Rend. Sem. Mat. Univ. Pol. Torino
Vol. 67, 2 (2009), 229 – 245
Second Conf. Pseudo-Differential Operators

A. Morando, P. Secchi and P. Trebeschi

CHARACTERISTIC INITIAL BOUNDARY VALUE PROBLEMS FOR SYMMETRIZABLE SYSTEMS

Dedicated to Professor Luigi Rodino on the occasion of his 60th birthday

Abstract. We consider an initial-boundary value problem for a linear Friedrichs symmetrizable system, with characteristic boundary of constant rank. Assuming that the problem is L^2 well posed, we show the regularity of the L^2 solution, for sufficiently smooth data, in the framework of anisotropic Sobolev spaces.

1. Introduction

We consider an initial boundary value problem for a linear Friedrichs symmetrizable system, with characteristic boundary of constant rank. It is well-known that for solutions of symmetric or symmetrizable hyperbolic systems with characteristic boundary full regularity (i.e. solvability in the usual Sobolev spaces H^m) cannot be expected generally because of the possible loss of derivatives in the normal direction to the boundary, see [23, 12].

The natural space is the anisotropic Sobolev space H^m_*, which comes from the observation that the one-order gain of normal differentiation should be compensated by two-order loss of tangential differentiation (cf. [4]). The theory has been developed mostly for characteristic boundaries of *constant multiplicity* (see the definition in assumption (B)) and *maximally nonnegative* boundary conditions, see [4, 5, 11, 16, 17, 18, 19, 21].

However, there are important characteristic problems of physical interest where boundary conditions are not maximally nonnegative. Under the more general *Kreiss-Lopatinski condition* (KL), the theory has been developed for problems satisfying the *uniform* KL condition with *uniformly* characteristic boundaries (when the boundary matrix has constant rank in a neighborhood of the boundary), see [8, 1] and references therein.

In this paper we are interested in the problem of the regularity. We assume the existence of the strong L^2 solution, satisfying a suitable energy estimate, without assuming any structural assumption sufficient for existence, such as the fact that the boundary conditions are maximally dissipative or satisfy the Kreiss–Lopatinski condition. We show that this is enough in order to get the regularity of solutions, in the natural framework of weighted anisotropic Sobolev spaces H^m_*, provided the data are sufficiently smooth. Obviously, the present results contain in particular what has been previously obtained for maximally nonnegative boundary conditions.

Let Ω be an open bounded subset of $\mathbb{R}^n$ (for a fixed integer $n \geq 2$), lying locally on one side of its smooth, connected boundary $\Gamma := \partial\Omega$. For any real $T > 0$, we set $Q_T := \Omega \times]0,T[$ and $\Sigma_T := \Gamma \times]0,T[$; in addition we define $Q_\infty := \Omega \times [0,+\infty[$, $\Sigma_\infty := \partial\Omega \times [0,+\infty[$, $Q := \Omega \times \mathbb{R}$ and $\Sigma := \partial\Omega \times \mathbb{R}$. We are interested in the following initial boundary value problem (written in the sequel IBVP)

$$Lu = F, \quad \text{in } Q_T \tag{1}$$

$$Mu = G, \quad \text{on } \Sigma_T \tag{2}$$

$$u_{|t=0} = f, \quad \text{in } \Omega, \tag{3}$$

where L is the first order linear partial differential operator

$$L = \partial_t + \sum_{i=1}^{n} A_i(x,t)\partial_i + B(x,t), \tag{4}$$

$\partial_t := \frac{\partial}{\partial t}$, $\partial_i := \frac{\partial}{\partial x_i}$, $i = 1,\dots,n$ and $A_i(x,t), B(x,t)$ are $N \times N$ real matrix-valued functions of (x,t), for a given integer size $N \geq 1$, defined over Q_∞. The unknown $u = u(x,t)$ and the data $F = F(x,t)$, $f = f(x)$ are real vector-valued functions with N components, defined on $\overline{Q}_T$ and $\overline{\Omega}$ respectively. In the boundary conditions (2), M is a smooth $d \times N$ matrix-valued function of (x,t), defined on Σ_∞, with maximal constant rank d. The boundary datum $G = G(x,t)$ is a d-vector valued function, defined on $\overline{\Sigma}_T$.

Let us denote by $\nu(x) := (\nu_1(x),\dots,\nu_n(x))$ the unit outward normal to Γ at the point $x \in \Gamma$; then

$$A_\nu(x,t) = \sum_{i=1}^{n} A_i(x,t)\nu_i(x), \quad (x,t) \in \Sigma_\infty, \tag{5}$$

is the *boundary matrix*. Let $P(x,t)$ be the orthogonal projection onto the orthogonal complement of $\ker A_\nu(x,t)$, denoted $\ker A_\nu(x,t)^\perp$; it is defined by

$$P(x,t) = \frac{1}{2\pi i}\int_{C(x,t)} (\lambda - A_\nu(x,t))^{-1}\, d\lambda, \quad (x,t) \in \Sigma_\infty, \tag{6}$$

where $C(x,t)$ is a closed rectifiable Jordan curve with positive orientation in the complex plane, enclosing all and only all non-zero eingenvalues of $A_\nu(x,t)$. Denoting again by P an arbitrary smooth extension on $\overline{Q}_\infty$ of the above projection, Pu and $(I-P)u$ are called respectively the *non characteristic* and the *characteristic* components of the vector field $u = u(x,t)$.

We study the problem (1)–(3) under the following assumptions:

(A) The operator L is Friedrichs symmetrizable, meaning that for all $(x,t) \in \overline{Q}_\infty$ there exists a symmetric positive definite matrix $S_0(x,t)$ such that the matrices $S_0(x,t)A_i(x,t)$, $i = 1,\cdots,n$, are also real symmetric; this implies, in particular, that the *symbol* $A(x,t,\xi) = \sum_{i=1}^{n} A_i(x,t)\xi_i$ is diagonalizable with real eigenvalues, whenever $(x,t,\xi) \in \overline{Q}_\infty \times \mathbb{R}^n$.

(B) The boundary is *characteristic, with constant rank*, namely the boundary matrix A_ν is singular on Σ_∞ and has constant rank $0 < r := \mathrm{rank} A_\nu(x,t) < N$ for all $(x,t) \in \Sigma_\infty$; this assumption, together with the symmetrizability of L and that Γ is connected, yields that the number of negative eigenvalues of A_ν (the so-called *incoming modes*) remains constant on Σ_∞.

(C) $\ker A_\nu(x,t) \subseteq \ker M(x,t)$, for all $(x,t) \in \Sigma_\infty$; moreover $d = \mathrm{rank} M(x,t)$ must equal the number of negative eigenvalues of $A_\nu(x,t)$.

(D) The orthogonal projection $P(x,t)$ onto $\ker A_\nu(x,t)^\perp$, $(x,t) \in \Sigma_\infty$, can be extended as a matrix-valued C^∞ function over $\overline{Q}_\infty$.

