the bounded components of an elliptic curve of the following type

$$y^2 + \frac{1}{4}x^4 + (\varepsilon-\lambda)x^2 + \mu x + \nu = 0.$$

We notice that the extremal curves of the the action functional

$$2\lambda\mathfrak{L}_0 + 2\varepsilon\mu\mathfrak{L}_1 + \mathfrak{L}_2. \tag{17}$$

satisfies (15). In other words, stationary curves of the second order Goldstein–Petrich flow arise as the critical points of linear combinations of the basic distinguished functionals $\mathfrak{L}_0, \mathfrak{L}_1$ and $\mathfrak{L}_2$.

REMARK 6. The value $\mu = 0$ yields *elastic curves* in $\mathcal{S}^2_\varepsilon$, i.e. the critical points of the total squared curvature functional with respect to variations with fixed length. These curves have been extensively studied in [2, 9, 17, 18]. If, in addition $\lambda = 0$ the corresponding curves are *free elasticæ*, i.e. the critical points of the total squared curvature functional. Free elasticæ have a special interest in differential geometry because of their interrelations with Willmore surfaces [3, 9, 13, 26].

We note that (15) can be written in terms of a Lax pair. To prove the assertion we set

$$\mathbf{H}_\lambda =: \mathbf{Q} + (\lambda - \varepsilon)\mathbf{K} : \mathbf{J}(\mathbb{R},\mathbb{R}) \to \mathfrak{so}_\varepsilon(3), \tag{18}$$

where $\mathbf{Q}$ and $\mathbf{K}$ are defined as in (8) and (9), with $\mathbf{V} = -u_{(1)}$, that is

$$\mathbf{H}_\lambda = \begin{pmatrix} 0 & -\varepsilon(\lambda - \frac{u^2}{2}) & \varepsilon u_{(1)} \\ \lambda - \frac{u^2}{2} & 0 & u_{(2)} + \frac{u^3}{2} - \lambda u \\ -u_{(1)} & -u_{(2)} - \frac{u^3}{2} + \lambda u & 0 \end{pmatrix}.$$

It is then a computational matter to verify that (15) holds if and only if

$$\delta(\mathbf{H}_\lambda|_{j(\kappa)}) = [\mathbf{H},\mathbf{K}]|_{j(\kappa)}. \tag{19}$$

The main consequence of (19) is the integrability by quadratures of the Golstein-Petrich curves. From the point of view of symplectic geometry, the Lax equation (19) is equivalent to the Noether theorem of the conservation of the momentum map along the extremal curves of the distiguished functionals (17).

3. Numerical solutions and examples

3.1. Numerical solutions

We now show how to implement standard numerical routines in our geometrical setting.

Step 1. Define the curvature ε of $\mathcal{S}^2_\varepsilon$, the coefficients λ and μ of (15), the initial conditions $P_0 = \kappa(0)$, $P_1 = \kappa'(0)$ and the interval $I = (a,b)$:

```
ε:=1;   λ:=1 − 2;   μ:=0.626;   P0:= − 0.2;   P1:=2;   a:= − 8;   b:=8;
```

Step 2. Solve (15)

```
s[0]:=NDSolve[{k”[t] + 1/2 k[t]^3 + (ε − λ) ∗ k[t] + μ == 0,k[0] == P0,k′[0]==P1},
{k},{t,a − 0.5,b}];
K[t_]:=({k[t]}/.s[0])[[1,1]];
```

Step 3. Solve the Frenet–Serret linear system with curvature κ:

```
s[1]:=NDSolve [{x′[s] == y[s],x[0] == ε,y′[s] == K[s] ∗ z[s] − ε ∗ x[s],
 y[0] == 0,z′[s] == −K[s] ∗ y[s],z[0] == 0} , {x,y,z}, {s,a,b}] ;
```

```
s[2]:=NDSolve[{x'[s] == y[s], x[0] == 0, y'[s] == K[s]*z[s] - ε*x[s],
 y[0] == 1, z'[s] == -K[s]*y[s], z[0] == 0}, {x, y, z}, {s, a, b}];
s[3]:=NDSolve[{x'[s] == y[s], x[0] == 0, y'[s] == K[s]*z[s] - ε*x[s],
 y[0] == 0, z'[s] == -K[s]*y[s], z[0] == 1}, {x, y, z}, {s, a, b}];
S[1][s_]:={x[s], y[s], z[s]}/.s[1];
S[2][s_]:={x[s], y[s], z[s]}/.s[2];
S[3][s_]:={x[s], y[s], z[s]}/.s[3];
```

Step 4. Define the corresponding GP curve (γ*S*=spherical GP curve, γE= Euclidean GP curve, γH=hyperbolic GP curve)

```
γS[t_]:={S[1][t][[1]][[1]], S[2][t][[1]][[1]], S[3][t][[1]][[1]]};
γH[t_]:=1/(S[1][t][[1]][[1]]) {S[2][t][[1]][[1]], S[3][t][[1]][[1]]};
γE[t_]:={S[2][t][[1]][[1]], S[3][t][[1]][[1]]};
```

Step 5. Visualize the curvature and the signature

```
SIGNATURE:=ParametricPlot[Evaluate[{K[t], D[K[t], t]}], {t, 0, b},
PlotPoints → 300, AspectRatio → Automatic,
Axes → True, PlotStyle → {Thickness[0.01], Black}, ImageSize → {400, 400},
Background → GrayLevel[0.8]];
HYPERBOLICCURVE:=Show[Graphics[{GrayLevel[0.6], Disk[{0, 0}, 1]}],
ParametricPlot[Evaluate[γH[s]], {s, a, b},
PlotStyle → {{Thickness[0.01], Black}}, Axes → False,
AspectRatio → Automatic, PlotPoints → 140, PlotRange → All],
Background → GrayLevel[0.8]];
EUCLIDEANCURVE:=ParametricPlot[Evaluate[γE[s]], {s, a, b},
Background → GrayLevel[0.8], PlotStyle → {{Thickness[0.01], Black}},
Axes → False, AspectRatio → Automatic, PlotPoints → 140,
PlotRange → All, ImageSize → {400, 400}];
SPHERICALCURVE:=
Show[Graphics3D[{Opacity[0.5], GrayLevel[0.6], Sphere[{0, 0, 0}]},
Lighting → "Neutral"], ParametricPlot3D[Evaluate[γS[s]],
{s, a, b}, PlotPoints → 300, Boxed → False, AspectRatio → Automatic,
Axes → False,
PlotStyle → {Thickness[0.01], Black}], Boxed → False,
Background → GrayLevel[0.8],
PlotRange → {{-1, 1}, {-1, 1}, {-1, 1}}, ImageSize → {400, 400}];
```

3.2. Wave-like Goldstein–Petrich curves

The values

$$\varepsilon = 0, \quad \lambda = 0, \quad \mu \approx 1.103, \quad P_0 = 2, \quad P_1 = 0.2$$

and

$$\varepsilon = 0, \quad \lambda = -1, \quad \mu \approx 1.103, \quad P_0 = 2.1904, \quad P_1 = 0.2$$

give simple wave-like GP curves in $\mathbb{R}^2$ with symmetry groups $\mathbb{Z}_8$ and $\mathbb{Z}_{11}$ respectively (see Figure 1). The values

$$\varepsilon = -1, \quad \lambda = 0, \quad \mu \approx -9.987, \quad P_0 = -3, \quad P_1 = 1$$

and

$$\varepsilon = -1, \quad \lambda = 0, \quad \mu \approx -12.05, \quad P_0 = -3, \quad P_1 = 1$$

correspond to simple wave-like GP curves in $\mathbb{H}^2$ with symmetry groups $\mathbb{Z}_6$ and $\mathbb{Z}_8$ respectively (see Figure 2).

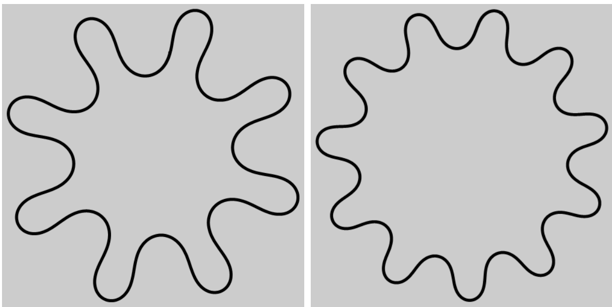

Figure 1: Simple wave-like GP curves in $\mathbb{R}^2$

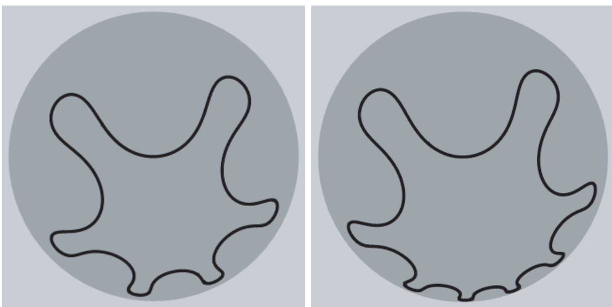

Figure 2: Simple wave-like GP curves in $\mathbb{H}^2$

3.3. Orbit-like Goldstein–Petrich curves

The values

$$\varepsilon = 0 \quad \lambda = -2 \quad \mu \approx -4.03, \quad P_0 = 2, \quad P_1 = 1,$$

and

$$\varepsilon = 1 \quad \lambda = -1 \quad \mu \approx -4.03, \quad P_0 = 2, \quad P_1 = 1$$

give rise to closed orbit-like GP curves in $\mathbb{R}^2$ and S^2 with symmetry group $\mathbb{Z}_7$ (see Figure 3).

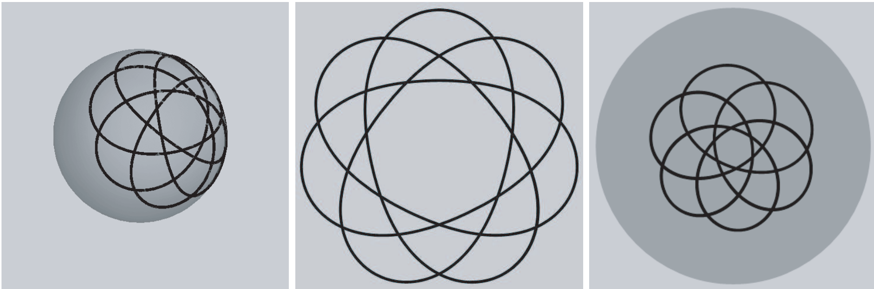

Figure 3: Orbit-like GP curves

Setting

$$\varepsilon = -1 \quad \lambda = 1, \quad \mu \approx -3.8936, \quad P_0 = 2, \quad P_1 = 1$$

we obtain an orbit-like curve in the hyperbolic plane with symmetry group $\mathbb{Z}_6$ and 24 points of self-intersection (see Figure 3).

4. Spherical free elasticæ

4.1. Spin frames

Consider the special unitary group

$$\mathrm{SU}(2) = \{\mathbf{V} = (V_1, V_2) \in \mathfrak{gl}(2,\mathbb{C}) : \mathbf{V}\cdot\overline{\mathbf{V}}^T = 1, \quad \det(\mathbf{V}) = 1\}.$$

We take

$$\mathbf{i} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \mathbf{j} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad \mathbf{k} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

as standard infinitesimal generators of the Lie algebra $\mathfrak{su}(2)$. We consider the Euclidean inner product

$$\|\mathbf{x}\|^2 = -\frac{1}{2}\mathrm{Tr}(\mathbf{x}^2), \quad \forall \mathbf{x} \in \mathfrak{su}(2).$$

Thus, $(\mathbf{i},\mathbf{j},\mathbf{k})$ is a orthogonal basis and

$$(x,y,z) \in \mathbb{R}^3 \to x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \in \mathfrak{su}(2)$$

gives an explicit identification of $\mathfrak{su}(2)$ with the Euclidean 3-space. Consequently, the 2-dimensional sphere S^2 is identified with $\{\mathbf{x} \in \mathfrak{su}(2) : \|\mathbf{x}\|^2 = 1\}$. With these notations at hand, the bundle of spin-frames of S^2 is defined by

$$\pi : \mathbf{V} \in \mathrm{SU}(2) \to \mathbf{V}\cdot\mathbf{i}\cdot\overline{\mathbf{V}}^T \in S^2$$

and, the 2 : 1 spin covering map $\mathrm{SU}(2) \to \mathrm{SO}(3)$ is given by

$$\mathbf{V} \in \mathrm{SU}(2) \to (\mathbf{V}\cdot\mathbf{i}\cdot\overline{\mathbf{V}}^T, \mathbf{V}\cdot\mathbf{j}\cdot\overline{\mathbf{V}}^T, \mathbf{V}\cdot\mathbf{k}\cdot\overline{\mathbf{V}}^T) \in \mathrm{SO}(3).$$

DEFINITION 1. *Let $\gamma : I \longrightarrow S^2$ be a smooth curve parameterized by the arc-length and with curvature κ. A spin frame field along γ is a map*

$$\mathbf{G} = (\Gamma, \Gamma^*) : I \to \mathrm{SU}(2)$$

such that

$$\gamma = \mathbf{G}\cdot\mathbf{i}\cdot\overline{\mathbf{G}}^T, \quad \Gamma' = \frac{1}{2}(i\kappa\Gamma + \Gamma^*), \qquad \Gamma^{*\prime} = -\frac{1}{2}(\Gamma + i\kappa\Gamma^*). \tag{20}$$

REMARK 7. The spin frame $\mathbf{G} : I \to \mathrm{SU}(2)$ is just a lift to $\mathrm{SU}(2)$ of the Frenet frame $\mathrm{F} : I \to \mathrm{SO}(3)$ along γ.

4.2. The monodromy

A spin frame $\mathbf{G} : \mathbb{R} \to \mathrm{SU}(2)$ with non-constant periodic curvature κ and initial condition $\mathbf{G}(0) = \mathbf{1}$ is the solution of linear system with periodic coefficients

$$2\mathbf{G}' = \mathbf{G}\cdot\begin{pmatrix} i\kappa & -1 \\ 1 & -i\kappa \end{pmatrix}, \qquad \mathbf{G}(0) = \mathbf{1}. \tag{21}$$

The *monodromy* of (21) is defined by

$$\mathbf{M} := \mathbf{G}(\omega) \in \mathrm{SU}(2),$$

where ω is the minimal period of κ. The two eigenvalues of $\mathbf{M}$ are

$$\mu_\pm = \mathrm{Re}\left(\Gamma_1^1(\omega)\right) \pm \sqrt{\mathrm{Re}\left(\Gamma_1^1(\omega)\right)^2 - 1} = e^{\pm 2\pi i\theta} \in S^1, \tag{22}$$

where $\theta \in [0,1)$. By the Floquet theorem for linear systems of o.d.e with periodic coefficients (cfr. [1]) it follows that $\mathbf{G}$ is a periodic solution of (21) if and only if

$$\theta = p/q \in [0,1), \quad p,q \in \mathbb{N}, \quad \gcd(p,q) = 1.$$

DEFINITION 2. *The minimal period of the spin frame $\mathbf{G}$ is $\ell_s = q\omega$ and the minimal period ℓ of the corresponding spherical curve $\gamma = \pi\circ\mathbf{G}$ can be either ℓ_s or else $\ell_s/2$. In the first case we say that γ is a spherical curve with spin 1 while in the second case we say that γ is a spherical curve with spin $1/2$.*

REMARK 8. For a spherical curve of spin 1 the integer q is odd and gives the order of its symmetry group. For a curve of spin $1/2$ the integer q is even and the order of the symmetry group is $q/2$.

