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UNIVERSAL-EXISTENTIAL AXIOM SYSTEMS FOR GEOMETRIES EXPRESSED WITH PIERI'S ISOSCELES TRIANGLE AS SINGLE PRIMITIVE NOTION

Abstract. We prove that, building upon the universal-existential orthogonality-based axiom system for metric planes presented in [28], one can provide universal-existential axiom systems – expressed solely in terms of the ternary predicate I, with $I(abc)$ standing for '*ab* is congruent to *ac*', which Pieri has introduced 100 years ago – for metric planes, for absolute geometry with the circle axiom, for Euclidean planes, for Euclidean geometry with the circle axiom, for Klingenberg's generalized hyperbolic planes, for plane elementary hyperbolic geometry, as well as for all the finite-dimensional versions of these geometries.

1. Introduction

Popular accounts of the history of mathematics present the problem of the axiomatic foundation of elementary geometry as having been solved with the publication of Hilbert's [11]. The underlying assumption in these accounts is that the problem to be solved was that of closing certain "gaps" in Euclid's *Elements*, by offering a rigorous foundation for two- or three-dimensional geometry.

The history of the axiomatic foundation of geometry since 1899 reveals an entirely different picture. Instead of looking at an axiom system as the solution to an axiomatization problem, the literature on this subject is concerned with questions of a metamathematical nature, regarding all possible axiomatizations of that theory.

Hilbert's axiom system for three-dimensional geometry required three sorts of individual variables, for points, lines, and planes, as well as (i) binary predicates for point-line, point-plane, and line-plane incidence, (ii) a ternary predicate for betweenness, (iii) a quaternary predicate for segment congruence, and (iv) a sexternary predicate for angle congruence.

The question arose whether one could provide simpler axiomatizations for the same theory. Among the many options to define simplicity (see [17] for a survey of simplicity criteria), there are several *purely syntactic* ones, some of which ask for the language in which the axiom system is expressed to be simple, whereas others ask for simple axioms. When asking for simple languages, simpler means having primitives (predicate and operation symbols) that are both few in number and of the lowest possible arity, where minimal arity takes precedence over scarcity of primitive symbols. Thus a language with five binary predicates is simpler than one with only one ternary predicate. When looking at the axioms themselves, a criterion for simplicity is to ask

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for the fewest number of quantifier alternations in each axiom, which is considered to be written in prenex form. In this sense an axiom containing only universal quantifiers is simpler than one containing a string of universal quantifiers followed by a string of existential quantifiers, and so on. Another criterion of simplicity regards the number of quantifiers (or of variables) appearing in each axiom. In this sense one tries to minimize the number of variables appearing in the axiom(s) containing the largest number of variables. There is no reason to believe that there exists for every theory an axiom system that is the simplest possible one according to all of the above criteria. It is to be expected that, in the simplest possible language, expressing the axioms will turn out to be complicated in both the quantifier complexity sense and in the number of quantifiers sense.

It is all the more remarkable that the task of achieving *the utmost simplicity* of the language of Euclidean geometry was accomplished in one step, by one person, in one paper. The person is Mario Pieri, who showed in [31] that Euclidean geometry can be axiomatized in terms of one sort of variables, to be interpreted as *points*, and a ternary relation I, with $I(abc)$ to be read as '*ab* is congruent to *ac*'. That this is the simplest possible language in which one can axiomatize Euclidean geometry was shown in [15, Remarks following Theorem 6] (see also [37, Theorem II.4.34]): No finite set of binary predicates with points as individual variables can be used to axiomatize Euclidean geometry. Nor is Euclidean geometry with a unit measure or with finitely many fixed distances axiomatizable in a language containing only binary relations with individual variables to be interpreted as points, as shown in [33] (see also [37, Theorem II.4.67]). If the logic in which the axiom system is expressed goes beyond first-order logic, by allowing the formation of denumerably infinite disjunctions of formulas, a logic designated by $L_{\omega_1\omega}$, then the single binary relation of *unit distance* can be used to axiomatize Archimedean ordered Euclidean geometry with the Circle Axiom — an axiom stating that a segment with one endpoint inside and one outside of a circle must intersect that circle.[1] This was shown for the 2-dimensional case in [19] and for the n-dimensional case, with $n \geq 2$ in [41] and [42]. It is also not possible to axiomatize Euclidean geometry by means of a finite set of binary operations on points, as shown in [4], although one can axiomatize hyperbolic geometry with points as variables by means of the single binary operation of midpoint, as shown in [26].

