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EVOLUTION FOR OVERDETERMINED SYSTEMS IN (SMALL) GEVREY CLASSES

Dedicated to Professor Luigi Rodino on the occasion of his 60th birthday

Abstract. Given a system of linear partial differential operators with constant coefficients whose affine algebraic varieties $V(\check{\wp})$ have dimension 1, we establish in which classes of (small) Gevrey functions the associated Cauchy problem admits at least one solution, looking at the Puiseux series expansions on the branches at infinity of the algebraic curves $V(\check{\wp})$. We focus, in particular, on the case of two variables, giving some examples.

1. Introduction and main theorems

Let $A_0(D)$ be an $a_1 \times a_0$ matrix of linear partial differential operators with constant coefficients in the N indeterminates $z_1,\ldots,z_N$.

To allow different scales of regularity in the time-variables t and in the space-variables x, we split $\mathbb{R}^N_z \simeq \mathbb{R}^k_t \times \mathbb{R}^n_x$ and consider then the spaces of (ultra-)differentiable functions of Beurling type

$$\begin{aligned}\mathcal{E}_\omega(\mathbb{R}^N) = \ &\{f \in \mathcal{E}(\mathbb{R}^N) : \forall K \subset\subset \mathbb{R}^N\ \forall \varepsilon > 0\ \exists c > 0 : \\ &\sup_K |D_t^\beta D_x^\gamma f(t,x)| \leq c\varepsilon^{|\gamma|+|\beta|}(\beta!)^{1/\alpha_1}(\gamma!)^{1/\alpha_2}\ \forall \gamma \in \mathbb{N}_0^n, \beta \in \mathbb{N}_0^k\},\end{aligned}$$

where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $0 \leq \alpha_j < 1$. If $\alpha_1 = \alpha_2 = 1/s$ with $s > 1$ this space coincides with the space of (small) Gevrey functions of order s. If $\alpha_1 = \alpha_2 = 0$ it is identified with the space $\mathcal{E}(\mathbb{R}^N)$ of smooth functions. We assume in the following that $\alpha_1 = 0$ if $\alpha_2 = 0$, so that we allow ultradifferentiability in all variables or only in the space-variables, but not only in the time-variables.

We want to consider the Cauchy problem for $A_0(D)$ in these classes of (ultra-)differentiable functions with initial data on $\{(t,x) \in \mathbb{R}^k \times \mathbb{R}^n : t = 0\}$. In order to avoid the problem of formal coherence of the initial data, which can be particularly intricate if the system is overdetermined, we allow Whitney functions as initial data, which means that we give functions with all their normal derivatives on $\{t = 0\}$. By Whitney's extension theorem it is not restrictive to give zero initial-data, so that we are concerned with the following (overdetermined) Cauchy problem:

$$(1)\qquad \begin{cases} \text{given } f \in \mathcal{E}_\omega(\mathbb{R}^N)^{a_1} \\ \text{find } \varphi \in \mathcal{E}_\omega(\mathbb{R}^N)^{a_0} \text{ such that} \\ A_0(D)\varphi = f \\ D_t^\alpha \varphi(0,x) = 0 \quad \forall \alpha \in \mathbb{N}_0^k, \forall x \in \mathbb{R}^n. \end{cases}$$

Let $\theta = (\tau,\zeta) \in \mathbb{C}^k \times \mathbb{C}^n$ be the dual coordinates of $z = (t,x)$ and denote by $\mathcal{P} = \mathbb{C}[\theta_1,\ldots,\theta_N]$ the ring of complex polynomials in the N indeterminates $\theta_1,\ldots,\theta_N$. By the formal substitution $\theta_j \leftrightarrow D_j = \frac{1}{i}\frac{\partial}{\partial z_j}$ we can associate to the operator $A_0(D)$ the $\mathcal{P}$-homomorphism $A_0(\theta)$ and insert it into a Hilbert resolution of the $\mathcal{P}$-module $\mathcal{M} = \operatorname{coker}{}^t A_0(\theta)$:

$$0 \to \mathcal{P}^{a_d} \xrightarrow{{}^tA_{d-1}(\theta)} \mathcal{P}^{a_{d-1}} \longrightarrow \ldots \longrightarrow \mathcal{P}^{a_2} \xrightarrow{{}^tA_1(\theta)} \mathcal{P}^{a_1} \xrightarrow{{}^tA_0(\theta)} \mathcal{P}^{a_0} \longrightarrow \mathcal{M} \to 0.$$