4.3. The monodromy of free elasticae in S^2

The curvature of a spherical free elastica satisfies

$$\kappa'' + \frac{1}{2}(\kappa^2 + 2)\kappa = 0. \tag{23}$$

The general periodic solution of (23) is

$$\kappa(s,\lambda) = 2\sqrt{\frac{\lambda}{1-2\lambda}}\mathrm{JacobiCN}\left(\frac{s}{\sqrt{1-2\lambda}},\lambda\right) \tag{24}$$

where $\lambda \in (0,1/2)$ and $\mathrm{JacobiCN}(z,\lambda)$ denotes the Jacobi cn-function with modulus $\sqrt{\lambda}$. The minimal period of $k(s,\lambda)$ is given by

$$\omega_\lambda = 4\sqrt{1-2\lambda} \quad \mathrm{EllipticK}(\lambda), \tag{25}$$

where

$$\mathrm{EllipticK}(\lambda) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-\lambda\sin^2(\theta)}}.$$

For each $\lambda \in (0,1/2)$ we denote by

$$\mathbf{G} : \mathbb{R} \times (0,1/2) \to \mathbf{G}(s,\lambda) \in \mathrm{SU}(2)$$

the spin frame field with curvature $\kappa(-,\lambda)$ and initial condition $\mathbf{G}_\lambda(0) = \mathbf{1}$. We then define the *Floquet map*

$$m : \lambda \in (0,1/2) \to m(\lambda) \in S^1 \tag{26}$$

by

$$m(\lambda) = \mathrm{Re}(\Gamma_1^1(\omega_\lambda,\lambda)) + \sqrt{\mathrm{Re}(\Gamma_1^1(\omega_\lambda,\lambda))^2 - 1} = e^{2\pi i\theta(\lambda)},$$

where $\theta(\lambda) \in (0,1)$. The Floquet map can be computed in closed form by means of the Heuman's Lambda function [19] or with numerical methods. We now exhibit the code of the numerical evaluation of the Floquet map:

Step 1. Define $\kappa(s,\lambda)$, the period ω_λ :

$\mathsf{K}[\mathsf{t\_},\lambda\mathsf{\_}]\text{:=}\sqrt{\frac{4*\lambda}{1-2*\lambda}}\mathsf{JacobiCN}\left[\frac{\mathsf{t}}{\sqrt{(1-2*\lambda)}},\lambda\right];$

$\omega[\lambda\mathsf{\_}]\text{:=}\left(4*\sqrt{(1-2*\lambda)}\right)*\mathsf{EllipticK}[\lambda];$

Step 2. Solve numerically the linear system (20) :

```
A[1]:={1,0};
A[2]:={0,1};
sol[1][λ_]:=
```

```
NDSolve[{x'[t] == (1/2)(i*K[t,λ]*x[t]+y[t]),x[0] == A[1][[1]],
y'[t] == (1/2)(-i*K[t,λ]*y[t]-x[t]),y[0] == A[2][[1]]},{x,y},
{t,-0.5,ω[λ]}];
sol[2][λ_]:=
NDSolve[{x'[t] == (1/2)(i*K[t,λ]*x[t]+y[t]),x[0] == A[1][[2]],
y'[t] == (1/2)(-i*K[t,λ]*y[t]-x[t]),y[0] == A[2][[2]]},{x,y},
{t,-0.5,ω[λ]}];
S[1][t_,λ_]:={x[t],y[t]}/.sol[1][λ];
S[2][t_,λ_]:={x[t],y[t]}/.sol[2][λ];
M[t_,λ_]:=Transpose[{{S[1][t,λ][[1]][[1]],S[2][t,λ][[1]][[1]]},
{-Conjugate[S[2][t,λ][[1]][[1]]],Conjugate[S[1][t,λ][[1]][[1]]]}}];
Monodromy[λ_]:=M[ω[λ],λ];
μ[λ_]:=Re[S[1][ω[λ],λ][[1]][[1]]]+√(Re[S[1][ω[λ],λ][[1]][[1]]]^2-1);
```

Step 3. Compute the monodromy and the Floquet map

```
M[t_,λ_]:=Transpose[{{S[1][t,λ][[1]][[1]],S[2][t,λ][[1]][[1]]},
{-Conjugate[S[2][t,λ][[1]][[1]]],Conjugate[S[1][t,λ][[1]][[1]]]}}];
Monodromy[λ_]:=M[ω[λ],λ];
μ[λ_]:=Re[S[1][ω[λ],λ][[1]][[1]]]+√(Re[S[1][ω[λ],λ][[1]][[1]]]^2-1);
```

Step 4. Plot and visualize the image of the Floquet map and the graph of its real part

```
α[λ_]:={Re[μ[λ]],Im[μ[λ]]};
FLOQUET:=Show[Graphics[{GrayLevel[0.8],Disk[{0,0}]}],
ParametricPlot[α[λ],{λ,0,0.499},Background → GrayLevel[0.8],
PlotStyle → {{Thickness[0.02],Black}},
PlotRange → {{-1.2,1.2},{-1.2,1.2}},PlotPoints → 200,
AspectRatio → Automatic,Axes → False]];
```

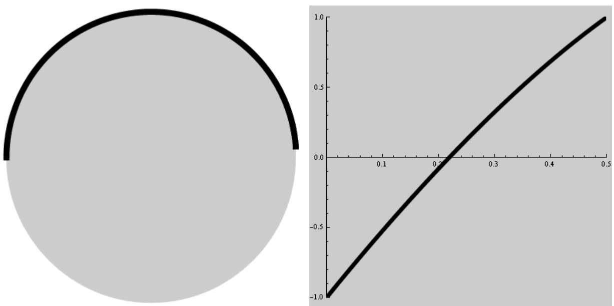

Figure 4: The Floquet map and its real part

Plotting the image of the Floquet map and the graph of its real part (Figure 4) we infer that m is a bijection of $(0,1/2)$ onto $S^1_+ = \{e^{it} : t \in (0,\pi)\}$. We set

$$\phi : \tau \in (0,1/2) \to m^{-1}(e^{2\pi i\tau}) \in (0,1/2) \tag{27}$$

and, for every $\tau \in (0,1/2)$ we let γ_τ be the free elastica with parameter $\lambda = \phi(\tau)$. Summarizing the discussion, we have

PROPOSITION 1. *The curve γ_τ is a closed free elastica if and only if $\tau = p/q$, where $p,q \in \mathbb{N}$, $\gcd(p,q) = 1$ and $0 < p/q < 1/2$.*

To evaluate $\phi(p/q)$ we proceed as follows : first we note that $\phi(p/q)$ is the unique zero of the function

$$\Phi_{p/q} : \lambda \in (0,1/2) \to \|m(\lambda) - e^{2\pi i \frac{p}{q}}\|^2 \in \mathbb{R}^+.$$

Plotting the graph of $\Phi_{p/q}$ we have a first rough estimate of the location of $\phi(p/q)$ (Figure 5).

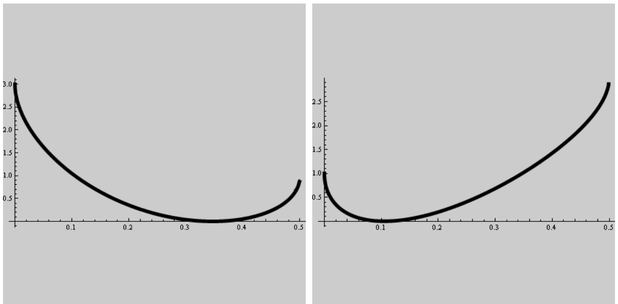

Figure 5: The graphs of $\Phi_{1/6}$ and $\Phi_{1/3}$

We then choose an initial value $\lambda_1 \in (0,1/2)$ sufficiently near to $\phi(p/q)$ and we start to search the minimum λ_2 of $\Phi_{p/q}$ inside the interval $[\lambda_1 - \delta, \lambda_1 + \delta]$, among a finite set D_1 of k elements. We repeat the procedure by taking λ_2 as a new initial value and searching for the minimum of $\Phi_{p/q}$ in the interval $[\lambda_2 - \delta/2, \lambda_2 + \delta/2]$ among a finite set D_2 of $2k$ elements. Proceeding recursively we find, after n steps, a value λ_n which approximates $\phi(p/q)$ with the desired precision. In practice, a good approximation of $\phi(p/q)$ is given by any λ_n such that $\Phi_{p/q}(\lambda_n) < 10^{-h}$, with $h \geq 8$.

Step 1. Compute $\Phi_{p/q}$

$\Psi[\lambda\_, \mathrm{p}\_, \mathrm{q}\_] := \mathrm{Abs}\left[\mu[\lambda] - \mathrm{Exp}\left[2 * \mathrm{Pi} * \mathrm{i} * \frac{\mathrm{p}}{\mathrm{q}}\right]\right] \hat{}2;$

and visualize its graph

```
DISTRIBUTION:=Plot[Ψ[λ, p, q], {λ, −3, 3},
PlotStyle → {{Thickness[0.01], Black}}, PlotRange → All,
PlotPoints → 200, AspectRatio → 1/2, Axes → True, ImageSize → {400, 250},
Background → GrayLevel[0.8], PlotRange → {{−3, 3}, {0, 4}}]
```

Step 2. Search of the approximated values of $\min(\Phi_{p/q})$:

```
p:=1; q:=4; τ1:=0.4;
internalparameter[1]:=1/12;
internalparameter[2]:=20;
steps:=8;
Q[y_, δ_, k_]:=First[Sort[Table[{Ψ[λ, p, q], λ}, {λ, y − δ, y + δ, 1/k}]]];
S[1, y_, δ_, k_]:=Q[y, δ, k];
S[m_, y_, δ_, k_]:=S [m − 1, S[m − 1, y, δ, k][[2]], δ/(2^(m−1)) , k ∗ (2^(m−1))] ;
S[steps, τ1, internalparameter[1], internalparameter[2]];
```

4.4. Examples

The approximated values of $\phi(1/2h)$, for $h = 2, \ldots, 7$ are given by

$$\phi(1/4) \approx 0.219105, \quad \phi(1/6) \approx 0.347017, \quad \phi(1/8) \approx 0.406183,$$

and by

$$\phi(1/10) \approx 0.437212, \quad \phi(1/12) \approx 0.455251, \quad \phi(1/14) \approx 0.466582.$$

The corresponding free elasticæ are reproduced in Figures 6 and 7. All these curves are simple, with symmetry groups $\mathbb{Z}_h$ and subdivide S^2 into two congruent pieces.

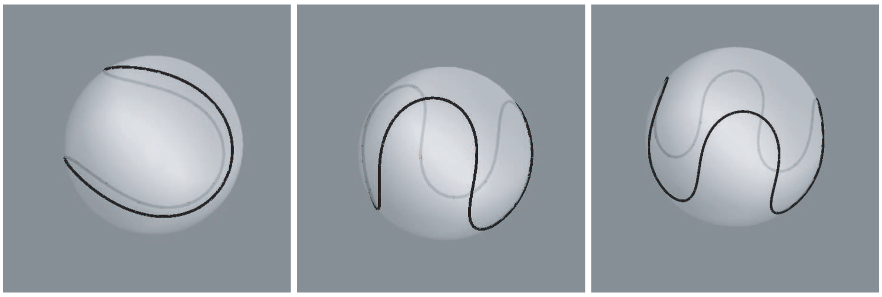

Figure 6: The free elastica $\gamma_{1/2h}$, $h = 2, 3, 4$
.

The first approximated values of $\phi(p/7)$ are

$$\phi(1/7) \approx 0.381729, \quad \phi(2/7) \approx 0.166961, \quad \phi(3/7) \approx 0.0201934.$$

The associated free elasticæ are closed, with symmetry group $\mathbb{Z}_7$ and with 7, 21 and 35 points of self-intersection, respectively. The curves are reproduced in Figure 8.
The first three approximated values of $\phi(p/16)$ are

$$\phi(3/16) \approx 0.315366, \quad \phi(5/16) \approx 0.130811, \quad \phi(7/16) \approx 0.0015496.$$

The associated free elasticæ are closed, with symmetry group $\mathbb{Z}_8$ and with 8, 32 and 58 points of self-intersection, respectively. The curves are reproduced in Figure 9.

Figure 7: The free elastica $\gamma_{1/2h}$, $h = 5, 6, 7$
.

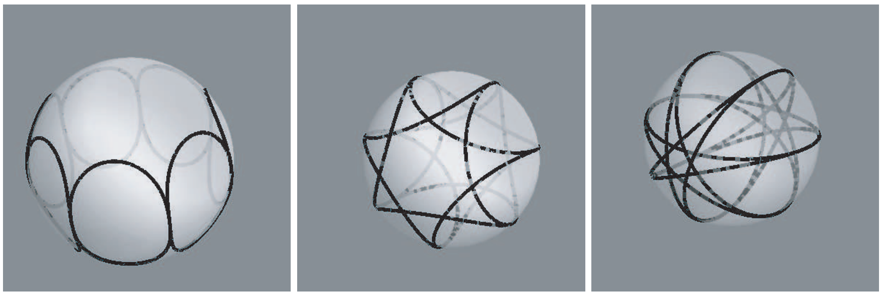

Figure 8: The free elastica $\gamma_{p/7}$, $p = 1, 2, 3$
.

4.5. Conclusions

The examples above and the evidence of other numerical experiments suggest the following geometrical facts*.

PROPOSITION 2. *Let* $\gamma_{p/q}$ *be a closed free elastica in* S^2 *with* $p, q \in \mathbb{N}$, $p/q \in (0, 1/2)$ *and* $\gcd(p, q) = 1$, *then*

- *if q is even* ($q = 2q'$) *the elastica* $\gamma_{p/2q'}$ *has spin* $1/2$, *symmetry group* $\mathbb{Z}_{q'}$ *and possesses* $(p-1)q'$ *points of self-intersection;*
- *if q is odd* ($q = 2q'+1$), *the elastica* $\gamma_{p/(2q'+1)}$ *has spin* 1, *symmetry group* $\mathbb{Z}_{2q'+1}$ *and possesses* $(2p-1)q$ *points of self-intersection;*
- $\gamma_{p/q}$ *is a simple curve if and only if* $p = 1$ *and* $q \in 2\mathbb{N}$.

*See ref. [18] for similar results in the case of free elasticæ in the Poincaré disk.

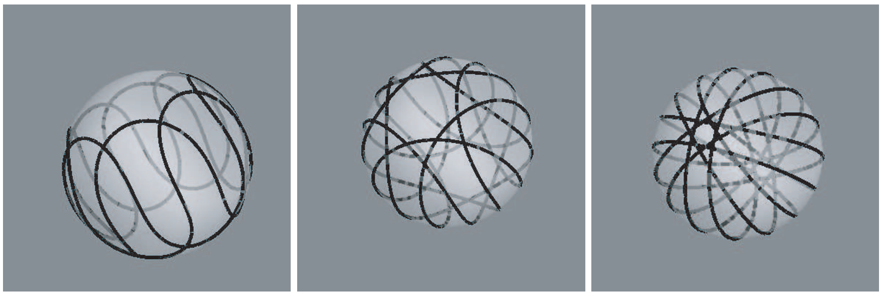

Figure 9: The free elastica $\gamma_{p/16}$, $p = 3, 5, 7$
.

4.6. Embedded Pinkall's tori

If $\mathbf{G} : I \to \mathrm{SU}(2)$ is a spin frame field of a unit-speed spherical curve $\gamma : I \to S^2$ with curvature κ hen the map

$$f : (s, \vartheta) \in I \times \mathbb{R} \to e^{\frac{i}{2}(\vartheta - \int \kappa(u) du)} \mathbf{G}(s) \in S^3 \subset \mathbb{C}^2$$

is a flat immersion into the unit 3-sphere which is called the *Hopf immersion* associated to γ. The first and second fundamental forms of f are

$$I = \frac{1}{4}(ds^2 + d\vartheta^2), \quad II = \frac{1}{2}(\kappa ds^2 - ds d\vartheta).$$

Thus, κ gives the mean curvature H of the corresponding Hopf immersion. Therefore, if γ is a Goldstein–Petrich curve the mean curvature of the Hopf immersion satisfies

$$\triangle(H) + 2\left(H^2 + \frac{a}{b} - K\right) H = p, \tag{28}$$

where $\triangle$ is the Laplace-Beltrami operator of the induced Riemannian metric and a, b, p are constants. Immersions satisfying (28) are the critical points of the Hooke's energy

$$E(f) = \int (bH^2 + a) dA.$$

They are known as *elastic surfaces* [13]. Physically, a is the surface tension, b the bending energy and p is the pressure. The case $a = b = 1$ and $p = 0$ has a particular geometrical interest; the curve γ is a free elastica and f parameterizes a *Willmore surface*. One of the key features of Willmore immersions is their invariance with respect to the group of Möbius (conformal) transformations of S^3. Using this construction and the results of Langer–Singer [17], U.Pinkall [26] discovered the first examples of embedded Willmore tori which are not Möbius equivalent to any minimal surface in S^3. The stereographic projections of these surfaces are called *Pinkall's tori* of $\mathbb{R}^3$. Our experiments show that Pinkall tori are associated to the free elastic curves $\gamma_{1/2n}$, $n \in \mathbb{N}$ and $n > 2$. If we choose appropriately the pole of the stereographic projection, the Pinkall surface defined by $\gamma_{1/2n}$ has a symmetry group isomorphic to $\mathbb{Z}_n$ (see Figure 10).