Concerning the solvability of the IBVP (1)–(3), we state the following well-posedness assumption:

(E) *Existence of the L^2 weak solution.* Assume that $S_0,\ A_i \in \mathrm{Lip}(\overline{Q}_\infty)$ for $i = 1,\dots,n$. For all $T > 0$ and all matrices $B \in L^\infty(\overline{Q}_T)$, there exist constants $\gamma_0 \geq 1$ and $C_0 > 0$ such that for all $F \in L^2(Q_T)$, $G \in L^2(\Sigma_T)$, $f \in L^2(\Omega)$ there exists a unique solution $u \in L^2(Q_T)$ of (1)–(3), with data (F,G,f), satisfying the following properties:

i. $u \in C([0,T];L^2(\Omega))$;

ii. $Pu_{|\Sigma_T} \in L^2(\Sigma_T)$;

iii. for all $\gamma \geq \gamma_0$ and $0 < \tau \leq T$ the solution u enjoys the following a priori estimate

$$\begin{aligned} & e^{-2\gamma\tau}\|u(\tau)\|^2_{L^2(\Omega)} + \gamma\int_0^\tau e^{-2\gamma t}\|u(t)\|^2_{L^2(\Omega)}\,dt \\ & \qquad + \int_0^\tau e^{-2\gamma t}\|Pu_{|\partial\Omega}(t)\|^2_{L^2(\partial\Omega)}\,dt \\ & \leq C_0\left(\|f\|^2_{L^2(\Omega)} + \int_0^\tau e^{-2\gamma t}\left(\frac{1}{\gamma}\|F(t)\|^2_{L^2(\Omega)} + \|G(t)\|^2_{L^2(\partial\Omega)}\right)dt\right). \end{aligned} \tag{7}$$

When the IBVP (1)–(3) admits an a priori estimate of type (7), with $F = Lu$, $G = Mu$, for all $\tau > 0$ and all sufficiently smooth functions u, one says that the problem is *strongly L^2 well posed*, see e.g. [1]. A necessary condition for (7) is the validity of the *uniform Kreiss-Lopatinski condition* (UKL) (an estimate of type (7) has been obtained by Rauch [13]). On the other hand, UKL is not sufficient for the well posedness and other structural assumptions have to be taken into account, see [1].

Finally, we require the following technical assumption that for C^∞ approximations of problem (1)–(3) one still has the existence of L^2 solutions. This stability property holds true for maximally nonnegative boundary conditions and for uniform KL conditions.

(F) Given matrices $(S_0, A_i, B) \in \mathcal{C}_T(H^\sigma_*) \times \mathcal{C}_T(H^\sigma_*) \times \mathcal{C}_T(H^{\sigma-2}_*)$, where $\sigma \geq [\frac{n+1}{2}]+4$, enjoying properties (A)–(E), let $(S_0^{(k)}, A_i^{(k)}, B^{(k)})$ be C^∞ matrix-valued functions converging to (S_0, A_i, B) in $\mathcal{C}_T(H^\sigma_*) \times \mathcal{C}_T(H^\sigma_*) \times \mathcal{C}_T(H^{\sigma-2}_*)$ as $k \to \infty$, and satisfying properties (A)–(D). Then, for k sufficiently large, property (E) holds also for the approximating problems with coefficients $(S_0^{(k)}, A_i^{(k)}, B^{(k)})$.

The solution of (1)–(3), considered in the statements (E), (F), must be intended in the sense of Rauch [15]. This means that for all $v \in H^1(Q_T)$ such that $v_{|\Sigma_T} \in (A_\nu(\ker M))^\perp$ and $v(T, \cdot) = 0$ in Ω, there holds:

$$\int_0^T \langle u(t), L^* v(t)\rangle\, dt = \int_0^T \langle F(t), v(t)\rangle\, dt - \int_{\Sigma_T} \langle A_\nu g, v\rangle\, d\sigma_x\, dt + \int_\Omega \langle f, v(0)\rangle\, dx,$$

where L^* is the adjoint operator of L and g is a function defined on Σ_T such that $Mg = G$. Notice also that for such a weak solution to (1)–(3), the boundary condition (2) makes sense. Indeed, in [15, Theorem 1] it is shown that for any $u \in L^2(Q_T)$, with $Lu \in L^2(Q_T)$, the trace of $A_\nu u$ on Σ_T exists in $H^{-1/2}(\Sigma_T)$. Moreover, for a given boundary matrix $M(x,t)$ satisfying assumption (C), there exists another matrix $M_0(x,t)$ such that $M(x,t) = M_0(x,t) A_\nu(x,t)$ for all $(x,t) \in \Sigma_\infty$. Therefore, for L^2 solutions of (1) one has

$$Mu = G \quad \text{on } \Sigma_T \iff M_0 A_\nu u_{|\Sigma_T} = G \quad \text{on } \Sigma_T. \tag{8}$$

In order to study the regularity of solutions to the IBVP (1)–(3), the data F, G, f need to satisfy some compatibility conditions. The compatibility conditions are defined in the usual way (see [14]). Given the IBVP (1)–(3), we recursively define $f^{(h)}$ by formally taking $h-1$ time derivatives of $Lu = F$, solving for $\partial_t^h u$ and evaluating it at $t = 0$; for $h = 0$ we set $f^{(0)} := f$. The *compatibility condition* of order $k \geq 0$ for the IBVP reads as

$$\sum_{h=0}^{p} \binom{p}{h} (\partial_t^{p-h} M)_{|t=0} f^{(h)} = \partial_t^h G_{|t=0}, \quad \text{on } \Gamma,\ p = 0, \ldots, k. \tag{9}$$

In the framework of the preceding assumptions, we are able to prove the following theorem.

THEOREM 1. *Let $m \in \mathbb{N}$ and $s = \max\{m, [\frac{n+1}{2}] + 5\}$. Assume that $S_0, A_i \in \mathcal{C}_T(H^s_*)$, for $i = 1, \ldots, n$, and that $B \in \mathcal{C}_T(H^{s-1}_*)$ (or $B \in \mathcal{C}_T(H^s_*)$ if $m = s$). Assume also that problem* (1)–(3) *obeys the assumptions* (A)–(F). *Then for all $F \in H^m_*(Q_T)$, $G \in H^m(\Sigma_T)$, $f \in H^m_*(\Omega)$, with $f^{(h)} \in H^{m-h}_*(\Omega)$ for $h = 1, \ldots, m$, satisfying the compatibility condition* (9) *of order $m-1$, the unique solution u to* (1)–(3)*, with data (F, G, f), belongs to $\mathcal{C}_T(H^m_*)$ and $Pu_{|\Sigma_T} \in H^m(\Sigma_T)$. Moreover u satisfies the a priori estimate*

$$\|u\|_{\mathcal{C}_T(H^m_*)} + \|Pu_{|\Sigma_T}\|_{H^m(\Sigma_T)} \leq C_m \big(|||f|||_{m,*} + \|F\|_{H^m_*(Q_T)} + \|G\|_{H^m(\Sigma_T)}\big), \tag{10}$$

with a constant $C_m > 0$ depending only on A_i, B.

The function spaces involved in the statement above (cf. also the assumption (F)), and the norms appearing in the energy estimate (10) are introduced in the next section.

2. Function spaces

For every integer $m \geq 1$, $H^m(\Omega)$, $H^m(Q_T)$ denote the usual Sobolev spaces of order m over Ω and Q_T respectively.

In order to define the anisotropic Sobolev spaces, first we need to introduce the differential operators in *tangential direction*. Throughout the paper, for every $j = 1,2,\dots,n$, the differential operator Z_j is defined by

$$Z_1 := x_1\partial_1\,, \quad Z_j := \partial_j\,, \text{ for } j = 2,\dots,n\,.$$

Then, for every multi-index $\alpha = (\alpha_1,\dots,\alpha_n) \in \mathbb{N}^n$, the tangential differential operator Z^α of order $|\alpha| = \alpha_1 + \cdots + \alpha_n$ is defined by setting

$$Z^\alpha := Z_1^{\alpha_1}\dots Z_n^{\alpha_n}$$

(we also write, with the standard multi-index notation, $\partial^\alpha = \partial_1^{\alpha_1}\dots\partial_n^{\alpha_n}$).