If the intended interpretation of the individual variables is not *points*, then it *is* possible to axiomatize both Euclidean and hyperbolic geometry with *spheres* as individual variables by means of the single binary relation of *sphere tangency* in all finite dimensions $n \geq 2$, as shown in [24]. With *lines* as individual variables one can axiomatize Euclidean geometry of any finite dimension $n \geq 4$, and hyperbolic geometry of any dimension $n \geq 2, n \neq 3$, in terms of the single binary predicate of line orthogonality, as shown in [36], [13], [23], [29], [26].

The entire literature cited above, and much more, such as Tarski's axiom system for Euclidean geometry (see [40]), as well as the research into other ternary relations among points that can serve as single primitive notions for Euclidean geometry, such

[1] The geometry in question is a Euclidean geometry in which the coordinate field is both Archimedean ordered and Euclidean, i. e. every positive element has a square root.

as the symmetrized version of Pieri's relation, 'three points a, b, c form in any order the vertices of a (possibly degenerate) isosceles triangle', the notion of equilaterality, the symmetrized perpendicularity notion, 'three points a, b, c form, in any order, a right triangle', surveyed in [37, Part II], were inspired by Pieri's 1908 memoir.

Recently, there has been renewed interest in it, leading to its first English translation in [16, Chapter 3], and to the expository note [8]. Both [16] and [8] address the question regarding the quantifier complexity of Euclidean geometry expressed in terms of points as the only individual variables, and Pieri's ternary relation I. In [16, 3.10.6, 5.2.4] the authors mention that Pieri's form of the Pasch axiom "is a $\Pi\Sigma\Pi\Sigma$ sentence that would require more than fifteen lines and three hundred characters!"[2], whereas in Tarski's formulation it is a universal-existential sentence. It is also pointed out that "most of the complexity of Pieri's Pasch postulate is the result of his having to substitute for the occurrences of the betweenness relation their corresponding definitions in terms of equidistance." In [8, p. 141], after having expressed in formal language some of Pieri's axioms, we are told that "although it is possible to express [...] Pieri's axioms only by means of [the] primitive notion, it would be extremely hard to do this" given their "growing complexity".[3] These remarks refer only to Pieri's axioms as he stated them, leaving open the possibility that a different axiom system, based on the same notion I, would consist entirely of $\forall\exists$-axioms, which would turn it into *the simplest axiom system* according to two syntactic criteria of simplicity: simplest language and simplest quantifier complexity, since there can be no universal axiom system expressed in a language without operation symbols for any geometry. The aim of this paper is to show that it is indeed possible to provide axiom systems in terms of I, consisting entirely of $\forall\exists$-axioms, by which we mean sentences written in prenex form, in which all universal quantifiers (if any) precede all existential quantifiers (if any), not only for finite-dimensional Euclidean geometry, but for a wide range of other finite-dimensional geometries. We caution that this fact does not imply that the complexity of axiom systems based on Pieri's I is the same as that based on Tarski's two relations B, with $B(abc)$ to be read as 'b lies between a and c', and $\equiv$, with $ab \equiv cd$ to be read as 'ab is congruent to cd', as there are other measures of complexity, under which the I-based axiom systems very likely have higher complexity. One such measure is that of the number of variables or quantifiers that each axiom requires, when written in prenex form. Euclidean n-dimensional geometry over Euclidean ordered fields expressed in terms of B and $\equiv$, with points as variables, can be axiomatized (as shown in [18], [20]) by an axiom system, all of whose axioms are prenex sentences about at most 5 points for $n = 2$, and about at most $n+2$ points for $n \geq 3$. Plane hyperbolic geometry can be axiomatized (as shown in [25]) in the same language by means of axioms that are prenex sentences about at most 6 points. The minimal number of variables that are needed to axiomatize in terms of I these geometries is not known, but it is likely to be *significantly higher*.

The reasons why Pieri's original axiom system would have such a high quanti-

[2] In this notation Π stands for $\forall$, and Σ for $\exists$.

[3] Although Pieri's I is treated at length in [37, II.4], the question of the quantifier complexity of axiom systems expressed in terms of I is not addressed.

fier complexity if expressed solely in terms of I, i. e. by replacing the defined notions which appear in some of the axioms by their I-definiens, are that it uses an $\forall$-definition for collinearity in terms of I, that collinearity shows up in the definition of betweenness, and that the axiom system contains axioms for betweenness, in particular the Pasch axiom. Pieri's definitions of collinearity λ, with $\lambda(abx)$ to be read as 'points a,b, and x are collinear (though not necessarily distinct)', midpoint μ, with $\mu(axb)$ to be read as 'point x is the midpoint of the segment ab, if $a \neq b$, and coincides with a if $a = b$', and betweenness B, with $B(axb)$ to be read as 'point x lies between points a and b (and may be equal to a or b)', are:

(1) $\lambda(abx) \quad :\Leftrightarrow \quad (\forall x')\,[a = b \vee (I(axx') \wedge I(bxx') \rightarrow x = x')]$,

(2) $\mu(axb) \quad :\Leftrightarrow \quad (a \neq b \vee x = a) \wedge \lambda(abx) \wedge I(xab)$,

(3) $B(axb) \quad :\Leftrightarrow \quad (\exists muv)\,[\lambda(abx) \wedge (\mu(amb) \wedge I(mua) \wedge I(mva) \wedge \mu(uxv))]$.

Since no $\exists$-definition of λ in terms of I is known in the dimension-free case, i. e. a definition that would hold inside geometries in which there is only a lower-dimension axiom, stipulating that the dimension be ≥ 2, but no upper-dimension axiom (see [37, Part II] for more on dimension-free geometries), Pieri's definition (1) is the best one we have for λ in the dimension-free case. If, however, we are interested in axiomatizing an n-dimensional geometry in terms of I, then we do have existential definitions of λ in terms of I: in two-dimensional geometry λ can be defined by:

(4) $$\lambda(a_1a_2a_3) :\Leftrightarrow (\exists uv)\,[u \neq v \wedge \bigwedge_{i=1}^{3} I(a_iuv)],$$

and in the three-dimensional case, by the formula provided in [37, Theorem II.4.14]. Such definitions exist in all finite-dimensions based on the same Leibnizian idea. If we used an $\exists$-definiens (such as (4)) for λ instead of (1), then (3) would turn into an $\exists$-definition of B in terms of I (after having replaced μ by its definiens (2)).[4] We can also $\exists$-define the equidistance relation $\equiv$ in terms of I as in [37, II.4.18] by

(5) $$ab \equiv cd \quad :\Leftrightarrow \quad (\exists ef)\,[\mu(bec) \wedge \mu(aef) \wedge I(cfd)],$$

and then replace all occurrences of B and $\equiv$ in an axiom system for n-dimensional plane absolute geometry written just with B and $\equiv$ as primitive notions (such as the axiom system A1-A9 in [37] for $n = 2$) to get an axiom system entirely in terms of I. Adding

(6) $$I(abc) \leftrightarrow ab \equiv cd$$

to that axiom system, where $\equiv$ has been replaced by its definiens in terms of I, we get an $\forall\exists$-axiom system for plane absolute geometry with the Circle Axiom, in which the

[4] Another $\exists$-definition of B in terms of I can be obtained as in [37, II.4.24], by first defining the notion of orthogonality $\perp_0$, by $\perp_0 (abc) :\Leftrightarrow (\exists d)\,[\mu(cad) \wedge I(bcd)]$, to be read as '$ab$ is perpendicular to ac' (where aa is considered to be perpendicular to ax for all x), and then B by $B(abc) :\Leftrightarrow (\exists d)\,[\lambda(abc) \wedge \perp_0 (bad) \wedge \perp_0 (dac)]$.

coordinate field must be Euclidean. That the axiom system axiomatizes plane absolute geometry follows from the fact that the defined notions B and $\equiv$ satisfy the required axioms; that I has the desired interpretation follows from the axiom (6) (or the corresponding two axioms, if we split the $\leftrightarrow$ occurring in (6) into $\leftarrow$ and $\rightarrow$) describing I in terms of $\equiv$. That the Circle Axiom will hold as well, although we did not adopt its I-translate as an axiom, follows from the fact that $\lambda(abc) \rightarrow (B(abc) \vee B(bca) \vee B(cab))$ is a theorem of plane absolute geometry, and since this has to hold with our definition of B (either (3) or the one stated in the footnote), the coordinate field must be Euclidean (see Section 4 for details). Starting from absolute geometry, we can also proceed by adding axioms to obtain $\forall\exists$-axiom systems for Euclidean and hyperbolic geometry.

Thus, if all we wanted to know was whether absolute, Euclidean, and hyperbolic geometry of fixed finite dimension are $\forall\exists$-axiomatizable in terms of I, then the simple observation that there is an $\exists$-definition of λ — and thus of B and $\equiv$ — in terms of I, and $\forall\exists$-axioms systems for these geometries in terms of B and $\equiv I$, is all we need.