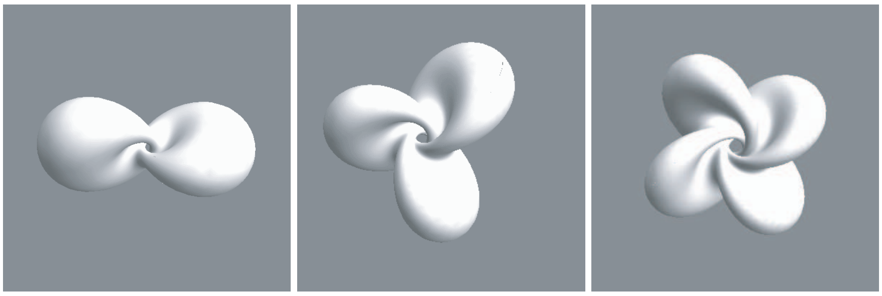

Figure 10: Pinkall's tori with symmetry groups $\mathbb{Z}_2$, $\mathbb{Z}_3$ and $\mathbb{Z}_4$
.

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In Memoriam Aristide Sanini

A. Sanini

GRASSMANN BUNDLES AND HARMONIC MAPS*

Introduction

The "classical" Gauss map Υ associates to any point x of an oriented surface M, immersed in $\mathbb{R}^3$, the unit normal vector N_x applied at a point O in $\mathbb{R}^3$, and so determines a mapping from M to the unit sphere S^2. It is therefore called the *spherical representation* of M.

This representation enables one to obtain plenty of information on various geometrical aspects of the surface. In particular, it provides an "extrinsic" interpretation of the Gaussian curvature of M. A classical result is expressed by the following:

THEOREM 1. *The Gauss map $\Upsilon : M \to S^2$ is conformal if and only if*

1. M is a minimal surface, or

2. M is contained in a sphere.

In both cases, Υ is a harmonic map. The analysis of Gauss maps, extended in a suitable way, is the focal point of many interesting research topics. Particular attention is dedicated to the conditions under which those maps are conformal or harmonic.

In his article [19], R. Osserman provides an excellent overview on the evolution of the concept of Gauss map and the information regarding the geometry of submanifolds that can be deduced from it.

Building on the classical concept of Gauss map, it is possibile to define the "generalized" one, which associates to each point x of an m-dimensional manifold isometrically immersed in $\mathbb{R}^n$, the subspace of $\mathbb{R}^n$ parallel to T_xM, i.e.,

$$\Upsilon : M \to G_m(n),$$

where $G_m(n)$ is the Grassmannian of m-planes in $\mathbb{R}^n$, having a well-known structure of a homogeneous (indeed, symmetric) Riemannian space.

Subsequently, M. Obata constructed in [18] a Gauss map for an m-dimensional Riemannian manifold M isometrically immersed in a simply connected space N with constant sectional curvature. Such a construction of a Gauss map is based on the mapping of $x \in M$ to the m-dimensional totally geodesic submanifold of N tangent to M at x and leads to several particularly significant results regarding conformality conditions.

The more recent theory of harmonic maps (see for example the extensive reports of J. Eells–L. Lemaire [8, 9]) immediately reveals remarkable points in contact with the theory of Gauss maps. We mainly refer to those results (see [7]) generalizing the theorem cited above (see [4]):

*Translation of a *Rapporto Interno, Politecnico di Torino, 1988.*

THEOREM 2 (Chern [2]). *Consider an isometric immersion f of an orientable surface M inside $\mathbb{R}^n$. Then f is harmonic (i.e., minimal) if and only if the Gauss map from M to $\tilde{G}_2(n)$ is anti-holomorphic, where $\tilde{G}_2(n)$ is the Grassmannian of oriented 2-planes in $\mathbb{R}^n$ which can be identified with the complex quadric Q_{n-2} in $\mathbb{CP}^{n-1}$.*

THEOREM 3 (Ruh–Vilms [21]). *A submanifold M of $\mathbb{R}^n$ has parallel mean curvature vector if and only if the Gauss map $\Upsilon : M \to G_m(n)$ is harmonic.*

More recently, C.M. Wood [24] and G.R. Jensen–M. Rigoli [11], considering a submanifold M of a generic Riemannian manifold N, define the Gauss map as the map from M to the Grassmann bundle $G_m(TN)$ of m-planes tangent to N endowed with a suitable metric. They analyse several aspects of the harmonic and conformal conditions, extending the previous results.

From another point of view, S.S. Chern and R.K. Lashof [6], considering a submanifold M isometrically immersed in $\mathbb{R}^n$, define the "spherical" Gauss map (another extension of the classical concept) as the correspondence $\nu : M \to S^{n-1}$ that associates to each unit vector v, normal to M in a point $x \in M$, a point in S^{n-1} obtained by parallel transport of v to the origin of $\mathbb{R}^n$.

In the article cited above, Jensen and Rigoli study the analogous problem in the case of a manifold M isometrically immersed in a generic Riemannian manifold N, associating to any unit vector normal to M the same element in the unit tangent bundle T_1N of N. They analyse also several problems related to the harmonicity of the map.

The present report aims also to expose some recent proper achievements regarding the subject and is divided in three parts.

The first part, entitled "Grassmann bundles and distributions", can be summarized as follows. Section 1 describes the construction of the Riemannian metric on the Grassmann bundle $G_p(TM)$ of p-planes tangent to a a manifold M. This is due to Jensen–Rigoli, and has been already applied by E. Musso–F. Tricerri [17] in the case of unit tangent bundles. The fibres of the Riemannian submersion $G_p(TM) \to M$ are totally geodesic and isometric to the Grassmannian $G_p(m)$ endowed with the standard metric.

Section 2 analyses some aspects related to the curvature of $G_p(TM)$.

Any given p-dimensional distribution over M singles out a section ϕ of $G_p(TM)$, and in Section 3 we determine the conditions under which ϕ is harmonic. Section 4 contains some examples of such a situation in the case in which M is the sphere S^3, the Heisenberg group or another Lie group admitting a left-invariant metric.

In the second part "Isometric immersions and maps between Grassman bundles" we analyse (starting in Section 5) the map F from $G_p(TM)$ to $G_p(TN)$ induced by an isometric immersion f of M inside N. If $p = \dim M$ then $G_p(TM)$ can be identified with M and F coincides with the Gauss map Υ.

Then in Section 6 we define the tension field of F and the conditions under which it is harmonic. We exhibit a significant example of a minimal surface M of the Heisenberg group H for which the Gauss map is conformal but not harmonic.

Section 7 develops a detailed analysis of the harmonic properties of F under the hypothesis that N has constant sectional curvature. The results are completely analogous to those obtained by the author (see [22]) in the case of the map induced between the unit tangent bundles by a Riemannian immersion of M to N. Furthermore, when F coincides with the Gauss map, the results we achieve are compared to those of E. Ruh–J. Vilms and T. Ishihara described in [10].

The third part is dedicated to the “Spherical Gauss map”. In Section 8 we introduce the Riemannian metric on the unit normal bundle $T_1^{\perp}M$ of a manifold M isometrically immersed in N. Later, in Section 9, we study the harmonicity of the spherical Gauss map $\nu : T_1^{\perp}M \to T_1^{\perp}N$ applying a technique analogous to the one adopted in the Second part (ν has already been analysed by Jensen–Rigoli with another method). In Section 10, we add some remarks and examples in the case in which N has a constant sectional curvature.

The present report contains two appendices:

– Appendix A, in which we recall several facts regarding the bundle of Darboux frames and the classical conditions (Gauss, Codazzi, Ricci) on the curvature tensors on a manifold.

– Appendix B, which describes the computation of the tension field of a map between Riemannian manifolds in terms of orthonormal coframes, following the method adopted by S.S. Chern–S.I. Goldberg [5].

I. GRASSMANN BUNDLES AND DISTRIBUTIONS

1. The Grassmann bundle of a Riemannian manifold

Let (M,g) be a Riemannian manifold of dimension m. The bundle of orthogonal frames of M, which has the orthogonal group $O(m)$ as a structure group, is characterized by the $\mathbb{R}^m$-valued canonical form $\theta = (\theta^i)$ and the $\mathfrak{o}(m)$-valued 1-form $\omega = (\omega^i_j)$ determined by the Levi-Civita connection.

Denoting by R_a right translation on $O(m)$ determined by an element a of $O(m)$, we have

$$(R_a^*\theta)^i = (a^{-1})^i_h \theta^h, \tag{1}$$

$$(R_a^*\omega)^i_j = (a^{-1})^i_h \omega^h_k a^k_j. \tag{2}$$

Furthermore

$$d\theta^i = -\omega^i_j \wedge \theta^j \qquad (\omega^i_j + \omega^j_i = 0), \tag{3}$$

$$d\omega^i_j = -\omega^i_k \wedge \omega^k_j + \frac{1}{2} R^M_{ijhk} \theta^h \wedge \theta^k, \tag{4}$$

where R^M_{ijhk} are the curvature functions on $O(M)$ associated to the Riemannian curvature tensor R^M of g; i.e.,

(5) $$R^M_{ijhk}(u) = R^M(u_i, u_j, u_h, u_k) = ((\nabla_{[u_i,u_j]} - \nabla_{u_i}\nabla_{u_j} + \nabla_{u_j}\nabla_{u_i})u_h, u_k),$$

$u = (x, u_1, \dots u_m)$ is an element of $O(M)$.

DEFINITION 1. *The Grassmann bundle of p-planes in the tangent spaces of M is the bundle on M associated to $O(M)$ with fibre the Grassmannian of p-planes in $\mathbb{R}^m$:*

$$G_p(m) = \frac{O(m)}{O(p) \times O(m-p)}.$$

In other words,

(6) $$G_p(TM) = O(M) \times_{O(M)} G_p(m).$$

The bundle $G_p(TM)$ can be defined in the following equivalent way (we refer the reader to [13, vol. I, Prop. 5.5, p. 57]):

(7) $$G_p(TM) = \frac{O(M)}{O(p) \times O(m-p)},$$

where $O(M)$ is a principal bundle over $G_p(TM)$ with structure group $O(p) \times O(m-p)$ identified with a subgroup of $O(m)$ as follows:

$$(a_1, a_2) \in O(p) \times O(m-p) \ \mapsto \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} \in \ O(m).$$

From now on, we shall exploit the representation of $G_p(TM)$ defined by (6).

The canonical projection $\psi : O(M) \to G_p(TM)$ is given by

$$\psi(u) = [u_1, \dots, u_p]_x$$

where $u = (x, u_1, \dots, u_p, u_{p+1}, \dots, u_m) \in O(M)$ and $[u_1, \dots, u_p]_x$ denotes the subspace of T_xM generated by the orthonormal vectors $u_1, \dots, u_p$.

Consider on $O(M)$ the quadratic semidefinite positive form

(8) $$Q = \sum(\theta^i)^2 + \lambda^2 \sum(\omega^a_r)^2$$

with $r = 1, \dots, p$, $a = p+1, \dots, m$, and λ an arbitrary real positive constant. The following facts are well known:

(i) The quadratic form Q is $O(p) \times O(m-p)$-invariant: this follows directly from (1) and (2) with $a = (a_1, a_2) \in O(p) \times O(m-p)$.

(ii) The bilinear form on M associated to Q, i.e.,

$$Q(X,Y) = \sum \theta^i(X)\theta^i(Y) + \lambda^2 \sum \omega^a_r(X)\omega^a_r(Y),$$

vanishes if and only if X or Y are tangent to the fibres of the submersion $\psi : O(M) \to G_p(TM)$.

For this reason (see also [17]), as the rank of the form Q is $m+p(m-p)$ and equals to the dimension of $G_p(TM)$, there exists a unique Riemannian metric ds^2_λ on $G_p(TM)$ such that:

$$\psi^* ds^2_\lambda = Q.$$

In a sequel to this article, we shall consider $G_p(TM)$ endowed with the Riemannian matric ds^2_λ defined by Jensen–Rigoli in [11].

Observe that if we consider on $O(M)$ the Riemannian metric

$$\tilde{g} = \sum(\theta^i)^2 + \frac{1}{2}\lambda^2 \sum(\omega^i_j)^2,$$

one has that ψ is a Riemannian submersion with totally geodesic fibres of $(O(M),\tilde{g})$ over $(G_p(TM), ds^2_\lambda)$.

Let U denote an open set of $G_p(TM)$ and $\sigma : U \to O(M)$ a local section of the bundle $O(M) \xrightarrow{\psi} G_p(TM)$. Thus σ associates to each p-dimensional subspace $[\pi] \subset T_xM$ an orthonormal basis in $x \in M$ such that its first p vectors belong to $[\pi]$.

The $m+p(m-p)$ 1-forms

$$\rho^i = \sigma^*\theta^i, \qquad \rho^{ar} = \lambda\sigma^*\omega^a_r \tag{9}$$

yield an orthonormal coframe on U with respect to the metric ds^2_λ. The forms associated to the Levi-Civita connection with respect to the frame in question are determined by the conditions:

$$\begin{aligned} d\rho^i &= -\rho^i_j \wedge \rho^j - \rho^i_{ar} \wedge \rho^{ar}, \\ d\rho^{ar} &= -\rho^{ar}_j \wedge \rho^j - \rho^{ar}_{bs} \wedge \rho^{bs}, \end{aligned}$$

imposing also skew-symmetry.

A standard computation using (3) and (4), leads to

$$\begin{cases} \rho^i_j = -\rho^j_i = \sigma^*\{\omega^i_j + \frac{1}{2}\lambda^2 R^M_{arji}\omega^a_r\} \\ \rho^i_{ar} = -\rho^{ar}_i = \sigma^*\{\frac{1}{2}\lambda R^M_{arji}\theta^j\} \\ \rho^{ar}_{bs} = -\rho^{bs}_{ar} = \sigma^*\{\delta^a_b\omega^r_s + \delta^r_s\omega^a_b\}. \end{cases} \tag{10}$$

Equation (8) implies also that the natural projection $\Gamma : (G_p(TM), ds^2_\lambda) \to (M,g)$ is a *Riemannian submersion* with *totally geodesic fibres*.

This property can be verified directly using (10). Indeed, let us denote by $\{E_i, E_{ar}\}$ the dual basis of the orthonormal coframe (9), so $\{E_{ar}\}$ is the basis of the vertical distribution V tangent to the fibres and $\{E_i\}$ the basis of the horizontal one H. We have:

$$(\nabla_{E_{bs}} E_{ar}, E_i) = \rho^i_{ar}(E_{bs}) = 0.$$

In the next sections the horizontal and the vertical component of a vector field X, tangent to $G_p(TM)$, will be denoted respectively by X^H and X^V, so

$$X = X^H + X^V.$$

Finally, with a suitable choice of the constants, each fibre of $G_p(TM)$ is isometric to $G_p(m)$ endowed with the canonical metric (we refer the reader to [13, vol. II, p. 272]). Indeed, if we consider

$$G_p(m) = \frac{O(m)}{H}, \qquad H = O(p) \times O(m-p),$$

and the decomposition of the Lie algebra

$$\mathfrak{o} = \mathfrak{h} + \mathfrak{m}$$

with

$$\mathfrak{m} = \begin{pmatrix} 0 & -X^T \\ X & 0 \end{pmatrix} \qquad X \in M(m-p, p, \mathbb{R}),$$

one has that $Ad(H)\mathfrak{m} \subset \mathfrak{m}$. The scalar product on $\mathfrak{m}$ obtained by restriction to $\mathfrak{m}$ of the inner product

$$(11) \qquad (A,B) = -\frac{1}{2}\lambda^2 Tr(AB)$$

on $\mathfrak{o}(m)$ defines a metric $d\bar{s}^2_\lambda$ on $G_p(m)$, invariant under the left action of $O(m)$ on $G_p(m)$. The choice of the same arbitrary positive constant λ in (8) and (11) implies that the isometry between $\mathbb{R}^m$ and T_xM, determined by an orthonormal frame in $x \in M$, extends to an isometry from $(G_p(m), d\bar{s}^2_\lambda)$ to $G_p(T_xM)$ (the fibre of $(G_p(TM), ds^2_\lambda)$ corresponding to x).

In particular, we have a *Riemannian product*:

$$(12) \qquad (G_p(T\mathbb{R}^m), ds^2_\lambda) \cong \mathbb{R}^m \times (G_p(m), d\bar{s}^2_\lambda).$$

Recall that by *Vilms' Theorem* [1, (9.59), p. 249], ds^2_λ is the unique Riemannian metric on $G_p(TM)$ for which the projection $\Gamma : (G_p(TM), ds^2_\lambda) \to (M, g)$ is a Riemannian submersion with completely geodesic fibres isometric to $(G_p(m), d\bar{s}^2_\lambda)$ and a horizontal distribution associated to the Levi-Civita connection.