We denote by $\mathbb{R}^n_+$ the n-dimensional positive half-space $\mathbb{R}^n_+ := \{x = (x_1,x') \in \mathbb{R}^n : \; x_1 > 0 \; x' := (x_2,\dots,x_n) \in \mathbb{R}^{n-1}\}$. For every positive integer m, the *tangential* (or *conormal*) Sobolev space $H^m_{tan}(\mathbb{R}^n_+)$ and the *anisotropic* Sobolev space $H^m_*(\mathbb{R}^n_+)$ are defined respectively by:

(11) $$H^m_{tan}(\mathbb{R}^n_+) := \{w \in L^2(\mathbb{R}^n_+) \,:\, Z^\alpha w \in L^2(\mathbb{R}^n_+)\,,\; |\alpha| \leq m\},$$

(12) $$H^m_*(\mathbb{R}^n_+) := \{w \in L^2(\mathbb{R}^n_+) \,:\, Z^\alpha \partial_1^k w \in L^2(\mathbb{R}^n_+)\,,\; |\alpha| + 2k \leq m\},$$

and equipped respectively with norms

(13) $$\|w\|^2_{H^m_{tan}(\mathbb{R}^n_+)} := \sum_{|\alpha|\leq m} \|Z^\alpha w\|^2_{L^2(\mathbb{R}^n_+)}\,,$$

(14) $$\|w\|^2_{H^m_*(\mathbb{R}^n_+)} := \sum_{|\alpha|+2k\leq m} \|Z^\alpha \partial_1^k w\|^2_{L^2(\mathbb{R}^n_+)}\,.$$

To extend the definition of the above spaces to an open bounded subset Ω of $\mathbb{R}^n$ (fulfilling the assumptions made at the beginning of the previous section), we proceed as follows. First, we take an open covering $\{U_j\}_{j=0}^l$ of $\overline{\Omega}$ such that $U_j \cap \overline{\Omega}$, $j = 1,\dots,l$, are diffeomorphic to $\overline{\mathbb{B}}_+ := \{x_1 \geq 0, |x| < 1\}$, with Γ corresponding to $\partial\mathbb{B}_+ := \{x_1 = 0, |x| < 1\}$, and $U_0 \subset\subset \Omega$. Next we choose a smooth partition of unity $\{\psi_j\}_{j=0}^l$ subordinate to the covering $\{U_j\}_{j=0}^l$. We say that a distribution u belongs to $H^m_{tan}(\Omega)$, if and only if $\psi_0 u \in H^m(\mathbb{R}^n)$ and, for all $j = 1,\dots,l$, $\psi_j u \in H^m_{tan}(\mathbb{R}^n_+)$, in local coordinates in U_j. The space $H^m_{tan}(\Omega)$ is provided with the norm

(15) $$\|u\|^2_{H^m_{tan}(\Omega)} := \|\psi_0 u\|^2_{H^m(\mathbb{R}^n)} + \sum_{j=1}^l \|\psi_j u\|^2_{H^m_{tan}(\mathbb{R}^n_+)}.$$

The anisotropic Sobolev space $H^m_*(\Omega)$ is defined in a completely similar way as the set of distributions u in Ω such that $\psi_0 u \in H^m(\mathbb{R}^n)$ and $\psi_j u \in H^m_*(\mathbb{R}^n_+)$, in local coordinates in U_j, for all $j = 1, \ldots, l$; it is provided with the norm

$$\|u\|^2_{H^m_*(\Omega)} := \|\psi_0 u\|^2_{H^m(\mathbb{R}^n)} + \sum_{j=1}^{l} \|\psi_j u\|^2_{H^m_*(\mathbb{R}^n_+)} . \tag{16}$$

The definitions of $H^m_{tan}(\Omega)$ and $H^m_*(\Omega)$ do not depend on the choice of the coordinate patches $\{U_j\}_{j=0}^l$ and the corresponding partition of unity $\{\psi_j\}_{j=0}^l$, and the norms arising from different choices of U_j, ψ_j are equivalent.

For an extensive study of the anisotropic Sobolev spaces, we refer the reader to [24], [20]; here we just remark that the continuous imbeddings

$$\begin{aligned} &H^m_{tan}(\Omega) \hookrightarrow H^p_{tan}(\Omega), \quad H^m_*(\Omega) \hookrightarrow H^p_*(\Omega), \quad \forall m \geq p \geq 1, \\ &H^m(\Omega) \hookrightarrow H^m_*(\Omega) \hookrightarrow H^m_{tan}(\Omega), \quad \forall m \geq 1, \\ &H^m_*(\Omega) \hookrightarrow H^{[m/2]}(\Omega), \quad H^1_*(\Omega) = H^1_{tan}(\Omega) \end{aligned} \tag{17}$$

hold true. For the sake of convenience, we also set $H^0_*(\Omega) = H^0_{tan}(\Omega) = L^2(\Omega)$. The spaces $H^m_{tan}(\Omega)$, $H^m_*(\Omega)$, endowed with their norms (15), (16), become Hilbert spaces. Analogously, we define the spaces $H^m_{tan}(Q_T)$ and $H^m_*(Q_T)$.

Let $C^m([0,T];X)$ denote the set of all m-times continuously differentiable functions over $[0,T]$, taking values in a Banach space X. We define the spaces

$$\mathcal{C}_T(H^m_{tan}) := \bigcap_{j=0}^{m} C^j([0,T]; H^{m-j}_{tan}(\Omega)), \quad \mathcal{C}_T(H^m_*) := \bigcap_{j=0}^{m} C^j([0,T]; H^{m-j}_*(\Omega)),$$

equipped respectively with the norms

$$\begin{aligned} \|u\|^2_{\mathcal{C}_T(H^m_{tan})} &:= \sum_{j=0}^{m} \sup_{t\in[0,T]} \|\partial_t^j u(t)\|^2_{H^{m-j}_{tan}(\Omega)}, \\ \|u\|^2_{\mathcal{C}_T(H^m_*)} &:= \sum_{j=0}^{m} \sup_{t\in[0,T]} \|\partial_t^j u(t)\|^2_{H^{m-j}_*(\Omega)}. \end{aligned} \tag{18}$$

For the initial datum f we set

$$|||f|||^2_{m,*} := \sum_{j=0}^{m} \|f^{(j)}\|^2_{H^{m-j}_*(\Omega)} .$$

3. The scheme of the proof of Theorem 1

The proof of Theorem 1 is made of several steps.

In order to simplify the forthcoming analysis, hereafter we only consider the case when the operator L has smooth coefficients. For the general case of coefficients with the finite regularity prescribed in Theorem 1, we refer the reader to [9]; this case is treated by a reduction to the smooth coefficients case, based upon the stability assumption (F). Thus, from now on, we assume that S_0, A_i, B are given functions in $C^\infty(\overline{Q}_\infty)$.

Just for simplicity, we even assume that the coefficients A_i of L are symmetric matrices (in this case the matrix S_0 reduces to I_N, the identity matrix of size N); the case of a symmetrizable operator can be easily reduced to this one, just by the application of the symmetrizer S_0 to system (1) (see [9] for details).

Below, we introduce the new unknown $u_\gamma(x,t) := e^{-\gamma t}u(x,t)$ and the new data $F_\gamma(x,t) := e^{-\gamma t}F(x,t)$, $G_\gamma(x,t) = e^{-\gamma t}G(x,t)$. Then problem (1)–(3) becomes equivalent to

$$\begin{aligned} (\gamma+L)u_\gamma &= F_\gamma && \text{in } Q_T\,, \\ Mu_\gamma &= G_\gamma && \text{on } \Sigma_T\,, \\ u_{\gamma|\,t=0} &= f && \text{in } \Omega\,. \end{aligned} \tag{19}$$

Let us now summarize the main steps of the proof of Theorem 1.

1. We firstly consider the *homogeneous* IBVP

$$\begin{aligned} (\gamma+L)u_\gamma &= F_\gamma && \text{in } Q_T\,, \\ Mu_\gamma &= G_\gamma && \text{on } \Sigma_T\,, \\ u_{\gamma|t=0} &= 0 && \text{in } \Omega\,. \end{aligned} \tag{20}$$

   We study (20), by reducing it to a *stationary* boundary value problem (see (26)), for which we deduce the *tangential* regularity. From the tangential regularity of this stationary problem, we deduce the tangential regularity of the homogeneous problem (20) (see the next Theorem 2).