When the map $A_1(\theta)$ is not trivial the system (1) is overdetermined and in order to be solvable f must satisfy the compatibility conditions

$$\begin{cases} A_1(D)f = 0 \\ D_t^\alpha f(0,x) = 0 \quad \forall\alpha \in \mathbb{N}_0^k,\ \forall x \in \mathbb{R}^n. \end{cases} \tag{2}$$

We say that the pair $(\mathbb{R}^n_x, \mathbb{R}^k_t \times \mathbb{R}^n_x)$ is *of evolution* for $A_0(D)$ (or for $\mathcal{M}$) in the class $\mathcal{E}_\omega$ if the Cauchy problem (1) admits at least one solution φ for each datum f satisfying the compatibility conditions (2).

Let us denote by $V = V(\check{\wp})$, for $\wp \in \operatorname{Ass}(\mathcal{M})$, the algebraic variety

$$V(\check{\wp}) = \{\theta \in \mathbb{C}^N : \ p(-\theta) = 0\ \forall p \in \wp\}.$$

It was proved in [3], [4] that evolution is equivalent to the validity of the following Phragmén-Lindelöf principle for every $V = V(\check{\wp})$ with $\wp \in \operatorname{Ass}(\mathcal{M})$:

$$PL(\omega) \quad \begin{cases} \exists A > 0 \text{ such that } \forall v \in \mathrm{PSH}(V) \text{ satisfying, for some } \alpha_v > 0, \\ \begin{cases} (\alpha) \quad v(\tau,\zeta) \le |\operatorname{Im}\tau| + |\operatorname{Im}\zeta| + \omega(\tau,\zeta) \quad \forall(\tau,\zeta) \in V \\ (\beta) \quad v(\tau,\zeta) \le \alpha_v(|\operatorname{Im}\zeta| + \omega(\tau,\zeta) + 1) \quad \forall(\tau,\zeta) \in V \end{cases} \\ \text{then } v \text{ must also satisfy} \\ (\gamma) \quad v(\tau,\zeta) \le A(|\operatorname{Im}\zeta| + \omega(\tau,\zeta) + 1) \quad \forall(\tau,\zeta) \in V, \end{cases}$$

where $\mathrm{PSH}(V)$ is the set of plurisubharmonic functions on V (cf. [2]), and $\omega(\tau,\zeta) = \sigma_{\alpha_1}(|\tau|) + \sigma_{\alpha_2}(|\zeta|)$ is the *weight function* defined, for $0 \le \alpha_1, \alpha_2 < 1$ and $t \ge 0$, by

$$\sigma_\alpha(t) = \begin{cases} t^\alpha & \text{if } 0 < \alpha < 1 \\ \log(1+t) & \text{if } \alpha = 0. \end{cases}$$

When the algebraic variety V has dimension one, i.e. is an algebraic curve, we can describe its branches at infinity by means of Puiseux series expansions. It turns out that the orders α_1, α_2 for which V satisfies $PL(\omega)$ are strictly related to the coefficients and the exponents of the Puiseux series expansions on its branches at infinity. This seems particularly useful since Puiseux series expansions can be computed by several programs, such as MAPLE, for instance.

Given an algebraic curve $V \subset \mathbb{C}^N \simeq \mathbb{C}^k_\tau \times \mathbb{C}^n_\zeta$ with cone of limiting directions

$$V^h = \bigcup_{j=1}^{\ell} V_j = \bigcup_{j=1}^{\ell} \mathbb{C} \cdot v_j$$

for $v_j = (\tau_j^o, \zeta_j^o) \in (\mathbb{C}^k \times \mathbb{C}^n) \setminus \{(0,0)\}$, there are two kinds of Puiseux series expansions on the branches of V near infinity, depending on whether their cone of limiting directions V_j is contained in $\mathbb{C}^k \times \{0\}$ or not. More precisely (cf. Lemma 3.6 of [2]):

