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$L^p(\mathbb{R})$ BOUNDEDNESS AND COMPACTNESS OF LOCALIZATION OPERATORS ASSOCIATED WITH THE STOCKWELL TRANSFORM

Dedicated to Professor Luigi Rodino on the occasion of his 60th birthday

Abstract. In this article, we prove the boundedness and compactness of localization operators associated with Stockwell transforms, which depend on a symbol and two windows, on $L^p(\mathbb{R})$, $1 \le p \le \infty$.

1. Introduction

1.1. The Stockwell transform

The Stockwell transform, which was defined in [13], is a hybrid of the Gabor transform and the wavelet transform. For a signal $f \in L^2(\mathbb{R})$, the Stockwell transform $S_\varphi f$ with respect to the window $\varphi \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ is given by

$$(1) \qquad S_\varphi f(b,\xi) = (2\pi)^{-1/2}|\xi| \int_{-\infty}^{\infty} e^{-ix\xi} f(x)\overline{\varphi(\xi(x-b))}dx, \quad b \in \mathbb{R},\ \xi \in \mathbb{R}.$$

More precisely,

$$S_\varphi f(b,\xi) = (f, \varphi^{b,\xi}),$$

where

$$(2) \qquad \varphi^{b,\xi} = (2\pi)^{-1/2}|\xi| e^{ix\xi}\varphi(\xi(x-b)),$$

or

$$\varphi^{b,\xi} = (2\pi)^{-1/2} M_\xi T_{-b} D_\xi \varphi,$$

and $(\ ,\ )$ is the inner product in $L^2(\mathbb{R})$. Here, M_ξ, T_{-b} and D_ξ are the modulation operator, the translation operator and the dilation operator, defined by

$$(M_\xi h)(x) = e^{ix\xi}h(x),$$

$$(T_{-b}h)(x) = h(x-b),$$

$$(D_\xi h)(x) = |\xi| h(\xi x),$$

for all $x \in \mathbb{R}$ and all measurable function h on $\mathbb{R}$.

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A great amount of articles use the Stockwell transform to study applied problems, covering areas as geophysics, engineering or biomedicine (see the references list in the papers [9] and [14]). Some mathematical aspects of such a transform are studied or expanded in the papers [2, 8, 9, 10, 11, 14].

1.2. Reconstruction formula

In an attempt to reconstruct a signal f from its Stockwell spectrum $\{S_\varphi f(b,\xi) : b,\xi \in \mathbb{R}\}$, we have the following result in [8].

THEOREM 1. *Let* $\varphi \in L^2(\mathbb{R})$ *be such that* $\|\varphi\|_{L^2(\mathbb{R})} = 1$ *and*

$$\int_{-\infty}^{\infty} \frac{|\hat{\varphi}(\xi-1)|^2}{|\xi|} d\xi < \infty. \tag{3}$$

Then for all signals f *and* g *in* $L^2(\mathbb{R})$,

$$(f,g)_{L^2(\mathbb{R})} = \frac{1}{c_\varphi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} S_\varphi f(b,\xi) \overline{S_\varphi g(b,\xi)} \, \frac{db\, d\xi}{|\xi|}, \tag{4}$$

where

$$c_\varphi = \int_{-\infty}^{\infty} \frac{|\hat{\varphi}(\xi-1)|^2}{|\xi|} d\xi, \tag{5}$$

and ˆ *denotes the Fourier transform defined by*

$$\hat{F}(\zeta) = (2\pi)^{-N/2} \int_{\mathbb{R}^N} e^{-ix\cdot\zeta} F(x)\, dx$$

for all F *in* $L^1(\mathbb{R}^N)$.

REMARK 1. Theorem 1 is known as the Plancherel formula or the resolution of the identity formula for the one-dimensional Stockwell transform. The integrability condition (3) is the admissibility condition for a function φ in $L^2(\mathbb{R})$ to be a window. An important corollary of Theorem 1 is that every signal f can be reconstructed from its Stockwell spectrum by means of the inversion formula

$$f = \frac{1}{c_\varphi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (f, \varphi^{b,\xi})_{L^2(\mathbb{R})} \varphi^{b,\xi} \, \frac{db\, d\xi}{|\xi|}. \tag{6}$$

That the admissibility condition (3) is a necessary condition for the inversion formula for the Stockwell transform can be seen by letting $f = g = \varphi$ in (4). Details can be found in [7].