2. We study the general problem (19). The anisotropic regularity, stated in Theorem 1, is obtained in two steps.

   2.i Firstly, from the tangential regularity of problem (20) above, we deduce the *anisotropic* regularity of (19) at order $m = 1$.

   2.ii Eventually, we obtain the anisotropic regularity of (19), at any order $m > 1$, by an induction argument.

3.1. The homogeneous IBVP. Tangential regularity

In this section, we concentrate on the study of the tangential regularity of solutions to the IBVP (19), where the initial datum f is identically zero, and the compatibility conditions are fulfilled in a more restrictive form than the one given in (9). More precisely, we consider the *homogeneous* IBVP (20) where, for a given integer $m \geq 1$, we assume that the data F_γ, G_γ satisfy the following conditions:

$$\partial_t^h F_{\gamma|\,t=0} = 0\,, \quad \partial_t^h G_{\gamma|\,t=0} = 0\,, \quad h = 0,\ldots,m-1\,. \tag{21}$$

One can prove that conditions (21) imply the compatibility conditions (9) of order $m-1$, in the case $f = 0$.

THEOREM 2. *Assume that A_i, B, for $i = 1, \dots, n$, are in $C^\infty(\overline{Q}_\infty)$, and that problem* (20) *satisfies assumptions* (A)–(E)*; then for all $T > 0$ and $m \in \mathbb{N}$ there exist constants $C_m > 0$ and γ_m, with $\gamma_m \geq \gamma_{m-1}$, such that for all $\gamma \geq \gamma_m$, for all $F_\gamma \in H^m_{tan}(Q_T)$ and all $G_\gamma \in H^m(\Sigma_T)$ satisfying* (21) *the unique solution u_γ to* (20) *belongs to $H^m_{tan}(Q_T)$, the trace of Pu_γ on Σ_T belongs to $H^m(\Sigma_T)$ and the a priori estimate*

$$\gamma \|u_\gamma\|^2_{H^m_{tan}(Q_T)} + \|Pu_{\gamma|\Sigma_T}\|^2_{H^m(\Sigma_T)} \leq C_m \left(\frac{1}{\gamma} \|F_\gamma\|^2_{H^m_{tan}(Q_T)} + \|G_\gamma\|^2_{H^m(\Sigma_T)} \right) \tag{22}$$

is fulfilled.

The first step to prove Theorem 2 is reducing the original mixed *evolution* problem (20) to a *stationary* boundary value problem, where the time is allowed to span the whole real line and it is treated then as an additional tangential variable. To make this reduction, we extend the data F_γ, G_γ and the unknown u_γ of (20) to all positive and negative times, by following methods similar to those of [1, Ch.9]. In the sequel, for the sake of simplicity, we remove the subscript γ from the unknown u_γ and the data F_γ, G_γ.

Because of (21), we extend F and G through $]-\infty, 0]$, by setting them equal to zero for all negative times; then for times $t > T$, we extend them by "reflection", following Lions–Magenes [7, Theorem 2.2]. Let us denote by $\breve{F}$ and $\breve{G}$ the resulting extensions of F and G respectively; by construction, $\breve{F} \in H^m_{tan}(Q)$ and $\breve{G} \in H^m(\Sigma)$.

As we did for the data, the solution u to (20) is extended to all negative times, by setting it equal to zero. To extend u also for times $t > T$, we exploit the assumption (E). More precisely, for every $T' > T$ we consider the mixed problem

$$\begin{aligned} &(\gamma + L)u &&= \breve{F}_{|]0,T'[} &&\text{in } Q_{T'}\,, \\ &Mu &&= \breve{G}_{|]0,T'[}\,, &&\text{on } \Sigma_{T'}\,, \\ &u_{|\,t=0} &&= 0\,, &&\text{in } \Omega \end{aligned} \tag{23}$$

Assumption (E) yields that (23) admits a unique solution $u_{T'} \in C([0,T']; L^2(\Omega))$, such that $Pu_{T'} \in L^2(\Sigma_{T'})$ and the energy estimate

$$\begin{aligned} &\|u_{T'}(T')\|^2_{L^2(\Omega)} + \gamma \|u_{T'}\|^2_{L^2(Q_{T'})} + \|Pu_{T'\,|\Sigma_{T'}}\|^2_{L^2(\Sigma_{T'})} \\ &\leq C' \left(\frac{1}{\gamma} \|\breve{F}_{|]0,T'[}\|^2_{L^2(Q_{T'})} + \|\breve{G}_{|]0,T'[}\|^2_{L^2(\Sigma_{T'})} \right) \end{aligned} \tag{24}$$

is satisfied for all $\gamma \geq \gamma'$ and some constants $\gamma' \geq 1$ and $C' > 0$ depending only on T' (and the norms $\|A_i\|_{\mathrm{Lip}(Q_{T'})}$, $\|B\|_{L^\infty(Q_{T'})}$).

From the uniqueness of the L^2 solution, we infer that for arbitrary $T'' > T' \geq T$ we have $u_{T''} = u_{T'}$ ($u_T := u$) over $]0, T'[$. Therefore, we may extend u beyond T, by setting it equal to the unique solution of (23) over $]0, T'[$ for all $T' > T$. Thus we define

$$\breve{u}(t) := \begin{cases} u_{T'}(t)\,, & \forall t \in]0, T'[\,,\ \forall T' > T\,, \\ 0\,, & \forall t < 0\,. \end{cases} \tag{25}$$

Since $\breve{u}$, $\breve{F}$, $\breve{G}$ are all identically zero for negative times, we can take arbitrary smooth extensions of the coefficients of the differential operator L and the boundary operator M (originally defined on Q_∞ and Σ_∞) on Q and Σ respectively, with the only care to preserve rank $A_\nu = r$ and rank $M = d$ and $\ker A_\nu \subseteq \ker M$ for all $t < 0$. This extensions, that we fix once and for all, are denoted again by A_i, B, M. Moreover, we denote by L the corresponding extension on Q of the differential operator (4).

By construction, we have that $\breve{u}$ solves the boundary value problem (BVP)

$$
\begin{aligned}
(\gamma+L)u &= \breve{F} \qquad \text{in } Q\,,\\
Mu &= \breve{G}\,, \qquad \text{on } \Sigma\,.
\end{aligned}
\tag{26}
$$

Using the estimate (24), for all $T' > T$, and noticing that the extended data $\breve{F}$, $\breve{G}$, as well as the solution $\breve{u}$, vanish identically for large $t > 0$, we derive that $\breve{u}$ enjoys the following estimate

$$
\gamma\|\breve{u}\|^2_{L^2(Q)} + \|P\breve{u}_{|\Sigma}\|^2_{L^2(\Sigma)} \le \breve{C}\left(\frac{1}{\gamma}\|\breve{F}\|^2_{L^2(Q)} + \|\breve{G}\|^2_{L^2(\Sigma)}\right),
\tag{27}
$$

for all $\gamma \ge \breve{\gamma}$, and suitable constants $\breve{\gamma} \ge 1$, $\breve{C} > 0$.

For the sake of simplicity, in the sequel we remove the superscript from the unknown $\breve{u}$ and the data $\breve{F}$, $\breve{G}$ of (26).