1) If $V_j \not\subset \mathbb{C}^k \times \{0\}$ and, for instance, the first component of ζ_j^o is not zero, then on the branches W of V with cone of limiting directions V_j we have a Puiseux series expansion of the form

$$(3) \qquad (\tau, \zeta_1, \zeta') = (\tau_j^o, 1, a)\zeta_1 + \sum_{\nu=-\infty}^{\kappa} (D_\nu, 0, E_\nu)\zeta_1^{\nu/m}, \qquad |\zeta_1| \gg 1$$

where $\zeta' = (\zeta_2, \ldots, \zeta_n)$, $m \in \mathbb{N}$, $\kappa \in \mathbb{Z} \cup \{-\infty\}$, $\kappa < m$, $D_\nu \in \mathbb{C}^k$, $a, E_\nu \in \mathbb{C}^{n-1}$ for all $\nu \leq \kappa$.

2) If $V_j \subset (\mathbb{C}^k \setminus \{0\}) \times \{0\}$ and, for instance, the first component of τ_j^o is not zero, then on the branches W of V with cone of limiting directions V_j we have a Puiseux series expansion of the form

$$(4) \qquad (\tau_1, \tau', \zeta) = (1, 0, 0)\tau_1 + \sum_{\nu=-\infty}^{p'} (0, F_\nu, G_\nu)\tau_1^{\nu/q}, \qquad |\tau_1| \gg 1$$

where $\tau' = (\tau_2, \ldots, \tau_k)$, $q \in \mathbb{N}$, $p' \in \mathbb{Z} \cup \{-\infty\}$, $p' < q$, $F_\nu \in \mathbb{C}^{k-1}$, $G_\nu \in \mathbb{C}^n$ for all $\nu \leq p'$.

Note that all the indices and the coefficients in (3) and (4) depend on the branches W (cf. [2]), so that we should write $\kappa = \kappa(W)$, $p' = p'(W)$, etc.

Moreover, we can multiply the coefficients D_ν, E_ν in (3) by ω_m^ν and the coefficients F_ν, G_ν in (4) by ω_q^ν (where ω_m and ω_q are, respectively, any m-th root and any q-th root of unity), obtaining an equivalent representation of W.

On each of these branches we have several necessary and/or sufficient conditions for $PL(\omega)$ to be valid (cf. [2]). In the case of one time-variable (and one or more space-variables) these necessary and sufficient conditions perfectly fit, so that we have a complete characterization of systems which are of evolution in $\mathcal{E}_\omega$. In this case the Puiseux series expansion (3) is of the form

$$(5) \qquad \begin{cases} \tau(\zeta_1) = \tau_j^o \zeta_1 + \displaystyle\sum_{\nu=-\infty}^{s} D_\nu \zeta_1^{\nu/m} \\ \zeta'(\zeta_1) = a\zeta_1 + \displaystyle\sum_{\nu=-\infty}^{t} E_\nu \zeta_1^{\nu/m} \end{cases} \qquad |\zeta_1| \gg 1$$

where $s = \max\{\nu \leq \kappa :\ D_\nu \neq 0\}$, $t = \max\{\nu \leq \kappa :\ E_\nu \neq 0\}$ (and the maximum of the empty set is defined, here and in the following, as $-\infty$).

The Puiseux expansion (4) is of the form

$$(6) \qquad \zeta(\tau) = \sum_{\nu=-\infty}^{p} G_\nu \tau^{\nu/q}, \qquad |\tau| \gg 1$$

with $p = \max\{\nu \le p' : G_\nu \neq 0\}$. If $G_p \in \mathbb{C}\mathbb{R}^n$ then we can assume, up to a real linear change of variables, that $G_p = (G_{p,1}, 0, \dots, 0)$ and hence obtain from (6) (cf. [1], [2]):

$$(7) \qquad \tau(\zeta_1) = \sum_{\nu=-\infty}^{q} A_\nu \zeta_1^{\nu/p} = \sum_{\nu=-\infty}^{q} A_\nu \zeta_1^{a\nu/r}, \qquad |\zeta_1| \gg 1,\ A_q = \left(\frac{1}{G_{p,1}^{1/p}}\right)^q$$

and, for $n \ge 2$,

$$\zeta'(\zeta_1) = \sum_{\nu=-\infty}^{P} B_\nu \zeta_1^{\nu/Q} = \sum_{\nu=-\infty}^{P} B_\nu \zeta_1^{b\nu/r} \qquad |\zeta_1| \gg 1$$

for $P \in \mathbb{Z}$, $Q \in \mathbb{N}$, $P < Q$, $r = ap = bQ$ the least common multiple of p and Q ($r := p$ if $n = 1$).