REMARKS 1. The canonical map of $G_p(TM)$ to $G_{m-p}(TM)$ which associates to each p-plane in T_xM the orthogonal $(m-p)$-plane is an isometry (with the same choice of the constant λ). This follows from (8) exchanging the indices a and r.

2. The unit tangent bundle T_1M of M can be identified (see [17]) with

$$T_1M = \frac{O(M)}{O(m-1)}.$$

Its metric is determined by (8) with $p = 1$ and coincides with the Sasaki metric if we assume $\lambda = 1$. Let us denote by $G_1(TM)$ the quotient of T_1M with respect to the equivalence relation identifying opposite unit vectors.

3. Obviously M can be identified with $G_m(TM)$, and from (8) it follows that this identification is an isometry.

2. The curvature of a Grassmann bundle

We denote by $\rho^X = (\rho^i, \rho^{ar})$ the forms belonging to the orthonormal coframe (9) of $G_p(TM)$, with $E_X = (E_i, E_{ar})$ the dual basis and with ρ^X_Y the forms associated to the Levi-Civita connection of $G_p(TM)$ determined by (10).

Starting from the structure equations

$$d\rho^X_Y + \rho^X_Z \wedge \rho^Z_Y = \frac{1}{2} R^G_{XYZT} \rho^Z \wedge \rho^T \tag{13}$$

by an elementary computation one can determine the components of the curvature tensor R^G of $G_p(TM)$:

$$\begin{aligned}
R^G_{ijhk}(\pi) &= \{R^M_{ijhk} + \tfrac{1}{2}\lambda^2 R^M_{arji} R^M_{arhk} - \tfrac{1}{4}\lambda^2 R^M_{arhi} R^M_{arkj} + \tfrac{1}{4}\lambda^2 R^M_{arki} R^M_{arhj}\}(\sigma[\pi]),\\
R^G_{ijh(ar)}(\pi) &= \tfrac{1}{2}\lambda\{\nabla_h R^M_{arji}\}(\sigma[\pi]),\\
R^G_{ij(ar)(bs)}(\pi) &= \{R^M_{rsji}\delta_{ab} - R^M_{abji}\delta_{rs} + \tfrac{1}{4}\lambda^2 R^M_{arki} R^M_{bsjk} - \tfrac{1}{4}\lambda^2 R^M_{bski} R^M_{arjk}\}(\sigma[\pi]),\\
R^G_{i(ar)h(bs)}(\pi) &= \{\tfrac{1}{2}\lambda R^M_{srhi}\delta_{ab} - \tfrac{1}{2}\lambda R^M_{abhi}\delta_{rs} - \tfrac{1}{4}\lambda^2 R^M_{bsji} R^M_{arhj}\}(\sigma[\pi]),\\
R^G_{i(ar)(bs)(ct)}(\pi) &= 0,\\
R^G_{(ar)(bs)(ct)(du)}(\pi) &= \tfrac{1}{\lambda^2}\{\delta_{ab}\delta_{cd}(\delta_{rt}\delta_{su} - \delta_{ru}\delta_{st}) + \delta_{rs}\delta_{tu}(\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc})\}(\sigma[\pi]).
\end{aligned}$$

From these expressions we obtain the components of the Ricci tensor Ric^G of $G_p(TM)$:

$$\mathrm{Ric}^G_{ih}(\pi) = \left\{\mathrm{Ric}^M_{ih} + \frac{1}{2}\lambda^2 R^M_{arji} R^M_{arhj}\right\}(\sigma[\pi]), \tag{14}$$

$$\mathrm{Ric}^G_{i(ar)}(\pi) = -\frac{1}{2}\lambda \nabla_j R^M_{jiar}(\sigma[\pi]), \tag{15}$$

$$\mathrm{Ric}^G_{(ar)(bs)}(\pi) = \left\{\frac{m-2}{\lambda^2}\delta_{ab}\delta_{rs} + \frac{1}{4}\lambda^2 R^M_{arji} R^M_{bsji}\right\}(\sigma[\pi]), \tag{16}$$

where the notation (ar) etc. is used only to separate the indices.

From (15) follows that the horizontal and the vertical distribution are orthogonal with respect to Ric^G and thus H is a Yang–Mills distribution (see [1, p. 243–244]) if M has harmonic curvature.

In the sequel, we shall assume that M has constant sectional curvature c, i.e.,

$$R^M_{ijhk} = c(\delta_{ih}\delta_{jk} - \delta_{ik}\delta_{jh}),$$

and we will examine the conditions under which $G_p(TM)$ is Einstein. From (14) and

(16) follows that the non-zero components of the tensor Ric^G are:

$$\mathrm{Ric}^G_{st} = \{c(m-1) - \tfrac{1}{2}\lambda^2 c^2 (m-p)\}\delta_{st},$$

$$\mathrm{Ric}^G_{ab} = \{c(m-1) - \tfrac{1}{2}\lambda^2 c^2 p\}\delta_{ab},$$

$$\mathrm{Ric}^G_{(ar)(bs)} = \{\tfrac{m-2}{\lambda^2} + \tfrac{1}{2}\lambda^2 c^2\}\delta_{ab}\delta_{rs}.$$

These last equations imply immediately that $G_p(TM)$ is Einstein if and only if

$$m = 2p, \tag{17}$$

$$c^2\lambda^4(p+1) - 2c\lambda^2(2p-1) + 4p - 4 = 0. \tag{18}$$

Equation (18) is consistent if and only if $p = 1$ and either $c\lambda^2 = 0$ or $c\lambda^2 = 1$. So we have:

PROPOSITION 1. *If M is a Riemannian manifold with constant sectional curvature, its Grassmann bundle $G_p(TM)$ is Einstein if and only if M is a quotient of the plane or the sphere S^2 and $p = 1$.*

On the other hand, it is well-known (see [12]) that $T_1(S^2)$, of which $G^1(S^2)$ is obviously a quotient, is isometric to $\mathbb{RP}^3$.

3. Sections of the Grassmann bundle

A distribution D of rank p on M determines a section ϕ of the Grassmann bundle

$$G_p(TM) \xrightarrow{\Gamma} M$$

in a natural way. It therefore appears reasonable to seek a relationship between geometrical properties of the distribution D, and those of the map ϕ between the Riemannian manifolds (M,g) and $(G_p(TM), ds^2_\lambda)$.

Afterwards, we will determine the conditions under which ϕ is harmonic.

Let us consider, as in Section 1, a section σ of the bundle $O(M) \xrightarrow{\Psi} G_p(TM)$. The distribution D determines a section

$$\sigma\cdot\phi : M \to O(M),$$

which means

$$(\sigma\cdot\phi)(x) = (x, \overline{e}_1, \ldots, \overline{e}_m)$$

where $(\overline{e}_1, \ldots, \overline{e}_m)$ is an orthonormal frame of T_xM in which the first p elements belong to $D_x \subset T_xM$.

In relation to the orthonormal coframe (9) of $G_p(TM)$, we have

$$\phi^*\rho^i = \phi^*\sigma^*\theta^i = \overline{\omega}^i, \tag{19}$$

$$\phi^*\rho^{ar} = \lambda\phi^*\sigma^*\omega^a_r = \lambda\overline{\Gamma}^a_{jr}\overline{\omega}^j. \tag{20}$$

($i = 1,\dots,m$, $r = 1,\dots,p$, $a = p+1,\dots,m$), where $(\overline{\omega}^i)$ is the coframe dual to $(\overline{e}_i)$, and $\overline{\Gamma}^i_{jk}$ are the components of the Levi-Civita connection with respect to $(\overline{e}_i)$, i.e.,

$$\overline{\Gamma}^i_{jk} = (\nabla^M_{\overline{e}_j}\overline{e}_k, \overline{e}_i).$$

From (19) and (20), it immediately follows that

$$\phi^*(ds^2_\lambda) = \sum(\overline{\omega}^i)^2 + \lambda^2 \sum(\overline{\Gamma}^a_{jr}\overline{\omega}^j)^2.$$

Thus ϕ is an isometric immersion if and only if

$$\overline{\Gamma}^a_{jr} = 0,$$

or in other words ∇^M_X maps D into D for all $X \in TM$. Then we set

$$\rho^X = (\rho^i, \rho^{ar}), \qquad \phi^*(\rho^X) = a^X_j \overline{\omega}^j$$

and (19) and (20) directly imply that

$$a^i_j = \delta^i_j, \qquad a^{ar}_j = \lambda\overline{\Gamma}^a_{jr}.$$

We indicate the tension field of ϕ by

$$\tau(\phi) = \tau^i(\phi)E_i + \tau^{ar}(\phi)E_{ar},$$

with

$$\tau^H(\phi) = \tau^i(\phi)E_i, \qquad \tau^V = \tau^{ar}(\phi)E_{ar};$$

its components are determined following the method described in Appendix B, exploiting in particular (10). A simple computation leads to

$$\tau^i(\phi) = \lambda^2 R^M_{arji}\overline{\Gamma}^a_{jr}, \tag{21}$$

$$\tau^{ar}(\phi) = \lambda\{\overline{e}_j(\overline{\Gamma}^a_{jr}) - \overline{\Gamma}^a_{hr}\overline{\Gamma}^h_{jj} - \overline{\Gamma}^a_{js}\overline{\Gamma}^s_{jr} + \overline{\Gamma}^a_{jb}\overline{\Gamma}^b_{jr}\}. \tag{22}$$

From these relations we observe that if $\overline{\Gamma}^a_{jr} = 0$, i.e., $\nabla^M_X D \subseteq D$. The map ϕ, being isometric, is also harmonic and thus minimal.

The following section will give several examples of harmonic maps from M into $G_p(TM)$ which are non-trivial in the sense that they correspond to distributions that are not parallel. It is important to keep in mind:

PROPOSITION 2. *If the map $\phi : M \to G_p(TM)$ determines a harmonic distribution D, then the map $\phi^\perp : M \to G_{m-p}(TM)$ determined by the distribution $D^\perp$ is also harmonic.*

This result follows directly from (21) and (22) exchanging the role of the indices $a, b = p+1,\dots,m$ with $r, s = 1,\dots,p$.

The condition $\tau^{ar}(\phi)=0$ on its own characterizes the *vertically harmonic distributions* studied by C.M. Wood [25]. With a simple computation one can prove that the vanishing of (22) is equivalent (see [25], Theorem 1.11) to

$$\bar{\nabla}^*\bar{\nabla}d^{\perp}|D=0 \quad (=\bar{\nabla}^*\bar{\nabla}d|D^{\perp}),$$

where d and $d^{\perp}$ are respectively the projections on D and $D^{\perp}$, and $\bar{\nabla}$ is the connection determined over the vector bundles D and $D^{\perp}$ by the Levi-Civita connection on M, i.e.,

$$\bar{\nabla}_X v=\begin{cases} d(\nabla^M_X v) & \text{if } v\in D,\\ d^{\perp}(\nabla^M_X v) & \text{if } v\in D^{\perp}.\end{cases}$$

4. Examples of distributions with harmonic map into the Grassmann bundle

Example 1. The sphere S^3.

Consider the sphere S^3 in $\mathbb{R}^4$ given by

$$x_1^2+x_2^2+x_3^2+x_4^2=1,$$

and the orthonormal basis of S^3 formed by the vectors

$$e_1=(-x_2,x_1,x_4,-x_3),\quad e_2=(-x_3,-x_4,x_1,x_2),\quad e_3=(-x_4,x_3,-x_2,x_1).$$

Denoting by $(\omega^1,\omega^2,\omega^3)$ the dual basis, and by (ω^i_j) the matrix of the Levi-Civita connection, we easily obtain

$$\omega_1^2=\omega^3,\quad \omega_1^3=-\omega^2,\quad \omega_2^3=\omega^1,$$

exploiting mainly the fact that

$$[e_1,e_2]=2e_3,\quad [e_3,e_1]=2e_2,\quad [e_2,e_3]=2e_1.$$

Referring to (21) and (22), we have:

– the one-dimensional distributions determined by e_1,e_2,e_3 respectively are (non-trivial) harmonic sections of $G_1(TS^3)$;

– the two-dimensional distributions $\{e_1,e_2\}$, $\{e_1,e_3\}$, $\{e_2,e_3\}$ determine harmonic sections of $G_2(TS^3)$, in accordance with Proposition 2.

As S^3 can be identified with the group $Sp(1)$ of unit quaternions, it is easy to prove that e_1,e_2,e_3 form a basis of left-invariant vector fields. The metric of S^3 is bi-invariant and the Levi-Civita connection is given by

$$\nabla_X Y=\frac{1}{2}[X,Y],$$

where X and Y are left-invariant vector fields. Furthermore, it is easy to prove that every unit left-invariant vector field u, and so the two-dimensional distribution orthogonal to u, determines a harmonic section of $G_1(TS^3)$ and one of $G_2(TS^3)$.

Example 2. The three-dimensional Heisenberg group.

Let us consider the Heisenberg group (we refer the reader to [23, p. 72–74] for example), i.e., the subgroup of $GL(3,\mathbb{R})$ formed by the matrices

$$\begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix}, \tag{23}$$

endowed with the left-invariant metric

$$g = dx^2 + dz^2 + (dy - xdz)^2. \tag{24}$$

Considering the orthonormal coframe

$$\omega^1 = dx, \qquad \omega^2 = dz, \qquad \omega^3 = dy - xdz \tag{25}$$

with dual frame

$$e_1 = \frac{\partial}{\partial X}, \qquad e_2 = \frac{\partial}{\partial Z} + X\frac{\partial}{\partial Y}, \qquad e_3 = \frac{\partial}{\partial Y} \tag{26}$$

we easily obtain:

$$[e_1, e_2] = e_3, \qquad [e_2, e_3] = [e_3, e_1] = 0.$$

The connection forms of the Levi-Civita connection are

$$\omega_1^2 = -\frac{1}{2}\omega^3, \quad \omega_1^3 = -\frac{1}{2}\omega^2, \quad \omega_2^3 = \frac{1}{2}\omega^1$$

and the non-vanishing components of the curvature tensor are

$$R_{1212} = -\frac{3}{4}, \quad R_{1313} = R_{2323} = \frac{1}{4}. \tag{27}$$

Comparing with (21) and (22) we can prove that

– the one-dimensional distributions determined by e_1, e_2, e_3 and their orthogonal complements induce harmonic maps from H to $G_1(TH)$ and $G_2(TH)$;

– (with some computations) the only left-invariant unit vector fields that determine harmonic sections of $G_1(TH)$ are $\pm e_3$ and all the unit vectors of the plane $\{e_1, e_2\}$ (a situation quite different from the case of S^3).

Observe that $\{e_1, e_2\}$ describes a contact distribution on H that has been extensively studied for its remarkable geometric properties (we refer the reader to [15] and [20]).

Example 3. Three-dimensional unimodular Lie groups.

The groups S^3 and H are examples of three-dimensional unimodular Lie groups. The classification of these groups has been provided by J. Milnor [16]. For such a group G there exists a basis of left-invariant vector fields $\{e_1, e_2, e_3\}$ such that:

$$[e_2, e_3] = \lambda_1 e_1, \quad [e_3, e_1] = \lambda_2 e_2, \quad [e_1, e_2] = \lambda_3 e_3.$$

Considering on G a left-invariant metric with respect to which $\{e_1, e_2, e_3\}$ is an orthonormal basis and denoting by $\{\omega^1, \omega^3, \omega^3\}$ the dual basis, it is easy to prove that the Levi-Civita connection forms are

$$\omega_1^2 = \tfrac{1}{2}(\lambda_1 + \lambda_2 - \lambda_3)\omega^3,$$
$$\omega_1^3 = \tfrac{1}{2}(-\lambda_1 + \lambda_2 - \lambda_3)\omega^2,$$
$$\omega_2^3 = \tfrac{1}{2}(\lambda_2 + \lambda_3 - \lambda_1)\omega^1 .$$

Some computations determine the non-zero components of the curvature tensor

$$R_{ijij} = \frac{1}{4}\{\lambda_i^2 + \lambda_j^2 - 3\lambda_k^2 + 2\lambda_i\lambda_k + 2\lambda_j\lambda_k - 2\lambda_i\lambda_j\},$$

where $i \neq j \neq k$ assume the values $1, 2, 3$.
Referring to (21) and (22), it is easy to prove that e_1, e_2, e_3 (and the corresponding orthogonal distributions) determine harmonic maps from G to $G_1(TG)$ and $G_2(TG)$.

