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THE KERNEL THEOREM IN ULTRADISTRIBUTIONS: MICROLOCAL REGULARITY OF THE KERNEL

Dedicated to Professor Luigi Rodino on the occasion of his 60th birthday

Abstract. In this paper we study kernels associated with continuous operators between spaces of Gevrey ultradistributions. The existence of such kernels has been established, in analogy with the kernel theorem of L. Schwartz for classical distributions, by H. Komatsu, and our aim here is to study these kernels from a microlocal point of view. The main results, which are the theorems 2, 3 below, show that there is a significant difference between the results which hold true in the case of Beurling ultradistributions and the results valid for Roumieu ultradistributions.

1. Introduction

The Schwartz kernel theorem states that the linear continuous operators T mapping $\mathcal{D}(U)$ to $\mathcal{D}'(V)$ are precisely the operators for which there is $\mathcal{K} \in \mathcal{D}'(V \times U)$ such that

$$Tu(\varphi) = \mathcal{K}(\varphi \otimes u), \quad u \in \mathcal{D}(U),\ \varphi \in \mathcal{D}(V). \tag{1}$$

(Cf. L. Schwartz, [17].) $\mathcal{K}$ is called the "kernel" of T and in this situation we write $Tu(x) = \int_U \mathcal{K}(x,y)u(y)dy$. Here U and V are open sets in $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively, $\mathcal{D}(U)$ is the space of $C_0^\infty(U)$ functions endowed with the Schwartz topology and $\mathcal{D}'(W)$ the space of distributions on W, with $W = V$ or $W = V \times U$. The Schwartz theorem has been extended to the case of ultradistributions by H. Komatsu and both L. Schwartz and H. Komatsu have also studied linear continuous operators defined on compactly supported distributions, respectively ultradistributions, to distributions or ultradistributions. We shall consider for the moment only the distribution case. The problem is then to consider a linear continuous operator $T : \mathcal{E}'(U) \to \mathcal{D}'(V)$, where $\mathcal{E}'(U)$ is the space of compactly supported distributions on U. T induces a linear continuous operator on $\mathcal{D}(U)$ and therefore it has a distributional kernel $\mathcal{K} \in \mathcal{D}'(V \times U)$. The relation (1) associates a separately continuous bilinear form $(\varphi,u) \mapsto \mathcal{K}(\varphi \otimes u)$ on $\mathcal{D}(V) \times \mathcal{D}(U)$ with T whereas the initial operator defined on $\mathcal{E}'(U)$ is associated with the bilinear form $(\varphi,u) \mapsto T(u)(\varphi)$ defined on $\mathcal{D}(V) \times \mathcal{E}'(U)$. If we want to understand the class of kernels $\mathcal{K} \in \mathcal{D}'(V \times U)$ which correspond to linear continuous operators $\mathcal{E}'(U) \to \mathcal{D}'(V)$, we may then just study the bilinear form $(\varphi,u) \mapsto \mathcal{K}(\varphi \otimes u)$ as a form on $\mathcal{D}(V) \times \mathcal{E}'(U)$. This has led to a sophisticated theory of tensor products of topological vector spaces in which the notion of "nuclear" spaces (introduced by

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A. Grothendieck) plays a central role. It turns out that most common spaces of distributions or ultradistributions are nuclear and the central result concerning the kernel theorem in distributions is that the operator $T : \mathcal{D}(U) \to \mathcal{D}'(V)$ associated with some $\mathcal{K} \in \mathcal{D}'(V \times U)$ can be extended to a linear continuous operator $\mathcal{E}'(U) \to \mathcal{D}'(V)$ if and only if $\mathcal{K}$ can be identified in a natural way with an element in $[\mathcal{D}(V)\hat{\otimes}\mathcal{E}'(U)]'$, where $\mathcal{D}(V)\hat{\otimes}\mathcal{E}'(U)$ is, say, the ε topological tensor product of $\mathcal{D}(V)$ with $\mathcal{E}'(U)$. Since the spaces under consideration are nuclear, we may as well work with the π tensor product. For definitions and details we refer to [2] and [19]. There is also an interpretation of this in terms of C^∞ functions with distributional values.

The theory of tensor products of topological vector spaces is very powerful and it explains, among other things, why kernel theorems in Banach spaces of (possibly generalized) functions must typically be more complicated than those in distributions (see e.g., [1] for some examples of kernel theorems in Lebesgue spaces): infinite dimensional Banach spaces are never nuclear. On the other hand, when one wants to consider kernel theorems in hyperfunctions, this kind of approach is not usable in practice since hyperfunctions have no reasonable topology. One may then try another approach, which has been worked out in microlocal analysis. The central notion is this time the “wave front set” of a distribution, ultradistribution, or hyperfunction (introduced in 1969 by M. Sato for hyperfunction, [15] and in 1970 by L. Hörmander for distributions, [3]). The main condition is then

$$\{(x,y,0,\eta); x \in V, y \in U, \eta \neq 0\} \cap \mathrm{WF}(\mathcal{K}) = \emptyset. \tag{2}$$

When $\mathcal{K}$ is a distribution, $\mathrm{WF}(\mathcal{K})$ stands for the C^∞ wave front set and if (2) holds then microlocal analysis gives a natural meaning to $\int_U \mathcal{K}(x,y)u(y)dy$ when $u \in \mathcal{E}'(U)$. (See [3], [20].) The same is true also in hyperfunctions if WF denotes the analytic wave front set: there is a natural meaning for $\int_U \mathcal{K}(x,y)u(y)dy$ when u is a real-analytic functional on U. Integration is then defined in terms of “integration along fibers” and $\int_U \mathcal{K}(x,y)u(y)dy$ has a meaning in hyperfunctions: see e.g., [16], [5] for details.

There is now however a fundamental difference between the two main cases contemplated by microlocal analysis, the distributional and the hyperfunctional one.

It is in fact not difficult to see that the condition (2) is not equivalent to the fact that $\mathcal{K} \in [\mathcal{D}(V)\hat{\otimes}\mathcal{E}'(U)]'$. This means that (2) is not a necessary condition when we want $\mathcal{K}$ to define a continuous operator from $\mathcal{E}'(U)$ to $\mathcal{D}'(V)$. On the other hand, it is part of the results described in [10], [11], that for hyperfunctions a reasonable operator acting from some space of analytic functionals to the space of hyperfunctions can only be defined in presence of condition (2). It seemed then natural to the present authors to look into the case of Gevrey ultradistributions and to study if microlocal conditions of type (2) are necessary for reasonable operators in ultradistributions to exist. It came, at least at first, as a surprise, that the answer depends on which type of ultradistributions one is considering: for ultradistributions of Beurling type, one may work with weaker conditions than the ones corresponding to (2), whereas for ultradistributions of Roumieu type such conditions are also necessary: see section 2 for the terminology and the theorems 2, 3 for the precise statements.

2. Definitions and main results

For the convenience of the reader, we shall now recall some of the definitions related to Gevrey-ultradistributions. (For most of the notions considered here, cf. e.g., Lions–Magenes, vol.3, section 1.3, or [14].)

Consider $s > 1$, $L > 0$, U open in $\mathbb{R}^n$ and let K be a compact set in U. We shall denote by $f \mapsto |f|_{s,L,K}$ the quasinorm

$$|f|_{s,L,K} = \sup_{\alpha\in\mathbb{N}^n} \sup_{x\in K} \frac{|(\partial/\partial x)^\alpha f(x)|}{L^{|\alpha|}(\alpha!)^s}, \tag{3}$$

defined on $\mathcal{C}^\infty(U)$. We further denote by

- $\mathcal{D}^{s,L}(K)$ the space of $\mathcal{C}^\infty$ functions f on $\mathbb{R}^n$ which vanish outside K such that for them $|f|_{s,L,K} < \infty$,
- $\mathcal{D}^{(s)}(K) = \bigcap_{L>0} \mathcal{D}^{s,L}(K)$, $\mathcal{D}^{\{s\}}(K) = \bigcup_{L>0} \mathcal{D}^{s,L}(K)$,
- $\mathcal{D}^{\{s\}}(U) = \bigcup_{K\subset U} \mathcal{D}^{\{s\}}(K)$, respectively $\mathcal{D}^{(s)}(U) = \bigcup_{K\subset U} \mathcal{D}^{(s)}(K)$,
- $\mathcal{E}^{(s)}(U) = \{f \in \mathcal{C}^\infty(U); \forall K \Subset U, \forall L > 0, |f|_{s,L,K} < \infty\}$, respectively $\mathcal{E}^{\{s\}}(U) = \{f \in \mathcal{C}^\infty(U); \forall K \Subset U, \exists L > 0, |f|_{s,L,K} < \infty\}$.

The functions in $\mathcal{E}^{\{s\}}(U)$, are called "ultradifferentiable" of Roumieu type, and those in $\mathcal{E}^{(s)}(U)$, ultradifferentiable of Beurling type, with Gevrey index s. Since we shall often encounter statements for the two types of classes which are quite similar, we now introduce the convention that we shall write $\mathcal{D}^*(U)$ when we give a statement which refers to both the case $* = (s)$ and the case $* = \{s\}$. The same convention also applies for other spaces associated with the two cases.
All the spaces mentioned above carry natural topologies:

- $\mathcal{D}^{s,L}(K)$ is a Banach space when endowed with $|\cdot|_{s,L,K}$ as a norm,
- $\mathcal{D}^{(s)}(K)$ is the projective limit (for "$L \to 0+$") of the spaces $\mathcal{D}^{s,L}(K)$, whereas $\mathcal{D}^{\{s\}}(K)$ is the inductive limit (for "$L \to \infty$") of the same spaces. The spaces $\mathcal{D}^{(s)}(K)$ are FS (i.e., Fréchet-Schwartz), whereas the spaces $\mathcal{D}^{\{s\}}(K)$ are DFS (duals of Fréchet–Schwartz). (The topological properties of these spaces are studied in [6].)
- $\mathcal{D}^{\{s\}}(U)$ is the inductive limit (for $K \subset U$) of the spaces $\mathcal{D}^{\{s\}}(K)$, whereas $\mathcal{D}^{(s)}(U)$ is the inductive limit (again for $K \subset U$) of the spaces $\mathcal{D}^{(s)}(K)$.
- We shall define topologies on $\mathcal{E}^{(s)}(U)$ and $\mathcal{E}^{\{s\}}(U)$ as follows. At first we define for $K \Subset U$ and $L > 0$ the space $Y_{K,L}$ of restrictions to K of functions in $\mathcal{C}^\infty(U)$, which satisfy $|f|_{s,L,K} < \infty$, endowed with the topology given by the semi-norm $|\cdot|_{s,L,K}$. Then,

$$\mathcal{E}^{(s)}(U) = \varprojlim_{K\Subset U} \varprojlim_{L>0} Y_{K,L}, \quad \mathcal{E}^{\{s\}}(U) = \varprojlim_{K\Subset U} \varinjlim_{L>0} Y_{K,L}.$$

We have continuous inclusions $\mathcal{D}^*(U) \subset \mathcal{E}^*(U)$ with dense image and $\mathcal{D}^*(K)$ is the subspace of $\mathcal{E}^*(U)$ ($K \subset U$) consisting of the functions with compact support lying in K.