1.3. Localization operators

Let φ, ψ be measurable functions on $\mathbb{R}$, σ be measurable function on $\mathbb{R}^2$, then for all functions $f \in L^p(\mathbb{R})$, we define the localization operator $L_{\sigma,\varphi,\psi} f$, by

$$L_{\sigma,\varphi,\psi} f = \int_{\mathbb{R}} \int_{\mathbb{R}} \sigma(b,\xi)(S_\varphi f)(b,\xi)\psi^{b,\xi} \frac{dbd\xi}{|\xi|} = \int_{\mathbb{R}^2} \sigma(b,\xi)(f,\varphi^{b,\xi})\psi^{b,\xi} \frac{dbd\xi}{|\xi|}. \tag{7}$$

REMARK 2. The symbol can be understood as a filter of the Stockwell spectrum. Formula (6) reconstructs the signal using the Stockwell spectrum $\{S_\varphi f(b,\xi) : b,\xi \in \mathbb{R}\}$ with respect to the window component $\varphi^{b,\xi}$. The localization operator using the filtered Stockwell spectrum $\{\sigma(b,\xi)S_\varphi f(b,\xi) : b,\xi \in \mathbb{R}\}$ may be defined by

$$T_{\sigma,\varphi} f = f = \frac{1}{c_\varphi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sigma(b,\xi)(f,\varphi^{b,\xi})_{L^2(\mathbb{R})} \varphi^{b,\xi} \frac{db\,d\xi}{|\xi|}.$$

However, in order to allow some linearity properties with respect to the windows, we consider the localization operator designed in the original way (7).

In accordance with the different choices ot the symbols $\sigma(b,\xi)$ and the different continuities required, we need to impose different conditions on φ and ψ. And then we obtain an operator on $L^p(\mathbb{R})$.

In the paper [15] by Wong, the L^p-boundedness of localization operators associated to left regular representations is studied for $1 \leq p \leq \infty$. L^p-boundedness and L^p-compactness of two-wavelet localization operators on the Weyl-Heisenberg group can be found in the papers [4] by Boggiatto and Wong, and [3] by Boggiatto, Oliaro and Wong. The aim of this paper is to give another set of results on the L^p-boundedness and also L^p-compactness of the localization operators defined by (7).

In Section 2, we prove that the localization operator associated with the Stockwell transform, with symbols in $L^1(\mathbb{R})$ and windows $\varphi \in L^{p'}(\mathbb{R})$ and $\psi \in L^p(\mathbb{R})$ are bounded linear operators on $L^p(\mathbb{R})$, $1 \leq p \leq \infty$. Herein, p' is the conjugate of p, such that

$$\frac{1}{p} + \frac{1}{p'} = 1. \tag{8}$$

If the symbols are in $L^r(\mathbb{R}^2)$, $1 \leq r \leq 2$, and the admissible windows φ, ψ are in $L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, then the localization operators are proved in Section 3 to be bounded linear operators on $L^p(\mathbb{R})$, $r \leq p \leq r'$. Section 4 deals with the compactness for symbols in $L^1(\mathbb{R}^2)$. The last section treats the localization operators associated to the generalized Stockwell transform defined in [10] and [11]. Due to the close relation between the Stockwell transform and generalized Stockwell transform, all our conclusions obtained in Section 2, Section 3 and Section 4 can be applied to these localization operators.

2. Symbols in $L^1(\mathbb{R}^n)$

For $1 \le p \le \infty$, let $\sigma \in L^1(\mathbb{R}^2)$, $\varphi \in L^{p'}(\mathbb{R})$ and $\psi \in L^p(\mathbb{R})$. We are going to show that $L_{\sigma,\varphi,\psi}$ is a bounded linear operator on $L^p(\mathbb{R})$.

Let us start with the following estimates:

PROPOSITION 1. *For $1 \le p \le \infty$, let $\psi \in L^p(\mathbb{R})$ and $f \in L^{p'}(\mathbb{R})$, where p' is the conjugate of p. Then*

$$\|\psi^{b,\xi}\|_p = (2\pi)^{-1/2}|\xi|^{1/p'}\|\psi\|_p, \tag{9}$$

and

$$|S_\psi f(b,\xi)| \le (2\pi)^{-1/2}|\xi|^{1/p'}\|\psi\|_p\|f\|_{p'}. \tag{10}$$

Proof. For $p = \infty$, the first equality is trivial. For $p \ne \infty$, by Fubini's theorem, we have

$$\begin{aligned}
\|\psi^{b,\xi}\|_p &= \left\{\int \left|(2\pi)^{-1/2}|\xi|e^{ix\xi}\psi(\xi(x-b))\right|^p dx\right\}^{1/p} \\
&= (2\pi)^{-1/2}|\xi|\left\{\int |\psi(\xi(x-b))|^p dx\right\}^{1/p} \\
&= (2\pi)^{-1/2}|\xi|^{1/p'}\|\psi\|_p.
\end{aligned}$$

Applying Hölder's inequality and (9), we have

$$|S_\psi f(b,\xi)| = |(f,\psi^{b\xi})| \le \|f\|_{p'}\|\psi^{b,\xi}\|_p = (2\pi)^{-1/2}|\xi|^{1/p'}\|f\|_{p'}\|\psi\|_p.$$

□

In the following we denote with $\|\cdot\|_{B(L^p(\mathbb{R}))}$ the operator norm in the Banach space $B(L^p)$ of bounded linear operators on $L^p, 1 \le p \le \infty$.

