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RAMIFICATION OF QUATERNION ALGEBRAS OVER STABLE ELLIPTIC SURFACES

Abstract. The aim of this work is to study the ramification of quaternion algebras over the function field of a stable elliptic surface, in particular over the field of complex numbers. Over number fields there are criteria for the ramification of quaternion algebras such as the tame symbol formula. We study how this formula can be interpreted in a geometrical way, and how the ramification relates to the geometry of the surface. In particular, we consider stable complex elliptic surfaces that have four 2-torsion sections.

1. Introduction

Let X be an (absolutely) irreducible variety over a field k . Let BrX be the Brauer group of the variety X , that is, the group of equivalence classes of Azumaya algebras on X . We have various descriptions of this group. One of these is given by means of the étale cohomology of X : by [2, Th. 1.1.8], we have an injective homomorphism

$$BrX \longrightarrow H^2(X, \mathcal{O}_X^*)_{tors} = H^2_{\text{ét}}(X, \mathbb{G}_m).$$

We will denote $K := k(X)$ the function field of X , and BrK its Brauer group, that is, the group of equivalence classes of central simple K -algebras. As shown in [4, Ch. 4, §4 and §5], BrK is a torsion group and we have the following description of this group by means of Galois cohomology:

$$BrK \simeq \varinjlim H^2(Gal(L/K), L^*),$$

where the direct limit is taken over the family of finite Galois extensions L of K . For later use, we will denote $Br_n K$ the n -torsion part of BrK . We will be concerned with $Br_2 K$, and, in particular, with equivalence classes of quaternion algebras over K .

The Brauer groups of X and K are deeply related: if \mathcal{A} is an Azumaya algebra, then, by definition, its stalk \mathcal{A}_{η} at the generic point η of X is a central simple K -algebra. This gives a morphism

$$BrX \longrightarrow BrK, \quad [\mathcal{A}] \longrightarrow [\mathcal{A}_{\eta}],$$

which is injective when X is regular (see [1]). Thus, we can think of an Azumaya algebra on X as a central simple K -algebra which extends to an Azumaya algebra on X . We will make this statement precise in Section 3.

The problem is the following. Let us suppose that we are able to give an explicit element in BrK . Does this element extend to an Azumaya algebra over X ? If we have a positive answer to this question, is the Azumaya algebra obtained in this way non-trivial?

Our strategy will be the following. Let $\pi : X \rightarrow \mathbb{P}_{\mathbb{C}}^1$ be a complex elliptic surface. By [6, Ch. III, Rem. 3.1 and Prop. 3.8], the generic fiber E_X of π is an elliptic curve over the field $\mathbb{C}(t)$. In the next section, we describe a way to construct quaternion algebras over E_X . Then we study when these extend to Azumaya algebras over X .

2. The Brauer group of an elliptic curve

We recollect some basic facts about the 2-torsion of the Brauer group of an elliptic curve E over a field k . Let us suppose that the 2-torsion points of E are k -rational, and fix an isomorphism $E(k)[2] \simeq (\mathbb{Z}/2\mathbb{Z})^2$. Let

$$Br^0 E := \ker(BrE \xrightarrow{\varepsilon_0^*} Brk)$$

be the kernel of the restriction of an Azumaya algebra to the origin of E , where $\varepsilon_0 : Spec(k) \rightarrow E$ is the point $0 \in E$. We have:

PROPOSITION 1. *There is a canonical isomorphism $Br^0 E \simeq H^1(k, E)$.*

Proof. See [9, Lemma 2.1]. □

The Kummer sequence for the 2-torsion part of the elliptic curve E is

$$0 \longrightarrow E[2] \longrightarrow E \xrightarrow{\cdot^2} E \longrightarrow 0.$$

Using the previous proposition we get the following exact sequence:

$$0 \longrightarrow E(k)/2E(k) \xrightarrow{\delta} (k^*/(k^*)^2)^2 \xrightarrow{\gamma} Br_2^0 E \longrightarrow 0$$

(see [5, Ch. VIII, §2]).