For a systematic study of the topological properties of these spaces we refer to [13], [6]. We shall however strive to use only a minimum of results on the topological structure of the spaces we shall consider. On the other hand, we shall consider later a new class of spaces in which we can state results which can serve as a common background for both the Roumieu and the Beurling case.

Finally, we shall denote by $\mathcal{D}^{\{s\}\prime}(U)$, $\mathcal{D}^{(s)\prime}(U)$, $\mathcal{E}^{\{s\}\prime}(U)$, $\mathcal{E}^{(s)\prime}(U)$, the strong dual spaces (called Gevrey-ultradistributions of Roumieu, respectively Beurling type) of the spaces $\mathcal{D}^{\{s\}}(U)$, $\mathcal{D}^{(s)}(U)$, $\mathcal{E}^{\{s\}}(U)$, $\mathcal{E}^{(s)}(U)$.

We then also have by duality continuous inclusions

$$\mathcal{E}^{*\prime}(U) \subset \mathcal{D}^{*\prime}(U). \tag{4}$$

As for integral operators, the following remark is easy to check (cf. [6]):

- assume $\mathcal{K} \in \mathcal{D}^{*\prime}(V \times U)$. Then the prescription $T(\varphi)(\psi) = \mathcal{K}(\psi \otimes \varphi)$ defines a linear continuous operator T from $\mathcal{D}^*(U)$ to $\mathcal{D}^{*\prime}(V)$.

We shall write this as

$$(T\varphi)(x) = \int_V \mathcal{K}(x,y)\varphi(y)dy, \ \varphi \in \mathcal{D}^*(U).$$

It is part of the results proved in [6], [7], [8], that also the converse is true:

THEOREM 1 (Komatsu). *a) Any linear continuous operator $T : \mathcal{D}^*(U) \to \mathcal{D}^{*\prime}(V)$ is of form $T(\varphi)(\psi) = \mathcal{K}(\psi \otimes \varphi)$ for some $\mathcal{K} \in \mathcal{D}^{*\prime}(V \times U)$.*

b) (See [8], page 655.) Any linear continuous operator $T : \mathcal{E}^{*\prime}(U) \to \mathcal{D}^{*\prime}(V)$ is of form $T(\varphi)(\psi) = \mathcal{K}(\psi \otimes \varphi)$ for some $\mathcal{K} \in \mathcal{D}^{*\prime}(V) \otimes_\varepsilon \mathcal{E}^*(U)$. ("$\otimes_\varepsilon$" is the ε-tensor product.)

Before we can state our own results, we must still introduce the notions of Gevrey wave front sets. In order to justify them, we start from the following straightforward (and standard) result, which is in fact also central in the calculations:

REMARK 1 (See e.g., [6]). Let B be a closed ball in $\mathbb{R}^n$ (or $\mathbb{R}^m$; in the case $\mathbb{R}^m$, notation should be changed slightly).

There are constants $c > 0, c' > 0$, such that for $f \in C_0^\infty(B)$ we have

$$\sup_{\xi\in\mathbb{R}^n} |\hat{f}(\xi)| \exp[c'(|\xi|/L)^{1/s}] \leq c|f|_{s,L,B}, \ |f|_{s,c'L,B} \leq c \sup_{\xi\in\mathbb{R}^n} |\hat{f}(\xi)| \exp[(|\xi|/L)^{1/s}]. \tag{5}$$

"Hats" will denote the Fourier transform, which we define by

$$\hat{f}(\xi) = \mathcal{F} f(\xi) = \int_{\mathbb{R}^n} f(x) \exp[-i\langle x,\xi\rangle]dx.$$

The relation (5) is based on the following elementary inequality, which is valid for $d|\xi| \geq 1$:

(6) $$|\xi|^{|\alpha|} \exp[-d|\xi|^{1/s}] \leq |\xi|^{|\alpha|} \inf_{\beta} |\beta|^{|\beta|}/(d|\xi|^{1/s})^{|\beta|} \leq (4s)^{s|\alpha|} d^{-s|\alpha|} |\alpha|^{s|\alpha|};$$

the last inequality is obtained by evaluating the function $F(\beta) = |\beta|^{|\beta|} \times (d|\xi|^{1/s})^{-|\beta|}$ for $|\beta| = [s|\alpha|] + 1$, where $[s|\alpha|]$ is the integer part of $s|\alpha|$. (The factor "$4^{s|\alpha|}$" appears because of the "integer part".)
We have the following relations:

- A function $f \in C_0^\infty(\mathbb{R}^n)$ lies in $\mathcal{D}^{\{s\}}(\mathbb{R}^n)$ precisely if there are constants $c, d > 0$, such that $|\hat{f}(\xi)| \leq c \exp[-d|\xi|^{1/s}]$.
- A function $f \in C_0^\infty(\mathbb{R}^n)$ lies in $\mathcal{D}^{(s)}(\mathbb{R}^n)$ precisely if there is c and a function $\ell : \mathbb{R}^n \to \mathbb{R}_+$ such that

  (7) $$\forall d > 0, \exists c' \quad \text{s.t.} \quad d|\xi|^{1/s} \leq \ell(\xi) + c', \forall \xi \in \mathbb{R}^n,$$

  and such that

  (8) $$|\hat{f}(\xi)| \leq c \exp[-\ell(\xi)], \forall \xi \in \mathbb{R}^n.$$

- A real analytic functional u lies in $\mathcal{E}^{(s)\prime}(\mathbb{R}^n)$ if there are constants $c, d > 0$ such that $|\hat{u}(\xi)| \leq c \exp[d|\xi|^{1/s}]$.
- A real analytic functional u lies in $\mathcal{E}^{\{s\}\prime}(\mathbb{R}^n)$ if for every $d > 0$ there is a constant c such that $|\hat{u}(\xi)| \leq c \exp[d|\xi|^{1/s}]$.

A useful remark is the sub-additivity of the function $\xi \mapsto |\xi|^{1/s}$ for $s \geq 1$, that is,

(9) $$|\xi + \theta|^{1/s} \leq |\xi|^{1/s} + |\theta|^{1/s}, \quad \forall \xi, \theta \in \mathbb{R}^n.$$

We now introduce the wave front sets corresponding to the ultradistribution spaces considered above. (Cf., e.g., [4], [9], [14].)

DEFINITION 1. *a) Let $u \in \mathcal{D}^{(s)\prime}(U)$ and consider $(x^0, \xi^0) \in U \times \dot{\mathbb{R}}^n$. We shall say that $(x^0, \xi^0) \notin \mathrm{WF}_{(s)}(u)$, if we can find $\varepsilon > 0$, $v \in \mathcal{E}^{(s)\prime}(\mathbb{R}^n)$, an open convex cone Γ which contains ξ^0, $c > 0$ and a function ℓ as in (7) with the following properties:*

(10) $$u \equiv v \ \text{ on } \ |x - x^0| < \varepsilon, \quad |\hat{v}(\xi)| \leq c \exp[-\ell(\xi)] \ \text{ for } \ \xi \in \Gamma.$$

b) Let $u \in \mathcal{D}^{\{s\}\prime}(U)$. We shall say that $(x^0, \xi^0) \notin \mathrm{WF}_{\{s\}}(u)$, if we can find $\varepsilon > 0$, $v \in \mathcal{E}^{\{s\}\prime}(\mathbb{R}^n)$, an open convex cone Γ which contains ξ^0 and $c, d > 0$ such that

(11) $$u \equiv v \ \text{ on } \ |x - x^0| < \varepsilon, \quad |\hat{v}(\xi)| \leq c \exp[-d|\xi|^{1/s}] \ \text{ for } \ \xi \in \Gamma.$$

The $\mathrm{WF}_{\{s\}}(u)$, $\mathrm{WF}_{(s)}(u)$ are the Gevrey wave front sets of u of Roumieu, respectively Beurling, type with Gevrey index s.

We now state the main results.

THEOREM 2. *Let $V \times U$ be an open set in $\mathbb{R}^n \times \mathbb{R}^m$ and consider a linear continuous map $T : \mathcal{D}^{\{s\}}(U) \to \mathcal{D}^{\{s\}\prime}(V)$ given by some kernel $\mathcal{K} \in \mathcal{D}^{\{s\}\prime}(V \times U)$. Then the following statements are equivalent:*

i) T can be extended to a continuous and linear map $T : \mathcal{E}^{\{s\}\prime}(U) \to \mathcal{D}^{\{s\}\prime}(V)$.

ii) $\mathcal{K}$ satisfies the Gevrey wave front set condition of Roumieu type:

$$\mathrm{WF}_{\{s\}}(\mathcal{K}) \cap \{(x,y,0,\eta); \eta \neq 0\} = \emptyset.$$

THEOREM 3. *With V and U as before, consider a linear continuous map $T : \mathcal{D}^{(s)}(U) \to \mathcal{D}^{(s)\prime}(V)$ given by some kernel $\mathcal{K} \in \mathcal{D}^{(s)\prime}(V \times U)$. Then the following statements are equivalent:*

a) T can be extended to a continuous and linear map $T : \mathcal{E}^{(s)\prime}(U) \to \mathcal{D}^{(s)\prime}(V)$.

b) For every $(x^0,y^0) \in V \times U$ and for all $d > 0$, $\exists \varepsilon > 0$, $\exists c, \exists c_1$, and $\exists \mathcal{K}' \in \mathcal{E}^{(s)\prime}(V \times U)$ such that $\mathcal{K}' = \mathcal{K}$ on $|(x,y) - (x^0,y^0)| < \varepsilon$ and

$$|(\mathcal{F}\mathcal{K}')(\xi,\eta)| \leq c_1 \exp[-d|\eta|^{1/s}] \text{ for } |\xi| \leq c|\eta|. \tag{12}$$

REMARK 2. A comparison of condition b) in Theorem 3 with part a) of Definition 1 shows that $\mathrm{WF}_{(s)}(\mathcal{K}) \cap \{(x,y,0,\eta); \eta \neq 0\} = \emptyset$ implies b) in the theorem. We shall see later on that the converse is not true: there are kernels which satisfy condition b), but do not satisfy the wave front set condition $\mathrm{WF}_{(s)}(\mathcal{K}) \cap \{(x,y,0,\eta); \eta \neq 0\} = \emptyset$.