Let us define, for $\zeta_o \in \{-1, 1\}$, for any branch f_m of the m-th root and for any branch f_r of the r-th root:

$$\begin{aligned}
&u(\zeta_o, f_m) = \max\{\nu \le t : \ \mathrm{Im}(E_\nu f_m(\zeta_o)^\nu) \neq 0\}\\
&w(\zeta_o, f_m) = \max\{\nu \le s : \ \mathrm{Im}(D_\nu f_m(\zeta_o)^\nu) \neq 0\}\\
&w_0 = \max\{w(\zeta_o, f_m) : \zeta_o \in \{-1,1\}, f_m \text{ a branch of the } m\text{-th root},\\
&\qquad\qquad\qquad\qquad w(\zeta_o, f_m) > \max\{0, u(\zeta_o, f_m)\}\}\\
&\mu(\zeta_o, f_r) = \max\{\nu < q : \ \mathrm{Im}(A_\nu f_r(\zeta_o)^{a\nu}) \neq 0\}\\
&\mu^* = \max\Big\{\mu(\zeta_o, f_r) : \ \zeta_o \in \{-1,1\},\ f_r \text{ a branch of the } r\text{-th root},\\
&\qquad\qquad \mu(\zeta_o, f_r) > q - p, \text{ and}\\
&\qquad\qquad \mathrm{Im}(B_\nu f_r(\zeta_o)^{b\nu}) = 0 \quad \forall \nu \ge Q\left(1 - \frac{q - \mu(\zeta_o, f_r)}{p}\right)\Big\},
\end{aligned}$$

where we mean, in the definition of μ^*, that we do not place any requirement on the B_ν if $n = 1$. Here again everything depends on the branch W of V that has V_j as cone of limiting directions (cf. [2]), so that we should write $w_0 = w_0(W)$, $\mu^* = \mu^*(W)$, etc.

We can then state the following theorem (cf. Theorem 5.16 of [2]):

THEOREM 1. *Let V be an algebraic curve in $\mathbb{C}_\tau \times \mathbb{C}^n_\zeta$ with cone of limiting directions*

$$V^h = \bigcup_{j=1}^{\ell} V_j = \bigcup_{j=1}^{\ell} \mathbb{C} \cdot v_j$$

for $v_j = (\tau_j^o, \zeta_j^o) \in (\mathbb{C} \times \mathbb{C}^n) \setminus \{(0,0)\}$, and let $\omega(\tau, \zeta) = \sigma_{\alpha_1}(|\tau|) + \sigma_{\alpha_2}(|\zeta|)$ be a given weight function. Then the following conditions are equivalent:

(1) V satisfies $PL(\omega)$.

(2) For each $j \in \{1, \dots, \ell\}$ and for each branch W of V with cone of limiting directions V_j, one of the following conditions holds (where we write p, q, etc. instead of $p(W), q(W)$, etc.):

(i) $\zeta_j^o \notin \mathbb{C}\mathbb{R}^n$;

(ii) $v_j = (\tau_j^o, \zeta_j^o) \in (\mathbb{R}\setminus\{0\}) \times (\mathbb{R}^n \setminus \{0\})$ *and*

$$\max\{\alpha_1, \alpha_2\} \geq w_0/m;$$

(iii) $v_j = (0, \zeta_j^o) \in \{0\} \times (\mathbb{R}^n \setminus \{0\})$ *and*

$$\max\left\{\frac{s}{m}\alpha_1, \alpha_2\right\} \geq \frac{w_0}{m};$$

(iv) $v_j = (\tau_j^o, 0) \in (\mathbb{R}\setminus\{0\}) \times \{0\}$, $p \leq 0$ *or* $G_p \notin \mathbb{C}\mathbb{R}^n$;

(v) $v_j = (\tau_j^o, 0) \in (\mathbb{R}\setminus\{0\}) \times \{0\}$, $p > 0$, $G_p \in \lambda\mathbb{R}^n$ *for some* $\lambda \in \mathbb{C}$, $q/p \notin \mathbb{N}$, $\alpha_1 \geq p/q$;