Before going on, we fix some notation. A quaternion algebra over a field k is a four dimensional K -algebra A with center k , for which we can find elements $i, j, l \in A$ such that $A = k \oplus i \cdot k \oplus j \cdot k \oplus l \cdot k$, and such that $ij = -ji = l$. These elements being fixed, we denote A with the standard symbol $(a, b)_k$, where $a = i^2$ and $b = j^2$ are elements in k^* . When no confusion on this symbol is possible, we will drop k , and we will simply write (a, b) for the corresponding quaternion algebra. The same symbol will be used for its equivalence class in the Brauer group of k .

REMARK 1. If $a, b \in k^*$, the quaternion algebras $(a, b)_k$, $(b, a)_k$, $(a, -ab)_k$ are all isomorphic. Moreover, for any $c \in k^*$, $(a, bc^2)_k$ is equivalent (in Brk) to $(a, b)_k$, so that $(a, b)_k$, $(1/a, b)_k$, $(a, 1/b)_k$ and $(1/a, 1/b)_k$ are all equivalent in Brk . Finally, the quaternion algebra $(1, a)_k$ is trivial in Brk .

Now fix a Weierstrass equation for E :

$$y^2 = x(x - p)(x - q),$$

where $p, q \in k$, then $R = (0, 0)$, $P = (p, 0)$ are generators of $E(k)[2]$. Let also $Q = (q, 0)$.

PROPOSITION 2. Let $M \in E(k)/2E(k)$. Then $\delta(M) \in (k^*/(k^*)^2)^2$ is given by:

$$\delta(M) = \begin{cases} (x(M), x(M) - p) & \text{if } M \neq R, P \\ (q/p, -p) & \text{if } M = R \\ (p, (p-q)/p) & \text{if } M = P \\ (1, 1) & \text{if } M = O. \end{cases}$$

Let $f, g \in k^*/(k^*)^2$. Then $\gamma(f, g) = (x, f)_k \otimes (x-p, g)_k \in Br_2^0 E$.

Proof. For δ see [5, Ch. X, Prop. 1.4]. For γ see [9, Prop. 2.2]. Note that since we have chosen R and P as generators of $E(k)/2E(k)$, the equations of δ and γ are different from what is written in [9] (where the basis is given by P and Q). \square

Let $\pi : X \rightarrow \mathbb{P}_{\mathbb{C}}^1$ be a complex elliptic surface which has four 2-torsion sections s_R, s_P, s_Q and O . The generic fiber E_X of π has therefore Weierstrass equation:

$$y^2 = x(x-p)(x-q),$$

with $p, q \in \mathbb{C}(t)$, and we assume that s_R, s_P and s_Q correspond to the points $R = (0, 0)$, $P = (p, 0)$, $Q = (q, 0)$ (and, clearly, O corresponds to the origin of E_X). By Proposition 2, $(x-p, f)_{\mathbb{C}(t)}, (x-q, g)_{\mathbb{C}(t)}, (x, h)_{\mathbb{C}(t)} \in Br_2^0 E_X$ for every $f, g, h \in \mathbb{C}(t)$. For example, by Remark 1 we have that $(x-p, f)_{\mathbb{C}(t)} = \gamma(1, f)$. Since the function field K of the surface X is generated by the rational functions x, y, t , we see that $(x-p, f)_{\mathbb{C}(t)}$ (as well as the others) can be viewed as an element in BrK . These will be the quaternion algebras we will study.