The next step is to move from BVP (26) to a similar BVP posed in the $(n+1)$-dimensional positive half-space $\mathbb{R}^{n+1}_+ := \{(x_1, x', t) : x_1 > 0, (x', t) \in \mathbb{R}^n\}$. To make this reduction into a problem in $\mathbb{R}^{n+1}_+$, we follow a standard localization procedure of the problem (26) near the boundary of the spatial domain Ω; this is done by taking a covering $\{U_j\}_{j=0}^l$ of $\overline{\Omega}$ and a partition of unity $\{\psi_j\}_{j=0}^l$ subordinate to this covering, as in Section 2. Assuming that each patch U_j, $j = 1, \ldots, l$, is sufficiently small, we can write the resulting localized problem in the form

$$
\begin{aligned}
(\gamma+L)u &= F \qquad \text{in } \mathbb{R}^{n+1}_+\,,\\
Mu &= G\,, \qquad \text{on } \mathbb{R}^n\,.
\end{aligned}
\tag{28}
$$

As a consequence of the localization, the data F and G of the problem (28) are functions in $H^m_{tan}(\mathbb{R}^{n+1}_+)$ and $H^m(\mathbb{R}^n)$ respectively; without loss of generality, we may also assume that the forcing term F and the solution u are supported in the set $\overline{\mathbb{B}}_+ \times [0, +\infty[$, and the boundary datum G is supported in $\partial\mathbb{B}_+ \times [0, +\infty[$. In $(28)_1$, L is now a differential operator in $\mathbb{R}^{n+1}$ of the form

$$
L = \partial_t + \sum_{i=1}^{n} A_i(x,t)\partial_i + B(x,t)\,,
\tag{29}
$$

where the coefficients A_i, B are matrix-valued functions of (x,t) belonging to the space $C^\infty_{(0)}(\mathbb{R}^{n+1}_+)$ of the restrictions onto $\mathbb{R}^{n+1}_+$ of (matrix-valued) functions in $C^\infty_0(\mathbb{R}^{n+1})$. Let us remark that the boundary matrix of (28) is now $-A_{1\,|\,\{x_1=0\}}$. It is a crucial step that the previously described localization process can be performed in such a way that A_1

has the following block structure

$$A_1(x,t) = \begin{pmatrix} A_1^{I,I} & A_1^{I,II} \\ A_1^{II,I} & A_1^{II,II} \end{pmatrix}, \quad (x,t) \in \mathbb{R}^{n+1}_+, \tag{30}$$

where $A_1^{I,I}, A_1^{I,II}, A_1^{II,I}, A_1^{II,II}$ are respectively $r\times r$, $r\times(N-r)$, $(N-r)\times r$, $(N-r)\times(N-r)$ sub-matrices. Moreover, $A_1^{I,I}(x,t)$ is invertible over the support of $u(x,t)$ and we have

$$A_1^{I,II} = 0, \quad A_1^{II,I} = 0, \quad A_1^{II,II} = 0, \quad \text{in} \quad \{x_1 = 0\} \times \mathbb{R}^n_{x',t}. \tag{31}$$

In view of assumption (C), we may even assume that the matrix M in the boundary condition $(28)_2$ is just $M = (I_d, 0)$, where I_d is the identity matrix of size d. According to (30), let us decompose the unknown u as $u = (u^I, u^{II})$; then we have $Pu = (u^I, 0)$.

Following the arguments of [3], one can prove that a local counterpart of the global estimate (27), associated to the stationary problem (26), can be attached to the local problem (28). More precisely, there exist constants $C_0 > 0$ and $\gamma_0 \geq 1$ such that for all $\varphi \in L^2(\mathbb{R}^{n+1}_+)$, supported in $\overline{\mathbb{B}}_+ \times [0,+\infty[$, such that $L\varphi \in L^2(\mathbb{R}^{n+1}_+)$ and $\gamma \geq \gamma_0$, we have

$$\begin{aligned} &\gamma\|\varphi\|^2_{L^2(\mathbb{R}^{n+1}_+)} + \|\varphi^I_{|\{x_1=0\}}\|^2_{L^2(\mathbb{R}^n)} \\ &\leq C_0\left(\frac{1}{\gamma}\|(\gamma+L)\varphi\|^2_{L^2(\mathbb{R}^{n+1}_+)} + \|M\varphi_{|\{x_1=0\}}\|^2_{L^2(\mathbb{R}^n)}\right). \end{aligned} \tag{32}$$

Regularity of the stationary problem (28)

The analysis performed in the previous section shows that the tangential regularity of the homogeneous IBVP (20) can be deduced from the study of the regularity of the stationary BVP (28).

For this stationary problem, we are able to show that if the data F and G belong to $H^m_{tan}(\mathbb{R}^{n+1}_+)$ and $H^m(\mathbb{R}^n)$ respectively, and the L^2 a priori estimate (32) is fulfilled, then the L^2 solution of the problem (28) belongs to $H^m_{tan}(\mathbb{R}^{n+1}_+)$, the trace of its non-characteristic part u^I belongs to $H^m(\mathbb{R}^n)$ and the estimate of order m

$$\gamma\|u\|^2_{H^m_{tan}(\mathbb{R}^{n+1}_+)} + \|u^I_{|\{x_1=0\}}\|^2_{H^m(\mathbb{R}^n)} \leq C_m\left(\frac{1}{\gamma}\|F\|^2_{H^m_{tan}(\mathbb{R}^{n+1}_+)} + \|G\|^2_{H^m(\mathbb{R}^n)}\right) \tag{33}$$

is satisfied with some constants $C_m > 0$, $\gamma_m \geq 1$ and for all $\gamma \geq \gamma_m$.

Then we recover the tangential regularity of the solution u to problem (26), posed on $Q = \Omega \times \mathbb{R}$, and we find an associated estimate of order m analogous to (33). Recalling that the solution u to (26) is the extension of the solution u_γ of the homogeneous IBVP (20), from the tangential regularity of u we can now derive the tangential regularity of u_γ, namely that $u_\gamma \in H^m_{\tan}(Q_T)$ and $Pu_{\gamma\,|\,\Sigma_T} \in H^m(\Sigma_T)$. To get the energy estimate (22), we observe that the extended data $\breve{F}$ and $\breve{G}$ are defined in such a way that

$$\|\breve{F}\|_{H^m_{\tan}(Q)} \leq C\|F_\gamma\|_{H^m_{\tan}(Q_T)}, \quad \|\breve{G}\|_{H^m(\Sigma)} \leq C\|G_\gamma\|_{H^m(\Sigma_T)},$$

with positive constant C independent of F_γ, G_γ and γ.

In order to prove the announced tangential regularity of the BVP (28), we adapt the classical technique of Friedrichs' mollifiers to our setting. More precisely, following Nishitani and Takayama [10], we introduce a "tangential" mollifier J_ε well suited to the tangential Sobolev spaces. Let χ be a function in $C_0^\infty(\mathbb{R}^{n+1})$. For all $0<\varepsilon<1$, we set $\chi_\varepsilon(y) := \varepsilon^{-(n+1)}\chi(y/\varepsilon)$. We define $J_\varepsilon : L^2(\mathbb{R}^{n+1}_+) \to L^2(\mathbb{R}^{n+1}_+)$ by

$$J_\varepsilon w(x) := \int_{\mathbb{R}^{n+1}} w(x_1 e^{-y_1}, x'-y')e^{-y_1/2}\chi_\varepsilon(y)dy\,, \tag{34}$$

which differs from the one introduced in Rauch [15] by the factor $e^{-y_1/2}$. Using J_ε we follow the same lines in Tartakoff [22], Nishitani and Takayama [10] to get regularity of the weak solution u.

Starting from a classical characterization of the ordinary Sobolev spaces given in [6, Theorem 2.4.1], the following characterization of tangential Sobolev spaces $H^m_{tan}(\mathbb{R}^{n+1}_+)$ by means of J_ε can be proved.