We start with the result about the boundedness of $L_{\sigma,\varphi,\psi}$ on $L^1(\mathbb{R})$.

PROPOSITION 2. *Let $\sigma \in L^1(\mathbb{R}^2)$ and $\varphi \in L^\infty(\mathbb{R}), \psi \in L^1(\mathbb{R})$. Then $L_{\sigma,\varphi,\psi}$: $L^1(\mathbb{R}) \to L^1(\mathbb{R})$ is a bounded linear operator and*

$$\|L_{\sigma,\varphi,\psi}\|_{B(L^1(\mathbb{R}))} \le \frac{1}{2\pi}\|\sigma\|_1\|\varphi\|_\infty\|\psi\|_1.$$

Proof. For any $f \in L^1(\mathbb{R})$, by (7), (2), and (10), we have

$$\begin{aligned}
&\|L_{\sigma,\varphi,\psi} f\|_1 = \int \Big| \iint \sigma(b,\xi) S_\varphi f(b,\xi) \psi^{b,\xi}(x) \frac{db\,d\xi}{|\xi|} \Big| dx \\
\leq\ & \iiint |\sigma(b,\xi)| \Big((2\pi)^{-1/2} |\xi| \|f\|_1 \|\varphi\|_\infty \Big) \Big((2\pi)^{-1/2} |\xi| |\psi(\xi(x-b))| \Big) \frac{db\,d\xi}{|\xi|} dx \\
\leq\ & \frac{1}{2\pi} \|f\|_1 \|\varphi\|_\infty \iiint |\sigma(b,\xi)| |\psi(\xi(x-b))| |\xi|\, db\,d\xi\,dx \\
=\ & \frac{1}{2\pi} \|f\|_1 \|\varphi\|_\infty \iint |\sigma(b,\xi)| \Big(\int |\xi| |\psi(\xi(x-b))|\, dx \Big) db\,d\xi \\
=\ & \Big(\frac{1}{2\pi} \|\varphi\|_\infty \|\sigma\|_1 \|\psi\|_1 \Big) \|f\|_1,
\end{aligned}$$

which completes our proof. □

For $p \neq 1$, we have the following conclusion about the boundedness of $L_{\sigma,\varphi,\psi}$.

PROPOSITION 3. *Let $\sigma \in L^1(\mathbb{R}^2)$, $\varphi \in L^{p'}(\mathbb{R})$ and $\psi \in L^p(\mathbb{R})$. Then $L_{\sigma,\varphi,\psi} : L^p(\mathbb{R}) \to L^p(\mathbb{R})$ is a bounded linear operator for $1 \leq p < \infty$ and*

$$\|L_{\sigma,\varphi,\psi}\|_{B(L^p(\mathbb{R}))} \leq \frac{1}{2\pi} \|\sigma\|_1 \|\varphi\|_{p'} \|\psi\|_p.$$

Proof. For any $f \in L^p(\mathbb{R})$, consider the linear functional

$$T_f : L^{p'}(\mathbb{R}) \to \mathbb{C}, \quad g \mapsto (g, L_{\sigma,\varphi,\psi} f).$$

By (7), we have

$$\begin{aligned}
|(g, L_{\sigma,\varphi,\psi} f)| &= |(L_{\sigma,\varphi,\psi} f, g)| \\
&= \Big| \int \sigma(b,\xi) S_\varphi f(b,\xi) \overline{S_\psi g(b,\xi)} \frac{db\,d\xi}{|\xi|} \Big| \\
&= \int |\sigma| |S_\varphi f(b,\xi)| |S_\psi g(b,\xi)| \frac{db\,d\xi}{|\xi|}.
\end{aligned}$$

Applying Proposition 1, we have

$$\begin{aligned}
&|(g, L_{\sigma,\varphi,\psi} f)| \\
\leq\ & \int |\sigma(b,\xi)| \Big((2\pi)^{-1/2} |\xi|^{1/p} \|f\|_p \|\varphi\|_{p'} \Big) \Big((2\pi)^{-1/2} |\xi|^{1/p'} \|g\|_{p'} \|\psi\|_p \Big) \frac{db\,d\xi}{|\xi|} \\
=\ & \Big(\frac{1}{2\pi} \|\sigma(b,\xi)\|_1 \|\varphi\|_{p'} \|\psi\|_p \|f\|_p \Big) \|g\|_{p'}
\end{aligned}$$

which implies that T_f is a continuous linear functional on $L^{p'}(\mathbb{R})$, and the operator norm

$$\|T_f\|_{B(L^{p'}(\mathbb{R}))} \leq \frac{1}{2\pi} \|\sigma\|_1 \|\varphi\|_{p'} \|\psi\|_p \|f\|_p.$$

Since $T_f g = (g, L_{\sigma,\varphi,\psi} f)$, by the Riesz representation theorem, we have

$$\|L_{\sigma,\varphi,\psi} f\|_p = \|T_f\|_{B(L^{p'}(\mathbb{R}))} \leq \frac{1}{2\pi}\|\sigma\|_1\|\varphi\|_{p'}\|\psi\|_p\|f\|_p,$$

which establishes the proposition. □

To sum up the two propositions above, we have the following theorem.