If we want to show that the $\forall\exists$-axiomatizability in question holds for a wider class of geometries, in which midpoints do not necessarily exist for all pairs of points, then we have to adopt another strategy. In the process, we will also obtain axiom systems in which the axioms are all statements reflecting genuinely metric thoughts. One can say that, unlike those obtained from an axiomatization in terms of B and $\equiv$ by replacing these predicates by their I-definiens, our axiomatizations represent a first step on the road to an axiom system *conceived* in terms of I.

2. Metric geometries

We now turn to axiom systems expressed entirely in purely metric terms, i. e. in terms of the quaternary equidistance relation $\equiv$ or of the ternary relation $\perp$, with $\perp(abc)$ to be read as 'a,b,c are the vertices of a right triangle with right angle at a' (a symmetrized notion, in which the vertex with the right angle is not singled out, was considered in [35]). The first such axiom system, in terms of $\equiv$, with all axioms $\forall\exists$-sentences (and we consider those in which $\exists^{=1}$ appears in the axioms in this category, as every statement $(\forall\overline{x}\exists^{=1}y)\varphi(\overline{x},y)$ can be split into two $\forall\exists$-sentences, namely $(\forall\overline{x}\exists y)\varphi(\overline{x},y)$ and $\varphi(\overline{x},y)\wedge\varphi(\overline{x},y') \rightarrow y = y'$, where by $\overline{x}$ we have denoted a finite sequence of variables $x_1, x_2, \ldots, x_n$) was the axiom system in [34] for plane Euclidean geometry with arbitrary fields of characteristic $\neq 2$ that are not quadratically closed as coordinate fields. It was followed by another axiomatization of the same theory in [7], by an axiomatization of non-elliptic metric planes in [39], and by an axiomatization of three-dimensional Euclidean spaces in [32]. The last axiom system of this nature was the one in [28] in terms of $\perp$ for metric planes, including the elliptic case. In the Euclidean two- and three-dimensional case, one may turn the $\forall\exists$-axiom systems in [34], [7], [32] into $\forall\exists$-axiom systems in terms of Pieri's I, by replacing $\equiv$ in all axioms by its definition (5) (see [37, II.4.18]) in terms of I, and adding (6) as an axiom.

Our concern will be to show that the $\forall\exists$-axiomatizations are possible, and not that the axiom systems involved are independent, or that they are in any other way optimal, as they most likely are not. Turning these into independent axiom systems would certainly require a significant amount of work.

There is a certain expectation coming from mathematicians encountering axiom systems only at the beginning of a lecture, that these ought to be short, contain few axioms, and be pleasing to the eye and to the mind. The current investigation is not motivated by such classroom-use aims, but has as its object of study the first-order theory of geometry expressed in terms of Pieri's I, and asks metamathematical questions regarding the syntactic form of possible axiom systems for that theory.

3. Universal-existential axiomatization for metric planes

Since we want to show that Pieri's I can serve as primitive notion, with $\forall\exists$-axiom systems, for a wide class of geometries, we will first provide the method for obtaining $\forall\exists$-axiom systems in terms of I for non-elliptic metric planes. The concept of a *metric plane*, intended to provide the metric skeleton of the classical plane geometries (Euclidean, hyperbolic, elliptic), grew out of the work of Hessenberg, Hjelmslev, and A. Schmidt, and was provided with a simple group-theoretic axiomatics by F. Bachmann ([2, §3,2, p. 33]). Other axiomatizations were presented in [39], [22], [27], and [28]. If the axioms of geometry one aims to I-axiomatize imply that every segment has a midpoint, then one can start with the axiom system in [39] (see also its formalization in terms of λ and $\equiv$ in [21]) and define by (4) (or a higher-dimensional version thereof), (2), and (5) the two primitive notions used there, namely λ and $\equiv$, in terms of I.

If the axioms of the geometry one aims to I-axiomatize do not imply that every segment has a midpoint, then we have to start with the $\forall\exists$-axiom system from [28] in terms of $\perp$. The predicate $\perp$ is both Σ-definable and Π-definable in terms of I, and thus Δ-definable in terms of I. The definitions are

$$(7) \qquad \perp(abc) \quad :\Leftrightarrow \quad (\exists b'c')\,[a \neq b \wedge a \neq c \wedge b \neq b' \wedge c \neq c' \wedge I(abb') \wedge I(acc') \wedge I(cbb') \wedge I(c'bb') \wedge I(bcc')],$$

$$(8) \qquad \perp(abc) \quad :\Leftrightarrow \quad (\forall b'c')\,[a \neq b \wedge a \neq c \wedge (I(abb') \wedge I(acc') \wedge I(bcc') \wedge I(b'cc') \wedge c \neq c' \rightarrow I(cbb'))].$$

We can now proceed by replacing every occurrence of $\perp$ (or of $\neg\perp$) in its axioms — which we take to be written in prenex normal form, with their matrix, i. e. their quantifier-free part, in conjunctive normal form — by its Σ-definition (7) (or by $\perp$'s Π-definition (9)), to obtain a set of $\forall\exists$-sentences expressed solely in terms of I.