PROPOSITION 1. *Assume that* $\chi \in C_0^\infty(\mathbb{R}^{n+1})$ *satisfies the following conditions:*

$$\widehat{\chi}(\xi) = O(|\xi|^p) \qquad \textit{as } \xi \to 0, \qquad \textit{for some } p\in\mathbb{N}; \tag{35}$$

$$\widehat{\chi}(t\xi) = 0\,, \qquad \textit{for all } t\in\mathbb{R}\,, \qquad \textit{implies } \xi = 0. \tag{36}$$

Then for all $m\in\mathbb{N}$ *with* $m<p$*, we have that* $u\in H^m_{tan}(\mathbb{R}^{n+1}_+)$ *if and only if*

a. $u \in H^{m-1}_{tan}(\mathbb{R}^{n+1}_+)$;

b. $\displaystyle\int_0^1 \|J_\varepsilon u\|^2_{L^2(\mathbb{R}^{n+1}_+)}\varepsilon^{-2m}\left(1+\frac{\delta^2}{\varepsilon^2}\right)^{-1}\frac{d\varepsilon}{\varepsilon}$ *is uniformly bounded for* $0<\delta\le 1$.

In view of Proposition 1, showing that the solution $u\in H^{m-1}_{tan}(\mathbb{R}^{n+1}_+)$ of (28) actually belongs to $H^m_{tan}(\mathbb{R}^{n+1}_+)$ amounts to provide a uniform bound, with respect to δ, for the integral quantity appearing in b., computed for the *mollified* solution $J_\varepsilon u$. To get this bound, the scheme is the following:

1. We notice that $J_\varepsilon u$ solves the following BVP

   $$\begin{aligned} &(\gamma+L)J_\varepsilon u = J_\varepsilon F + [L,J_\varepsilon]u\,, \ \text{in } \mathbb{R}^{n+1}_+\,,\\ &MJ_\varepsilon u = G_\varepsilon\,, \quad \text{on } \mathbb{R}^n\,, \end{aligned} \tag{37}$$

   where $[L,J_\varepsilon]$ is the commutator between the operators L and J_ε, and G_ε is a suitable boundary datum that can be computed from the original datum G and the function χ_ε involved in (34) (see [9]).

2. Since the BVP (37) is the same as (28), with data $J_\varepsilon F+[L,J_\varepsilon]u$ and G_ε, the L^2 estimate (32) applied to (37) gives that the L^2 norm of $J_\varepsilon u$ can be estimated by

   $$\begin{aligned} &\gamma\,\|J_\varepsilon u\|^2_{L^2(\mathbb{R}^{n+1}_+)} + \|J_\varepsilon u^I_{|\{x_1=0\}}\|^2_{L^2(\mathbb{R}^n)}\\ &\le C_0\left(\frac{1}{\gamma}\|J_\varepsilon F+[L,J_\varepsilon]u\|^2_{L^2(\mathbb{R}^{n+1}_+)} + \|G_\varepsilon\|^2_{L^2(\mathbb{R}^n)}\right). \end{aligned} \tag{38}$$

3. From the preceding estimate, we immediately derive, for the integral quantity in b. and the analogous integral quantity associated to the trace of non characteristic part of the solution, the following bound

$$
\begin{aligned}
&\gamma\int_0^1 \|J_\varepsilon u\|^2_{L^2(\mathbb{R}^{n+1}_+)} \varepsilon^{-2m}\left(1+\tfrac{\delta^2}{\varepsilon^2}\right)^{-1}\frac{d\varepsilon}{\varepsilon} \\
&\qquad + \int_0^1 \|J_\varepsilon u^I_{|\{x_1=0\}}\|^2_{L^2(\mathbb{R}^n)} \varepsilon^{-2m}\left(1+\tfrac{\delta^2}{\varepsilon^2}\right)^{-1}\frac{d\varepsilon}{\varepsilon} \\
&\le C_0\left(\frac{1}{\gamma}\int_0^1 \|J_\varepsilon F\|^2_{L^2(\mathbb{R}^{n+1}_+)} \varepsilon^{-2m}\left(1+\tfrac{\delta^2}{\varepsilon^2}\right)^{-1}\frac{d\varepsilon}{\varepsilon}\right. \\
&\qquad + \frac{1}{\gamma}\int_0^1 \|[L,J_\varepsilon]u\|^2_{L^2(\mathbb{R}^{n+1}_+)} \varepsilon^{-2m}\left(1+\tfrac{\delta^2}{\varepsilon^2}\right)^{-1}\frac{d\varepsilon}{\varepsilon} \\
&\qquad \left. + \int_0^1 \|G_\varepsilon\|^2_{L^2(\mathbb{R}^n)} \varepsilon^{-2m}\left(1+\tfrac{\delta^2}{\varepsilon^2}\right)^{-1}\frac{d\varepsilon}{\varepsilon}\right).
\end{aligned}
\tag{39}
$$

Since $F\in H^m_{tan}(\mathbb{R}^{n+1}_+)$ and $G\in H^m(\mathbb{R}^n)$, the first and the last integrals in the right-hand side of (39) can be estimated by $\|F\|^2_{H^m_{tan}(\mathbb{R}^{n+1}_+)}$ and $\|G\|^2_{H^m(\mathbb{R}^n)}$ respectively.

It remains to provide a uniform estimate for the middle integral involving the commutator $[L,J_\varepsilon]u$. For this term we get the following estimate

$$
\begin{aligned}
&\int_0^1 \|[L,J_\varepsilon]u\|^2_{L^2(\mathbb{R}^{n+1}_+)} \varepsilon^{-2m}\left(1+\tfrac{\delta^2}{\varepsilon^2}\right)^{-1}\frac{d\varepsilon}{\varepsilon} \\
&\le C\int_0^1 \|J_\varepsilon u\|^2_{L^2(\mathbb{R}^{n+1}_+)} \varepsilon^{-2m}\left(1+\tfrac{\delta^2}{\varepsilon^2}\right)^{-1}\frac{d\varepsilon}{\varepsilon} \\
&\qquad + C\gamma^2\|u\|^2_{H^{m-1}_{tan}(\mathbb{R}^{n+1}_+)} + C\|F\|^2_{H^m_{tan}(\mathbb{R}^{n+1}_+)}.
\end{aligned}
\tag{40}
$$

The estimate (40) is obtained by treating separately the different contributions to the commutator $[L,J_\varepsilon]$ associated to the different terms in the expression (29) of L (see [9] for details). The terms of the the commutator involving the tangential derivatives $[A_i\partial_i,J_\varepsilon]$, for $i=2,\dots,n$ (note that $[\partial_t,J_\varepsilon]=0$) and the zero-th order term $[B,J_\varepsilon]$ are estimated by using [10, Lemma 9.2]. The term $[A_1\partial_1,J_\varepsilon]$, involving the normal derivative ∂_1, needs a more careful analysis; to estimate it, it is essential to make use of the structure (30), (31) of the boundary matrix in (28). Actually, by inverting $A_1^{I,I}$ in $(28)_1$, we can write $\partial_1 u^I$ as the sum of *space-time* tangential derivatives by

$$\partial_1 u^I = \Lambda Z u + R,$$

where

$$
\begin{aligned}
&\Lambda Z u = -(A_1^{I,I})^{-1}\left[\left(\partial_t u^I + \sum_{j=2}^n A_j Z_j u\right)^I + A_1^{I,II}\partial_1 u^{II}\right], \\
&R = (A_1^{I,I})^{-1}(F-\gamma u - Bu)^I.
\end{aligned}
$$

Here, we use the fact that, if a matrix A vanishes on $\{x_1=0\}$, we can write $A\partial_1 u = HZ_1 u$, where H is a suitable matrix; this trick transforms some normal derivatives into tangential derivatives.