THEOREM 2. *Let $\sigma \in L^1(\mathbb{R}^2)$, $\varphi \in L^{p'}(\mathbb{R})$, $\psi \in L^p(\mathbb{R})$. Then $L_{\sigma,\varphi,\psi} : L^p(\mathbb{R}) \to L^p(\mathbb{R})$ is bounded linear operator for $1 \leq p \leq \infty$ and*

$$\|L_{\sigma,\varphi,\psi}\|_{B(L^p(\mathbb{R}))} \leq \frac{1}{2\pi}\|\sigma\|_1\|\varphi\|_{p'}\|\psi\|_p.$$

3. Symbols in $L^r(\mathbb{R})$, $1 \leq r \leq 2$

In this section, we study the localization operators $L_{\sigma,\varphi,\psi}$ for symbols $\sigma \in L^r(\mathbb{R})$, $1 \leq r \leq 2$.

PROPOSITION 4. *Let ψ and φ be admissible windows, $\psi \in L^2(\mathbb{R})$ and $\varphi \in L^2(\mathbb{R})$, $\sigma \in L^2(\mathbb{R}^2)$. Then $L_{\sigma,\varphi,\psi} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is a bounded linear operator and*

$$\|L_{\sigma,\varphi,\psi}\|_{B(L^2(\mathbb{R}))} \leq \left(\frac{\sqrt{c_\varphi c_\psi}}{2\pi}\|\varphi\|_2\|\psi\|_2\right)^{1/2}\|\sigma\|_2.$$

To prove the proposition, let us start with the following lemma.

LEMMA 1. *Let ψ and φ be admissible windows, $\psi \in L^2(\mathbb{R})$ and $\varphi \in L^2(\mathbb{R})$, $\sigma \in L^\infty(\mathbb{R}^2)$. Then $L_{\sigma,\varphi,\psi} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is a bounded linear operator and*

$$\|L_{\sigma,\varphi,\psi}\|_{B(L^2(\mathbb{R}))} \leq \sqrt{c_\varphi c_\psi}\,\|\sigma\|_\infty.$$

Proof. For any $f, g \in L^2(\mathbb{R})$, by (7) and Hölder's inequality, we have

$$\begin{aligned}
|(L_{\sigma,\varphi,\psi} f, g)| &= \left|\int_{\mathbb{R}^2}\sigma(b,\xi)S_\varphi f(b,\xi)\overline{S_\psi g(b,\xi)}\frac{db\,d\xi}{|\xi|}\right| \\
&\leq \|\sigma\|_\infty\int_{\mathbb{R}^2}|S_\varphi f(b,\xi)||\overline{S_\psi g(b,\xi)}|\frac{db\,d\xi}{|\xi|} \\
&\leq \|\sigma\|_\infty\left(\int_{\mathbb{R}^2}|S_\varphi f(b,\xi)|^2\frac{db\,d\xi}{|\xi|}\right)^{1/2}\left(\int_{\mathbb{R}^2}|\overline{S_\psi g(b,\xi)}|^2\frac{db\,d\xi}{|\xi|}\right)^{1/2}.
\end{aligned}$$

By Theorem 1, we have

$$\begin{aligned}
|(L_{\sigma,\varphi,\psi} f, g)| &\leq \|\sigma\|_\infty(c_\varphi)^{1/2}(c_\psi)^{1/2}\|f\|_2\|g\|_2 \\
&= \sqrt{c_\varphi c_\psi}\,\|\sigma\|_\infty\|f\|_2\|g\|_2,
\end{aligned}$$

which completes the proof. □

Proof of Proposition 4. For any fixed $f \in L^2(\mathbb{R})$, admissible windows $\varphi, \psi \in L^2(\mathbb{R})$, we define a linear map from $L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ to $L^2(\mathbb{R})$ by

$$T(\sigma) = L_{\sigma,\varphi,\psi} f.$$

From the above lemma we have

$$\|T(\sigma)\|_2 \leq \sqrt{c_\varphi c_\psi}\|f\|_2\|\sigma\|_\infty, \tag{11}$$

and let $p = 2$ in Theorem 2, we have

$$\|T(\sigma)\|_2 \leq \left(\frac{1}{2\pi}\|f\|_2\|\varphi\|_2\|\psi\|_2\right)\|\sigma\|_1. \tag{12}$$