(vi) $v_j = (\tau_j^o, 0) \in (\mathbb{R}\setminus\{0\}) \times \{0\}$, $p > 0$, $G_p \in \lambda\mathbb{R}^n$, $q/p \in \mathbb{N}$, $\lambda/|\lambda| \notin \{e^{ik\pi p/q} : k \in \mathbb{Z}\}$, $\alpha_1 \geq p/q$;

(vii) $v_j = (\tau_j^o, 0) \in (\mathbb{R}\setminus\{0\}) \times \{0\}$, $p > 0$, $G_p \in \lambda\mathbb{R}^n$, $q/p \in \mathbb{N}$, $\lambda/|\lambda| \in \{e^{ik\pi p/q} : k \in \mathbb{Z}\}$,

$$\max\left\{\frac{q}{p}\alpha_1, \alpha_2\right\} \geq 1 - \frac{q-\mu^*}{p}.$$

We now want to find a more explicit formulation of this theorem in the case of two variables, i.e. $k = n = 1$. In this case there exists a polynomial $P \in \mathbb{C}[\tau, \zeta]$ of degree $m' > 0$ such that

$$V = V(P) = \{(\tau, \zeta) \in \mathbb{C}^2 : \ P(\tau, \zeta) = 0\},$$
$$V^h = V(P_{m'}) = \{(\tau, \zeta) \in \mathbb{C}^2 : \ P_{m'}(\tau, \zeta) = 0\},$$

where $P_{m'}$ is the principal part of P and is of the form

$$P_{m'}(\tau, \zeta) = b\tau^\nu \zeta^\mu \prod_{j=1}^{\sigma} (\tau - a_j \zeta)^{m_j}, \qquad (\tau, \zeta) \in \mathbb{C}^2$$

for some $\mu, \nu, \sigma \in \mathbb{N}_0$, $b \in \mathbb{C}\setminus\{0\}$, and $m_j \in \mathbb{N}_0$, $a_j \in \mathbb{C}\setminus\{0\}$ for $1 \leq j \leq \sigma$.

Therefore the Puiseux series expansions (5) reduce to

(8) $$\tau(\zeta) = A\zeta + \sum_{\nu=-\infty}^{s} D_\nu \zeta^{\nu/m}, \qquad |\zeta| \gg 1,$$

with $A = 0$ or $A = a_j$ for some $j \in \{1, \ldots, \sigma\}$.

The series expansions (6) and (7) are of the form:

(9) $$\zeta(\tau) = \sum_{\nu=-\infty}^{p} G_\nu \tau^{\nu/q}, \qquad |\tau| \gg 1,$$

(10) $$\tau(\zeta) = \sum_{\nu=-\infty}^{q} A_\nu \zeta^{\nu/p}, \qquad |\zeta| \gg 1,$$

for $G_\nu \in \mathbb{C}$ and $A_q = (1/G_p^{1/p})^q$.

Now we check what this specialization means for the conditions $(i)-(vii)$ in *(2)* of Theorem 1. Obviously, the condition (i) is empty when $n=1$.

Let us look at the conditions *(2)(ii)* and *(2)(iii)* for $n=1$. We first prove that if $s>0$ then $w_0=s$. To this aim we choose the branch $g(\rho e^{i\varphi}) = \rho^{1/m}\exp(i\varphi/m)$ of the m-th root. Then, for $D_s = re^{i\psi}$, we have that

$$D_s g(1)^s = re^{i\psi} \in \mathbb{R} \qquad \text{iff} \qquad \psi = h\pi,\ h \in \mathbb{Z};$$

in this case

$$D_s g(-1)^s = re^{i(\psi+\pi s/m)} = re^{i(h\pi+\pi s/m)} = \pm re^{i\frac{s}{m}\pi} \notin \mathbb{R}$$

since $s/m \notin \mathbb{Z}$ for $0<s<m$. This means that we can find $\zeta_o \in \{-1,1\}$ and a branch $f_m = g$ of the m-th root such that $w(\zeta_o, f_m) = s > 0$. Since $n=1$ we have $u(\zeta_o, f_m) = -\infty$ and hence $w_0 = s$. Therefore the conditions *(2)(ii)* and *(2)(iii)* of Theorem 1 become, respectively:

(ii)′ $v_j = (\tau_j^o, \zeta_j^o) \in (\mathbb{R}\setminus\{0\}) \times (\mathbb{R}\setminus\{0\})$ and

$$\max\{\alpha_1, \alpha_2\} \geq s/m;$$

(iii)′ $v_j = (0, \zeta_j^o) \in \{0\} \times (\mathbb{R}\setminus\{0\})$ and $\alpha_2 \geq s/m$.