Example 4. Certainly the three-dimensional unimodular Lie groups do not exhaust the examples of groups with left-invariant metrics admitting harmonic distributions.

This is the case of a four-dimensional group with orthonormal left-invariant basis e_1, e_3, e_3, e_4 such that

$$[e_1, e_3] = e_4, \qquad [e_1, e_4] = -e_3$$

and all other commutators vanishing. It is easy to prove that this group is flat ($R = 0$) and the unique non-zero connection form is $\omega_3^4 = \omega^1$. We can verify for example that the two-dimensional distribution determined by $\{e_1, e_3\}$ is a non-parallel harmonic section of $G_2(TG)$.

II. ISOMETRIC IMMERSIONS, MAPS BETWEEN GRASSMANN BUNDLES

5. The map induced by an isometry between Grassmann bundles

A Riemannian immersion $f : M \to N$ induces in a natural way an immersion

$$F : G_p(TM) \to G_p(TN)$$

which associates to each p-plane tangent to M in a point x its image in $T_{f(x)}N$ via the differential of f. In the special case in which $p = m = \dim M$, F coincides with the Gauss map $\Upsilon : M \to G_m(TN)$.

We can define on $G_p(TN)$ (with $p \leq \dim M$) a metric $d\tilde{s}_\lambda^2$ in a way that is completely analogous to the one described in Section 1 in the case of $G_p(TM)$. The choice for the metric on $G_p(TN)$ with the same constant λ as on $G_p(TM)$ corresponds to rendering an isometry the inclusion of the fibre of $G_p(TM)$ into the fibre of $G_p(TN)$ relative to the same point $x \in M$. Recalling the discussion at the end of Section 1, this means that the immersion of $G_p(m)$ into $G_p(n)$ induced by the natural immersion of $\mathbb{R}^m$ in $\mathbb{R}^n$, namely $\mathbb{R}^m \to (\mathbb{R}^m, O) \subset \mathbb{R}^n$, is isometric.

Let us denote by $\tilde{\theta} = (\tilde{\theta}^A)$ and $\tilde{\omega} = (\tilde{\omega}^A_B)$ the $\mathbb{R}^n$-valued canonical form and the $\mathfrak{o}(n)$-valued form associated to the Levi-Civita connection defined on $O(N)$. Let $\tilde{\sigma}$ denote a section of the bundle

$$O(N) \xrightarrow{\tilde{\psi}} G_p(TN).$$

The Riemannian metric $d\tilde{s}^2_\lambda$ on $G_p(TN)$ is determined by the orthonormal coframe

$$\tilde{\rho}^A = \tilde{\sigma}^*\tilde{\theta}^A, \qquad \tilde{\rho}^{ar} = \lambda\tilde{\sigma}^*\tilde{\omega}^a_r, \qquad \tilde{\rho}^{\alpha r} = \lambda\tilde{\sigma}^*\tilde{\omega}^\alpha_r. \tag{28}$$

The indices involved in the previous equations for the entire second part of the present report will vary as follows:

$$\begin{aligned} &A,B,\ldots = 1,\ldots,n, \qquad i,j,\ldots = 1,\ldots,m, \qquad r,s,\ldots = 1,\ldots,p,\\ &a,b,\ldots = p+1,\ldots,m, \quad \alpha,\beta,\ldots = m+1,\ldots,n. \end{aligned}$$

In analogy to equation (10), the Levi-Civita connection forms for $(G_p(TN), d\tilde{s}^2_\lambda)$ are given by:

$$\begin{cases} \tilde{\rho}^A_B = -\tilde{\rho}^B_A = \tilde{\sigma}^*(\tilde{\omega}^A_B + \frac{1}{2}\lambda^2 R^N_{arBA}\tilde{\omega}^a_r + \frac{1}{2}\lambda^2 R^N_{\alpha rBA}\tilde{\omega}^\alpha_r)\\ \tilde{\rho}^A_{ar} = -\tilde{\rho}^{ar}_A = \tilde{\sigma}^*(\frac{1}{2}\lambda R^N_{arBA}\tilde{\theta}^B)\\ \tilde{\rho}^A_{\alpha r} = -\tilde{\rho}^{\alpha r}_A = \tilde{\sigma}^*(\frac{1}{2}\lambda R^N_{\alpha rBA}\tilde{\theta}^B)\\ \tilde{\rho}^{ar}_{bs} = -\tilde{\rho}^{bs}_{ar} = \tilde{\sigma}^*(\delta^a_b\tilde{\omega}^r_s + \delta^r_s\tilde{\omega}^a_b)\\ \tilde{\rho}^{ar}_{\beta s} = -\tilde{\rho}^{\beta s}_{ar} = \tilde{\sigma}^*(\delta^r_s\tilde{\omega}^a_\beta)\\ \tilde{\rho}^{\alpha r}_{\beta s} = -\tilde{\rho}^{\beta s}_{\alpha r} = \tilde{\sigma}^*(\delta^\alpha_\beta\tilde{\omega}^r_s + \delta^r_s\tilde{\omega}^\alpha_\beta). \end{cases} \tag{29}$$

Since we wish to explore the geometrical implications of an Riemannian immersion $f : M \to N$, we need to exploit the bundle of Darboux frames $O(N,M)$ along f (for more details see Appendix A).

It will be helpful to keep in mind the following diagram:

$$\begin{tikzcd}
O(M) \arrow[d, shift right, "\psi"'] & O(N,M) \arrow[l, "s"'] \arrow[r, "k"] \arrow[dl, shift left, "\eta"] & O(N) \arrow[d, shift left, "\tilde{\psi}"] \\
G_pTM \arrow[u, shift right, "\sigma"'] \arrow[ur, shift left, "\chi"] \arrow[rr, "F"] \arrow[d] & & G_p(TN) \arrow[u, shift left, "\tilde{\sigma}"] \arrow[d] \\
M \arrow[rr, "f"] & & N
\end{tikzcd}$$

Here, $\eta = \psi \circ s$ is the submersion which associates to each adapted orthonormal frame $u = (x, u_1, \ldots, u_p, u_{p+1}, \ldots, u_m, u_{m+1}, \ldots, u_n)$ the subspace $[u_1, \ldots, u_p] \subset T_xM$.

Let χ be a local section of the bundle

$$O(N,M) \xrightarrow{\eta} G_p(TM)$$

defined on an open subset U of $G_p(TM)$; it determines a local section σ of

$$O(M) \xrightarrow{\psi} G_p(TM)$$

such that

$$\sigma = s \circ \chi; \tag{30}$$

let then $\tilde{\sigma}$ denote a local section of $O(N) \xrightarrow{\tilde{\psi}} G_p(TN)$ such that

$$\tilde{\sigma} \circ F = k \circ \chi. \tag{31}$$

Consider

$$\tilde{\rho}^{\Sigma} = (\tilde{\rho}^A, \tilde{\rho}^{ar}, \tilde{\rho}^{\alpha r}), \qquad \rho^X = (\rho^i, \rho^{ar}),$$

the 1-forms corresponding to the orthonormal coframes of $G_p(TN)$ and $G_p(TM)$ as in (28) and (9), and set

$$F^* \tilde{\rho}^{\Sigma} = a^{\Sigma}_X \rho_X. \tag{32}$$

In the sequel we will denote by $\overline{\theta}^A$ and $\overline{\omega}^A_B$ the forms induced on $O(N,M)$ by the forms $\tilde{\theta}^A$ and $\tilde{\omega}^A_B$ defined on $O(N)$ via the injection k, i.e.,

$$\overline{\theta}^A = k^* \tilde{\theta}^A, \qquad \overline{\omega}^A_B = k^* \tilde{\omega}^A_B. \tag{33}$$

Using equation (31) we obtain for example

$$\chi^* \overline{\theta}^i = \chi^* k^* \tilde{\theta}^i = F^* \tilde{\sigma}^* \tilde{\theta}^i = F^* \tilde{\rho}^i,$$

and from (30) we obtain

$$\chi^*\overline{\theta^i} = \chi^* s^* \theta^i = \sigma^* \theta^i = \rho^i,$$

which implies

$$F^*\tilde{\rho}^i = \rho^i.$$

Analogously

$$F^*\tilde{\rho}^\alpha = 0, \qquad F^*\tilde{\rho}^{ar} = \rho^{ar}, \qquad F^*\tilde{\rho}^{\alpha r} = \lambda\chi^*(h^\alpha_{rj}\overline{\theta}^j) = \lambda(\chi^* h^\alpha_{rj}\rho^j),$$

where the functions

$$\chi^* h^\alpha_{rj} = h^\alpha_{rj} \cdot \chi \tag{34}$$

evaluated on an element $[\pi] \in G_p(TM)$, are the components of the second fundamental form of the immersion f with respect to the adapted frame $\chi([\pi])$.

It follows that the coefficients a^Σ_X in (32) are given by

$$\begin{cases} a^i_j = \delta^i_j, & a^i_{ar} = 0 \\ a^\alpha_j = 0, & a^\alpha_{ar} = 0 \\ a^{ar}_j, & a^{ar}_{bs} = \delta^a_b \delta^r_s \\ a^{\alpha r}_j = \lambda\chi^*(h^\alpha_{rj}), & a^{\alpha r}_{bs} = 0. \end{cases} \tag{35}$$

Equations (35) imply that

$$F^* d\tilde{s}^2_\lambda = \sum(\rho^i)^2 + \sum(\rho^{ar})^2 + \lambda^2\left(\sum h^\alpha_{ri} h^\alpha_{rj} \cdot \chi\right)\rho^i\rho^j. \tag{36}$$

Since the metric on $G_p(TM)$ is given by

$$ds^2_\lambda = \sum(\rho^i)^2 + \sum(\rho^{ar})^2, \tag{37}$$

the map F is an *isometric immersion* of $(G_p(TM), ds^2_\lambda)$ in $(G_p(TN), d\tilde{s}^2_\lambda)$ *if and only if* f is *totally geodesic* (i.e., $h = 0$).

Furthermore, if $p < m$, the forms (36) and (37) are proportional if and only if they coincide and this occurs only in the case $h = 0$.

If $p = m$, from (36) and (37) (in which the forms ρ^{ar} do not appear any more) occurs that that $F = \Upsilon$ is *conformal* if and only if there exists a function ℓ on M such that

$$\sum_{k=1}^m h^\alpha_{ik} k^\alpha_{jk} = \ell^2 \delta_{ij}$$

so, setting

$$L(X,Y) = \sum_{k=1}^m (h(u_k, X), h(u_k, Y)), \quad X, Y \in TM, \tag{38}$$

we have

$$L(X,Y) = \ell^2 g(X,Y). \tag{39}$$

Observe that equation (39) is independent from the frame and equivalent to:

$$L(X,Y)=0, \qquad X \perp Y. \tag{40}$$

Keeping in mind the Gauss equations (see (100)) we obtain

$$R^N(u_k,u_i,u_k,u_j)=\mathrm{Ric}^M(u_i,u_j)+L(u_i,u_j)-mH\cdot h(u_i,u_j). \tag{41}$$

The bilinear form $\tilde{Q}_\lambda$ associated to $\Upsilon^* d\tilde{s}^2_\lambda$ is therefore given by:

$$\tilde{Q}_\lambda(X,Y)=(X,Y)+\lambda^2\{mH\cdot h(X,Y)-\mathrm{Ric}^M(X,Y)+R^N(u_k,X,u_k,Y)\}, \tag{42}$$

with $X,Y\in TM$.

Equation (42), and the conditions implying that the Gauss map Υ is conformal, are developed in the article of Jensen–Rigoli that we have already cited and also in [24]. Hence we get an extension of the results obtained by Obata in [18] expressed by

THEOREM 4 (Obata). *Assume that N has constant sectional curvature and*

1. Υ is conformal,

2. M is Einstein,

3. M is pseudo-umbilical, i.e., $h(X,Y)\cdot H=h(X,Y)|H|^2$;

then two of the above conditions imply the third.

In the case in which M is a surface in $\mathbb{R}^3$ (and thus $\mathrm{Ric}^M(X,Y)=Kg(X,Y)$ where K is the Gaussian curvature) we obtain the classical result, as observed in the Introduction of this report:

The Gauss map $M\to S^2$ is conformal if either M is a minimal surface, or M is contained in a sphere. In fact, these conditions are equivalent to being pseudo-umbilical in the case of surfaces in $\mathbb{R}^3$.

6. Tension field of the map induced between Grassmann bundles

For the computation of the tension field of F, we exploit the method described in Appendix B, and we set

$$Da^\Sigma_X \equiv da^\Sigma_X - a^\Sigma_Y\rho^Y_X + a^\Omega_X F^*\tilde{\rho}^\Sigma_\Omega = a^\Sigma_{XY}\rho^Y,$$

where a^Σ_X, ρ^Y_X and $\tilde{\rho}^\Sigma_\Omega$ are given by (35), (10) and (29), respectively. Recalling (10), (30), (33), we obtain

$$\begin{cases} \rho^i_j=\chi^*\left\{\overline{\omega}^i_j+\frac{1}{2}\lambda^2 R^M_{arji}\overline{\omega}^a_r\right\} \\ \\ \rho^i_{ar}=\chi^*\left\{\frac{1}{2}\lambda R^M_{arji}\overline{\theta}^j\right\} \\ \\ \rho^{ar}_{bs}=\chi^*\left\{\delta^a_b\overline{\omega}^r_s+\delta^r_s\overline{\omega}^a_b\right\}. \end{cases} \tag{43}$$

Then (29), (31), (33) imply

$$(44)\qquad \begin{cases} F^*\tilde{\rho}^A_B = \chi^*\{\overline{\omega}^A_B + \frac{1}{2}\lambda^2 R^N_{arBA}\overline{\omega}^a_r + \frac{1}{2}\lambda^2 R^N_{\alpha rBA}\overline{\omega}^\alpha_r\} \\ F^*\tilde{\rho}^A_{ar} = \chi^*\{\frac{1}{2}\lambda R^N_{arjA}\overline{\theta}^j\} \\ F^*\tilde{\rho}^A_{\alpha r} = \chi^*\{\frac{1}{2}\lambda R^N_{\alpha rjA}\overline{\theta}^j\} \\ F^*\tilde{\rho}^{ar}_{bs} = \chi^*\{\delta^a_b\overline{\omega}^r_s + \delta^r_s\overline{\omega}^a_b\} \\ F^*\tilde{\rho}^{ar}_{\beta s} = \chi^*\{\delta^r_s\overline{\omega}^\alpha_\beta\} \\ F^*\tilde{\rho}^{\alpha r}_{\beta s} = \chi^*\{\delta^r_s\overline{\omega}^\alpha_\beta + \delta^\alpha_\beta\overline{\omega}^r_s\}. \end{cases}$$

The components of the tension field $\tau(F)$ with respect to the orthonormal basis $\tilde{E}_\Sigma = \{\tilde{E}_i, \tilde{E}_\alpha, \tilde{E}_{ar}\tilde{E}_{\alpha r}\}$ dual of the basis (28) of $G_p(TN)$ are given by

$$\tau^\Sigma(F) = a^\Sigma_{XX}$$

and so, using equations (35), (43) and (44) we get

$$(45)\qquad \tau^i(F) = \lambda^2\chi^*(R^N_{\alpha rji}h^\alpha_{rj})$$

$$(46)\qquad \tau^\alpha(F) = \chi^*(h^\alpha_{jj} + \lambda^2 R^N_{\beta rj\alpha}h^\beta_{rj})$$

$$(47)\qquad \tau^{ar}(F) = -\lambda\chi^*(h^\alpha_{rj}h^\alpha_{aj})$$

$$(48)\qquad \tau^{\alpha r}(F) = (h^\alpha_{rjj}).$$

The computations leading to the previous equations are simple except for case (48), which we display explicitly:

$$(49)\qquad \begin{aligned} Da^{\alpha r}_j = {} & da^{\alpha r}_j - a^{\alpha r}_i\rho^i_j + a^i_j F^*\tilde{\rho}^{\alpha r}_i + a^{\beta s}_j F^*\tilde{\rho}^{\alpha r}_{\beta s} \\ = {} & \lambda\chi^*(dh^\alpha_{rj}) - \lambda\chi^*(h^\alpha_{ri})\chi^*(\overline{\omega}^i_j + \tfrac{1}{2}\lambda^2 R^M_{arji}\overline{\omega}^a_r) - \tfrac{1}{2}\lambda\chi^*(R^N_{\alpha rkj}\overline{\theta}^k) \\ & + \lambda\chi^*(h^\beta_{sj})\chi^*(\delta^r_s\overline{\omega}^\alpha_\beta + \delta^\alpha_\beta\overline{\omega}^r_s) \\ = {} & a^{\alpha r}_{jk}\rho^k + a^{\alpha r}_{j(bs)\rho^{bs}}. \end{aligned}$$

$$(50)\qquad \begin{aligned} Da^{\alpha r}_{bs} = {} & da^{\alpha r}_{bs} - a^{\alpha r}_j\rho^j_{bs} + a^{ct}_{bs}F^*\tilde{\rho}^{\alpha r}_{ct} \\ = {} & -\lambda\chi^*(h^\alpha_{rj})\chi^*(\tfrac{1}{2}\lambda R^M_{bskj}\overline{\theta}^k) + \chi^*(\delta^r_s h^\alpha_{bk}\overline{\theta}^k) \\ = {} & a^{\alpha r}_{(bs)k}\rho^k + a^{\alpha r}_{(bs)(ct)}\rho^{ct}. \end{aligned}$$