3. The Brauer group of a smooth surface

In this section, X will be a smooth surface over an algebraically closed field k of characteristic zero. In the introduction, we saw that there is an injective homomorphism $BrX \xrightarrow{\rho} BrK$. We give a description of the elements in the image of ρ . The main result we will use is the following:

THEOREM 1. There is an exact sequence:

$$0 \longrightarrow BrX \xrightarrow{\rho} BrK \xrightarrow{\alpha} \bigoplus_{C \subseteq X} H^1(k(C), \mathbb{Q}/\mathbb{Z}),$$

where the direct sum is taken over the set of irreducible curves in X . The map α is defined in the following way: let η be the generic point of an irreducible curve C on X , $\mathcal{O}_{X,\eta} \subseteq K$ the local ring of X in η , and let D be a central division algebra over $\mathcal{O}_{X,\eta}$. Then $\alpha(D)$ will be the cyclic extension L of $k(C)$ associated to D .

Proof. See [1, Ch. 3, Th. 1], or [8, Lemma 4.1]. \square

Let $D = (a, b)$ be a quaternion algebra over K , so that a and b are rational functions on X . Let us denote $\alpha(D)_C$ the component of $\alpha(D)$ in $H^1(k(C), \mathbb{Q}/\mathbb{Z})$. By Theorem 1, there can only be a finite number of irreducible curves $C_1, \dots, C_n \subseteq X$ such that $\alpha(D)_{C_i} \neq 0$. The curve C_D given by the union of these C_i will be called *ramification curve* of D . By Theorem 1, D extends to an Azumaya algebra over X if and only if its ramification curve C_D is empty. But how to find this ramification curve?

Let V be the closed subset of X given by the curves on which a or b have zeroes or poles, and let U be its complement. Over this open subset we can define the following locally free sheaf $\mathcal{A}_{a,b}$: if $W \subseteq U$ is an open in U , we define

$$\mathcal{A}_{a,b}(W) := \mathcal{O}_X(W) \oplus i \cdot \mathcal{O}_X(W) \oplus j \cdot \mathcal{O}_X(W) \oplus l \cdot \mathcal{O}_X(W),$$

where $i^2 = a|_W$, $j^2 = b|_W$ and $ij = -ji = l$. Moreover, $\mathcal{O}_X(W)$ is the center of this algebra. Since $a|_W$ and $b|_W$ are regular functions without zero on W , $\mathcal{A}_{a,b}$ is a sheaf of \mathcal{O}_U -modules which has the structure of Azumaya algebra on U . Thus, $\mathcal{A}_{a,b}$ is the extension of the quaternion algebra (a, b) to U , and the ramification curve of (a, b) is contained in V . To determine this curve we have to study how $\mathcal{A}_{a,b}$ behaves on the irreducible components of V . We use the following well-known tame symbol formula and, for completeness sake, we give a proof:

PROPOSITION 3. *Let X be a smooth surface over an algebraically closed field k of characteristic 0, and let $(a, b) \in Br_2(k(X))$ be a quaternion algebra over $k(X)$. Let U and V be as above, and C be an irreducible component of V . Then $\mathcal{A}_{a,b}$ extends to an Azumaya algebra over $U \cup C$ if and only if the following rational function on C (called tame symbol):*

$$c = (-1)^{v(a)v(b)} a^{v(b)} b^{-v(a)},$$

is a square in $k(C)^$, where v is the valuation for the discrete valuation ring $\mathcal{O}_{X,\eta}$ at the generic point η of C .*

Proof. First, choose a function t such that C is given by $t = 0$ and $v(t) = 1$, so that $\mathcal{O}_{X,\eta}/(t) = k(C)$. Then we can write $a = \alpha t^{v(a)}$, $b = \beta t^{v(b)}$, where $v(\alpha) = v(\beta) = 0$. By Remark 1, it suffices to study only where a or b have zeroes, that is, we can take $v(a), v(b) \geq 0$. Since k is algebraically closed, c is a square in $k(C)^*$ if and only if $\alpha^{v(b)} \beta^{-v(a)}$ is a square. This is possible if one of the following is satisfied:

1. $v(a)$ and $v(b)$ are even,
2. $v(a)$ (resp. $v(b)$) is even and α (resp. β) is a square in $k(C)$,
3. $v(a)$ and $v(b)$ are odd, and $\alpha\beta$ is a square in $k(C)$.