Since it is not clear that, inside the set of sentences thus obtained, we would be able to show that I has the desired interpretation, we add four axioms, the first two of which state that the two definitions of $\perp$ in terms of I are equivalent, i. e. that the definiens of (9) implies the definiens of (7) and vice-versa (the former implication producing an $\forall\exists$-sentence, the latter a universal sentence), the other two of which which describe the relation I in terms of $\perp$ — which we consider here as an abbreviation for its definienses in (7) and (9), the substitutions being made such that the resulting axioms turn into $\forall\exists$-sentences in terms of I. To state these two additional axioms we first need several defined notions in terms of ⊥, all definitions being existential. The first defined notion is the quaternary predicate L, with $L_e(abc)$ standing for 'a,b,c are three collinear points, with a different from b and c, and a,b,e are the vertices of a right triangle with right angle at a', while the second notion, φ stands for the reflection of a point in a line, $\varphi(abpmnoqq'rr'uvp')$ standing, in case p is not the pole of the line ab, for 'point p' is the reflection of p in line ab', the additional points in φ helping to achieve the construction of the reflected point p' (see Figure 1). The last defined notion is the quaternary predicate M, with $M_u(bac)$ standing for 'a is the midpoint of the segment bc, with $b \neq c$, and u is a point on the perpendicular in a on ab, which is not the pole of ab' (which we express by means of φ, by stating that there is a point u, such that au is perpendicular bc, and such that the reflection of b in au is c).

$$\begin{aligned}
L_e(abc) \quad &:\Leftrightarrow \quad \perp(abe) \wedge \perp(ace),\\
\varphi(abpmnoqq'rr'uvp') \quad &:\Leftrightarrow \quad a \neq b \wedge (o = a \vee \perp(oap)) \wedge (o = b \vee \perp(obp))\\
&\qquad \wedge L_o(pqr) \wedge q \neq r \wedge L_o(mqq') \wedge q \neq q'\\
&\qquad \wedge L_o(nrr') \wedge r \neq r' \wedge L_o(p'q'r') \wedge L_m(opp') \wedge m \neq n\\
&\qquad \wedge L_p(omn) \wedge L_u(orq') \wedge L_v(oqr'),\\
M_u(bac) \quad &:\Leftrightarrow \quad (\exists mnoqq'rr'vw)[\perp(abu) \wedge \neg \perp(bau)\\
&\qquad \wedge \varphi(aubmnaqq'rr'vwc)].
\end{aligned}$$

With these existentially defined notions L, φ, M, which we consider as abbreviations of their existential I-definiens, we can formulate the following axiom, that can be split into two axioms, by splitting the ↔ appearing in it into ← and →, each an ∀∃-axiom:

$$\begin{aligned}
(9) \qquad I(abc) \quad \leftrightarrow \quad &(\exists u)[b = c \vee (\perp(bac) \wedge \perp(cab)) \vee M_u(bac)\\
&\vee(\neg(\perp(bac) \wedge \perp(cab)) \wedge M_a(buc) \wedge \perp(uab))].
\end{aligned}$$

This axiom states that $I(abc)$ holds if and only if $b = c$, or a is the pole of line bc (i. e. both ab and ac are perpendiculars from a to bc), or a is the midpoint of bc, or a is not the pole of line bc, and the foot of the perpendicular from a to bc is the midpoint u of the segment bc.

Let Σ denote the axiom system, with all axioms ∀∃-sentences, for metric planes from [28], in which all occurrences of ⊥ have been replaced by their I-definiens in such a way that all axioms turn out to be ∀∃-sentences (some axioms will be dropped altogether, being implicit in the definition of ⊥), to which the four axioms mentioned above are added. If we consider the axiom system formed by the I-translations of the axioms from [28] as ⊥-axioms (which we can do by translating back into the language of ⊥ every occurrence of (9) or (7), given that we have two axioms telling us that the two definitions are equivalent), then we know that ⊥ must have its intended interpretation and that all models must be metric planes. The two axioms into which (9) has been split now tell us that I must have the intended interpretation as well, given that it is equivalent to its definiens, in which we now know the meaning of ⊥. Thus, all models of Σ are metric planes and I has the intended interpretation.