REMARK 3. Note that, taking into account Theorem 1, the conditions ii) and b) in the preceding theorems may be regarded as characterizations of the respective spaces $\mathcal{D}^{\{s\}\prime}(V) \otimes_\varepsilon \mathcal{E}^{\{s\}}(U)$ and $\mathcal{D}^{(s)\prime}(V) \otimes_\varepsilon \mathcal{E}^{(s)}(U)$, as subspaces of $\mathcal{D}^{\{s\}\prime}(V \times U)$ and $\mathcal{D}^{(s)\prime}(V \times U)$.

The following remark is immediate.

REMARK 4. Let $*$ denote (s) or $\{s\}$ with $s > 1$, and consider $\chi_1 \in \mathcal{D}^*(V)$, $\chi_2 \in \mathcal{D}^*(U)$. We denote by B_1 the support of χ_1 and by B_2 the support of χ_2. If $T : \mathcal{E}^{*\prime}(U) \to \mathcal{D}^{*\prime}(V)$ is a linear continuous operator, then so is $T_1 : \mathcal{E}^{*\prime}(B_2) \to \mathcal{E}^{*\prime}(B_1)$ defined by $T_1 u = \chi_1 T(\chi_2 u)$. Conversely, if all operators obtained in this way are continuous for some linear operator $T : \mathcal{E}^{*\prime}(U) \to \mathcal{D}^{*\prime}(V)$, then T is continuous. Note that if T corresponds to a kernel $\mathcal{K}(x,y)$, then T_1 corresponds to the kernel $\chi_1(x)\chi_2(y)\mathcal{K}(x,y)$. In view of this remark we may assume henceforth without loss of generality that $U = \mathbb{R}^m$, respectively that $V = \mathbb{R}^n$, and that

$$\operatorname{supp} \mathcal{K} \subset B' \times B, \tag{13}$$

for some closed balls $B \subset \mathbb{R}^m$, $B' \subset \mathbb{R}^n$.

REMARK 5. If $\mathcal{K} \in \mathcal{D}^{*\prime}(\mathbb{R}^{n+m})$ satisfies (13), then $\operatorname{supp} Tg \subset B'$ for every $g \in \mathcal{D}^*(\mathbb{R}^m)$. Conversely, if $\operatorname{supp} Tg \subset B'$ for every $g \in \mathcal{D}^*(\mathbb{R}^m)$, then $\operatorname{supp} \mathcal{K} \subset B' \times \mathbb{R}^m$.

3. Intermediate spaces and weight functions

In this section we define spaces which are intermediate between Roumieu and Beurling ultradistributions. We fix a closed ball B and consider for $f \in \mathcal{C}_0^\infty(\mathbb{R}^n)$, $u \in \mathcal{A}'(\mathbb{R}^n)$, a new set of quasinorms

$$\|f\|_{s,d} = \sup_{\xi \in \mathbb{R}^n} |\hat{f}(\xi)| \exp[d|\xi|^{1/s}],$$

$$\|u\|^{s,d} = \sup_{\xi \in \mathbb{R}^n} |\hat{u}(\xi)| / \exp[d|\xi|^{1/s}]. \tag{14}$$

(Here $\mathcal{A}'(\mathbb{R}^n)$ denotes the real-analytic functionals on $\mathbb{R}^n$.) Thus formally, $\|u\|^{s,d} = \|u\|_{s,-d}$, but the two quasinorms refer to different situations, so we wanted to make the difference visible also notationally.

DEFINITION 2. *We denote by $\mathcal{G}^{s,d}(B)$ the space of $\mathcal{C}^\infty$ functions u with support in B such that $\|u\|_{s,d} < \infty$, endowed with the norm $\|u\|_{s,d}$. In a similar way, we consider the space $\mathcal{G}_d^{s,\prime}(B)$ of ultradistributions u with support in B for which $\|u\|^{s,d} < \infty$, endowed with the norm $\|u\|^{s,d}$.*

Also note that, using the estimates (5), we have for suitable constants c', c'', the following continuous inclusions:

$$\mathcal{G}^{s,1/L^{1/s}}(B) \subset \mathcal{D}^{s,c'L}(B) \subset \mathcal{G}^{s,c''/L^{1/s}}(B), \quad \text{if } L > 0. \tag{15}$$

Thus (for fixed s) the spaces $\mathcal{G}^{s,d}(B)$ form a scale (indexed by $d > 0$) of function spaces which is essentially equivalent with the scale $\mathcal{D}^{s,L}(B)$. For example, we have

$$\mathcal{D}^{\{s\}}(B) = \varinjlim_{d>0} \mathcal{G}^{s,d}(B) \tag{16}$$

as locally convex spaces. (Also see [6].)

REMARK 6. When $f \in \mathcal{D}^{*\prime}(\mathbb{R}^n)$ has compact support and $g \in \mathcal{D}^*(\mathbb{R}^n)$, we can calculate $f(g)$ by $f(g) = (2\pi)^{-n} \int_{\mathbb{R}^n} \hat{f}(\xi)\hat{g}(-\xi)d\xi$, where the integral is the standard Lebesgue integral. (See [7].)

We now mention that $\mathcal{G}_d^{s,\prime}(B)$ is not defined as a dual space and, in some sense, the norms $\|u\|^{s,d}$ are not optimal for duality arguments. We now state a lemma that will help us to bypass this shortcoming. This is typically used for the cut-off multiplier $\chi \in \mathcal{D}^{(s)}(B'')$ for balls $B' \Subset B''$, satisfying $\chi \equiv 1$ on B'.

LEMMA 1. *Consider $\chi \in \mathcal{D}^{(s)}(\mathbb{R}^n)$, $d > 0$. Then the constants $c_1 := \|\chi\|_{s,d}$ and $c_2 := \|\hat{\chi}(\xi)\exp[d|\xi|^{1/s}]\|_{\mathcal{L}^1(\mathbb{R}^n)}$ are finite.*
a) Moreover, we have

$$\|\chi f\|_{s,d} \leq (2\pi)^{-n} c_1 \|\hat{f}(\xi)\exp[d|\xi|^{1/s}]\|_{\mathcal{L}^1(\mathbb{R}^n)}. \tag{17}$$

b) In a similar vein, we also have

$$\|\chi f\|_{s,d} \leq (2\pi)^{-n} c_2 \|f\|_{s,d}.$$

c) Finally, if h is measurable,

$$|(\hat{\chi} * h)(\xi)| \leq c_2 \|h(\xi)\exp[-d|\xi|^{1/s}]\|_{\mathcal{L}^\infty(\mathbb{R}^n)} \cdot \exp[d|\xi|^{1/s}].$$

Proof. The finiteness of the constants comes from (8). For a), we have

$$\begin{aligned}
&(2\pi)^n |\mathcal{F}(\chi f)(\xi)| \cdot \exp[d|\xi|^{1/s}] = \Big|\int_{\mathbb{R}^n} \hat{\chi}(\xi-\theta)\hat{f}(\theta)d\theta\Big| \cdot \exp[d|\xi|^{1/s}] \\
&\leq \Big|\int_{\mathbb{R}^n} \hat{\chi}(\xi-\theta)\exp[d|\xi-\theta|^{1/s}] \cdot \hat{f}(\theta)\exp[d|\theta|^{1/s}] \\
&\qquad \times \exp[d|\xi|^{1/s} - d|\theta|^{1/s} - d|\xi-\theta|^{1/s}]d\theta\Big| \\
&\leq \|\chi\|_{s,d} \cdot \|\hat{f}(\theta)\exp[d|\theta|^{1/s}]\|_{\mathcal{L}^1(\mathbb{R}^n)}.
\end{aligned}$$

Here we used the inequality $|\xi|^{1/s} \leq |\xi-\theta|^{1/s} + |\theta|^{1/s}$. See (9).

Parts b) and c) are proved with a similar argument. □

A measurable and non-negative valued function on $\mathbb{R}^n$ is called a weight function. A weight function $\varphi(\xi)$ is said to be sub-linear if it satisfies

$$\sup_{\xi\in\mathbb{R}^n} (\varphi(\xi) - \varepsilon|\xi|) < +\infty, \quad \text{for any } \varepsilon > 0.$$

In this article, we only consider radial weight functions, and we say, by abuse of notation, that a weight function is increasing when it is an increasing function of $|\xi|$.

Now consider two sub-linear weight functions $\varphi, \psi : \mathbb{R}^n \to \mathbb{R}_+$ and assume that $\psi(\theta) - |\xi-\theta|^{1/s} \leq \varphi(\xi) + c$, $\forall\xi$, $\forall\theta$, in $\mathbb{R}^n$. If $\chi \in \mathcal{D}^{(s)}(\mathbb{R}^n)$, then there exists a constant c' such that

$$\|(\hat{\chi} * h)e^{\psi}\|_{\mathcal{L}^1(\mathbb{R}^n)} \leq c' \|he^{\varphi}\|_{\mathcal{L}^1(\mathbb{R}^n)} \tag{18}$$

holds for any measurable function h. Indeed, the left hand side of (18) is estimated from above by

$$\begin{aligned}
&\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |\hat{\chi}(\theta-\xi)h(\xi)|\exp[\psi(\theta)]d\xi d\theta \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{\psi(\theta)-|\xi-\theta|^{1/s}-\varphi(\xi)} \cdot |\hat{\chi}(\theta-\xi)|e^{|\xi-\theta|^{1/s}} \cdot |h(\xi)|e^{\varphi(\xi)} d\xi d\theta \\
&\leq e^c \|\hat{\chi}(\theta)e^{|\theta|^{1/s}}\|_{\mathcal{L}^1(\mathbb{R}^n)} \cdot \|he^{\varphi}\|_{\mathcal{L}^1(\mathbb{R}^n)}.
\end{aligned}$$

REMARK 7. Our next lemma is similar to Lemma 1, c), but is more abstract and therefore less precise. We also mention that in the proof of the lemma we consider Lebesgue-spaces associated with weights. We briefly recall the terminology. We assume that we are given a continuous weight function $\varphi : \mathbb{R}^n \to \mathbb{R}_+$ and say that two measurable functions on $\mathbb{R}^n$ are equivalent if they are equal except on a set of Lebesgue measure zero. Then we denote by $\mathcal{L}^1(\mathbb{R}^n, \varphi)$ the space of equivalence classes of measurable functions on $\mathbb{R}^n$ for which the integral $\int_{\mathbb{R}^n} |f(\xi)| \exp[\varphi(\xi)] d\xi$ is finite. The norm on this space is of course

$$f \mapsto \|f\|_{\mathcal{L}^1,\varphi} = \int_{\mathbb{R}^n} |f(\xi)| \exp[\varphi(\xi)] d\xi. \tag{19}$$

If $L : \mathcal{L}^1(\mathbb{R}^n, \varphi) \to \mathbb{C}$ is a linear continuous map, then there is a measurable function h defined on $\mathbb{R}^n$ such that $L(f) = \int_{\mathbb{R}^n} f(\xi) h(\xi) d\xi$, $\forall f \in \mathcal{L}^1(\mathbb{R}^n, \varphi)$ and we have $|h(\xi)| \le \|L\|_1 \exp[\varphi(\xi)]$, for almost all $\xi \in \mathbb{R}^n$, where $\|L\|_1$ is the norm of L as a functional on $\mathcal{L}^1(\mathbb{R}^n, \varphi)$.