The remaining case, namely $v(a)$ and $v(b)$ are odd and $\alpha\beta$ is not a square in $k(C)$, is studied in [1, Ch. 4, Prop. 2], where it is shown that (a, b) ramifies over C . So, we study the 3 cases listed above. Write $v(a) = 2m + e$, $v(b) = 2n + f$, where e and f can be 0 or 1. The strategy goes as follows: by purity, it suffices study the problem locally, so we take $p \in C$ and a neighborhood U' of p in X . We construct an Azumaya algebra \mathcal{D}

on U' and an isomorphism between $\mathcal{D}|_{U \cap U'}$ and $\mathcal{A}_{a,b}|_{U \cap U'}$. \mathcal{D} is the quaternion algebra $\mathcal{O}_{U'} \oplus i_1 \cdot \mathcal{O}_{U'} \oplus j_1 \cdot \mathcal{O}_{U'} \oplus i_1 j_1 \cdot \mathcal{O}_{U'}$, where i_1 and j_1 are sections of \mathcal{D} over U' such that $i_1 j_1 = -j_1 i_1$. Similarly, we have $\mathcal{A}_{a,b} = \mathcal{O}_U \oplus i \cdot \mathcal{O}_U \oplus j \cdot \mathcal{O}_U \oplus ij \cdot \mathcal{O}_U$ with $i^2 = a$ and $j^2 = b$. So let us study the 3 cases above:

1. If $e = f = 0$, we take $i_1 = t^{-m}i$, $j_1 = t^{-n}j$, and the isomorphism is given by $i_1 \mapsto i$, $j_1 \mapsto j$.
2. If $e = 0, f = 1$ we take $i_1 = t^{-m}i$, $j_1 = t^{-n}j$, so that $i_1^2 = \alpha$ is a square modulo t . We write $i_1^2 = -z^2 + ht$, $j_1^2 = \beta t$, where $v(h) \geq 0$. Now take $i_2 = i_1 - j_1 i_1$ and $j_2 = j_1$. In this way,

$$i_2^2 = i_1^2 + (j_1 i_1)^2 = (1 + j_1^2)i_1^2 = (1 + \beta t)(-z^2 + ht) = -z^2 + tw,$$

is unit, where $w = h - \beta z^2 + \beta ht$. Now take $i_3 = i_2$, $j_3 = t^{-1}(z - i_2)j_2$, so that $i_3^2 = -z^2 + tw$ and $j_3^2 = \beta w$. Since these two have valuation 0, we are done.

3. If $e = 1, f = 0$, we take $i_1 = t^{-n}j$, $j_1 = t^{-m}i$ and we go on as in the previous case.
4. If $e = 1, f = 1$, we take $i_1 = t^{-n-m-1}ij$, $j_1 = t^{-n}j$ and we go on as in the second case.

□

4. Criterion for ramification

In this section we give a criterion for the ramification of a quaternion algebra of the form $(x-p, f)$, $(x-q, g)$, or (x, h) over an elliptic surface $\pi : X \rightarrow \mathbb{P}_{\mathbb{C}}^1$ whose Weierstrass model $\tilde{\pi} : \tilde{X} \rightarrow \mathbb{P}_{\mathbb{C}}^1$ has equation:

$$y^2 = x(x-p)(x-q),$$

where $p, q \in \mathbb{C}[t]$. For simplicity's sake, we will only consider stable elliptic surfaces. We then need the following criterion, which establishes when X is stable: we study the fibers of $\tilde{\pi}$ and follow the Kodaira classification.