Applying interpolation theory, see [1] for instance, we have

$$\begin{aligned}\|T(\sigma)\|_2 &\leq (\sqrt{c_\varphi c_\psi}\|f\|_2)^{1/2}\left(\frac{1}{2\pi}\|f\|_2\|\varphi\|_2\|\psi\|_2\right)^{1/2}\|\sigma\|_2 \\ &= \left(\frac{\sqrt{c_\varphi c_\psi}}{2\pi}\|\varphi\|_2\|\psi\|_2\right)^{1/2}\|f\|_2\|\sigma\|_2.\end{aligned}$$

By the definition of $T(\sigma)$, we have

$$\|L_{\sigma,\varphi,\psi} f\|_2 \leq \left(\frac{\sqrt{c_\varphi c_\psi}}{2\pi}\|\varphi\|_2\|\psi\|_2\right)^{1/2}\|f\|_2\|\sigma\|_2.$$

Thus the proof is complete. □

THEOREM 3. *Let ψ and φ be admissible windows, $\psi \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and $\varphi \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$. Let $\sigma \in L^r(\mathbb{R}^2)$, $1 \leq r \leq 2$. Then there exists a unique bounded linear operator $L_{\sigma,\varphi,\psi} : L^p(\mathbb{R}) \to L^p(\mathbb{R})$ for all $p \in [r, r']$ such that*

$$\|L_{\sigma,\varphi,\psi}\|_{B(L^p(\mathbb{R}))} \leq M_1^{1-\theta} M_2^{\theta}\|\sigma\|_p, \tag{13}$$

where

$$M_1 = \left(\frac{1}{2\pi}\|\varphi\|_\infty\|\psi\|_1\right)^{\frac{2}{r}-1}\left(\frac{\sqrt{c_\varphi c_\psi}}{2\pi}\|\varphi\|_2\|\psi\|_2\right)^{\frac{1}{r'}},$$

$$M_2 = \left(\frac{1}{2\pi}\|\varphi\|_1\|\psi\|_\infty\right)^{\frac{2}{r}-1}\left(\frac{\sqrt{c_\varphi c_\psi}}{2\pi}\|\varphi\|_2\|\psi\|_2\right)^{\frac{1}{r'}}.$$

Proof. Let T be the bilinear mapping from $\{L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)\} \times \{L^1(\mathbb{R}) \cap L^2(\mathbb{R})\}$ to $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, defined by

$$T(\sigma, f) = L_{\sigma,\varphi,\psi} f. \tag{14}$$

By Proposition 2 and Propostion 3.1, we have

$$\|T(\sigma, f)\|_1 \leq \frac{1}{2\pi}\|\varphi\|_\infty\|\psi\|_1\|\sigma\|_1\|f\|_1,$$

$$\|T(\sigma, f)\|_2 \leq \frac{\sqrt{c_\varphi c_\psi}}{2\pi}\|\varphi\|_2\|\psi\|_2\|\sigma\|_2\|f\|_2.$$

By the multi-linear interpolation theory, see Section 10.1 in [5] for reference, we get a unique bounded linear operator $T(\sigma, f) : L^r(\mathbb{R}^2) \times L^r(\mathbb{R}) \to L^r(\mathbb{R})$ such that

$$\|T(\sigma, f)\|_r \le M_1 \|\sigma\|_r \|f\|_r, \tag{15}$$

where

$$M_1 = \left(\frac{1}{2\pi}\|\varphi\|_\infty\|\psi\|_1\right)^{1-\alpha}\left(\frac{\sqrt{c_\varphi c_\psi}}{2\pi}\|\varphi\|_2\|\psi\|_2\right)^{\alpha/2},$$

with

$$\frac{1-\alpha}{1} + \frac{\alpha}{2} = \frac{1}{r} \qquad or \qquad \alpha = 2 - \frac{2}{r}.$$

By the definition of T in (14), we have

$$\|L_{\sigma,\varphi,\psi}\|_{B(L^r(\mathbb{R}))} \le \left(\frac{1}{2\pi}\|\varphi\|_\infty\|\psi\|_1\right)^{\frac{2}{r}-1}\left(\frac{\sqrt{c_\varphi c_\psi}}{2\pi}\|\varphi\|_2\|\psi\|_2\right)^{\frac{1}{r'}}\|\sigma\|_r. \tag{16}$$