From (50), we obtain

$$a^{\alpha r}_{(bs)(ct)} = 0, \qquad a^{\alpha r}_{(bs)k} = \chi^*(h^\alpha_{bk}\delta^r_s - \frac{1}{2}\lambda^2 R^M_{bskj}h^\alpha_{rj}).$$

Since

$$a^{\alpha r}_{(bs)k} = a^{\alpha r}_{k(bs)},$$

substituting in (49) yields

$$\lambda\chi^*(dh^{\alpha}_{rj}) - \lambda\chi^*(h^{\alpha}_{ri}\overline{\omega}^i_j) - \tfrac{1}{2}\lambda\chi^*(R^N_{\alpha rkj}\overline{\theta}^k) + \lambda\chi^*(h^{\beta}_{rj}\overline{\omega}^{\alpha}_{\beta} - h^{\alpha}_{sj}\overline{\omega}^s_r)$$
$$= a^{\alpha r}_{jk}\chi^*(\overline{\theta}^k) + \chi^*(h^{\alpha}_{\beta j})\delta^r_s\chi^*(\lambda\overline{\omega}^b_s),$$

from which we get

$$a^{\alpha r}_{jk}\chi^*(\overline{\theta}^k) = \lambda\chi^*\left\{dh^{\alpha}_{rj} - h^{\alpha}_{ri}\overline{\omega}^i_j - h^{\alpha}_{ij}\overline{\omega}^i_r + h^{\beta}_{rj}\overline{\omega}^{\alpha}_{\beta} - \frac{1}{2}R^N_{\alpha rkj}\overline{\theta}^k\right\}.$$

From equation (103), it follows that

$$a^{\alpha r}_{jk} = \lambda\chi^*\left\{h^{\alpha}_{rjk} - \frac{1}{2}R^N_{\alpha rkj}\overline{\theta}^k\right\},$$

taking the trace of which we obtain equation (48).

With particular attention to (109), we also have

$$\tau^{\alpha r}(F) = \lambda\chi^*(m\nabla^{\perp}_r H^{\alpha} - R^N_{jrj\alpha}). \tag{51}$$

The vanishing of all the components of $\tau(F)$ corresponds to the fact that F is harmonic. On the other hand, the vanishing of $\tau^{ar}(F)$ and $\tau^{\alpha r}(F)$ is equivalent to being F *vertically harmonic*.

We postpone the analysis of these conditions in the relevant case in which N has constant sectional curvature to the following section. Here we provide an example of a Gauss map ($p = m$) which highlights the role of the curvature of N.

From equations (44) – (48) it follows naturally that, if M is a totally geodesic submanifold of N, the map F is harmonic irrespective of the curvature of N. For this reason, we will not consider this trivial case in the sequel.

Example.

In the three-dimensional Heisenberg group G (recall Section 4, **2**), we consider the isometrically immersed surface defined by the equation $y = 0$. In other words, this is the submanifold consisting of matrices of the type

$$\begin{pmatrix} 1 & x & 0 \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

which is therefore generated by the product of two one-parameter subgroups of H.

We will show that S is a *minimal surface* in H whose *Gauss map* Υ *is conformal but neither harmonic nor vertically harmonic*.

Adopting the same notation as in Section 4, we consider the orthonormal basis $u = \{u_1, u_2, u_3\}$ of H defined as

$$u_1 = e_1, \qquad u_2 = \frac{1}{\sqrt{1+x^2}}(e_2 - xe_3), \qquad u_3 = \frac{1}{\sqrt{1+x^2}}(xe_2 + e_3).$$

It is easy to prove that the restriction $u|_S$ yields a Darboux frame with u_3 unit normal vector.

Denoting by h the second quadratic form of S in H, we get

$$h(u_1,u_1)\cdot u_3 = 0, \qquad h(u_1,u_2)\cdot u_3 = \frac{x^2-1}{2(x^2+1)}, \qquad h(u_2,u_2)\cdot u_3 = 0,$$

which implies that S is a minimal surface *not* totally geodesic of H.

The components of the curvature tensor R with respect to the frame u are

$$R_{1212} = \frac{x^2-3}{4(x^2+1)}, \quad R_{1213} = -\frac{x}{1+x^2}, \quad R_{1223} = 0,$$
$$R_{1313} = \frac{1-3x^2}{4(1+x^2)}, \qquad R_{1323} = 0, \qquad R_{2323} = \frac{1}{4},$$

where $R_{1212} = R(u_1,u_2,u_1,u_2)$ etc. Bearing in mind (45), (46) and (48), for the tension field of the Gauss map of S in $G_2(TH)$ we have

$$\tau^1(\Upsilon) = \lambda^2 \frac{x(1-x^2)}{2(1+x^2)^2}, \quad \tau^2(\Upsilon) = 0, \quad \tau^3(\Upsilon) = 0,$$
$$\tau^{3,1}(\Upsilon) = 0, \qquad \tau^{3,2}(\Upsilon) = \lambda \frac{x}{1+x^2},$$

which implies that Υ is neither harmonic nor vertically harmonic.

7. Harmonicity of the map between Grassmann bundles

We examine the different cases that can occur for the harmonicity of the map

$$F : (G_p(TM), ds^2_\lambda) \to G_p(TN), d\bar{s}^2_\lambda)$$

induced by a Riemannian immersion f of M in N. We will distinguish the case $p < m$ from $p = m$, and we will discuss with particular attention the case of constant sectional curvature on N.

Case 1. $p < m = \dim M$.

The vanishing of the components $\tau^{ar}(F)$ given in (47) corresponds to the fact that, for each couple of orthogonal vectors X, Y tangent to M, we have

$$L(X,Y) = \sum h(u_j, X)\cdot h(u_j, Y) = 0, \qquad (X \perp Y). \tag{52}$$

This condition is equivalent (via (40)) to the fact that the Gauss map $\Upsilon : M \to G_p(TN)$ is harmonic.

Suppose that N has constant sectional curvature c, and let us distinguish further the case $c = 0$ from $c \neq 0$.

Subcase 1.1. $p < m$, $R^N = 0$.

We still obtain $\tau^i(F) = 0$; the condition $\tau^\alpha(F) = 0$ is equivalent to $H = 0$, i.e., that f is a minimal immersion. Under such hypothesis we have also $\tau^{ar} = 0$. Observe that if $H = 0$ and $R^N = 0$, equation (41) implies that

$$\mathrm{Ric}^M = -L \tag{53}$$

and for this reason the condition $\tau^{ar} = 0$ can be expressed by one of the following equivalent conditions:

– M is Einstein;

– the Gauss map $\Upsilon : M \to G_m(TN)$ is conformal (and thus recalling (53), homothetic).

This fact motivates the following:

PROPOSITION 3. *If N is a flat space, the map $F : G_p(TM) \to G_p(TN)$ with $p < \dim M$ is harmonic if and only if the following conditions are satisfied:*

– f is a minimal immersion;

– M is Einstein or (equivalently) the Gauss map $\Upsilon : M \to G_m(TN)$ is conformal (homothetic if $\dim M > 2$).

Under the same hypothesis, F is vertically conformal if and only if

– the mean curvature vector is parallel;

– the Gauss map is conformal.

Subcase 1.2. $p < m$, N has constant sectional curvature $c \neq 0$.

In this case the condition $\tau^i(F) = 0$ is still identically satisfied, but we have

$$\tau^\alpha(F) = mH^\alpha - c\lambda^2 \sum_{r=1}^{p} h^\alpha_{rr}.$$

For this reason, the condition $\tau^\alpha = 0$ is satisfied (independently of the choice of frame) if and only if for each vector X tangent to M we have that

$$h(X,X) = \frac{m}{cp\lambda^2} H.$$

Thus

$$\lambda^2 = \frac{m}{pc} \tag{54}$$

$$h(X,X) = H, \quad \forall X, \ |X| = 1, \tag{55}$$

conditions leading to $c > 0$ and then

$$h(X,Y) = g(X,Y)H, \qquad X,Y \in TM. \tag{56}$$

Equation (56) means that M should be a *totally umbilical submanifold* of N. The conditions $\tau^{ar}(F) = 0$ and $\tau^{\alpha r}(F) = 0$ are then identically satisfied (indeed M has constant curvature and $\nabla^{\perp}_X H = 0$, see for example [3, p. 50–51]).

Obviously a choice of λ different from (54) implies that F is harmonic only if $h = 0$, which means that M is a totally geodesic submanifold of N.

In conclusion,

PROPOSITION 4. *If N is a manifold with constant positive sectional curvature, the map $F : G_p(TM) \to G_p(TN)$ (with $p < \dim M$) is harmonic if either M is a totally geodesic submanifold of N, or the following conditions hold:*

– *M is a totally umbilical submanifold of N;*
– *the choice of the constant λ for the metric of $G_p(TN)$ is the same as in* (54)*.*

As in the case in which N is flat, *the map F is vertically conformal if and only if:*

– *the Gauss map is conformal;*
– *the mean curvature vector H is parallel.*

REMARK 1. The results we have obtained are substantially independent of the rank $p < \dim M$ of the Grassmannian bundles (except for the choice of the constant λ according to equation (54)). In this way we obtain results completely analogous to those proved in [22] for the case of unit tangent bundles.

Case 2. $p = m$ and thus $F = \Upsilon$.

In such a case, there is no component $\tau^{ar}(F)$. Assuming that N has constant sectional curvature, we distinguish the following subcases:

Subcase 2.1. $p = m$, $R^N = 0$.

We get always $\tau^i(F) = 0$ and $\tau^{\alpha}(F) = 0$ only if M is minimal, from which follows also $\tau^{\alpha r}(F) = 0$. From these considerations we deduce

PROPOSITION 5. *If N is flat, the Gauss map $\Upsilon : M \to G_m(TN)$ is harmonic if and only if M is a minimal submanifold of N.*

REMARK 2. If $N = \mathbb{R}^n$ it turns out that

$$(G_m(TN, d\tilde{s}^2_\lambda)) \cong \mathbb{R}^n \times (G_m(n), d\bar{s}^2_\lambda),$$

(recall from Section 1), where $F = \Upsilon = (f, \overline{\Upsilon})$ and $\overline{\Upsilon} : M \to G_m(n)$ is the generalized Gauss map. The *vertical* harmonicity of Υ coincides with the harmonicity of $\overline{\Upsilon}$ and may be expressed (recall (48)) by the condition $\nabla^{\perp} H = 0$. In such a way we recover the result of Ruh–Vilms presented in [21].

Subcase 2.2. $p = m$, N has constant sectional curvature $c \neq 0$.

The condition $\tau^i(F) = 0$ is always satisfied, and (46) implies that

$$\tau^\alpha(F) = m(1 - \lambda^2 c)H^\alpha$$

so either $H = 0$ and consequently $\tau^{\alpha r} = 0$, or

$$\lambda^2 = \frac{1}{c}, \qquad \nabla^\perp H = 0. \tag{57}$$

In conclusion,

PROPOSITION 6. *If N has non-zero constant sectional curvature, the Gauss map $\Upsilon : M \to G_m(TN)$ is harmonic if one of the following conditions holds:*

– M is a minimal submanifold of N;

– M has parallel mean curvature vector, N has positive curvature c and the metric of $G_m(TN)$ is obtained by setting $\lambda^2 = 1/c$.

Furthermore Υ is vertically harmonic if the mean curvature vector is parallel.

REMARK 3. When N is a sphere $S^n(r)$ (and so $c = 1/r^2$), besides the Gauss map $\Upsilon : M \to G_m(TS^n(r))$, T. Ishihara in [10] and M. Obata in [18] analyse other types of Gauss mappings related to the immersion of of $S^n(r)$ in $\mathbb{R}^{n+1}$. We can therefore consider the following mappings:

(i) $\Upsilon_1 : M \to G_m(n+1)$,
which associates to each point $x \in M$ the m-dimensional subspace of $\mathbb{R}^n$ parallel to the tangent space of T_xM;

(ii) $\Upsilon_2 : M \to G_{m+1}(n+1)$,
which associates to $x \in M$ the subspace of $\mathbb{R}^{n+1}$ singled out by the space tangent to M and the unit vector X/r; this is exactly the Gauss map introduced by Obata.

Regarding the harmonicity of these maps we have
(i)$'$ Υ_1 is harmonic if and only if the mean curvature vector H^* of M in $\mathbb{R}^{n+1}$ is parallel, in accordance with the Ruh–Vilms Theorem. Since the mean curvature H of M in $S^n(r)$ is determined by:

$$H = H^* + \frac{x}{r^2},$$

H^* parallel is equivalent to H parallel and thus Υ_1 *harmonic is equivalent to* Υ *vertically harmonic*.

(ii)$'$ The map Υ_2 is harmonic if and only if M is a minimal submanifold of $S^n(r)$ (see Theorem 4.8 in [10] where the Υ_1 is denoted by g_3). In conclusion, Υ_2 harmonic implies that Υ is harmonic.

III. THE SPHERICAL GAUSS MAP

A Riemannian immersion $f : M \to N$ induces a map $\nu : T_1^{\perp}M \to T_1N$ between the unit normal bundle of M and the unit tangent bundle of N, defined by

$$\nu(x,v) = (f(x),v).$$

Jensen and Rigoli (see [11]) examine in particular the conditions under which ν is harmonic, exploiting a method analogous to the one adopted in [22] for the map induced by f on the unit tangent bundles.

In this part of the report, the analysis of the harmonic properties of ν will be developed using the *repère mobile* (moving frame) method that we have already adopted in the previous sections.

If $N = \mathbb{R}^n$, then $T_1N \cong \mathbb{R}^n \times S^{n-1}$ and the map from $T_1^{\perp}M$ to S^{n-1} coincides with the map defined by Chern–Lashof in [6] for the study of the total curvature.

8. Riemannian structure of the normal unit bundle of a submanifold

Our aim is to examine the spherical Gauss map $\nu : T_1^{\perp}M \to T_1N$,

$$\nu(x,v) = (f(x),v), \tag{58}$$

induced by the isometric immersion f of an m-dimensional submanifold M in an n-dimensional manifold N, so first of all we specify the metrics on $T_1^{\perp}M$ and T_1N.
Since

$$T_1N = \frac{O(N)}{O(n-1)},$$

its metric is the one already introduced in Section 5 for $G_p(TN)$ with $p = 1$, which we re-propose with a slight variation of the notation.

Let $\psi_n : O(N) \to T_1N$ denote the canonical submersion

$$\psi_n(y,u_1,\dots,u_n) = (y,u_n), \tag{59}$$

and define on T_1N the metric $d\tilde{s}^2_\lambda$ so that

$$\psi_n^*\tilde{s}^2_\lambda = \sum(\tilde{\theta}^A)^2 + \lambda^2\sum(\tilde{\omega}^a_n)^2, \tag{60}$$

where $A = 1,\dots,n$, $a = 1,\dots,n-1$, and $(\tilde{\theta}^A)$, $(\tilde{\omega}^A_B)$ denote as usual the canonical form and the Levi-Civita connection form on $O(N)$.