PROPOSITION 4. *Using the same notation as above, let $p(t) = t^a p_1(t)$ and $q = t^b q_1(t)$, with $p_1(0), q_1(0) \neq 0$. Moreover, let $p(t) - q(t) = t^n h(t)$, with $h(0) \neq 0$. The fiber $\pi^{-1}(0)$ is stable if and only if $ab = 0$. In this case, the fiber is of type*

1. I_{2a} if $a > 0, b = 0$;
2. I_{2b} if $a = 0, b > 0$;
3. I_{2n} if $a = b = 0$.

Proof. This can easily be done using the Weierstrass form of the equation of \tilde{X} , that is $y^2 = x^3 + Ax + B$, where A and B depend only on p and q . Since $\pi^{-1}(0)$ is stable if and only if $A(0)B(0) \neq 0$ (see [3, Lecture 2, §3 and Lecture 3, §2, §3]), writing out the explicit formulas for A and B , we see that the fiber is stable if and only if $ab = 0$. The discriminant of this Weierstrass equation is $\Delta(t) = t^{2a+2b+2n} p_1(t) q_1(t) h(t)$, and we get the three possibilities listed above (see again [3, Lecture 2, §3 and Lecture 3, §2, §3]). \square

We use Proposition 2 to study the ramification of the quaternion algebras of the form (x, f) , $(x - q, g)$ or $(x - p, h)$. If we change coordinates by $x - q \mapsto x$ or $x - p \mapsto x$, we can always get to a quaternion algebra of the form (x, f) .

We have the following proposition:

PROPOSITION 5. *Let $\pi : X \longrightarrow \mathbb{P}_{\mathbb{C}}^1$ be a stable elliptic surface, whose Weierstrass model $\tilde{\pi} : \tilde{X} \longrightarrow \mathbb{P}_{\mathbb{C}}^1$ is given by the equation $y^2 = x(x-p)(x-q)$, with $p, q \in \mathbb{C}[t]$, $p \neq q$ and $m = \max\{\deg(p), \deg(q)\}$ even. Let $f \in \mathbb{C}(t)$. The quaternion algebra (x, f) extends to an Azumaya algebra over X if and only if for every zero or pole $t_0 \in \mathbb{P}_{\mathbb{C}}^1$ of f , we have $p(t_0)q(t_0) \neq 0$ and $p(t_0) = q(t_0)$.*

Proof. Using Remark 1, it is clear that it suffices to show the proposition in the case of $f(t) = t$, so that t_0 is 0 or $\infty \in \mathbb{P}_{\mathbb{C}}^1$.

It is clear that (x, t) extends to an Azumaya algebra at least over the open subset U of \tilde{X} where x and t do not have any zero or pole: we blow up any singularity in U and apply Proposition 3 (here there is no zero nor pole of t or x). Next, we have to study what happens on $\tilde{X} \setminus U$, which is given by $x = 0$ and by the fibers of $\tilde{\pi}$ over 0 and $\infty \in \mathbb{P}^1$. We start by studying the fiber over 0.

We begin with the case when $p(0)q(0) \neq 0$ and $p(0) \neq q(0)$, so that the fiber of $\tilde{\pi}$ over 0 is smooth. Here, we don't need to blow up (at least near this fiber). We have $v(t) = 1$, $v(x) = 0$. By Proposition 3, we get that (x, t) extends to an Azumaya algebra over $U \cup \tilde{\pi}^{-1}(0)$ if and only if x is a square in the function field of $\tilde{\pi}^{-1}(0)$. Since the equation of the fiber is $y^2 = x(x-p(0))(x-q(0))$, with $p(0), q(0) \neq 0$ and $p(0) \neq q(0)$, we see that this is not the case, so that (x, t) ramifies.