Since the adjoint of $L_{\sigma,\varphi,\psi}$ is $L_{\overline{\sigma},\overline{\psi},\overline{\varphi}}$, so $L_{\sigma,\varphi,\psi}$ is a bounded linear map on $L^{r'}(\mathbb{R})$, with its operator norm

$$\begin{aligned} \|L_{\sigma,\varphi,\psi}\|_{B(L^{r'}(\mathbb{R}))} &= \|L_{\overline{\sigma},\overline{\psi},\overline{\varphi}}\|_{B(L^r(\mathbb{R}))} \\ &\le \left(\frac{1}{2\pi}\|\varphi\|_1\|\psi\|_\infty\right)^{\frac{2}{r}-1}\left(\frac{\sqrt{c_\varphi c_\psi}}{2\pi}\|\varphi\|_2\|\psi\|_2\right)^{\frac{1}{r'}}\|\sigma\|_r. \end{aligned} \tag{17}$$

Using an interpolation of (16) and (17), we have that, for any $p \in [r, r']$,

$$\|L_{\sigma,\varphi,\psi}\|_{B(L^p(\mathbb{R}))} \le M_1^{1-\theta} M_2^{\theta}\|\sigma\|_p,$$

with

$$\frac{1-\theta}{r} + \frac{\theta}{r'} = \frac{1}{p} \qquad or \qquad \theta = \left(\frac{1}{r} - \frac{1}{p}\right)\Big/\left(\frac{1}{r} - \frac{1}{r'}\right).$$

□

4. Compact operators

In this section, we study the compactness of the localization operators $L_{\sigma,\varphi,\psi} : L^p(\mathbb{R}) \to L^p(\mathbb{R})$. We start with a simple case:

LEMMA 2. *For $1 \le p < \infty$, let $\varphi \in L^{p'}(\mathbb{R})$, σ and ψ be compactly supported and continuous. Then $L_{\sigma,\varphi,\psi} : L^p(\mathbb{R}) \to L^p(\mathbb{R})$ is compact.*

Proof. To prove that $L_{\sigma,\varphi,\psi}$ is compact, it is enough to show that the image of any bounded sequence has a convergent subsequence. Let $\{f_j\}_{j=1}^\infty$ be a sequence of functions in $L^p(\mathbb{R})$ such that

$$\|f_j\|_p \le 1, \qquad j = 1, 2, \ldots.$$

Because σ is compactly supported, we may assume that

$$\sigma(b, \xi) = 0, \qquad \text{for all } (b, \xi) \text{ such that } (|b|^2 + |\xi|^2)^{1/2} > M.$$

From Proposition 1 and the fact that ψ is continuous, we have

$$|\psi^{b,\xi}(x)| \le (2\pi)^{-1/2}|\xi|\|\psi\|_\infty,$$

$$|(S_\varphi f_j)(b,\xi)| \le (2\pi)^{-1/2}\|f_j\|_p\||\xi|^{1/p}\|\varphi\|_{p'} \le (2\pi)^{-1/2}|\xi|^{1/p}\|\varphi\|_{p'}.$$

Therefore

$$\begin{aligned} &|L_{\sigma,\varphi,\psi} f_j(x)| \\ &= \left| \iint_{\mathbb{R}^2} \sigma(b,\xi)(S_\varphi f_j)(b,\xi)\psi^{b,\xi}(x)\frac{db\,d\xi}{|\xi|}\right| \\ &\le \iint_{\substack{b\in\mathbb{R}\\ |\xi|\le M}} |\sigma(b,\xi)|((2\pi)^{-1/2}|\xi|^{1/p}\|\varphi\|_{p'})((2\pi)^{-1/2}|\xi|\|\psi\|_\infty)\frac{db\,d\xi}{|\xi|} \\ &\le \frac{1}{2\pi}\|\varphi\|_{p'}\|\psi\|_\infty \iint_{\substack{b\in\mathbb{R}\\ |\xi|\le M}} |\sigma(b,\xi)||\xi|^{1/p} db\,d\xi \\ &\le \frac{1}{2\pi}M^{1/p}\|\varphi\|_{p'}\|\psi\|_\infty\|\sigma\|_1, \end{aligned}$$

for all $j = 1,2,\ldots$. Thus the sequence $\{L_{\sigma,\varphi,\psi} f_j\}_{j=1}^\infty$ is uniformly bounded.

Let ε be any positive number. Since ψ is compactly supported and continuous, it is therefore uniformly continuous. So there exists $\delta_1 > 0$, such that

$$|\psi(x) - \psi(y)| \le \varepsilon, \qquad \text{for any } |x-y| < \delta_1.$$