Given a local section $\tilde{\sigma}$ of the bundle $O(N) \xrightarrow{\psi_n} T_1N$, the 1-forms

$$\rho^A = \tilde{\sigma}^*\tilde{\theta}^A, \qquad \tilde{\rho}^{an} = \lambda\tilde{\sigma}^*\tilde{\omega}^a_n \tag{61}$$

give an orthonormal coframe of T_1N. The forms associated to the Levi-Civita connection of $(T_1N, d\tilde{s}^2_\lambda)$ computed with respect to this coframe are:

$$\text{(62)}\qquad \begin{cases} \tilde{\rho}^A_B = -\tilde{\rho}^B_A = \tilde{\sigma}^*\{\tilde{\omega}^A_B + \frac{1}{2}\lambda^2 R^N_{anBA}\tilde{\omega}^a_n\} \\ \tilde{\rho}^A_{an} = -\tilde{\rho}^{an}_A = \tilde{\sigma}^*\{\frac{1}{2}\lambda R^N_{anBA}\tilde{\theta}^B\} \\ \tilde{\rho}^{an}_{bn} = -\tilde{\rho}^{bn}_{an} = \tilde{\sigma}^*(\tilde{\omega}^a_b)\,. \end{cases}$$

To determine the metric on $T_1^\perp M$, we consider the submersion π_n of the bundle of Darboux frames $O(N,M)$ on $T_1^\perp M$ defined by

$$\text{(63)}\qquad \pi_n(x, u_1, \dots, u_m, u_{m+1}, \dots, u_n) = (x, u_n)\,.$$

Observe that the fibres of π_n are diffeomorphic to $O(m)\times O(n-m-1)$ immersed in $O(n)$ as follows:

$$\text{(64)}\quad (a', a'') \mapsto a = \begin{pmatrix} a' & 0 & 0 \\ 0 & a'' & 0 \\ 0 & 0 & 1 \end{pmatrix},\ a' = (a^i_k) \in O(m),\ a'' = (a^\alpha_\beta) \in O(n-m-1),$$

with $i, k = 1, \dots, m$ and $\alpha, \beta = m+1, \dots, n-1$.

Therefore

$$T_1^\perp M = \frac{O(N,M)}{O(m)\times O(n-m-1)}.$$

Denoting by κ the canonical immersion $O(N,M)\to O(N)$ and setting

$$\overline{\theta}^A = \kappa^*\tilde{\theta}^A, \qquad \overline{\omega}^A_B = \kappa^*\tilde{\omega}^A_B,$$

we consider on $O(N,M)$ the quadratic form

$$\text{(65)}\qquad \overline{Q} = \sum(\overline{\theta}^i)^2 + \lambda^2\sum(\overline{\omega}^\alpha_n)^2.$$

For each $a \in O(m)\times O(n-m)$ of type (64), we have

$$\text{(66)}\qquad R_a\cdot\kappa = \kappa\cdot R_a.$$

If a is of the form specified by (64) then

$$R^*_a\overline{\theta}^i = (a^{-1})^i_k\overline{\theta}^k, \qquad R^*_a\overline{\omega}^\alpha_n = (a^{-1})^\alpha_\beta\overline{\omega}^\beta_n,$$

and hence

(i) $\overline{Q}$ is invariant under the right action of $O(m)\times O(n-m-1)$ on $O(N,M)$.
(ii) $\overline{Q}$ is a semidefinite positive form on $O(N,M)$ of rank equal to $n-1$, the dimension of $T_1^\perp M$.
(iii) The bilinear form associated to $\overline{Q}$ annihilates the vertical vector fields of the submersion π_n.

For this reason, there exists unique Riemannian metric ds^2_λ on $T_1^\perp M$ such that

$$\pi_n^* ds^2_\lambda = \overline{Q},$$

and this is the metric on $T_1^\perp M$ that we will refer to in the sequel of the section. Given a local section χ of $O(N,M) \xrightarrow{\pi_n} T_1 M$, i.e.,

$$\chi(x,v) = (x, u_1, \dots, u_m, u_{m+1}, \dots, u_{n-1}, v),$$

the 1-forms

$$\rho^i = \chi^* \overline{\theta}^i, \qquad \rho^{\alpha n} = \lambda \chi^* \overline{\omega}^\alpha_n \tag{67}$$

constitute an orthonormal basis of $T_1^\perp M$. Since we have

$$\overline{\omega}^\alpha_i = \kappa^* \tilde{\omega}^\alpha_i = h^\alpha_{ij} \overline{\theta}^j, \qquad \overline{\omega}^n_i = h^n_{ij} \overline{\theta}^j, \tag{68}$$

a standard computation leads to the following expression of the Levi-Civita connection forms on $(T^\perp M, ds^2_\lambda)$ with respect to the orthonormal frame (67):

$$\begin{cases} \rho^i_j = -\rho^j_i = \chi^* \{ \overline{\omega}^i_j + \frac{1}{2}\lambda^2 (R^N_{\alpha n j i} + h^\alpha_{kj} h^n_{ki} - h^\alpha_{ki} h^n_{kj}) \overline{\omega}^\alpha_n \} \\ \rho^i_{\alpha n} = -\rho^{\alpha n}_i = \frac{1}{2}\lambda \chi^* \{ (R^N_{\alpha n j i} + h^\alpha_{kj} h^n_{ki} - h^\alpha_{ki} h^n_{kj}) \overline{\theta}^j \} \\ \rho^{\alpha n}_{\beta n} = -\rho^{\beta n}_{\alpha n} = \chi^* (\overline{\omega}^\alpha_\beta). \end{cases} \tag{69}$$

Also, considering the normal curvature tensor $R^\perp$, and referring to (101),

$$\begin{cases} \rho^i_j = -\rho^j_i = \chi^* \{ \overline{\omega}^i_j + \frac{1}{2}\lambda^2 R^\perp_{\alpha n j i} \overline{\omega}^\alpha_n \} \\ \rho^i_{\alpha n} = -\rho^{\alpha n}_i = \chi^* \{ \frac{1}{2}\lambda R^\perp_{\alpha n j i} \overline{\theta}^j \} \\ \rho^{\alpha n}_{\beta n} = -\rho^{\beta n}_{\alpha n} = \chi^* (\overline{\omega}^\alpha_\beta). \end{cases} \tag{70}$$

REMARK 4. If M is a hypersurface of N, equations (67) imply directly that $T_1 M$ is isometric to M.

9. The tension field of the spherical Gauss map

Assume that the section $\tilde{\sigma}$ of $O(N) \xrightarrow{\psi_n} T_1 N$ is chosen in a way that

$$\tilde{\sigma} \circ \nu = \kappa \circ \nu \tag{71}$$

and so we are in the situation described by the following diagram:

$$\begin{tikzcd} O(N,M) \arrow[r, "\kappa"] \arrow[d, shift right, "\pi_n"'] & O(N) \arrow[d, shift left, "\psi_n"] \\ T_1^{\perp}M \arrow[r, "\nu"] \arrow[u, shift right, "\chi"'] \arrow[d] & T_1N \arrow[u, shift left, "\tilde{\sigma}"] \arrow[d] \\ M \arrow[r, "f"] & N. \end{tikzcd}$$

Considering (71), the orthonormal coframes of T_1N and $T^{\perp}M$ defined by (61) and (67) satisfy

$$\begin{aligned} &\nu^*\tilde{\rho}^i = \rho^i, \qquad \nu^*\tilde{\rho}^\alpha = 0, \qquad \nu^*\tilde{\rho}^n = 0, \\ &\nu^*\tilde{\rho}^{in} = -\lambda\chi^*\overline{\omega}_i^n = \lambda\chi^*(h_{ij}^n)\rho^j, \\ &\nu^*\tilde{\rho}^i = \lambda\chi^*\overline{\omega}_n^\alpha = \rho^{\alpha n}. \end{aligned} \tag{72}$$

In other words, denoting the coframes of T_1N and $T^{\perp}M$ by $\tilde{\rho}^\Sigma = (\tilde{\rho}^i, \tilde{\rho}^\alpha, \tilde{\rho}^n, \tilde{\rho}^{in}, \tilde{\rho}^{\alpha n})$ and $\rho^X = (\rho^i, \rho^{\alpha n})$, we set

$$\nu^*\tilde{\rho}^\Sigma = a_X^\Sigma \rho^X. \tag{73}$$

It follows that

$$\begin{cases} a_j^i = \delta_j^i, & a_{\alpha n}^i = 0 \\ a_j^\alpha = 0, & a_{\beta n}^\alpha = 0 \\ a_j^n = 0, & a_{\alpha n}^n = 0 \\ a_j^{in} = -\lambda\chi^*(h_{ij}^n), & a_{\alpha n}^{in} = 0 \\ a_j^{\alpha n} = 0 & a_{\alpha n}^{in} = \delta_\beta^\alpha. \end{cases} \tag{74}$$

From the above relations, we deduce that

$$\nu^* d\tilde{s}_\lambda^2 = \sum(\rho^i)^2 + \sum(\rho^{\alpha n})^2 + \lambda^2 \sum(h_{ij}^n h_{ik}^n \cdot \chi)\rho^j\rho_k. \tag{75}$$

Since $ds_\lambda^2 = \Sigma(\rho^i)^2 + \Sigma(\rho^{\alpha n})^2$ is the metric on $T_1^{\perp}M$, it follows that:

(i) ν is an isometry only if $h = 0$, i.e., M is a totally geodesic submanifold of N;

(ii) if $\dim N - \dim M \geq 2$ and if $h \neq 0$ the metric $\nu^* d\tilde{s}_\lambda^2$ cannot be conformal or in particular homothetic to ds_λ^2;

(iii) if M is a hypersurface (in such a case we should not consider the forms $\rho^{\alpha n}$) the metrics $\nu^* d\tilde{s}_\lambda^2$ and ds_λ^2 are mutually conformal if and only if on M there exists a function ℓ such that

$$h_{ij}^n h_{ik}^n = \ell^2 \delta_{jk}, \tag{76}$$

which is the same condition as (39).

Equation (76) is equivalent to asserting that the absolute values of the principal curvatures of M are equal. This can be easily proved using an orthonormal frame on M diagonalising the matrix (h^n_{ij}).

The tension field of the map ν is determined as usual following the method described in Appendix B, by setting

$$Da^{\Sigma}_X \equiv da^{\Sigma}_X - a^{\Sigma}_Y \rho^Y_X + a^{\Omega}_X \nu^*(\tilde{\rho}^{\Sigma}_{\Omega}) = a^{\Sigma}_{XY}\rho^Y. \tag{77}$$

Thus

$$\tau^{\Sigma}(\nu) = a^{\Sigma}_{XX}, \tag{78}$$

where the coefficients a^{Σ}_X are given by (74), ρ^Y_X are the coefficients of the Levi-Civita connection on $T^{\perp}M$ (see (69)) and the forms $\nu^*(\tilde{\rho}^{\Sigma}_{\Omega})$ can be computed starting from (62) and using (71), (72).

Simple computations lead to

$$\tau^i(\nu) = -\lambda^2\chi^*(R^N_{knji}h^n_{kj}) \tag{79}$$

$$\tau^{\alpha}(\nu) = \chi^*(h^{\alpha}_{jj} - \lambda^2 R^N_{inj\alpha}h^n_{ij}) \tag{80}$$

$$\tau^n(\nu) = \chi^*(h^n_{jj} - \lambda^2 R^N_{injn}h^n_{ij}) \tag{81}$$

$$\tau^{in}(\nu) = -\lambda\chi^*(h^n_{ijj}) = -\lambda\chi^*(m\nabla^{\perp}_i H^n - R^N_{jijn}) \tag{82}$$

$$\tau^{\alpha n}(\nu) = -\lambda\chi^*(h^n_{ij}h^{\alpha}_{ij}). \tag{83}$$

The computation of the components of $\tau(\nu)$ is simple. The only slightly more complicated case is (82), which is treated in a manner analogous to (48).

10. Harmonicity of the spherical Gauss map

We now examine the conditions under which the spherical Gauss map ν is harmonic, devoting particular attention to the case in which N has constant sectional curvature.

With the intention of interpreting the vanishing of the components $\tau^{\alpha n}(\nu)$ given by (83) (which makes sense only if $n - m \geq 2$), we consider for each element $v \in T^{\perp}M$ the symmetric 2-form

$$h_v = h \cdot v,$$

or better

$$h_v(X,Y) = h(X,Y)\cdot v. \tag{84}$$

We then obtain $\tau^{\alpha n}(\nu) = 0$ if and only if

$$h_v \cdot h_w = \sum h_v(u_i,u_j)\cdot h_w(u_i,u_j) = 0, \qquad \forall v,w \in T^{\perp}_1 M,\ \ v \perp w, \tag{85}$$

for any orthonormal frame $\{u_i\}$ on M. There should therefore exist a function μ on M such that

$$h_v \cdot h_w = \mu^2 (v, w), \qquad \forall v, w \in T_1^\perp M, \tag{86}$$

which implies in particular that

$$\|h\|^2 = \sum h_{v_\alpha} \cdot h_{v_\alpha} = (n - m)\mu^2,$$

where $\{v_\alpha\}$ is an orthonormal basis of $T_1^\perp M$. In the language introduced in [11], equation (86) means that *the second fundamental form is conformal*.

Assuming that N has constant sectional curvature c, then the tangential component $\tau^i(\nu)$ always vanishes, while

$$\tau^\alpha(\nu) = mH^\alpha, \qquad \tau^n(\nu) = m(1 - \lambda^2 c)H^n, \tag{87}$$

so both these expressions should be zero in order for ν to be harmonic. For this reason if $n - m \geq 2$, then $H = 0$, from which it follows that $\tau^{\alpha n}(\nu) = 0$. In conclusion:

PROPOSITION 7. *Let N be a manifold with constant sectional curvature. The spherical Gauss map $\nu : T_1^\perp M \to T_1 N$ induced by the isometric immersion f of M in N, with* $\dim N - \dim M \geq 2$, *is harmonic if and only the following conditions hold:*

1. *f is minimal;*
2. *the second fundamental form is conformal.*

Example.

If M is a minimal surface in N, with $\dim N \geq 4$, an orthonormal frame of M can be chosen in a way that $h^\alpha_{ij} = 0$ for $\alpha > 4$. The minimality conditions reduce to

$$h^3_{11} + h^3_{22} = 0, \qquad h^4_{11} + h^4_{22} = 0,$$

and with a suitable choice of an orthonormal basis of M we can assume $h^3_{12} = 0$. The condition that the second fundamental form is conformal leads to

$$h^4_{11} = h^4_{22} = 0, \qquad (h^3_{11})^2 = (h^4_{12})^2.$$

This implies that M should be a *minimal isotropic surface*, i.e., $|h(X,X)| =$ constant for $|X| = 1$.

Otherwise, if M is a *hypersurface* inside the manifold N with constant curvature c, the components of $\tau(\nu)$ are determined by (79), (81) and (82). The components not identically zero are:

$$\tau(\nu) = m(1 - \lambda^2 c)H^n, \qquad \tau^{in}(\nu) = -\lambda m \nabla^\perp_i H^n,$$

and so the spherical Gauss map ν is harmonic if and only if

$$(1 - \lambda^2 c)H = 0, \qquad \nabla^\perp_i H = 0. \tag{88}$$

We conclude that:

1. either N is flat and M is a minimal hypersurface; or
2. N is not flat,

$$c > 0, \qquad \lambda^2 = \frac{1}{c}, \qquad \nabla_i^\perp H = 0,$$

and so M is a hypersurface with mean curvature vector with constant norm, a condition equivalent in such a case to $\nabla_i^\perp H = 0$.

REMARK 5. The spherical Gauss map ν associated to the Riemannian submersion $T_1 N \to N$, is called *vertically harmonic* if the vertical component of $\tau(\nu)$ vanishes, i.e.,

$$\tau^{in}(\nu) = 0, \qquad \tau^{\alpha n}(\nu) = 0.$$

Equations (82) and (83) imply therefore that under the hypothesis that N has constant sectional curvature, ν is vertically harmonic if the following conditions are satisfied:

1. the mean curvature vector is parallel;
2. the second fundamental form is conformal.

If M is a hypersurface, the only condition is $|H| = \text{constant}$.

Appendix A. Darboux frames

Consider a Riemannian immersion f of an m-dimensional manifold M in an n-dimensional manifold N. We denote by $O(M)$ and $O(N)$ the principal bundles of orthonormal frames on M and N, with structure groups $O(m)$, $O(n)$.