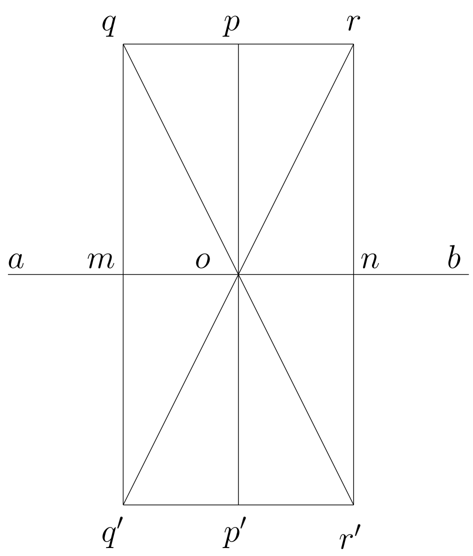

Figure 1: The reflection of p in the line ab obtained by means of φ

4. Universal-existential axiomatizations for several geometries

Starting with Σ, we can provide $\forall\exists$-axiom systems in terms of I for a wide range of two-dimensional and, more generally, for finite dimensional geometries. First, let us note that finite-dimensional metric geometries — first axiomatized, for the 3-dimensional case, by Ahrens [1], then, for the n-dimensional case by Kinder [12], and, for the dimension-free case, by Ewald [5] and simplified by Heimbeck [9], and J. T. Smith [38], from where axiom systems for the n-dimensional case can be obtained by adding axioms fixing the dimension — can be axiomatized in terms of I by means of $\forall\exists$-axioms, by stating that every plane is a metric plane ($(n-1)$-dimensional flats can be defined by two different points u and v in terms of I by stating that a point p belongs to the flat if and only if $I(puv)$; inside $(n-1)$-dimensional flats one can define in the same manner, by means of two different points belonging to the $(n-1)$-dimensional flat, $(n-2)$-dimensional flats, and so on, until we get, by means of existential definitions, to 2-dimensional flats) and that the dimension of the whole space is n. Plane metric-Euclidean geometry (see [2, §12,1]) can be axiomatized by adding an axiom stating the existence of a rectangle (Axiom R in [2, §6,7]), and Euclidean planes by adding the axiom (addition in the indices being modulo 3)

(10) $$(\forall a_1a_2a_3)(\exists uv)\left[\left(u \neq v \wedge \bigwedge_{i=1}^{3} I(a_iuv)\right) \vee \left(\bigwedge_{i=1}^{3} I(ua_ia_{i+1})\right)\right],$$

stating that every triangle has a circumcenter.

Metric planes with free mobility, a theory whose models have been described algebraically in [3], can be ∀∃-axiomatized by adding to Σ two axioms: one stating the existence of the midpoint of any given segment, i. e.:

$$(\forall abuv)(\exists c)\,[u \neq v \wedge I(auv) \wedge I(buv) \wedge a \neq b \rightarrow I(cab) \wedge I(cuv)], \tag{11}$$

and one stating that one can lay off any given segment on any line emanating from one of the endpoints of the segment, i. e.:

$$(\forall abcuv)(\exists d)\,[u \neq v \wedge I(auv) \wedge I(cuv) \wedge a \neq c \rightarrow I(duv) \wedge I(abd)]. \tag{12}$$

Hilbert planes in which the Circle Axiom holds, i. e. models of the plane axioms of groups I, II, III of Hilbert's [11] (or of axioms A1-A9 in [37]) together with the Circle Axiom (CA in [37, p. 15]), the coordinate fields of which must be Euclidean, can be ∀∃-axiomatized by adding to $\Sigma \cup \{(11),(12)\}$ two axioms: one stating the uniqueness of the perpendicular from a point outside of a line to that line, in order to exclude the elliptic case, and the other stating that, given two non-degenerate and non-congruent segments *ab* and *ac*, there must be a right triangle having one of them as hypotenuse and the other as side, i. e.:

$$(\forall abc)(\exists u)\,[a = b \vee a = c \vee I(abc) \vee (I(abu) \wedge \perp(cau)) \vee (I(acu) \wedge \perp(bau))]. \tag{13}$$

To see that this axiom system, with axioms very far removed from those of the axiom systems in [11] and [37], axiomatize Hilbert planes in which the Circle Axiom holds, it is enough to check that, in all non-elliptic models of $\Sigma \cup \{(11),(12)\}$, described algebraically in [3], the axiom (13) implies that the coordinate field is Euclidean, and thus the models are those described algebraically in [30].

To see this, we need to succinctly present the basic elements of the algebraic description of metric planes with free mobility.