Let $\delta = \min\left\{\frac{\delta_1}{1+M}, \frac{\varepsilon}{1+M}\right\}$. Then for any $|x-y| < \delta$, $|\xi| \le M$,

$$\begin{aligned} &|\psi^{b,\xi}(x) - \psi^{b,\xi}(y)| \\ &= (2\pi)^{-1/2}|\xi|\left|e^{ix\xi}\psi(\xi(x-b)) - e^{iy\xi}\psi(\xi(y-b))\right| \\ &\le (2\pi)^{-1/2}|\xi|\left(|e^{ix\xi}|\left|\psi(\xi(x-b)) - \psi(\xi(y-b))\right| + \left|e^{ix\xi} - e^{iy\xi}\right||\psi(\xi(y-b))|\right) \\ &\le (2\pi)^{-1/2}|\xi|(|\psi(\xi(x-b)) - \psi(\xi(y-b))| + |x-y||\xi|\|\psi\|_\infty) \\ &\le (2\pi)^{-1/2}|\xi|(\varepsilon + \|\psi\|_\infty\varepsilon), \end{aligned}$$

and thus for any $x, y \in \mathbb{R}$ such that $|x-y| < \delta$,

$$\begin{aligned} &|(L_{\sigma,\varphi,\psi} f_j)(x) - (L_{\sigma,\varphi,\psi} f_j)(y)| \\ &\le \frac{1}{c_{\varphi,\psi}} \iint_{\substack{b\in\mathbb{R}\\ |\xi|\le M}} |\sigma(b,\xi)||(S_\varphi f_j)(b,\xi)|\left|\psi^{b,\xi}(x) - \psi^{b,\xi}(y)\right|\frac{db\,d\xi}{|\xi|} \\ &\le \frac{1}{c_{\varphi,\psi}} \iint_{\substack{b\in\mathbb{R}|\\ \xi|\le M}} |\sigma(b,\xi)|\left((2\pi)^{-1/2}|\xi|^{1/p}\|\varphi\|_{p'}\right)\left((2\pi)^{-1/2}|\xi|(\varepsilon + \|\psi\|_\infty\varepsilon)\right)\frac{db\,d\xi}{|\xi|} \\ &\le \frac{1}{2\pi c_{\varphi,\psi}}\|\sigma\|_1\|\varphi\|_{p'}M^{1/p}(1 + \|\psi\|_\infty)\varepsilon. \end{aligned}$$

So $\{L_{\sigma,\varphi,\psi} f_j\}_{j=1}^{\infty}$ is equicontinuous on $\mathbb{R}$. Therefore for every compact subset K of $\mathbb{R}$, the Ascoli–Arzelà theorem ensures that $\{L_{\sigma,\varphi,\psi} f_j\}_{j=1}^{\infty}$ has a subsequence that converges uniformly on K. Thus by the Cantor diagonal procedure, we can find a subsequence $\{L_{\sigma,\varphi,\psi} f_{j_k}\}_{k=1}^{\infty}$ converging pointwise to a function g on $\mathbb{R}$. By (7) and (2), and the inequality (10), we have

$$|(L_{\sigma,\varphi,\psi} f_j)(x)|^p \leq ((2\pi)^{-1}\|\varphi\|_{p'})^p \left(\iint |\sigma(b,\xi)||\xi|^{1/p}|\psi(\xi(x-b))|\, db\, d\xi\right)^p.$$

Denote the function on the left hand side of the above inequality by h. By Hölder's inequality, we have

$$\begin{aligned}
&\int |h(x)|\,dx \\
&= C\int \left(\iint |\sigma(b,\xi)||\xi|^{1/p}|\psi(\xi(x-b))|\, db\, d\xi\right)^p dx \\
&= C\int \left(\iint_{|b|^2+|\xi|^2\leq M2} |\sigma(b,\xi)||\xi|^{1/p}|\psi(\xi(x-b))|\, db\, d\xi\right)^p dx \\
&\leq C\int \left(\iint \left(|\sigma(b,\xi)||\xi|^{1/p}|\psi(\xi(x-b))|\right)^p db\, d\xi\right)\cdot\left(\iint_{|b|^2+|\xi|^2\leq M2} 1^{p'}\, db\, d\xi\right)^{p/p'} dx \\
&= C(2\pi M^2)^{p/p'}(\|\sigma(b,\xi)\|_p\|\psi\|_p)^p < \infty,
\end{aligned}$$

where C is the constant $((2\pi)^{-1}\|\varphi\|_{p'})^p$. So by Lebesgue's dominated convergence theorem, the sequence $\{|L_{\sigma,\varphi,\psi} f_{j_k}|^p\}_{k=1}^{\infty}$ converges to $|g|^p$ in $L^1(\mathbb{R})$ as $k\to\infty$. And thus,

$$|L_{\sigma,\varphi,\psi} f_{j_k}(x)-g(x)|^p \leq 2^p(|L_{\sigma,\varphi,\psi} f_{j_k}(x)|^p+|g(x)|^p) \leq 2^{p+1}h(x),$$

and $|L_{\sigma,\varphi,\psi} f_{j_k}-g|^p$ converges to 0 pointwise, so by the Lebesgue's dominated convergence theorem, $\int |L_{\sigma,\varphi,\psi} f_{j_k}(x)-g(x)|^p dx$ converges to 0. Thus $\{L_{\sigma,\varphi,\psi} f_{j_k}\}_{k=1}^{\infty}$ converges to g in $L^p(\mathbb{R})$. Therefore $L_{\sigma,\varphi,\psi}$ is compact. □