We denote by $\theta = (\theta^i)$ and $\omega = (\omega^i_j)$ $(i, j = 1, \ldots, m)$ the canonical $\mathbb{R}^m$-valued form on $O(M)$ and the $\mathfrak{o}(m)$-valued form associated to the Levi-Civita connection on M respectively. Then $\tilde\theta = (\tilde\theta^i)$ and $\tilde\omega = (\tilde\omega^i_j)$, $(A, B = 1, \ldots, n)$ denote the analogous forms on $O(N)$. Hence,

$$d\theta^i = -\omega^i_j \wedge \theta^j \qquad (\omega^i_j + \omega^j_i = 0), \tag{89}$$

$$d\omega^i_j = -\omega^i_k \wedge \omega^k_j + \tfrac{1}{2} R^M_{ijhk} \theta^h \theta^k. \tag{90}$$

Given an orthonormal frame $u = (x, u_1, \ldots, u_m)$ of $O(M)$, we have

$$R^M_{ijhk}(u) = R^M(u_i, u_j, u_h, u_k) = ((\nabla^M_{[u_i, u_j]} - \nabla^M_{u_i} \nabla^M_{u_j} + \nabla^M_{u_j} \nabla^M_{u_i}) u_h, u_k). \tag{91}$$

Similarly,

$$d\tilde\theta^A = -\tilde\omega^A_B \wedge \tilde\theta^B, \qquad (\tilde\omega^A_B + \tilde\omega^B_A = 0), \tag{92}$$

$$d\tilde\omega^A_B = -\tilde\omega^A_C \wedge \tilde\omega^B_C + \tfrac{1}{2} R^N_{ABCD} \tilde\theta^C \wedge \tilde\theta^D. \tag{93}$$

Identifying M with its image $f(M)$ in N, the bundle of Darboux frames $O(N, M)$ along f is the bundle on M defined as follows. An element

$$u = (x, u_1, \ldots, u_m, u_{m+1}, \ldots, u_n), \quad x = \delta(u),$$

of $O(N,M)$ (where δ is the canonical projection of $O(N,M)$ on M) is such that

$$u' = (x, u_1, \dots u_m), \qquad u'' = (x, u_{m+1}, \dots, u_n)$$

are orthonormal frames of respectively T_xM and $T_xM^\perp$ (the subspace of T_xN orthogonal to T_xM with respect to the metric of T_xN).

The structure group $O(m)\times O(n-m)$ of $O(N,M)$ is naturally immersed in $O(n)$ as follows:

$$(a', a'') \in O(m)\times O(n-m) \mapsto \begin{pmatrix} a' & 0 \\ 0 & a'' \end{pmatrix} \in O(n).$$

Let $s : O(N,M) \to O(M)$ and $\kappa : O(N,M) \to O(N)$ denote the submersion and the natural immersion defined by the diagram

$$\begin{array}{ccccc} O(M) & \xleftarrow{s} & O(N,M) & \xrightarrow{\kappa} & N \\ \Big\downarrow \pi & & \Big\downarrow \delta & & \Big\downarrow \tilde{\pi} \\ M & \longleftarrow & M = f(M) & \longrightarrow & N \end{array}$$

Observe that (we refer the reader to [13, vol. II, p. 3–4]):

(94) $$\kappa^*\tilde{\theta}^i = s^*\theta^i = \overline{\theta}^i, \qquad i = 1, \dots, m,$$

(95) $$\kappa^*\tilde{\theta}^\alpha = 0, \qquad \alpha = m+1, \dots, n.$$

If we set

$$\kappa^*(\tilde{\omega}^A_B) = \overline{\omega}^A_B,$$

then differentiating (94), we obtain

$$\overline{\omega}^i_j = \kappa^*\tilde{\omega}^i_j = s^*\omega^i_j.$$

The forms $(\omega', \omega'') = (\overline{\omega}^i_j, \overline{\omega}^\alpha_\beta)$, representing the $\mathfrak{o}(m)$ and the $\mathfrak{o}(n-m)$ components of $\kappa^*\tilde{\omega}$, define a connection on $O(N,M)$. Given a local section χ of

$$O(N,M) \xrightarrow{\delta} M,$$

i.e., a local field of frames

(96) $$\chi(x) = (x, e_1, \dots, e_m, e_{m+1}, \dots, e_n)$$

adapted along the immersion f, it obviously follows that

$$(\chi^*\overline{\omega}^i_j)(X) = (\nabla^N_X e_j, e_i) = (\nabla^M_X e_j, e_i)$$
$$(\chi^*\overline{\omega}^\alpha_\beta)(X) = (\nabla^N_X e_\beta, e_\alpha) = (\nabla^\perp_X e_\beta, e_\alpha),$$

where $\nabla^\perp$ is the connection on $T^\perp M$ singled out by $\omega'' = (\overline{\omega}^\alpha_\beta)$.

Differentiating equation (95) we have

$$0=\kappa^*(d\tilde\theta^\alpha)=-\kappa^*\tilde\omega^\alpha_i\wedge\kappa^*\tilde\theta^i=\overline{\omega}^\alpha_i\wedge\overline{\theta}^i,$$

from which *Cartan's Lemma* implies that

$$\overline{\omega}^\alpha_i=h^\alpha_{ij}\overline{\theta}^j,\qquad h^\alpha_{ij}=h^\alpha_{ji},$$

where h^α_{ij} are the components of the second fundamental form h of the immersion f of M in N, i.e.,

$$h^\alpha_{ij}(u)=(\nabla^N_{u_i}u_j,u_\alpha).$$

Differentiating the equations

(97) $$\kappa^*\tilde\omega^i_j=\overline{\omega}^i_j$$

(98) $$\kappa^*\tilde\omega^\alpha_\beta=\overline{\omega}^\alpha_\beta$$

(99) $$\kappa^*\tilde\omega^\alpha_i=h^\alpha_{ij}\overline{\theta}^j,$$

we obtain the *equations of Gauss*, *Ricci*, *Codazzi*. Indeed

$$\begin{aligned}\kappa^*d\tilde\omega^i_j &= \kappa^*(-\tilde\omega^i_k\wedge\tilde\omega^k_j-\tilde\omega^i_\alpha\wedge\tilde\omega^\alpha_j+\tfrac12 R^N_{ijAB}\tilde\theta^A\wedge\tilde\theta^B)\\ &= -\overline{\omega}^i_k\wedge\overline{\omega}^k_j+h^\alpha_{ih}h^\alpha_{jk}\overline{\theta}^h\wedge\overline{\theta}^k+\tfrac12 R^N_{ijhk}\overline{\theta}^h\wedge\overline{\theta}^k.\end{aligned}$$

From another point of view,

$$\begin{aligned}\kappa^*d\tilde\omega^i_j &= s^*d\overline{\omega}^i_j=s^*(-\omega^i_k\wedge\omega^k_j+\tfrac12 R^N_{ijhk}\theta^h\wedge\theta^k)\\ &= -\overline{\omega}^i_k\wedge\overline{\omega}^k_j+\tfrac12 R^N_{ijhk}\overline{\theta}^h\wedge\overline{\theta}^k.\end{aligned}$$

Comparing the last expressions we obtain the following (Gauss equations):

(100) $$R^M_{ijhk}=R^N_{ijhk}+h^\alpha_{ih}h^\alpha_{jk}-h^\alpha_{ik}h^\alpha_{jh}.$$

Similarly, from (98), we have

$$\begin{aligned}\kappa^*d\tilde\omega^\alpha_\beta &= \kappa^*(-\tilde\omega^\alpha_i\wedge\tilde\omega^i_\beta-\tilde\omega^\alpha_\gamma\wedge\tilde\omega^\gamma_\beta+\tfrac12 R^N_{\alpha\beta AB}\tilde\theta^A\wedge\tilde\theta^B)\\ &= h^\alpha_{ih}h^\beta_{jk}\overline{\theta}^h\wedge\overline{\theta}^k-\overline{\omega}^\alpha_\gamma\wedge\overline{\omega}^\gamma_\beta+\tfrac12 R^N_{\alpha\beta hk}\overline{\theta}^h\wedge\overline{\theta}^k.\end{aligned}$$

Then setting

$$d\overline{\omega}^\alpha_\beta=-\overline{\omega}^\alpha_\gamma\wedge\overline{\omega}^\gamma_\beta+\frac12 R^\perp_{\alpha\beta hk}\overline{\theta}^h\wedge\overline{\theta}^k,$$

where $R^\perp$ is the curvature tensor of the connection $\nabla^\perp$ on $T^\perp M\to M$, we find the Ricci equations

(101) $$R^\perp_{\alpha\beta hk}=R^N_{\alpha\beta hk}+h^\alpha_{ih}h^\beta_{ik}-h^\alpha_{ik}h^\beta_{ih}.$$

Finally, differentiating equation (99), we have

$$\begin{aligned}\kappa^* d\tilde{\omega}_i^\alpha &= \kappa^*(-\tilde{\omega}_h^\alpha \wedge \tilde{\omega}_i^h - \tilde{\omega}_\beta^\alpha \wedge \tilde{\omega}_i^\beta + \tfrac{1}{2} R^N_{\alpha iAB} \tilde{\theta}^A \wedge \tilde{\theta}^B) \\ &= -h^\alpha_{hk}\overline{\theta}^k \wedge \overline{\omega}_i^h - h^\beta_{ik}\overline{\omega}^\alpha_\beta \wedge \overline{\theta}^k + \tfrac{1}{2} R^N_{\alpha ijk}\overline{\theta}^h \wedge \overline{\theta}^k.\end{aligned}$$

From another point of view, we have

$$d(h^\alpha_{ij}\overline{\theta}^j) = dh^\alpha_{ij} \wedge \overline{\theta}^j - h^\alpha_{ih}\overline{\omega}^h_j \wedge \overline{\theta}^j.$$

Comparing the last equations we obtain:

$$(dh^\alpha_{ik} - h^\alpha_{ik}\overline{\omega}^h_k - h^\alpha_{hk}\overline{\omega}^h_i + h^\beta_{ik}\overline{\omega}^\alpha_\beta - \frac{1}{2}R^N_{\alpha ihk}\overline{\theta}^h) \wedge \overline{\theta}^k = 0.$$

Cartan's Lemma implies

$$dh^\alpha_{ik} - h^\alpha_{ik}\overline{\omega}^h_k - h^\alpha_{hk}\overline{\omega}^h_i + h^\beta_{ik}\overline{\omega}^\alpha_\beta - \frac{1}{2}R^N_{\alpha ihk}\overline{\theta}^h = A^\alpha_{ikj}\overline{\theta}^j,$$

(102) $$A^\alpha_{ikj} = A^\alpha_{ijk}.$$

If we set

(103) $$dh^\alpha_{ik} - h^\alpha_{ik}\overline{\omega}^h_k - h^\alpha_{hk}\overline{\omega}^h_i + h^\beta_{ik}\overline{\omega}^\alpha_\beta = h^\alpha_{ikj}\overline{\theta}^j,$$

with $h^\alpha_{ikj} = h^\alpha_{kij}$ then, because of the symmetry of the second fundamental form, we have

(104) $$h^\alpha_{ikj} = A^\alpha_{ikj} + \frac{1}{2}R^N_{\alpha ijk}.$$

The relations expressed by (102) and (104) lead to the *Codazzi equations*

(105) $$h^\alpha_{ikj} = h^\alpha_{ijk} + R^N_{\alpha ijk}.$$

In fact, the first term of (103) gives an expression of the covariant differential of the second fundamental form h, since (given a frame $u \in O(N,M)$), (103) implies that

(106) $$\begin{aligned}h^\alpha_{ikj}(u) &= (\nabla^\perp_{u_j}(h(u_i,u_k)) - h(\nabla^M_{u_j}u_i,u_k) - h(u_i,\nabla^M_{u_j}u_k),\ u_\alpha) \\ &= ((\overline{\nabla}_{u_j}h)(u_i,u_k),\ u_\alpha).\end{aligned}$$

The mean curvature vector H of the Riemannian immersion f is given by

(107) $$H = \frac{1}{m}h(u_i,u_i) = \frac{1}{m}h^\alpha_{ii}u_\alpha,$$

and together with equation (106), we obtain

$$h^\alpha_{iij} = m(\nabla^\perp_{u_j}H, u_\alpha) = m\nabla^\perp_{u_j}H^\alpha.$$

It follows that H is parallel (in the normal bundle) if and only if

$$h^{\alpha}_{iij} = 0. \tag{108}$$

Equation (105), upon setting $k = i$ and summing over i, implies that

$$h^{\alpha}_{iij} = h^{\alpha}_{iji} + R^N_{\alpha iji} = h^{\alpha}_{jii} + R^N_{\alpha iji}. \tag{109}$$

For this reason, *if N has constant sectional curvature*, the conditions

$$h^{\alpha}_{iij} = 0, \qquad h^{\alpha}_{jii} = 0, \qquad \nabla^{\perp} H = 0 \tag{110}$$

are equivalent.

Appendix B. The tension field of a map

Let (M,g) and (N,h) be Riemannian manifolds of respective dimensions m,n. Let $\{e_i\}$ and $\{e_A\}$ be local orthonormal local frames on M and N, and let $\{\theta^i\}$, $\{\theta^A\}$ be the corresponding dual coframes and $\{\omega^i_j\}$, $\{\omega^A_B\}$ the local forms reppresenting the Levi-Civita connections with respect to these local frames. We therefore have

$$\begin{aligned} d\theta^i &= -\omega^i_j \wedge \theta^j, \qquad & (\omega^i_j + \omega^j_i) &= 0, \\ \theta^i(Z) &= (X, e_i), \qquad & \omega^i_j(X) &= (\nabla^N_X e_i, e_j). \end{aligned} \tag{111}$$

Similarly,

$$\begin{aligned} d\theta^A &= -\omega^A_B \wedge \theta^B, \qquad & (\omega^A_B + \omega^B_A) &= 0, \\ \theta^A(Z) &= (Z, e_A), \qquad & \omega^A_B(Z) &= (\nabla^N_Z e_B, e_A). \end{aligned} \tag{112}$$

Consider a smooth map $f : M \to N$. Its differential df can be viewed either as a map from TM to TN determined by

$$df(x,X) = (f(x), df_x(X)), \qquad X \in T_x M,$$

or as a $f^{-1}(TN)$-valued 1-form on M.

Setting

$$df(e_i) = a^A_i e_A, \tag{113}$$

it turns out that

$$f^*\theta^A = a^A_i \theta^i. \tag{114}$$

Differentiating equations (114) we have

$$f^*(-\omega^A_B \wedge \theta^B) = da^A_j \wedge \theta^j - a^A_j \omega^i_j \wedge \theta^j.$$

Equivalently,

$$(da_j^A - a_i^A\omega_j^i + a_j^B f^*\omega_B^A)\wedge\theta^j = 0,$$

from which by *Cartan's Lemma* we obtain

$$Da_j^A \equiv da_j^A - a_i^A\omega_j^i + a_j^B f^*\omega_B^A = a_{jk}^A\theta^k, \tag{115}$$

with

$$a_{jk}^A = a_{kj}^A.$$

The functions a_{jk}^A are the components of the covariant differential Ddf, also called the second fundamental quadratic form of the map f; the differential Ddf may also be introduced as the $f^{-1}(TN)$-valued symmetric 2-form defined by

$$Ddf(X,Y) = (D_Y df)X = D_Y(dfX) - df\nabla_Y^M X, \tag{116}$$

where D is a metric connection on $f^{-1}(TN)\to M$ induced by the Levi-Civita connection on N.

It is easy to see that equations (115) and (116) are actually equivalent; indeed (116) implies

$$\begin{aligned} Ddf(e_j,e_k) &= D_{e_k}(a_j^A e_A) - df(\omega_j^i(e_k)e_i) \\ &= e_k(a_j^A)e_A + a_j^B\omega_B^A(dfe_k)e_A - \omega_j^i(e_k)a_i^A e_A \\ &= (da_j^A + a_j^B f^*\omega_B^A - a_i^A\omega_j^i)(e_k)e_A = a_{jk}^A e_A, \end{aligned}$$

with the same coefficients a_{jk}^A as in equation (115). It is obvious that the computation of the covariant differential Ddf according to (115) (put in evidence by Chern–Goldberg [5]), is particulary useful when orthonormal coframes on the manifolds are chosen.

The map $f: M\to N$ is called *totally geodesic* if $Ddf = 0$, equivalently if $a_{jk}^A = 0$.

The *tension field* τf is the trace of Ddf, i.e., the $f^{-1}(TN)$-valued field on M defined by

$$\tau(f) = Ddf(e_j,e_j) = a_{jj}^A e_A. \tag{117}$$

Furthermore the map f is called *harmonic* if $\tau f = 0$, i.e., $a_{jj}^A = 0$.

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