LEMMA 2. *Let $B' \Subset B''$ be two balls in $\mathbb{R}^n$, χ a function in $\mathcal{D}^{(s)}(B'')$ satisfying $\chi \equiv 1$ on B', and $d > 0$. Consider two sub-linear weight functions $\varphi, \psi : \mathbb{R}^n \to \mathbb{R}_+$. Assume that*

$$\int_{\mathbb{R}^n} |\mathcal{F}(\chi f)(\xi)| \exp[\psi(\xi)] d\xi \le c_1 \int_{\mathbb{R}^n} |\hat{f}(\xi)| \exp[\varphi(\xi)] d\xi, \forall f \in \mathcal{L}^2(\mathbb{R}^n), \tag{20}$$

for some constant c_1, provided the right hand side in (20) is finite. Also denote by $\mathcal{N}(B'', \psi)$ the set

$$\mathcal{N}(B'', \psi) := \{g \in \mathcal{D}^{(s)}(B''); \int_{\mathbb{R}^n} |\hat{g}(\xi)| \exp[\psi(\xi)] d\xi \le 1\}.$$

Then there is a constant c_2 such that for any $v \in \mathcal{D}^{(s)\prime}(\mathbb{R}^n)$ with $\operatorname{supp} v \subset B'$ *we have that*

$$|\hat{v}(\xi)| \le c_2 \exp[\varphi(-\xi)] \sup_{g \in \mathcal{N}(B'', \psi)} |v(g)|.$$

Proof. We define the spaces Z and Y, $Y \subset Z$, by

$$Z = \{f \in \mathcal{L}^2(\mathbb{R}^n); \int_{\mathbb{R}^n} |\hat{f}(\xi)| \exp[\varphi(\xi)] d\xi < \infty\},$$
$$Y = \{f \in Z; \|f\|_{s,d'} < \infty \text{ for all } d'\}.$$

It is easy to see that Y is dense in Z if the latter is endowed with the norm defined by $f \mapsto \|\hat{f}\|_{\mathcal{L}^1,\varphi}$: if f is given in Z, then $k \mapsto f_k = \mathcal{F}^{-1}(\exp[-(1/k)|\xi|]\hat{f})$, $k = 1, 2, \ldots$ is a sequence of functions in Y which approximates f. Now, $Y \subset \mathcal{E}^{(s)}(\mathbb{R}^n)$ and we also observe that if $\mu \in C_0^\infty(\mathbb{R}^n)$, then $\mathcal{F}^{-1}\mu \in Z$.

It suffices to construct c_2 such that

$$|\hat{v}(\xi)| \le c_2 \exp[\varphi(-\xi)]$$

holds for any $v \in \mathcal{D}^{(s)\prime}(\mathbb{R}^n)$ satisfying $\operatorname{supp} v \subset B'$ and

$$\sup_{g \in \mathcal{N}(B'', \psi)} |v(g)| \le 1. \tag{21}$$

Now we fix such a v and consider the functional $f \mapsto v(\chi f)$, which is initially defined on $\mathcal{E}^{(s)}(\mathbb{R}^n)$. For $f \in Y$, we have

$$|v(\chi f)| \leq \int_{\mathbb{R}^n} |\mathcal{F}(\chi f)(\xi)| \exp[\psi(\xi)] d\xi \leq c_1 \|\hat{f}\|_{\mathcal{L}^1,\varphi},$$

where the first inequality follows from (21), and the second from (20). Therefore, this functional can be extended, by continuity, to a linear continuous functional L on Z. Next we introduce the space $\hat{Z} = \{f \in \mathcal{L}^2(\mathbb{R}^n); \int_{\mathbb{R}^n} |f(\xi)| \exp[\varphi(\xi)] d\xi < \infty\}$, which is the image of Z under the Fourier transform. We endow $\hat{Z}$ with the norm $f \mapsto \|f\|_{\mathcal{L}^1,\varphi}$; this is of course the norm induced by the norm of Z if we use the Fourier transform to identify Z and $\hat{Z}$. The map L gives rise in this way to a linear continuous map $\hat{L}$ on $\hat{Z}$ defined by $\hat{L}(f) = L(\mathcal{F}^{-1} f)$.

Finally, we can apply the Hahn-Banach theorem to extend $\hat{L}$ to a linear continuous map defined on the space $\mathcal{L}^1(\mathbb{R}^n, \varphi)$ introduced in Remark 7, with the norm not greater than c_1. (Instead of applying the Hahn-Banach theorem, we can also use the density of Z in $\mathcal{L}^1(\mathbb{R}^n, \varphi)$.) It follows therefore from Remark 7, that $\hat{L}$ is of form $\hat{L}(f) = \int_{\mathbb{R}^n} \hat{f}(\xi) h(\xi) d\xi$, for some suitable measurable function h on $\mathbb{R}^n$ which satisfies $|h(\xi)| \leq c_1 \exp[\varphi(\xi)]$ for almost all ξ. The proof of the lemma will come to an end if we can show that $\hat{v}(\xi) = (2\pi)^n h(-\xi)$. This is the case, since

$$\int_{\mathbb{R}^n} \mu(\xi) h(\xi) d\xi = \hat{L}(\mu) = L(\mathcal{F}^{-1}\mu) = v(\chi \mathcal{F}^{-1}\mu) = v(\mathcal{F}^{-1}\mu)$$
$$= (2\pi)^{-n} \int_{\mathbb{R}^n} \hat{v}(-\xi) \mu(\xi) d\xi$$

for $\mu \in C_0^\infty(\mathbb{R}^n)$, which means that $h(\xi)$ and $(2\pi)^{-n}\hat{v}(-\xi)$ coincide as distributions. Here we have used the fact that $\operatorname{supp} v \subset B'$ and that $\chi \equiv 1$ on B''. □

COROLLARY 1. *There is a constant c' for which we have the following implication for $v \in \mathcal{D}^{(s)\prime}(\mathbb{R}^n)$ satisfying* $\operatorname{supp} v \subset B'$*:*

(22) $\quad |v(f)| \leq 1$ *for all* $f \in \mathcal{D}^{(s)}(B'')$ *with* $\|f\|_{s,d} \leq 1$, *implies* $\|v\|^{s,2d} \leq c'$.

In other words, the quasinorm $v \mapsto \|v\|^{s,2d}$ can be estimated from above by the inequality

$$\|v\|^{s,2d} \leq c' \sup_{f \in \mathcal{M}} |v(f)|$$

using the bounded set $\mathcal{M} = \{f \in \mathcal{D}^{(s)}(B''); \|f\|_{s,d} \leq 1\}$ in $\mathcal{D}^{(s)}(B'')$, and a constant c' depending only on B', B'', and d. Since, in the opposite direction, we have

$$\sup_{f \in \mathcal{M}} |v(f)| \leq c'' \|v\|^{s,d/2}$$

for some constant c'' independent of v, it is clear that the topology induced on $\mathcal{E}^{*\prime}(B)$ as a subspace of $\mathcal{D}^{*\prime}(\mathbb{R}^n)$ is given as the inductive/projective limit of the spaces $\mathcal{G}_d^{s,\prime}(B)$.

The corollary follows from Lemma 2, if we also take into account Lemma 1.

REMARK 8. The statement in the corollary is meaningful also for $v \in \mathcal{D}^{\{s\}\prime}(\mathbb{R}^n)$. In this case, we know from the very beginning that there is a constant c', which may depend on v, with $\|v\|^{s,2d} \le c'$, and the lemma just gives an estimate by duality of the norm $\|v\|^{s,2d}$.

We now consider a sequence of numbers C_j which satisfies the condition

$$j^2 \le C_j, \tag{23}$$

(other conditions on the constants C_j will be introduced in a moment) and denote by ℓ the (increasing) function

$$\ell(\xi) = \sup_j (j|\xi|^{1/s} - C_j). \tag{24}$$

REMARK 9. a) The function ℓ is well-defined since $j|\xi|^{1/s} - C_j$ is negative for $|\xi| < j$. (This implies that the "sup" is finite for every ξ.) Somewhat more specifically, $j|\xi|^{1/s} - C_j \le -j(j - |\xi|^{1/s})$ tends to $-\infty$ for $j \to \infty$ when ξ is fixed, and therefore we also see that actually, $\ell(\xi) = \max_j (j|\xi|^{1/s} - C_j)$, i.e., the "sup" is actually a "max".

b) The function ℓ clearly satisfies (7).

c) Assume now that C_j also satisfies

$$C_j \ge 4C_{[j/2]+1}, [j/2] \text{ the integer part of } j/2. \tag{25}$$

Since $k|\xi|^{1/s} - C_k \le 4\left(([k/2]+1)|\xi/2|^{1/s} - C_{[k/2]+1}\right)$, we then also have

$$\ell(\xi) \le 4\ell(\xi/2). \tag{26}$$

We recall here the fact that when one defines function spaces by inequalities of type $|\hat{f}(\xi)| \le \exp[\varphi(\xi)]$, then conditions of type $\varphi(\xi) \le c\varphi(\xi/2)$ are used (for increasing weight functions) in relation to the requirement that the function space be stable under multiplication. (When the weight functions are not increasing, the formulation of the corresponding condition is somewhat more involved. We shall not use c) in this paper. Also cf. the "ring condition" in [12].)

The condition (23) is needed to show that the function ℓ is finite. We now put further conditions on the constants C_j to show that we can make ℓ sub-linear and Lipschitz-continuous. We should mention that while the fact that ℓ is sub-linear is essential, the fact that it is Lipschitz continuous is not strictly needed in this paper. Lipschitzianity is however needed as soon as one wants to develop a theory of pseudodifferential and Fourier integral operators in spaces related to weight functions and therefore we show also in this paper that we can choose the functions ℓ with this property. (See [12].)