PROPOSITION 5. *For* $1\leq p<\infty$, *let* $\sigma\in L^1(\mathbb{R}^2)$ *and* $\psi\in L^p(\mathbb{R}), \varphi\in L^{p'}(\mathbb{R})$. *Then* $L_{\sigma,\varphi,\psi}: L^p(\mathbb{R})\to L^p(\mathbb{R})$ *is compact.*

Proof. For any $\sigma,\tau\in L^1(\mathbb{R}^2)$, $\varphi\in L^{p'}(\mathbb{R})$ and $\psi,\phi\in L^p(\mathbb{R})$, by (7) and Theorem 2, we have

$$\begin{aligned}
\|L_{\sigma,\varphi,\psi}-L_{\tau,\varphi,\psi}\|_{B(L^p(\mathbb{R}))} &= \|L_{\sigma-\tau,\varphi,\psi}\|_{B(L^p(\mathbb{R}))} \\
&\leq (2\pi)^{-1}\|\sigma-\tau\|_1\|\varphi\|_{p'}\|\psi\|_p,
\end{aligned}$$

and

$$\begin{aligned}
\|L_{\sigma,\varphi,\psi}-L_{\sigma,\varphi,\phi}\|_{B(L^p(\mathbb{R}))} &= \|L_{\sigma,\varphi,\psi-\phi}\|_{B(L^p(\mathbb{R}))} \\
&\leq (2\pi)^{-1}\|\sigma\|_1\|\varphi\|_{p'}\|\psi-\phi\|_p.
\end{aligned}$$

By the above lemma, and the fact that $C_0(\mathbb{R}^2)$ is dense in $L^1(\mathbb{R}^2)$, and $C_0(\mathbb{R})$ is dense in $L^p(\mathbb{R})$ for $1 \le p < \infty$, and the fact that the set of compact operators is closed in $B(L^p(\mathbb{R}))$, the proposition holds. □

THEOREM 4. *Under the same hypotheses on* σ, φ, ψ *as Theorem 2, the bounded linear operator* $L_{\sigma,\varphi,\psi} : L^p(\mathbb{R}) \to L^p(\mathbb{R})$ *is compact for* $1 \le p \le \infty$.

Proof. From the previous proposition, we only need to show that the conclusion holds for $p = \infty$. In fact, the operator $L_{\sigma,\varphi,\psi} : L^\infty(\mathbb{R}) \to L^\infty(\mathbb{R})$ is the adjoint of the operator $L_{\bar{\sigma},\bar{\psi},\bar{\varphi}} : L^1(\mathbb{R}) \to L^1(\mathbb{R})$, which is compact by Proposition 5. Thus by the duality property, $L_{\sigma,\varphi,\psi} : L^\infty(\mathbb{R}) \to L^\infty(\mathbb{R})$ is compact. □

5. Localization operators associated with the modified Stockwell transform

In the papers [10, 11], the modified Stockwell transform is defined by

$$\begin{aligned} (S^s_\varphi f)(b,\xi) &= (2\pi)^{-1} \int f(x)\, e^{-ix\xi} |\xi|^{1/s} \overline{\varphi(\xi(x-b))}\, dx \\ &= (f, \varphi_s^{b,\xi}), \end{aligned} \tag{18}$$

where

$$\varphi_s^{b,\xi}(x) = e^{ix\xi} |\xi|^{1/s} \varphi(\xi(x-b)) = |\xi|^{1/s-1} \varphi^{b,\xi}(b,\xi)(x).$$

The connection between the modified Stockwell transform and Stockwell transform is

$$S^s_\varphi f = |\xi|^{1/s-1} S_\varphi f(b,\xi).$$

And so the localization operators associated with the modified Stockwell transform can be expressed by

$$\begin{aligned} L^s_{\sigma,\varphi,\psi} f &= \int_{\mathbb{R}} \int_{\mathbb{R}} \sigma(b,\xi)(S^s_\varphi f)(b,\xi) \varphi_s^{b,\xi} \frac{db\, d\xi}{|\xi|^{(2/s)-1}} \\ &= \iint_{\mathbb{R}^2} \sigma(b,\xi)(|\xi|^{1/s-1} S_\varphi f(b,\xi)(|\xi|^{1/s-1} \varphi^{b,\xi}) \frac{db d\xi}{|\xi|^{(2/s)-1}} \\ &= \iint \sigma(b,\xi) S_\varphi f(b,\xi) \psi^{b,\xi} \frac{db\, d\xi}{|\xi|} \\ &= L_{\sigma,\varphi,\psi} f. \end{aligned}$$

So our results in this paper can be extended to the localization operators associated with the modified Stockwell transform.

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