Let K be a Pythagorean field, i. e. every sum of squares is a square, and no square is -1, and k an element of K. By the *affine-metric plane* $\mathfrak{A}(K,k)$ (cf. [10, p.215]) over the field K with the *metric constant* k we mean the projective plane $\mathfrak{P}(K)$ over the field K from which the line $[0,0,1]$, as well as all the points on it have been removed (and denote by $\mathfrak{A}(K)$ the remaining point-set), for whose points of the form $(x,y,1)$ we shall write (x,y), and say that such a point is *incident with* a line $[u,v,w]$ if and only if

$$xu + yv + w = 0, \tag{14}$$

together with a notion of orthogonality, the lines $[u,v,w]$ and $[u',v',w']$ being *orthogonal* if and only if

$$uu' + vv' + kww' = 0. \tag{15}$$

If K is an ordered field, then one can order $\mathfrak{A}(K)$ in the usual way.

The algebraic characterization of metric planes with free mobility consists in specifying a point-set E of an affine-metric plane $\mathfrak{A}(K,k)$, which is the universe of the metric plane, and whose algebraic description is very intricate and is the main result in [3].

The congruence of two segments **ab** and **cd** will be given by the usual Euclidean formula $(a_1-b_1)^2+(a_2-b_2)^2=(c_1-d_1)^2+(c_2-d_2)^2$ if $E\subset\mathfrak{A}(K,0)$, and by

$$\frac{F(\mathbf{a},\mathbf{b})^2}{Q(\mathbf{a})Q(\mathbf{b})}=\frac{F(\mathbf{c},\mathbf{d})^2}{Q(\mathbf{c})Q(\mathbf{d})}, \tag{16}$$

if $E\subset\mathfrak{A}(K,k)$ with $k\neq 0$, where $F(\mathbf{x},\mathbf{y})=k(x_1y_1+x_2y_2)+1,\ Q(\mathbf{x})=F(\mathbf{x},\mathbf{x}),$ and $\mathbf{x}=(x_1,x_2),\mathbf{y}=(y_1,y_2)$.

The metric plane can always be represented such that it contains the point $(0,0)$, and is an *embedded* subplane of an affine-metric plane, i. e. it contains with every point all the lines of the projective-metric plane that are incident with it.

Suppose now that $\mathfrak{M}$ is a metric plane with free mobility in which (13) holds, and let E be its point set, K the coordinate field of the projective-metric plane in which it is embedded, and k be its metric constant.

Regardless of whether $k\neq 0$ or $k=0$, the fact that segments $(0,0)(x,0)$ and $(0,0)(0,y)$ are, whenever $x\neq\pm y$, such that one can be the hypotenuse and the other the side of a right triangle, imply that

$$x^2-y^2\in K^2 \text{ or } y^2-x^2\in K^2. \tag{17}$$

If a,b,c are three points such that there is a right triangle with ac as hypotenuse and ab as side, then we write $ab<ac$.

We can thus define a betweenness relation for collinear points a,b,c in $\mathfrak{M}$, by stipulating that $B_0(abc)$ holds if and only if $a=b$ or $b=c$ or $ab<ac$ and $cb<ca$. If $a,b,$ and c are collinear, then one and only one of $B_0(abc)$, $B_0(bca)$, and $B_0(cab)$ must hold. This can be seen, in case they are distinct points, by coordinatizing $\mathfrak{M}$ so that $a=(0,0)$, $b=(0,x)$, and $c=(0,y)$, and noticing, in the case in which k is 0, that one of the systems $x^2-y^2\in K^2$ and $x^2-(y-x)^2\in K^2$, $y^2-x^2\in K^2$ and $y^2-(y-x)^2\in K^2$, and $(y-x)^2-x^2\in K^2$ and $(y-x)^2-y^2\in K^2$ must hold. This follows from the fact that (17) must hold for any of the pairs involved, and the cases in which the differences of squares of the three involved pairs do not have a pair of difference listed in the systems above, lead to the the conclusion that the sum of two non-zero squares is 0, contradicting the Pythagorean nature of K. Analogously in case the metric constant is $\neq 0$.

Regardless of whether $k=0$ or $k\neq 0$, the foot of the orthogonal line through $(a,0)$ (a point in E) to the line $[-\lambda,1,0]$ (a line which belongs to $\mathfrak{M}$ for any $\lambda\in K$, since this line passes through the point $(0,0)$, and this is a point in E, and $\mathfrak{M}$ contains all the lines passing through a point of $\mathfrak{M}$) has coordinates $(\frac{a}{1+\lambda^2},\frac{a\lambda}{1+\lambda^2})$, as can be computed using (14) and (15). For $\lambda=a$, we get that $(\frac{a^2}{1+a^2},0)\in E$, and thus, since $1+a^2\in K^2$, we have that, for some $b\in K\setminus\{0\}$, $(b^2,0)\in E$, and thus also $(\frac{b^2\lambda}{1+\lambda^2},0)\in E$, for any $\lambda\in K$.