LEMMA 3. *a) Consider a sequence of constants $C_j \ge j^2$, and define a function $\tilde{\rho} : \mathbb{R}_+ \to \mathbb{R}_+$ by*

$$\tilde{\rho}(\tau) = \begin{cases} \sup_j (j\tau^{1/s} - C_j), & (\tau \ge 1) \\ \sup_j (j - C_j), & (\tau < 1). \end{cases} \tag{27}$$

Then $\tilde{\rho}$ *is finite. If* C_j *tends to infinity quick enough and is suitably chosen, then* $\tilde{\rho}$ *is sub-linear and Lipschitz. Moreover, we may assume that if* $s' > s$ *is fixed, then*

(28) $$\lim_{\tau\to\infty} \tilde{\rho}(\tau)/\tau^{1/s'} = 0.$$

b) Let $\tilde{\rho}$ *be as in the conclusion of part a) and denote by* $\rho : \mathbb{R}^n \to \mathbb{R}_+$ *the function* $\rho(\xi) = \tilde{\rho}(|\xi|)$. *Then* ρ *is sub-linear and Lipschitz.*

Proof. We choose a sequence of positive numbers $M_j \searrow 0$, with $M_1 = 1$. Further, we iteratively define numbers τ_j, $j \geq 0, C_j \geq j^2$, $j \geq 1$, and functions ρ_j with the following properties:

- $\tau_0 = 0, C_1 = 0, \rho_1(\tau) = \tau^{1/s}$,
- $\rho_j(\tau) = j\tau^{1/s} - C_j$,
- the sequence $j \mapsto (C_{j+1} - C_j)$ is strictly increasing,
- $\tau_j^{1/s} = C_{j+1} - C_j$, and therefore also the sequence $j \mapsto \tau_j$ is strictly increasing,
- $j(1/s)\tau^{-1+1/s} = \rho_j'(\tau) \leq M_j$ on $[\tau_{j-1}, \infty)$,
- $\rho_j(\tau) \geq \rho_{j-1}(\tau)$ for $\tau \geq \tau_{j-1}$, $\rho_j(\tau) \leq \rho_{j-1}(\tau)$ for $\tau \leq \tau_{j-1}$.

As a preparation for this, we notice that, independently of the way we choose the constants C_j, we shall have $\rho_j'(\tau) \geq \rho_{j+1}'(\tau)$, $\forall \tau$. Therefore, if τ_j is chosen with $\rho_j(\tau_j) = \rho_{j+1}(\tau_j)$, then we have $\rho_j(\tau) \geq \rho_{j+1}(\tau)$ for $\tau \leq \tau_j$, respectively $\rho_j(\tau) \leq \rho_{j+1}(\tau)$ for $\tau \geq \tau_j$. We now return to the construction of the C_j, τ_j. Note that, by our requirements, we have to set $\tau_0 = 0, C_1 = 0$. We next note that the functions $\rho_j(\tau)$ are concave and $\rho_2(\tau) = 2\tau^{1/s} - C_2$ is negative for $\tau > 0$ small, whatever the value of $C_2 > 0$ may be, whereas ρ_1 is positive. Moreover, when C_2 increases so does τ_1 given by $\tau_1^{1/s} = C_2 - C_1 = C_2$ and we fix some $C_2 \geq 2^2$ so that $2(1/s)\tau_1^{-1+1/s} \leq M_2$. This already defines ρ_2 by $\rho_2(\tau) = 2\tau^{1/s} - C_2$, and it is automatic that $\rho_2'(\tau) \leq M_2$ for every $\tau \geq \tau_1$. We may now assume that we have found C_j, τ_{j-1} and have set $\rho_j = j\tau^{1/s} - C_j$. In particular, $\rho_j(\tau) \geq \rho_{j-1}(\tau)$ for $\tau \geq \tau_{j-1}$ and $\rho_j'(\tau) \leq M_j$ for $\tau \geq \tau_j$. Next we fix $C_{j+1} \geq (j+1)^2$, large enough such that for $\tau_j^{1/s} = C_{j+1} - C_j$ we have $j(1/s)\tau_j^{-1+1/s} \leq M_j$ and set $\rho_{j+1}(\tau) = (j+1)\tau^{1/s} - C_{j+1}$.

This concludes the construction of the numbers τ_j, C_j, ρ_j by iteration. If we also want to have (28), then it suffices to choose τ_{j-1} so that $j(1/s)\tau^{1/s-1} \leq M_j\tau^{1/s'-1}$ on $[\tau_{j-1}, \infty)$.

It follows for these choices that

(29) $$\sup_k \rho_k(\tau) = \rho_j(\tau) \text{ on } [\tau_j, \tau_{j+1}] \text{ for } \tau \geq 1,$$

and we set $\tilde{\rho}(\tau) = \sup_k \rho_k(\tau)$ for such τ.

This shows that $\tilde{\rho}'(\tau) \to 0$ there where the derivative is defined (which is except the points $\tau = \tau_j$) when $\tau \to \infty$. The sub-linearity and the Lipschitz-continuity of ρ is then clear, so part a) of the lemma is proved. Part b) is an immediate consequence. □

LEMMA 4. *Let $\tilde{\ell} : \mathbb{R}^n \to \mathbb{R}_+$ be a function which satisfies (7) and denote $\mathcal{M} = \{f \in \mathcal{D}^{(s)}(B_2); \int_{\mathbb{R}^n} |\hat{f}(\xi)| \exp[\tilde{\ell}(\xi)] d\xi \leq 1\}$. Then $\mathcal{M}$ is a bounded set in $\mathcal{D}^{(s)}(\mathbb{R}^n)$.*

We apply this for "$\tilde{\ell} = \ell'/2$", where ℓ' will be constructed later on.

Proof. In view of the support condition in the definition of $\mathcal{M}$, we only need to estimate the derivatives of the elements in $\mathcal{M}$, and in fact show that for every j there is a constant $\tilde{c}_j$ such that $|(\partial/\partial x)^\alpha f(x)| \leq \tilde{c}_j j^{-s|\alpha|} |\alpha|^{s|\alpha|}$, for $f \in \mathcal{M}$. We write for this purpose for fixed j, α,

$$\begin{aligned} j^{s|\alpha|} \left| \frac{\partial^\alpha}{\partial x^\alpha} f(x) \right| &\leq j^{s|\alpha|} \sup_{\xi \in \mathbb{R}^n} |\xi^\alpha| \exp[-sj|\xi|^{1/s}] \cdot \int_{\mathbb{R}^n} |\hat{f}(\xi)| \exp[sj|\xi|^{1/s}] d\xi \\ &\leq |\alpha|^{s|\alpha|} \int_{\mathbb{R}^n} |\hat{f}(\xi)| \exp[\ell(\xi) + \ln c_j] d\xi \\ &\leq \tilde{c}_j |\alpha|^{s|\alpha|}, \end{aligned}$$

since $|\xi^\alpha| \exp[-sj|\xi|^{1/s}] \leq c' j^{-s|\alpha|} |\alpha|^{s|\alpha|}$, $\forall \alpha \in \mathbb{N}^n$. (See, e.g., the argument for studying (6). The point is that by analogy, $\exp[sj|\xi|^{1/s}] \geq (sj|\xi|^{1/s})^{s|\alpha|}/(s|\alpha|)^{|\alpha|}$. In the second inequality we have used $sj|\xi|^{1/s} \leq \ell(\xi) + \ln c_j$ for some constants c_j.) □

PROPOSITION 1. *Fix $\chi \in \mathcal{D}^{(s)}(B)$, and consider sequences of constants C_j, C'_j. Assume that $\int_{\mathbb{R}^n} |\hat{f}(\xi)| \exp[2j|\xi|^{1/s}] d\xi \leq C_{2j}$ implies $\|\chi f\|_{s,j} \leq C'_j$. (See Lemma 1.) Assume further that both sequences satisfy the condition $\ln C_j \geq j^2, \ln C'_j \geq j^2$.*

We now denote by $\ell, \ell' : \mathbb{R}^n \to \mathbb{R}_+$ the functions $\ell(\xi) = \sup_j (j|\xi|^{1/s} - \ln C_j)$, $\ell'(\xi) = \sup_j (j|\xi|^{1/s} - \ln C'_j)$. Then we have that

$$|\mathcal{F}(\chi f)(\xi)| \leq \exp[-\ell'(\xi)] \cdot \|\hat{f}(\xi) \exp \ell(\xi)\|_{\mathcal{L}^1(\mathbb{R}^n)} \tag{30}$$

and

$$\int_{\mathbb{R}^n} |\mathcal{F}(\chi f)(\xi)| \exp[\ell'(\xi)/2] d\xi \leq \|\hat{f}(\xi) \exp \ell(\xi)\|_{\mathcal{L}^1(\mathbb{R}^n)} \cdot \int_{\mathbb{R}^n} \exp[-\ell'(\xi)/2] d\xi.$$

Proof. It suffices to argue for the case $\|\hat{f}(\xi) \exp \ell(\xi)\|_{\mathcal{L}^1(\mathbb{R}^n)} = 1$. Thus, f satisfies $\|\hat{f}(\xi) \exp[j|\xi|^{1/s} - \ln C_j]\|_{\mathcal{L}^1(\mathbb{R}^n)} \leq 1$, so it follows from the assumption on C_j, that $\|\mathcal{F}(\chi f)(\xi)\|_{s,j} \leq C'_j$ for every j. This shows that

$$|\mathcal{F}(\chi f)(\xi)| \leq \inf_j \exp[-j|\xi|^{1/s} + \ln C'_j] = \exp[-\ell'(\xi)].$$

Since $\exp[-\ell'(\xi)/2]$ is integrable, we also obtain the last inequality. □

4. Kernels and the spaces $\mathcal{G}^s$

It seems natural to study the integral operator $Tu(x) = \int_{\mathbb{R}^m} \mathcal{K}(x,y)u(y)dy$ in the frame of the spaces $\mathcal{G}^s$. The conditions which we use for $\mathcal{K}$ in this section are motivated by the following considerations:

- let $\mathcal{K} \in \mathcal{D}^{(s)\prime}(\mathbb{R}^{n+m})$ have compact support. Then there is $d > 0$ and $c > 0$ such that

  (31) $$|\hat{\mathcal{K}}(\xi,\eta)| \leq c\exp[d(|\xi|^{1/s} + |\eta|^{1/s})], \forall(\xi,\eta) \in \mathbb{R}^{n+m}.$$

- From (31), condition b) in Theorem 3 is equivalent to the following:

  (32) $$\forall d'', \exists d' > 0, \exists c, \text{ s.t. } |\hat{\mathcal{K}}(\xi,\eta)| \leq c\exp[d'|\xi|^{1/s} - d''|\eta|^{1/s}].$$

Most of our arguments are based on the following simple relation:

(33) $$\mathcal{K}(\psi) = (2\pi)^{-n-m}\int_{\mathbb{R}^{n+m}} \hat{\mathcal{K}}(\xi,\eta)\hat{\psi}(-\xi,-\eta)d\xi d\eta, \psi \in \mathcal{D}^{(s)}(\mathbb{R}^{n+m}),$$

the integral being the Lebesgue integral as above. (See Remark 6.) It follows that

(34) $$\mathcal{F}(Tg)(\xi) = (2\pi)^{-m}\int_{\mathbb{R}^m} \hat{\mathcal{K}}(\xi,\eta)\hat{g}(-\eta)d\eta.$$

PROPOSITION 2. *a) Let $\mathcal{K} \in \mathcal{D}^{(s)\prime}(\mathbb{R}^{n+m})$ satisfy (13) and assume that (31) holds for some $d > 0$. Also consider $\tilde{d} > d$. Then*

$$(\varphi,u) \mapsto (Tu)(\varphi) := \int_{\mathbb{R}^{n+m}} \hat{\mathcal{K}}(\xi,\eta)\hat{\varphi}(-\xi)\hat{u}(-\eta)d\xi d\eta,$$

for $\varphi \in \mathcal{D}^{(s)}(\mathbb{R}^n)$ defines a continuous operator $T : \mathcal{G}^{s,\tilde{d}}(\mathbb{R}^m) \to \mathcal{G}_d^{s,\prime}(B')$.

b) Let $\mathcal{K}$ be a ultradistribution with support in $B' \times B$ which satisfies the estimate

(35) $$|\hat{\mathcal{K}}(\xi,\eta)| \leq \exp[d_1|\xi|^{1/s} - d_2|\eta|^{1/s}], \text{ for some constants } d_1 > 0, d_2 > 0.$$

Also fix $d_3 < d_2$, $B_1 \Supset B$. Then the correspondence

$$g \mapsto Tg(x) := \int_U \mathcal{K}(x,y)g(y)dy,$$

for $g \in \mathcal{D}^{(s)}(B_1)$, can be extended to a continuous operator $\mathcal{G}_{d_3}^{s,\prime}(B) \to \mathcal{G}_{d_1}^{s,\prime}(B')$.