As second case, we consider when $p(0)q(0) \neq 0$ and $p(0) = q(0)$, so $\tilde{\pi}^{-1}(0)$ has equation $y^2 = x(x-p(0))^2$. The point $P_0 = (p(0), 0, 0)$ is singular. Let $p(t) - q(t) = t^n h(t)$ with $h(0) \neq 0$. We change coordinates in order to get $P_0 = (0, 0, 0)$, so the equation becomes:

$$y = x(x+p)(x+p-q),$$

and the quaternion algebra is $(x+p, t)$. We blow up the surface \tilde{X} in P_0 . We shall write down explicit equations for the blow-up: we work in the subvariety of $\mathbb{A}^3 \times \mathbb{P}^2$ with coordinates $(x, y, t, (x_1 : y_1 : t_1))$ defined by the equations $xy_1 - x_1y = 0$, $xt_1 - x_1t = 0$ and $yt_1 - y_1t = 0$. Where $x_1 \neq 0$ we have $y = y_1x$ and $t = t_1x$, so that the equation becomes

$$y_1^2 = t_1^n h(t_1x)x^n + t_1^n h(t_1x)p(t_1x)x^{n-1} + x + p(t_1x),$$

smooth near the exceptional fiber. The quaternion algebra is $(x + p, t_1 x)$, and we study it over the curves given by $t_1 = 0$ and $x = 0$. If $t_1 = 0$, then $y_1^2 = x + p(0)$, so that $v(t_1) = 1$, $v(x + p(0)) = 0$ and $x + p$ is a square modulo t . If $x = 0$, then $v(x) = 1$, but $v(x + p(0)) = 0$ and $p(0)$ is a square. In both cases $(x + p, t_1 x)$ extends to Azumaya algebra on this curve.

Where $y_1 \neq 0$ we have $x = x_1 y$, $t = t_1 y$, quaternion algebra $(x_1 y + p, t_1 y)$ and equation $1 = x_1(x_1 y + p(t_1 y))(x_1 + t_1^n y^{n-1} h(t_1 y))$. Calculations similar to those we did before show that the quaternion algebra extends to Azumaya algebra also over $t_1 y = 0$.

There remains the component given by $t_1 \neq 0$, where we have $x = x_1 t$ and $y = y_1 t$. The quaternion algebra becomes $(x_1 t + p, t)$ and the equation is

$$y_1^2 = x_1(tx_1 + p)(x_1 + t^{n-1}h).$$

If $n = 1$, this surface is smooth, while if $n > 1$, it has a singular point in $(0, 0, 0)$. Let $n = 1$: if $t = 0$, we have $v(t) = 1$, $v(x_1 t + p) = 0$. By Proposition 3, $(x_1 t + p, t)$ extends to an Azumaya algebra over this curve if and only if $x_1 t + p$ is a square modulo t , that is if and only if $p(0)$ is a square in \mathbb{C} , which is the case. This shows that if $n = 1$, then (x, t) extends to an Azumaya algebra over $\tilde{\pi}^{-1}(0)$. There remains $n > 1$: we must blow up in $(0, 0, 0)$ till we get a smooth surface and analyze the quaternion algebra we obtain as we did for the case $n = 1$. By induction on n , we study the ramification of the quaternion algebra $(x t^{n-1} + p, t)$ over a surface of equation

$$y^2 = x(xt^{n-1} + p)(x + th).$$

With the same notation as before, if $x_1 \neq 0$ we get the quaternion algebra $(x^n t_1^{n-1} + p(t_1 x), t_1 x)$ and equation

$$y_1^2 = (t_1^{n-1} x^n + p(t_1 x))(1 + t_1 h(t_1 x)),$$

which describes again a smooth surface. As before, the quaternion algebra doesn't ramify over the curve given by $t_1 x = 0$. If $y_1 \neq 0$, we get the quaternion algebra $(x_1 t_1^{n-1} y^n + p(t_1 y), t_1 y)$ and the equation

$$1 = x_1(x_1 t_1^{n-1} y^n + p(t_1 y))(x_1 + t_1 h(t_1 y)),$$

thus we don't have any ramification. If $t_1 \neq 0$, we get the quaternion algebra $(x_1 t^n + p, t)$ and the equation

$$y_1^2 = x_1(x_1 t^n + p)(x_1 + h),$$

which describes a smooth surface, and we have no ramification.