We want to show that K must be Euclidean. We know that K is Pythagorean, so K can be ordered. If it can be ordered in a unique way, then it is Euclidean, and we are done. Suppose K can be ordered in at least two ways, the orders being denoted by $<_1$ and $<_2$. Each of these orders defines a betweenness relation in $\mathfrak{M}$, which we denote by B_1 and B_2. Since in these ordered planes the hypotenuse is larger than the side of a right triangle, we must have that,

(18) If $B_0(abc)$ then $B_i(abc)$ for $i = 1,2$.

Since the two orders $<_1$ and $<_2$ are different, there must be a $\lambda \in K$, such that $\lambda >_1 0$ and $\lambda <_2 0$, and thus such that $\lambda b^2 >_1 0$ and $\lambda b^2 <_2 0$ for some $b \in K$, for which $(b^2,0) \in E$, thus $B_2((\lambda b^2,0)(0,0)(b^2,0))$ holds, but $B_1((\lambda b^2,0)(0,0)(b^2,0))$ does *not* hold. Exactly one of $B_0(xyz)$, $B_0(yzx)$, and $B_0(zxy)$ must hold for the points $x = (\lambda b^2,0)$, $y = (0,0)$, and $z = (\lambda b^2,0)$ and thus, by (18), for precisely the same ordered triple, the B_1 and B_2 relations must hold as well, a contradiction.

This shows that K can be ordered in only one way, and thus must be a Euclidean field.

The betweenness relation as defined by B_0 is now seen to satisfy all universal properties that the betweenness relation satisfies in absolute geometry. The only property we thus need to show it satisfies as well is the Pasch axiom. If $k \neq 0$, then we know from the main result in [3], namely [3, Satz 6.2], that $\mathfrak{M}$ is the intersection of all the *metric hulls* (see [3, §5] or [6] for a definition of the notion of metric bull) of $\mathfrak{M}$. However, since K has a unique order, there is only one metric hull and that one must be our $\mathfrak{M}$. The metric hull itself does satisfy the Pasch axiom (see [6, §1,5] or [3, §5]), and we are done.

If k is 0, then the set of coordinates of the points of E form, as shown in [2, §19,3, Satz 7], a submodule of K with all totally integer-majorizable elements as multipliers (an element x of K is called *totally integer-majorizable* if there exists an integer m such that $-m \leq x \leq m$ holds for every order $<$ of K). Since the Pasch axiom asks for the existence of some $c := \lambda \cdot a + (1-\lambda) \cdot b$, where a and b are in E, and $0 < \lambda < 1$, and the order is unique, c must be in E whenever a and b are in E, and thus the Pasch axiom holds in this case as well.

Adding to this axiom system the axiom (10) one gets an axiom system for plane Euclidean geometry with Euclidean fields as coordinate fields. Higher finite dimensional versions of this geometry are easily obtained by the same method mentioned for finite-dimensional metric geometries.

To get Klingenberg's generalized hyperbolic planes (see [2, §14]) one needs to add an axiom stating that there are limiting parallel lines, a statement which can be expressed as an existential statement in terms of $\perp$, by means of Bergau's criterion (see [2, §14,2, p. 224]), thus, using (7), as an existential statement in terms of I, as well as an axiom stating that from a point to a line there are no more than two limiting parallel lines, which can be written, using Bergau's criterion, as an ∀∃-statement. Adding an axiom stating that every segment has a midpoint one gets the elementary version of hyperbolic geometry, coordinatized by Euclidean fields, first axiomatized by Hilbert [11, Anhang III]. To get higher-dimensional versions of these axiom systems for the spaces axiomatized in [14] and for higher-dimensional hyperbolic geometry over Euclidean fields we use the same method mentioned in the case of finite-dimensional metric geometries.

As mentioned earlier, for all geometries in which all segments have midpoints, one obtains axiom systems with fewer variables in their axioms by starting with the axiom system for non-elliptic metric planes proposed by Sörensen [39], rather than the one in [28], and replacing therein every occurrence of $\equiv$ with its definiens in (5) in terms of I (replacing $ab \equiv ac$ with $I(abc)$ throughout) and λ by its I-definiens in (4).

It is not known whether the dimension-free versions of the geometries we have considered allow $\forall\exists$-axiomatizations in terms of I.

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