Proof. We only prove b). (Part a) is proved by similar arguments but is even simpler.) We have already observed in Remark 5 that $\operatorname{supp} Tg \subset B'$. When $g \in \mathcal{D}^{(s)}(B_1)$, then $\mathcal{F}(Tg)(\xi)$ is given (34). We claim that we have for some $c > 0$ the estimate

(36) $$\|Tg\|^{s,d_1} \leq c\|g\|^{s,d_3}, \forall g \in \mathcal{D}^{(s)}(B_1).$$

To prove this, we just argue as follows:

$$\begin{aligned}&\Big|\int_{\mathbb{R}^m}\hat{\mathcal{K}}(\xi,\eta)\hat{g}(-\eta)d\eta\Big|\\&\le \exp[d_1|\xi|^{1/s}]\int_{\mathbb{R}^m}\exp[-d_1|\xi|^{1/s}+d_2|\eta|^{1/s}]|\hat{\mathcal{K}}(\xi,\eta)|\exp[-d_2|\eta|^{1/s}]|\hat{g}(-\eta)|d\eta\end{aligned}$$

and notice that

$$\int_{\mathbb{R}^m}\exp[-d_2|\eta|^{1/s}]|\hat{g}(-\eta)|d\eta\le\|g\|^{s,d_3}\int_{\mathbb{R}^m}\exp[(-d_2+d_3)|\eta|^{1/s}]d\eta.$$

We have now proved (36) and can conclude the argument by observing that we can approximate elements in $\mathcal{G}^{s,\prime}_{d_3}(B)$ with functions in $\mathcal{D}^{(s)}(B_1)$ by convolution: we fix $\kappa\in\mathcal{D}^{(s)}(y;|y|\le 1)$ with $\hat{\kappa}(0)=1$ and approximate $\hat{u}$ by $\hat{\kappa}(\eta/j)\hat{u}$. We have then for j large that $\mathcal{F}^{-1}(\hat{\kappa}(\cdot/j))*u\in\mathcal{D}^{(s)}(B_1)$ and that $\sup_\eta\exp[-d_3|\eta|^{1/s}]|(1-\hat{\kappa}(\eta/j))\hat{u}(\eta)|\to 0$ as $j\to\infty$. □

REMARK 10. The proposition gives in particular the implications ii)⇒i) in Theorem 2 and b)⇒a) in Theorem 3. See Remark 4 and Corollary 1.

To establish the remaining implications in the theorems 2, 3, we first prove a lemma (part of which will be used only in section 6):

LEMMA 5. *Let* $\chi\in\mathcal{D}^{(s)}(B')$, $\kappa\in\mathcal{D}^{(s)}(B)$ *and fix* L, d. *Then there is* $c>0$ *such that*

$$|\exp[-i\langle x,\xi\rangle]|_{s,L,B}=\sup_\alpha\frac{|\xi^\alpha|}{L^{|\alpha|}(\alpha!)^s}\le\exp[c|\xi|^{1/s}/L^{1/s}], \tag{37}$$

$$\|\chi(x)\exp[-i\langle x,\xi\rangle]\|_{s,d}\le\|\chi\|_{s,d}\exp[d|\xi|^{1/s}], \tag{38}$$

and

$$\|\kappa(y)\exp[-i\langle y,\eta\rangle]\|^{s,d}\le\|\kappa\|_{s,d}\exp[-d|\eta|^{1/s}]. \tag{39}$$

Note that (39) is an estimate referring to the spaces $\mathcal{G}^{s,\prime}_d$, although the function $y\mapsto\kappa(y)\exp[-i\langle y,\eta\rangle]$ lies in $\mathcal{D}^{(s)}(\mathbb{R}^m)$.

Proof. (37) is a direct calculation.

For (38) we have to calculate $\sup_\theta|\mathcal{F}(\chi\exp[-i\langle x,\xi\rangle])(\theta)|\exp[d|\theta|^{1/s}]$. Since $\mathcal{F}(\chi\exp[-i\langle x,\xi\rangle])(\theta)=\hat{\chi}(\theta+\xi)$, it suffices to observe that

$$\begin{aligned}\sup_\theta|\hat{\chi}(\theta+\xi)|\exp[d|\theta|^{1/s}]&\le\sup_\theta\|\chi\|_{s,d}\exp[d|\theta|^{1/s}-d|\theta+\xi|^{1/s}]\\&\le\|\chi\|_{s,d}\exp[d|\xi|^{1/s}],\end{aligned}$$

where we used $|\theta|^{1/s} \le |\theta+\xi|^{1/s} + |\xi|^{1/s}$. As for (39), we can argue similarly as

$$\begin{aligned}\|\kappa(y)\exp[-i\langle y,\eta\rangle]\|^{s,d} &= \sup_\theta |\hat{\kappa}(\theta+\eta)|\exp[-d|\theta|^{1/s}] \\ &\le \sup_\theta \|\kappa\|_{s,d}\exp[-d|\theta+\eta|^{1/s} - d|\theta|^{1/s}] \\ &\le \|\kappa\|_{s,d}\exp[-d|\eta|^{1/s}].\end{aligned}$$

Here we again used (9). □

We can now prove the following converse to part b) in Proposition 2:

PROPOSITION 3. *Let $\mathcal{K}$ be a ultradistribution with support in $B' \times B$. Denote by T the operator $Tu(x) = \int_{\mathbb{R}^m} \mathcal{K}(x,y)u(y)dy$. Assume that there are constants c, d_1, d_2 and balls B_1, B_2, $B \Subset B_1, B' \Subset B_2$, such that T can be extended to a continuous operator $\mathcal{G}^{s,\prime}_{d_1}(B_1) \to \mathcal{G}^{s,\prime}_{d_2}(B_2)$. Then the Fourier transform of $\mathcal{K}$ satisfies the estimate*

$$|\hat{\mathcal{K}}(\xi,\eta)| \le c_1 \exp[d_2|\xi|^{1/s} - d_1|\eta|^{1/s}]. \tag{40}$$

In particular, we have $|\hat{\mathcal{K}}(\xi,\eta)| \le c_1 \exp[-d_1|\eta|^{1/s}/2]$ if $|\xi| \le d_1|\eta|/(2d_2)$.

Proof. We shall obtain (40) starting from the estimate

$$\|T(\kappa(y)\exp[-i\langle y,\eta\rangle])\|^{s,d_2} \le c_2\|\kappa(y)\exp[-i\langle y,\eta\rangle]\|^{s,d_1},$$

where $\kappa \in \mathcal{D}^{(s)}(B_1)$ is identically 1 on B. On the other hand, by fixing $\chi \in \mathcal{D}^{(s)}(B_2)$ identically one on B', we have that

$$\begin{aligned}&\|T(\kappa(y)\exp[-i\langle y,\eta\rangle])\|^{s,d_2} \\ &= \sup_\xi \exp[-d_2|\xi|^{1/s}]|\mathcal{F}_{x\to\xi}(T(\kappa(y)\exp[-i\langle y,\eta\rangle]))(\xi)| \\ &= \sup_\xi \exp[-d_2|\xi|^{1/s}]|\mathcal{K}(\chi(x)\kappa(y)\exp[-i\langle x,\xi\rangle - i\langle y,\eta\rangle])| \\ &= \sup_\xi \exp[-d_2|\xi|^{1/s}]|\hat{\mathcal{K}}(\xi,\eta)|.\end{aligned}$$

The last equality follows from the fact that $\chi(x)\kappa(y)$ is identically one on the support of $\mathcal{K}$. By applying (39), we now obtain that

$$\sup_\xi \exp[-d_2|\xi|^{1/s}]|\hat{\mathcal{K}}(\xi,\eta)| \le c_3\exp[-d_1|\eta|^{1/s}],$$

which is the estimate we wanted to prove. □

There is a result dual to Proposition 3 which we now consider.

PROPOSITION 4. *Let $\mathcal{K}$ be as in the previous proposition and assume that the map $S : \mathcal{G}^{s,d_1}(B_2) \to \mathcal{G}^{s,\prime}_{d_2}(B_1)$ such that*

$$\mathcal{F}(S\varphi)(\eta) = \int \hat{\mathcal{K}}(-\xi,\eta)\hat{\varphi}(\xi)d\xi$$

maps $\mathcal{G}^{s,d_1}(B_2)$ to $\mathcal{G}^{s,d_3}(B_1)$ and is continuous as a map $\mathcal{G}^{s,d_1}(B_2) \to \mathcal{G}^{s,d_3}(B_1)$. Then there is c such that

$$|\hat{\mathcal{K}}(-\xi,\eta)| \leq c\exp[d_1|\xi|^{1/s} - d_3|\eta|^{1/s}]. \tag{41}$$

REMARK 11. Proposition 4 can be reduced to Proposition 3 by tricks, but the proof is rather simple and does not seem worth the effort this would require.