Finally, we show that if $p(0)q(0) = 0$, then (x, t) ramifies over $t = 0$. We can suppose $q(0) = 0$ (and $p(0) \neq 0$ by Proposition 4), and we write $q(t) = t^b q_1(t)$, with $q_1(0) \neq 0$. We use the same notation as before for the blow-up.

Where $x_1 \neq 0$ we have $y = y_1 x$, $t = t_1 x$, quaternion algebra $(x, t_1 x)$ and equation

$$y_1^2 = (x - p(t_1 x))(1 - t_1^b q_1(t_1 x)x^{b-1})$$

which is smooth. Over the curve $t_1 = 0$ we have $v(t_1) = 1$, $v(x) = 0$ so that $(x, t_1 x)$ extends to an Azumaya algebra over this curve if and only if x is a square modulo t_1 . By the equation above, we get $x = y_1^2 + p(0)$, so that x is not a square modulo t_1 . Since this component does not change under further blow-ups, (x, t) ramifies over $\tilde{\pi}^{-1}(0)$.

Next, we study how (x, f) behaves on the fiber $\tilde{\pi}^{-1}(\infty)$. Since (x, t) is equivalent to $(x, 1/t)$, we can reduce to the previous case, but the equation is different: using $s = 1/t$, we get

$$y^2 = x(x - p(1/s))(x - q(1/s)).$$

Using $m = \max\{\deg(p), \deg(q)\}$, we have

$$s^{2m}y^2 = x(s^m x - p_1(s))(s^m x - p_2(s))$$

where $p_1(s) = s^m p(1/s)$, $q_1(s) = s^m q(1/s)$. Since m is even, we can define $Y = s^{3m/2}y$ and $X = s^m x$. Finally, we get the equation

$$Y^2 = X(X - p_1)(X - q_1),$$

and the quaternion algebra $(s^m X, s)$, which is equivalent to (X, s) since m is even. By the first part of the proof, we can study this quaternion algebra on the fiber of 0 and conclude as stated in the proposition.

In order to finish the proof, we shall study how (x, t) behaves over the curve $\{x = 0\}$. The only problem here can be given by singular points of the surface that lie on this curve. Let $t_0 \in \mathbb{P}^1$ be the image under $\tilde{\pi}$ of such a point, and assume $t_0 \neq 0, \infty$. We can change coordinates in order to get a quaternion algebra $(x, t + t_0)$ and the usual equation. We can use the same calculations as above for the blow-ups, and look to the transform of $x = 0$. In any case, we see that since $t_0 \neq 0$ is a square in \mathbb{C} , we do not have any ramification. \square

REMARK 2. As we saw in the proof, we often make use of the fact that we are over the field of complex numbers, as here any number is a square. We can modify the statement for any field k of characteristic 0 asking also that for every zero or pole t_0 of f , $p(t_0)$ and $q(t_0)$ are squares in k .

A natural question is whether (x, f) is non trivial in BrX . We have the following:

PROPOSITION 6. *Let $\pi : X \rightarrow \mathbb{P}^1$ be a stable complex elliptic surface whose Weierstrass model has equation $y^2 = x(x - p)(x - q)$, where $p, q \in \mathbb{C}[t]$, $p \neq q$. Assume that $rk(MW(X)) = 0$, where $MW(X)$ is the Mordell-Weil group of X . If $f \in \mathbb{C}(t)$ and $(x, f) \in BrX$ is trivial, then we are in one of the following three cases: f is a square in the function field of E_X ; $f = \lambda q$ and $q - p = \mu$; $f = -\lambda q$ and $p = -\mu$, where λ, μ are squares in the function field of E_X .*