Combining the inequalities (39) and (40), and arguing by finite induction on m to estimate $\|u\|_{H^{m-1}_{tan}(\mathbb{R}^{n+1}_+)}$ in the right-hand side of (40), we get the desired uniform bounds of the integrals

$$\int_0^1 \|J_\varepsilon u\|^2_{L^2(\mathbb{R}^{n+1}_+)} \varepsilon^{-2m} \left(1+\frac{\delta^2}{\varepsilon^2}\right)^{-1} \frac{d\varepsilon}{\varepsilon}\,,$$
$$\int_0^1 \|J_\varepsilon u^I_{|\{x_1=0\}}\|^2_{L^2(\mathbb{R}^n)} \varepsilon^{-2m} \left(1+\frac{\delta^2}{\varepsilon^2}\right)^{-1} \frac{d\varepsilon}{\varepsilon}\,,$$

appearing in the left-hand side of (39). From this, in view of Proposition 1 and [6, Theorem 2.4.1], we conclude that $u \in H^m_{tan}(\mathbb{R}^{n+1}_+)$ and $u^I \in H^m(\mathbb{R}^n)$. The a priori estimate (33) is deduced from (39), by following the same arguments.

3.2. The nonhomogeneous IBVP. Case $m = 1$

For *nonhomogeneous* IBVP, we mean the problem (1)–(3) where the initial datum f is different from zero.

As announced before, we firstly prove the statement of Theorem 1 for $m = 1$, namely we prove that, under the assumptions (A)–(F), for all $F \in H^1_*(Q_T)$, $G \in H^1(\Sigma_T)$ and $f \in H^1_*(\Omega)$, with $f^{(1)} \in L^2(\Omega)$, satisfying the compatibility condition $M_{|t=0} f_{|\partial\Omega} = G_{|t=0}$, the unique solution u to (1)–(3), with data (F, G, f), belongs to $\mathcal{C}_T(H^1_*)$ and $Pu_{|\Sigma_T} \in H^1(\Sigma_T)$; moreover, there exists a constant $C_1 > 0$ such that u satisfies the a priori estimate

$$\|u\|_{\mathcal{C}_T(H^1_*)} + \|Pu_{|\Sigma_T}\|_{H^1(\Sigma_T)} \leq C_1 \Big(|||f|||_{1,*} + \|F\|_{H^1_*(Q_T)} + \|G\|_{H^1(\Sigma_T)} \Big). \tag{41}$$

To this end, we approximate the data with regularized functions satisfying one more compatibility condition. In this regard we get the following result, for the proof of which we refer to [9] and the references therein.

LEMMA 1. *Assume that problem* (1)–(3) *obeys the assumptions* (A)–(E). *Let* $F \in H^1_*(Q_T)$, $G \in H^1(\Sigma_T)$, $f \in H^1_*(\Omega)$, *with* $f^{(1)} \in L^2(\Omega)$, *such that* $M_{|t=0} f_{|\partial\Omega} = G_{|t=0}$. *Then there exist* $F_k \in H^3(Q_T)$, $G_k \in H^3(\Sigma_T)$, $f_k \in H^3(\Omega)$, *such that* $M_{|t=0} f_k = G_{k|t=0}$, $\partial_t M_{|t=0} f_k + M_{|t=0} f^{(1)}_k = \partial_t G_{k|t=0}$ *on* $\partial\Omega$, *and such that* $F_k \to F$ *in* $H^1_*(Q_T)$, $G_k \to G$ *in* $H^1(\Sigma_T)$, $f_k \to f$ *in* $H^1_*(\Omega)$, $f^{(1)}_k \to f^{(1)}$ *in* $L^2(\Omega)$, *as* $k \to +\infty$.

Given the functions F_k, G_k, f_k as in Lemma 1, we first calculate through equation $Lu = F_k, u_{|t=0} = f_k$, the initial time derivatives $f^{(1)}_k \in H^2(\Omega)$, $f^{(2)}_k \in H^1(\Omega)$. Then we take a function $w_k \in H^3(Q_T)$ such that

$$w_{k|t=0} = f_k\,, \quad \partial_t w_{k|t=0} = f^{(1)}_k\,, \quad \partial^2_{tt} w_{k|t=0} = f^{(2)}_k\,.$$

Notice that this yields

$$(Lw_k)_{|t=0} = F_{k|t=0}\,, \quad \partial_t (Lw_k)_{|t=0} = \partial_t F_{k|t=0}\,. \tag{42}$$

Now we look for a solution u_k of problem (1)–(3), with data F_k, G_k, f_k, of the form $u_k = v_k + w_k$, where v_k is solution to

$$
\begin{aligned}
&Lv_k = F_k - Lw_k, && \text{in } Q_T\\
&Mv_k = G_k - Mw_k, && \text{on } \Sigma_T\\
&v_{k|t=0} = 0, && \text{in } \Omega.
\end{aligned}
\tag{43}
$$

Let us denote again $u_{k\gamma} = e^{-\gamma t} u_k$, $v_{k\gamma} = e^{-\gamma t} v_k$ and so on. Then (43) is equivalent to

$$
\begin{aligned}
&(\gamma + L)v_{k\gamma} = F_{k\gamma} - (\gamma + L)w_{k\gamma}, && \text{in } Q_T\\
&Mv_{k\gamma} = G_{k\gamma} - Mw_{k\gamma}, && \text{on } \Sigma_T\\
&v_{k\gamma|t=0} = 0, && \text{in } \Omega.
\end{aligned}
\tag{44}
$$

We easily verify that (42) yields

$$
\big(F_{k\gamma} - (\gamma + L)w_{k\gamma}\big)_{|t=0} = 0\,, \quad \partial_t\,\big(F_{k\gamma} - (\gamma + L)w_{k\gamma}\big)_{|t=0} = 0
$$

and $M_{|t=0} f_{k|\partial\Omega} = G_{k|t=0}$, $\partial_t M_{|t=0} f_{k|\partial\Omega} + M_{|t=0} f^{(1)}_{k|\partial\Omega} = \partial_t G_{k|t=0}$ yield

$$
(G_{k\gamma} - Mw_{k\gamma})_{|t=0} = 0, \quad \partial_t (G_{k\gamma} - Mw_{k\gamma})_{|t=0} = 0.
$$

Thus the data of problem (44) obey conditions (21) for $h = 0, 1$; then we may apply to (44) Theorem 2 for γ large enough and find $v_k \in H^2_{tan}(Q_T)$, with $Pv_{k|\Sigma_T} \in H^2(\Sigma_T)$. Accordingly, we infer that $u_k \in H^2_{tan}(Q_T) \hookrightarrow \mathcal{C}_T(H^1_*)$ and $Pu_{k|\Sigma_T} \in H^2(\Sigma_T)$. Moreover $u_k \in L^2(Q_T)$ solves

$$
\begin{aligned}
&Lu_k = F_k, && \text{in } Q_T\\
&Mu_k = G_k, && \text{on } \Sigma_T\\
&u_{k|t=0} = f_k, && \text{in } \Omega.
\end{aligned}
\tag{45}
$$

Arguing as in the previous section, we take a covering $\{U_j\}_{j=0}^{l}$ of $\overline{\Omega}$ and a related partition of unity $\{\psi_j\}_{j=0}^{l}$, and we reduce problem (45) into a corresponding problem posed in the positive half-space $\mathbb{R}^n_+$, with new data $F_k \in H^3(\mathbb{R}^n_+ \times]0,T[)$, $G_k \in H^3(\mathbb{R}^{n-1} \times]0,T[)$, $f_k \in H^3(\mathbb{R}^n_+)$, and boundary matrix $M = (I_d, 0)$. We also write the similar problem solved by the first order derivatives $Zu_k = (Z_1 u_k, \ldots, Z_{n+1} u_k) \in H^1_{tan}(Q_T) = H^1_*(Q_T)$ (where $Z_{n+1} = \partial_t$). Since assumption (E) is satisfied, applying the a priori estimate (7) to a difference of solutions $u_h - u_k$ of those problems readily gives

$$
\begin{aligned}
&\|u_k - u_h\|_{\mathcal{C}_T(H^1_*)} + \|P(u_k - u_h)_{|\Sigma_T}\|_{H^1(\Sigma_T)}\\
&\le C\Big(|||f_k - f_h|||_{1,*} + \|F_k - F_h\|_{H^1_*(Q_T)} + \|G_k - G_h\|_{H^1(\Sigma_T)}\Big).
\end{aligned}
$$