Proof of Proposition 4. Continuity of S means that there is a constant c' such that

$$\|S\varphi\|_{s,d_3} \leq c'\|\varphi\|_{s,d_1}, \forall \varphi \in \mathcal{G}^{s,d_1}(B_2). \tag{42}$$

We shall apply this for the family of functions $\varphi_{\tilde{\xi}}$ defined by

$$\varphi_{\tilde{\xi}}(x) := \chi(x)e^{-i\langle x,\tilde{\xi}\rangle},$$

where $\chi \in \mathcal{D}^{(s)}(V)$ is a fixed function with the property that $\chi \equiv 1$ on B'. Note that then $\hat{\varphi}_{\tilde{\xi}}(\xi) = \hat{\chi}(\xi+\tilde{\xi})$, so we also have $\mathcal{F}(S\varphi_{\tilde{\xi}})(\eta) = \int \hat{\mathcal{K}}(-\xi,\eta)\hat{\varphi}_{\tilde{\xi}}(\xi)d\xi = \int \hat{\mathcal{K}}(-\xi,\eta)\hat{\chi}(\xi+\tilde{\xi})d\xi$. Now, since $\chi \equiv 1$ on B', $\mathcal{F}(S\varphi_{\tilde{\xi}})(\eta)$ is just $\hat{\mathcal{K}}(\tilde{\xi},\eta)$. It follows from the continuity of S that

$$\sup_{\eta} |\hat{\mathcal{K}}(\tilde{\xi},\eta)| \exp[d_3|\eta|^{1/s}] \leq c'\|\varphi_{\tilde{\xi}}\|_{s,d_1}. \tag{43}$$

We can also write this as

$$|\hat{\mathcal{K}}(\tilde{\xi},\eta)| \leq c\exp[d_1|\tilde{\xi}|^{1/s} - d_3|\eta|^{1/s}], \tag{44}$$

if we also use (38) for $\|\varphi_{\tilde{\xi}}\|_{s,d_1}$. □

5. Proof of Theorem 3

In this section we apply Proposition 3 to prove a)⇒b) in Theorem 3. For the implication b)⇒a), see Remark 10.

As a preparation, we choose balls $B_2 \Supset B_1 \Supset B'$ in $\mathbb{R}^n$ and consider the spaces X, Y_d, where X is the space $\{v \in \mathcal{D}^{(s)\prime}(\mathbb{R}^n); \text{supp}\, v \subset B'\}$ and $Y_d = \mathcal{G}^{s,\prime}_d(B_1) = \{v; \text{supp}\, v \subset B_1, \|v\|^{s,d} < \infty\}$. The spaces Y_d are clearly Banach spaces with the natural norm and the inclusions $Y_d \subset Y_{d'}$ are continuous for $d < d'$. Moreover, $X \subset Y := \bigcup_d Y_d$. We endow X with the topology induced by $\mathcal{D}^{(s)\prime}(\mathbb{R}^n)$ and also Y with the inductive limit topology by $Y = \varinjlim_d Y_d$. It is then, in the terminology of [2], a LF-space.

We have the following result:

PROPOSITION 5. *a) The inclusion $Y \subset \mathcal{D}^{(s)\prime}(\mathbb{R}^n)$ is continuous.*
b) The inclusion $X \subset Y$ is continuous.

Proof. In all the argument we fix some $\chi \in \mathcal{D}^{(s)}(B_1)$ which is identically one on B'. Whenever we refer in the argument which follows to some result obtained in a previous section in which a cut-off function is used, it will be this one.

a) Let us first show that the inclusions $Y_d \subset \mathcal{D}^{(s)\prime}(\mathbb{R}^n)$ are continuous. Assume then that $v \mapsto \|v\|_q$ is a continuous semi-norm on $\mathcal{D}^{(s)\prime}(\mathbb{R}^n)$. There is no loss of generality to assume that it has the form $\|v\|_q = \sup_{f\in\mathcal{M}} |v(f)|$ for some bounded set $\mathcal{M} \subset \mathcal{D}^{(s)}(\mathbb{R}^n)$. It follows that there exists a ball $\tilde{B}$ such that $\mathcal{M} \subset \mathcal{D}^{(s)}(\tilde{B})$ and such that $\mathcal{M}$ is bounded in the space $\mathcal{G}^{s,d}(\tilde{B})$ for every $d > 0$.

Then from Lemma 1 b), we can see that the set $\mathcal{N} = \{\chi f; f \in \mathcal{M}\}$ is bounded in $\mathcal{G}^{s,d}(B_2)$ for every $d > 0$, and for $v \in Y_d$ we have

$$\begin{aligned}\|v\|_q = \sup_{f\in\mathcal{M}} |v(f)| = \sup_{f\in\mathcal{M}} |v(\chi f)| = \sup_{g\in\mathcal{N}} |v(g)| \\ \leq \|v\|^{s,d} \cdot \sup_{g\in\mathcal{N}} \|g\|_{s,2d} \cdot \int_{\mathbb{R}^n} \exp[-d|\xi|^{1/s}]d\xi.\end{aligned}$$

Here we used Remark 6 for the last inequality. Since the second and the last factor in the right hand side are bounded, the inclusion $Y_d \to \mathcal{D}^{(s)\prime}(\mathbb{R}^n)$ is continuous, as claimed.

b) Now let $\mathcal{U} \subset Y$ be a convex set such that its intersection with the space Y_d is a neighborhood of the origin for every $d > 0$. This means in particular that for every j we can find a constant $c''_j > 0$ such that $\{v \in Y_j; \|v\|^{s,j} \leq c''_j\} \subset \mathcal{U}$. (The constants c''_j will have to be, in general, small.) We now choose constants c_j such that $|h(\xi)| \leq c_j \exp[j|\xi|^{1/s}]$ implies that $|(\hat{\chi} * h)(\xi)| \leq 2^{-j}c''_{2j} \exp[2j|\xi|^{1/s}]$. (See Lemma 1.) Note that c_j must be small compared with c''_{2j}.

By using Corollary 1 we also see that there are constants C_j such that if $v \in \mathcal{G}^{s,\prime}_d(B_1)$ and if $f \in \mathcal{L}^2(\mathbb{R}^n)$, $\|f\|_{s,j} \leq C_j$ implies $|v(f)| \leq 1$, then $\|v\|^{s,2j} \leq c''_{2j}$ and hence $v \in \mathcal{U}$. The constants C_j will typically be large and once we have found such constants, we may increase them still further. We then assume that they are larger than $\max(1/c_j, \exp[j^2])$.

Next, we now consider an increasing sequence of positive constants C'_j for which the numbers $\ln C'_j$ satisfy (23) and for which we also have that for the sequence C_j chosen above, it follows from $\int_{\mathbb{R}^n} |\hat{f}(\xi)| \exp[2j|\xi|^{1/s}]d\xi \leq C_{2j}$ that $|\mathcal{F}(\chi f)(\xi)| \leq C'_j \exp[-j|\xi|^{1/s}]$. Again this can be obtained using Lemma 1. (In all this argument we denote "large constants" by capital letters and "small" ones, by small letters.)

We now denote $\ell(\xi) = \sup_j [j|\xi|^{1/s} - \ln C_j]$, $\ell'(\xi) = \sup_j [j|\xi|^{1/s} - \ln C'_j]$ and consider $\mathcal{M} = \{f \in \mathcal{D}^{(s)}(B_2); \int_{\mathbb{R}^n} |\hat{f}(\xi)| \exp[\ell'(\xi)/2]d\xi \leq 1\}$. $\mathcal{M}$ is then a bounded set in $\mathcal{D}^{(s)}(\mathbb{R}^n)$: see Lemma 4.

For a fixed positive constant $\tilde{c}$, it follows that the set

$$W = \{v \in X; |v(f)| \leq \tilde{c}, \forall f \in \mathcal{M}\}$$

is a neighborhood of the origin in X. To conclude the argument it will therefore suffice to show that $W \subset \mathcal{U}$ if $\tilde{c}$ is chosen suitably.

Assume then that $v \in W$, which means in particular that $v \in \mathcal{G}_d^{s,\prime}(B')$ for some d, since $X = \bigcup_d \mathcal{G}_d^{s,\prime}(B')$ as vector spaces.

Since $|v(f)| \leq \tilde{c}$ for all $f \in \mathcal{M}$ it follows combining Proposition 1 with Lemma 2 that $|\hat{v}(\xi)| \leq c''\tilde{c}\exp[\ell(\xi)]$ for some constant c'' which depends only on ℓ and ℓ'. We now put on $\tilde{c}$ the condition $c''\tilde{c} \leq 1$. Since we also know that $|\hat{v}(\xi)| \leq C\exp[d|\xi|^{1/s}]$ for some C and d, we conclude that

$$|\hat{v}(\xi)| \leq \exp[\min(\ell(\xi), d|\xi|^{1/s} + \ln C)],\ \forall \xi \in \mathbb{R}^n. \tag{45}$$

Note that the constants C and d depend on v. Now we choose a natural number $k > d+1$. If $|\xi|^{1/s}$ is large enough, say, larger than $\ln C + \ln C_k$, it follows that

$$d|\xi|^{1/s} + \ln C \leq k|\xi|^{1/s} - |\xi|^{1/s} - \ln C_k + \ln C_k + \ln C \leq k|\xi|^{1/s} - \ln C_k.$$

This shows that there is σ, which also depends on v, such that

$$|\hat{v}(\xi)| \leq \max_{j=1,\ldots,\sigma} \exp[j|\xi|^{1/s} - \ln C_j].$$

Indeed, for $|\xi|^{1/s} \geq \ln C + \ln C_k$, this is true by what we saw before if we assume $\sigma \geq k$, and for $|\xi|^{1/s} \leq \ln C + \ln C_k$, we have that $j|\xi|^{1/s} - \ln C_j \leq j(\ln C + \ln C_k) - \ln C_j \to -\infty$, with $j \to \infty$ (uniformly for the vectors ξ under consideration), such that $\ell(\xi) \leq \sup_{j \leq j^0}(j|\xi|^{1/s} - \ln C_j)$, for some j^0.

We can now find measurable functions h_j, $j = 1,\ldots,\sigma$, such that $\hat{v} = \sum_{j=1}^{\sigma} h_j$ and such that $|h_j(\xi)| \leq c_j \exp[j|\xi|^{1/s}]$. Multiplying $w_j = \mathcal{F}^{-1}h_j$ with the cut-off function χ, we obtain in this way ultradistributions $v_j = \chi w_j$, $j = 1,\ldots,\sigma$, such that $|\hat{v}_j(\xi)| \leq 2^{-j}c''_{2j}\exp[2j|\xi|^{1/s}]$ and such that $v = \sum_{j=1}^{\sigma} v_j$. Since the ultradistributions $2^j v_j$ lie in $\mathcal{U}$ and $\mathcal{U}$ is convex and contains the origin, it follows that $v \in \mathcal{U}$. This concludes the proof. □

We have now proved Proposition 5 and turn to the proof of Theorem 3. Recall that we may assume that $\operatorname{supp}\mathcal{K} \subset B' \times B$, with B and B' closed balls in $\mathbb{R}^m$, respectively $\mathbb{R}^n$. (See Remark 4.) Let us then assume that $T : \mathcal{D}^{(s)\prime}(\mathbb{R}^m) \to \mathcal{D}^{(s)\prime}(\mathbb{R}^n)$ is a continuous operator such that the restriction to $\mathcal{D}^{(s)}(\mathbb{R}^m)$ is given by the kernel $\mathcal{K}$. Since the inclusions $\mathcal{G}_d^{s,\prime}(B) \to \mathcal{D}^{(s)\prime}(\mathbb{R}^n)$ are continuous we obtain for every $d > 0$ a continuous map (denoted again T) $T : \mathcal{G}_d^{s,\prime}(B) \to \mathcal{D}^{(s)\prime}(\mathbb{R}^n)$ and consider $\chi \in \mathcal{D}^{(s)}(B_2)$ which is identically one on B'. On $\mathcal{G}_d^{s,\prime}(B)$ the operator T coincides with χT, so in particular it is trivial that T defines a continuous operator $T : \mathcal{G}_d^{s,\prime}(B) \to X$. By part b) of Proposition 5, it also defines a continuous operator $T : \mathcal{G}_d^{s,\prime}(B) \to Y$. It follows therefore from Grothendieck's theorem which we recall in a moment, that there is d' with $T(\mathcal{G}_d^{s,\prime}(B)) \subset Y_{d'}$ and such that the map $T : \mathcal{G}_d^{s,\prime}(B) \to Y_{d'}$ is continuous. At this moment we can essentially apply Proposition 3 to conclude the argument.