Proof. We can argue as follows: if (x, f) is trivial in BrX , then its restriction to the general fiber E_X of π is trivial in BrE_X . By Proposition 2, $(x, f) = \gamma([f], 1)$, where $[f]$ is f modulo squares in the function field $\mathbb{C}(t)$ of E_X . It is now clear that if $(x, f) = 0$ in BrX , then there must be a $\mathbb{C}(t)$ -rational point M of E_X such that $\delta(M) = ([f], 1)$. Since

the Mordell-Weil group of X is torsion, the only possible M can be P , Q , R and the origin O of E_X . By Proposition 2, we have only this four possibilities:

1. f is a square in the function field of E_X ;
2. $q = \lambda f$ and $p = \mu(f - 1)$;
3. $q = -\lambda f$ and $p = -\mu$;
4. $p = \lambda f$ and $q = \mu(f - 1)$

with λ, μ squares in the function field of E_X . Since the surface is stable, by Proposition 4 we see that the last case is impossible. The other cases agree with the hypothesis $(x, f) \in BrX$. \square

We give two examples. The first one, due to Olivier Wittenberg, is an elliptic K3 surface with Picard number 20. This is described in [9, Section 3], but here we can study it in a more geometrical way. The second one was suggested to me by Bert van Geemen, and shows how important the hypothesis on the annulation of the rank of the Mordell-Weil group of the elliptic surface is.

EXAMPLE 1. Let X be the elliptic complex surface over \mathbb{P}^1 whose Weierstrass model has equation

$$y^2 = x(x - p)(x - q),$$

where $p(t) = 3(t+1)^3(t-3)$ and $q(t) = 3(t-1)^3(t+3)$. It is easy to show that the only singular fibers are of type I_2 over $0, 3, -3$ and of type I_6 over $1, -1, \infty$. Thus X is stable, has Euler number $e(X) = 24$ (which implies that X is a K3 surface) and Picard number $\rho(X) = 20$. By the Shioda-Tate formula (see [3, Corollary VII.2.4]), $rk(MW(X)) = 0$. By Proposition 5, since $p(0) = q(0) = -9$, the quaternion algebra (x, t) does not ramify over 0. Over ∞ , following the proof of Proposition 5, we study the ramification over 0 of the quaternion algebra (x, s) for the surface of equation

$$y^2 = x(x - p_1)(x - q_1),$$

where $p_1(s) = 3(1+s)^3(1-3s)$ and $q_1 = 3(1-s)^3(1+3s)$. Since $p_1(0) = q_1(0) = 3$, $(x, t) \in BrX$. By Proposition 6, (x, t) is non trivial.

By Remark 2, we even get [9, Remark 3.7]: the quaternion algebra $(x, t) \in Br(\mathbb{Q}(X))$ is not ramified over the field $\mathbb{Q}(i, \sqrt{3})$ (we must have that -9 and 3 are square in the base field, which is the case for $\mathbb{Q}(i, \sqrt{3})$).

We can even show that $(x - p, t)$ ramifies, but $(x - p, (t - 1)(t + 3))$ is in BrX and it is non trivial. To do so, we can write $X = x - p$, so that the Weierstrass model of X becomes $y^2 = X(X + p)(X + p - q)$ and the quaternion algebras become (X, t) and $(X, (t - 1)(t + 3))$. Now we can apply Propositions 5 and 6. Using the same kind of arguments, we get the same conclusions for $(x - q, t)$ and $(x - q, (t + 1)(t - 3))$.

EXAMPLE 2. Let X be an elliptic complex surface over \mathbb{P}^1 whose Weierstrass model has equation

$$y^2 = x(x - p)(x - q),$$

where $p(t) = (t - 1)^2$ and $q(t) = t^2 + 1$. It is easy to show that X is rational: the only singular fibers are of type I_4 over 1, and of type I_2 over $i, -i, 0$ and ∞ , so that the Euler number of X is 12 and Picard number 10 (see [7, Lemma 10.2]). By Proposition 5, we get easily that $(x, t) \in BrX$. Since X is rational, $BrX = 0$. In this case, we see that the rank of the Mordell–Weil group of X is 1, by the Shioda–Tate formula, so that we cannot apply Proposition 6.

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