From Lemma 1, we infer that $\{u_k\}$ is a Cauchy sequence in $\mathcal{C}_T(H^1_*)$ and $\{Pu_{k|\Sigma_T}\}$ is a Cauchy sequence in $H^1(\Sigma_T)$. Therefore there exists a function in $\mathcal{C}_T(H^1_*)$ which is the limit of $\{u_k\}$. Passing to the limit in (45) as $k \to \infty$, we see that this function is a solution to (1)–(3). The uniqueness of the L^2 solution yields $u \in \mathcal{C}_T(H^1_*)$ and

$Pu_{|\Sigma_T} \in H^1(\Sigma_T)$. Applying the a priori estimate (7) to the solution u_k of (45) and its first order derivatives, and passing to the limit finally gives (41). This completes the proof of Theorem 1 for $m = 1$ in the case of C^∞ coefficients. As we already said, here we do not deal with the case of less regular coefficients, for which the reader is referred to [9, Sect. 5].

3.3. The nonhomogeneous IBVP. Proof for $m \geq 2$

Without entering in too many details (we still refer to [9, Sect. 6] for a more extensive discussion), we briefly describe the different steps of the proof, for the reader's convenience.

We proceed by finite induction on m. Assume that Theorem 1 is valid up to $m-1$. Let $f \in H^m_*(\Omega)$, $F \in H^m_*(Q_T)$, $G \in H^m(\Sigma_T)$, with $f^{(k)} \in H^{m-k}_*(\Omega)$, with $k = 1,\ldots,m$. Assume also that the compatibility conditions (9) hold at the order $m-1$. By the inductive hypothesis there exists a unique solution u of problem (1)–(3) such that $u \in \mathcal{C}_T(H^{m-1}_*)$.

In order to show that $u \in \mathcal{C}_T(H^m_*)$, we have to increase the regularity of u by order one, that is by one more tangential derivative and, if m is even, also by one more normal derivative. This can be done as in [16, 17], with the small change of the elimination of the auxiliary system (introduced in [16, 17]) as in [2, 19]. At every step, we can estimate some derivatives of u through equations, where in the right-hand side we can put other derivatives of u that have already been estimated at previous steps. The reason why the main idea in [16] works, even though here we do not have maximally nonnegative boundary conditions, is that for the increase of regularity we consider the problem of the type of (1)–(3), solved by the purely tangential derivatives, where we can use the inductive assumption, and other systems of equations solved by the mixed tangential and normal derivatives where the boundary matrix vanishes identically, so that no boundary condition is needed and we can apply an energy method, under the assumption of the symmetrizable system.

References

[1] BENZONI-GAVAGE S., SERRE D., *Multidimensional hyperbolic partial differential equations*, Oxford Mathematical Monographs. The Clarendon Press Oxford University Press, Oxford 2007.

[2] CASELLA E., SECCHI P., TREBESCHI P., *Non-homogeneous linear symmetric hyperbolic systems with characteristic boundary*, Differential Integral Equations **19** 1 (2006), 51–74.

[3] CHAZARAIN J., PIRIOU A., *Introduction to the theory of linear partial differential equations*, North-Holland Publishing Co., 1982.

[4] CHEN S., *Initial boundary value problems for quasilinear symmetric hyperbolic systems with characteristic boundary*, Front. Math. China **2** 1 (2007), 87–102. Translated from Chinese Ann. Math. **3** 2 (1982), 222–232.

[5] GUÈS O., *Problème mixte hyperbolique quasi-linéaire caractéristique*, Comm. Partial Differential Equations **15** 5 (1990), 595–645.

[6] HÖRMANDER L., *Linear partial differential operators*, Springer Verlag, Berlin, 1976.

[7] LIONS J.-L., MAGENES E., *Problèmes aux limites non homogènes et applications. Vol.1*, Travaux et Recherche Mathématiques, No 17. Dunod, Paris 1968.

[8] MAJDA A., OSHER S., *Initial-boundary value problems for hyperbolic equations with uniformly characteristic boundary*, Comm. Pure Appl. Math. **28** 5 (1975), 607–675.

[9] MORANDO A., SECCHI P., TREBESCHI P., *Regularity of solutions to characteristic initial-boundary value problems for symmetrizable systems*, J. Hyperbolic Differ. Equ., to appear.

[10] NISHITANI T., TAKAYAMA M., *Regularity of solutions to non-uniformly characteristic boundary value problems for symmetric systems*, Comm. Partial Differential Equations **25** 5-6 (2000), 987–1018.

[11] OHNO M., SHIZUTA Y., YANAGISAWA T., *The initial-boundary value problem for linear symmetric hyperbolic systems with boundary characteristic of constant multiplicity*, J. Math. Kyoto Univ. **35** 2 (1995), 143–210.

[12] OHNO M., SHIROTA T., *On the initial-boundary-value problem for the linearized equations of magnetohydrodynamics.*, Arch. Ration. Mech. Anal. **144** 3 (1998), 259–299.

[13] RAUCH J., *L^2 is a continuable initial condition for Kreiss' mixed problems*, Comm. Pure Appl. Math. **25** (1972), 265–285.

[14] RAUCH J., MASSEY F. J. III, *Differentiability of solutions to hyperbolic initial-boundary value problems*, Trans. Amer. Mat. Soc. **189** (1974), 303–318.

[15] RAUCH J., *Symmetric positive systems with boundary characteristic of constant multiplicity*, Trans. Amer. Mat. Soc. **291** 1 (1985), 167–187.

[16] SECCHI P., *Linear symmetric hyperbolic systems with characteristic boundary*, Math. Methods Appl. Sci. **18** 11 (1995), 855–870.

[17] SECCHI P., *The initial-boundary value problem for linear symmetric hyperbolic systems with characteristic boundary of constant multiplicity*, Differential Integral Equations **9** 4 (1996), 671–700.

[18] SECCHI P., *Well-posedness of characteristic symmetric hyperbolic systems*, Arch. Rational Mech. Anal. **134** 2 (1996), 155–197.

[19] SECCHI P., *Characteristic symmetric hyperbolic systems with dissipation: global existence and asymptotics*, Math. Methods Appl. Sci. **20** 7 (1997), 583–597.

[20] SECCHI P., *Some properties of anisotropic Sobolev spaces*, Arch. Math. (Basel) **75** 3 (2000), 207–216.

[21] SHIZUTA Y., *On the final form of the regularity theorem for solutions to the characteristic initial boundary value problem for symmetric hyperbolic systems*, Proc. Japan Acad. Ser. A Math. Sci. **76** 4 (2000), 47–50.

[22] TARTAKOFF D., *Regularity of solutions to boundary value problems for first order systems*, Indiana Univ. Math. J. **21** (1972), 1113–1129.

[23] TSUJI M., *Regularity of solutions of hyperbolic mixed problems with characteristic boundary*, Proc. Japan Acad. **48** (1972), 719–724.

[24] YANAGISAWA T., MATSUMURA A., *The fixed boundary value problems for the equations of ideal magnetohydrodynamics with a perfectly conducting wall condition*, Comm. Math. Phys. **136** 1 (1991), 119–140.

AMS Subject Classification: 35L40, 35L50, 35L45

Alessandro MORANDO, Paolo SECCHI, Paola TREBESCHI,
Dipartimento di Matematica, Università di Brescia,
Via Valotti 9, 25133 Brescia, ITALIA
e-mail: alessandro.morando@ing.unibs.it, paolo.secchi@ing.unibs.it,
paola.trebeschi@ing.unibs.it

Lavoro pervenuto in redazione il 07.05.2009