If, on the contrary, $s \leq 0$, then $w_0 = -\infty$ and the conditions *(2)(ii)* and *(2)(ii)'*, *(2)(iii)* and *(2)(iii)'* are empty, and hence coincide again.

In case of *(2)(iv)* we have only the condition $p \leq 0$, since $G_p \in \mathbb{CR}$ is always satisfied.

Let us now take $p=1$ and look at the conditions *(2)(v)–(vii)*. We have $q/p = q \in \mathbb{N}$ (hence the condition *(2)(v)* is empty) and

$$\frac{G_1}{|G_1|} := e^{i\phi} \in \{e^{ik\frac{p}{q}\pi} : k \in \mathbb{Z}\} = \{e^{i\frac{k\pi}{q}} : k \in \mathbb{Z}\}$$

if and only if $\phi q = k\pi$ for some $k \in \mathbb{Z}$, i.e. if and only if $G_1^q \in \mathbb{R}$. In this case $\mu^* = -\infty$, since the condition

$$q-1 = q-p < \mu(\zeta_o, f_p) < q$$

cannot be satisfied for any integer $\mu(\zeta_o, f_p)$. Therefore the condition *(2)(vii)* is empty.

If, on the contrary, $G_1^q \notin \mathbb{R}$ then we have the condition $\alpha_1 \geq p/q$ from *(2)(vi)*.

Let us now take $p=2$. If q is odd then $q/p \notin \mathbb{N}$ and we have the condition $\alpha_1 \geq p/q$ from *(2)(v)*.

If q is even then $q/p \in \mathbb{N}$ and

$$\frac{G_2}{|G_2|} := e^{i\phi} \in \{e^{ik\pi\frac{p}{q}} : k \in \mathbb{Z}\} = \{e^{i\frac{2k\pi}{q}} : k \in \mathbb{Z}\}$$

if and only if $\phi q = 2k\pi$ for some $k \in \mathbb{Z}$, i.e. if and only if $G_2^q > 0$. Let us now investigate μ^* in this case. If $A_{q-1} \neq 0$, then there exist $\zeta_o \in \{-1, 1\}$ and a branch f_2 of the square root such that

$$\operatorname{Im}(A_{q-1} f_2(\zeta_o)^{q-1}) \neq 0$$

since $q-1$ is odd. In this case $\mu^* = q-1 > q-2 = q-p$, and the condition *(2)(vii)* becomes

$$\max\left\{\frac{q}{2}\alpha_1, \alpha_2\right\} \geq 1 - \frac{q-(q-1)}{2} = \frac{1}{2}.$$

If, on the contrary, $A_{q-1} = 0$ then for any $\zeta_o \in \{-1, 1\}$ and any branch f_2 of the square root we have that $\mu(\zeta_o, f_2) < q-1$ and hence $\mu^* = -\infty$, because the condition

$$q-2 = q-p < \mu(\zeta_o, f_2) < q-1$$

cannot be satisfied for any integer $\mu(\zeta_o, f_2)$. In this case the condition *(2)(vii)* is therefore empty.

If we assume that $G_2^q \in \mathbb{C} \setminus \mathbb{R}$ or $G_2^q < 0$ then $G_2/|G_2| \notin \{e^{i\frac{2k\pi}{q}} : k \in \mathbb{Z}\}$. In this case we have the condition $\alpha_1 \geq p/q$ from *(2)(vi)*.

Let us finally remark that if $V(P)$ satisfies $PL(\omega)$, then also $V(P_{m'})$ satisfies $PL(\omega)$ because of Theorem 5.3 of [2]. Vice versa, if $V(P_{m'}) = V_1 \cup \ldots \cup V_\ell$ satisfies $PL(\omega)$, then every V_j, for $j \in \{1, \ldots, \ell\}$, admits a real generator $v_j = (\tau_j^o, \zeta_j^o) \in \mathbb{R}^2 \setminus \{0\}$ by Theorem 3.3 of [2].