THEOREM 4 (Grothendieck, [2]). *Let* $\cdots \to X_i \to X_{i+1} \to \cdots$ *be a sequence of Fréchet spaces and continuous maps. Denote by* X *the inductive limit of the spaces* X_i, *by* $f_i : X_i \to X$ *the natural maps and consider a continuous linear map* $T : F \to X$ *where* F *is a Fréchet space. Assume that* X *is Hausdorff. Then there is an index* i^0 *such that* $T(F) \subset f_{i^0}(X_{i^0})$. *Moreover if* f_{i^0} *is injective, then there is a continuous map* $T^0 : F \to X_{i^0}$ *such that* T *is factorized into* $F \xrightarrow{T^0} X_{i^0} \xrightarrow{f_{i^0}} X$.

6. Proof of Theorem 2

In this section we prove the implication i)⇒ii) in Theorem 2. For the implication ii)⇒i), see Remark 10.

PROPOSITION 6. *Let* $S : \mathcal{D}^{\{s\}}(B) \to \mathcal{D}^{\{s\}}(B')$ *be a continuous integral operator associated with a kernel* $\mathcal{K}$ *with support in* $B' \times B$, B, B', *balls in* $\mathbb{R}^m$, *respectively* $\mathbb{R}^n$, *and fix* $d > 0$. *Then there is* $d' > 0$ *such that* S *induces a continuous operator* $\mathcal{G}^{s,d}(B) \to \mathcal{G}^{s,d'}(B')$.

Proof. Using (16), we have a continuous operator from a Banach space to a countable inductive limit of Banach spaces:

$$\mathcal{G}^{s,d}(B) \to \varinjlim_{d>0} \mathcal{G}^{s,d}(B) \xrightarrow{S} \varinjlim_{j \in \mathbb{N}} \mathcal{G}^{s,j}(B'),$$

where the first map is the standard inclusion given by the definition of an inductive limit. Then the conclusion follows from Theorem 4. □

Proof of Theorem 2. The assumption is that $Tu(x) = \int \mathcal{K}(x,y)u(y)dy$ is a linear continuous operator $\mathcal{E}^{\{s\}\prime}(U) \to \mathcal{D}^{\{s\}\prime}(V)$. Since we can multiply with cut-off functions in the x and in the y variables, there is again no loss of generality to assume that $U = \mathbb{R}^m$, $V = \mathbb{R}^n$ and that $\operatorname{supp} \mathcal{K} \subset B' \times B$ for two balls $B' \subset \mathbb{R}^n$, $B \subset \mathbb{R}^m$. By duality, we obtain then a continuous operator $S : \mathcal{D}^{\{s\}}(\mathbb{R}^n) \to \mathcal{E}^{\{s\}}(\mathbb{R}^m)$ defined by

$$\varphi(x) \mapsto (S\varphi)(y) = \int \mathcal{K}(x,y)\varphi(x)dx.$$

From the support condition, the image of S is included in $\mathcal{D}^{\{s\}}(B)$, and S becomes a continuous operator

$$S : \mathcal{D}^{\{s\}}(B') \to \mathcal{D}^{\{s\}}(B),$$

since the topology of $\mathcal{D}^{\{s\}}(B)$ is equal to the one induced by the inclusion $\mathcal{D}^{\{s\}}(B) \subset \mathcal{E}^{\{s\}}(\mathbb{R}^m)$. It follows therefore from Proposition 6 that if we fix $d' > 0$, then there is $d > 0$ such that S induces a continuous operator $\mathcal{G}^{s,d'}(B') \to \mathcal{G}^{s,d}(B)$. The conclusion in the theorem is then a consequence of Proposition 4. □

7. An example and some comments

In this section we give an example of a distribution which satisfies condition b) in theorem 3, but does not satisfy a wave front set condition of form $\mathrm{WF}_{(s)}(\mathcal{K}) \cap \{(x,y,0,\eta);\ \eta \neq 0\} = \emptyset$.

We shall work for $n = m = 1$, on $V \times U = T^2 = T \times T$, T the one-dimensional torus. Since we are dealing with a non-quasianalytic setup, there is no real loss of generality in doing so. (We say something about this in Remark 12 below.) On the other hand, working on the torus makes the example a little bit simpler.

We denote $\exp[-k^{1+1/s}/j]$, for $j \in \mathbb{N}$, $k \in \mathbb{N}$, by a_{jk} and define the distribution $\mathcal{K}$ on T^2 by

$$\mathcal{K}(x,y) = (2\pi)^{-2} \sum_{j,k=1}^{\infty} a_{jk} \exp[i(jx+ky)]. \tag{46}$$

(The numbers a_{jk} are thus the Fourier coefficients of $\mathcal{K}$ and convergence in (46) is in the space of classical distributions.) It is immediate that $\mathcal{K}$ defines a continuous operator $L : \mathcal{D}^{(s)\prime}(T) \to \mathcal{D}^{(s)\prime}(T)$ by

$$Lu = (2\pi)^{-1} \sum_{j=1}^{\infty} b_j \exp[ijx], \quad b_j = \sum_{k=1}^{\infty} a_{jk}\hat{u}(-k) \tag{47}$$

where $\hat{u}(k) = u(\exp[-iyk])$ are the Fourier coefficients of u and and convergence in the first part of (47) is in the space of ultradistributions.

We claim that we have

PROPOSITION 7. *Let $\mathcal{K}$ be the kernel defined by (46). Then there is $(x^0,y^0) \in T^2$ such that $((x^0,y^0),(0,1)) \in \mathrm{WF}_{(s)}(\mathcal{K})$. (Also see Remark 13 below.)*

Thus $\mathcal{K}$ defines a continuous operator $\mathcal{D}^{(s)\prime}(T) \to \mathcal{D}^{(s)\prime}(T)$, but we do not have $\mathrm{WF}_{(s)}(\mathcal{K}) \cap \{(x,y,0,\eta); x \in T, y \in T, \eta \neq 0\} = \emptyset$.

To prove Proposition 7, we first state

PROPOSITION 8. *Consider $w \in \mathcal{D}^{(s)\prime}(T^2)$ and suppose that for some (x^0,y^0), $((x^0,y^0),(0,1)) \notin \mathrm{WF}_{(s)}(w)$. Then there is $\varepsilon > 0$ such that if $\chi \in \mathcal{D}^{(s)}(\mathbb{R}^2)$ is supported in an ε-neighborhood of (x^0,y^0), then $|\mathcal{F}(\chi w)(\xi,\eta)| \leq \exp[-\ell(\xi,\eta)]$ for some sublinear function ℓ as in (7) when (ξ,η) is in a suitably small conic neighborhood of $(0,1)$.*

The proof of this proposition is straightforward and is similar e.g., to the proof of lemma 1.7.3 in [14]. We omit details.

We can now prove Proposition 7. In fact, arguing by contradiction and using the preceding proposition, we can find a partition of unity formed of functions χ_i, $i = 1,\ldots,\sigma$, in $\mathcal{D}^{(s)}(T^2)$ such that for some conic neighborhood Γ of $(0,1)$ in $\mathbb{R}^2$ and some function ℓ as in (7) we have $|\mathcal{F}(\chi_i\mathcal{K})(\xi,\eta)| \leq \exp[-\ell(\xi,\eta)]$ for $(\xi,\eta) \in \Gamma$ and

$i = 1, \ldots, \sigma$. Since $a_{jk} = \sum_{i=1}^{\sigma} \mathcal{F}(\chi_i \mathcal{K})(j,k)$ it would follow that $|a_{jk}| \leq \sigma \exp[-\ell(j,k)]$ when $(j,k) \in \Gamma$, which is false.

REMARK 12. We have argued on the torus but we can now also immediately obtain from this an example of a kernel $\mathcal{K}'$ defined on $\mathbb{R} \times \mathbb{R}$ which satisfies condition b), but not the wave front set relation $\mathrm{WF}_{(s)}(\mathcal{K}) \cap \{(x,y,0,\eta); x \in \mathbb{R}, y \in \mathbb{R}, \eta \neq 0\} = \emptyset$. To simplify notations, we first observe that after a translation on the torus, it follows from above that there are kernels which define linear continuous maps $\mathcal{D}^{(s)\prime}(T) \to \mathcal{D}^{(s)\prime}(T)$, but with $((0,0),(0,1)) \in \mathrm{WF}_{(s)}(\mathcal{K})$. Next, pick $\psi \in \mathcal{D}^{(s)}(\mathbb{R}^2)$ which has support in a small neighborhood of $0 \in \mathbb{R}^2$ with $\psi \equiv 1$ in a still smaller neighborhood of 0. If $\mathcal{K} \in \mathcal{D}'(T^2)$ is the one just introduced above, then $\mathcal{K}' = \psi\mathcal{K}$ has a natural interpretation as a distribution on $\mathbb{R}^2$. Since $\mathcal{K}$ gave rise to a linear continuous operator $\mathcal{D}^{(s)\prime}(T) \to \mathcal{D}^{(s)\prime}(T)$, $\mathcal{K}'$ defines in a natural way a linear continuous operator $\mathcal{D}^{(s)\prime}(\mathbb{R}) \to \mathcal{D}^{(s)\prime}(\mathbb{R})$. It clearly does not satisfy the wave front set condition we would like to have.

REMARK 13. With a small extra effort, we can show that actually $((0,0),(0,1)) \in \mathrm{WF}_{(s)}(\mathcal{K})$, $\mathcal{K}$ the one defined in (46). To prove this it is essential that the coefficients a_{jk} are positive. We leave the details to the reader.

REMARK 14. The arguments in this paper can in principle be extended to more general classes of non-quasianalytic ultradistributions but we have not tried to work out such cases.

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