All the above considerations allow us to reformulate Theorem 1 in the case of two variables as follows:

THEOREM 2. *For $P \in \mathbb{C}[\tau, \zeta] \setminus \mathbb{C}$ with principal part $P_{m'}$ and a weight function $\omega(\tau, \zeta) = \sigma_{\alpha_1}(|\tau|) + \sigma_{\alpha_2}(|\zeta|)$ the algebraic curve $V(P)$ satisfies $PL(\omega)$ if and only if the following two conditions are satisfied:*

(1) $V(P_{m'})$ satisfies $PL(\omega)$.

(2) For each $j \in \{1, \ldots, \ell\}$ and for each branch W of V with cone of limiting directions V_j, one of the following conditions holds:

(i) $v_j = (\tau_j^o, \zeta_j^o) \in \mathbb{R} \times (\mathbb{R} \setminus \{0\})$ and

$$\begin{cases} \max\{\alpha_1, \alpha_2\} \geq \frac{s}{m} & \text{if } \tau_j^o \neq 0 \\ \alpha_2 \geq \frac{s}{m} & \text{if } \tau_j^o = 0; \end{cases}$$

(ii) $v_j = (\tau_j^o, 0) \in (\mathbb{R} \setminus \{0\}) \times \{0\}$, $p \leq 0$, or $p = 1$ and $G_1^q \in \mathbb{R}$, or $p = 2$, $G_2^q > 0$, q is even and $A_{q-1} = 0$;

(iii) $v_j = (\tau_j^o, 0) \in (\mathbb{R}\setminus\{0\}) \times \{0\}$, $p > 0$ *and*

$$\begin{cases} \alpha_1 \geq \frac{p}{q} & \text{if } p \in \{1,2\} \text{ and } G_p^q \in \mathbb{C}\setminus\mathbb{R} \\ & \text{or if } p = 2 \text{ and } G_2^q < 0 \\ & \text{or if } p = 2,\ G_2^q > 0,\ q \text{ odd} \\ & \text{or if } p \geq 3 \text{ and } \frac{q}{p} \notin \mathbb{N} \\ & \text{or if } p \geq 3,\ \frac{q}{p} \in \mathbb{N}, \text{ and} \\ & \quad \frac{G_p}{|G_p|} \notin \left\{ e^{ik\pi\frac{p}{q}} : k \in \mathbb{Z} \right\} \\ \max\left\{\frac{q}{2}\alpha_1, \alpha_2\right\} \geq \frac{1}{2} & \text{if } p = 2,\ G_2^q > 0, \\ & \quad q \text{ even, and } A_{q-1} \neq 0 \\ \max\left\{\frac{q}{p}\alpha_1, \alpha_2\right\} \geq 1 - \frac{q-\mu^*}{p} & \text{if } p \geq 3,\ \frac{q}{p} \in \mathbb{N}, \text{ and} \\ & \quad \frac{G_p}{|G_p|} \in \left\{ e^{ik\pi\frac{p}{q}} : k \in \mathbb{Z} \right\}. \end{cases}$$

REMARK 1. Theorem 2 corrects [1], Theorem 4.16, which is not correct, due to a mistake in the proof of part (1) of Lemma 4.10 in [1]. However, the arguments for this part of Lemma 4.10 are right whenever $(p,q) = 1$. Therefore Theorem 2 concides with Theorem 4.16 of [1] if $(p,q) = 1$ on every branch W of $V(P)$. Note that [1], Theorem 4.16, is also correct if $V(P)$ has no branches W for which $p \geq 3$, $q/p \in \mathbb{N}$ and $G_p/|G_p| \in \{e^{ik\pi\frac{p}{q}} : k \in \mathbb{Z}\}$.

2. Examples

EXAMPLE 1. Let us consider the algebraic curve

$$V = \{(\tau,\zeta) \in \mathbb{C}^2 : P(\tau,\zeta) = \zeta^6 + 3\zeta^2\tau^2 + \tau^2 - 3\zeta^4\tau - 6\zeta\tau^2 - 2\zeta^3\tau - \tau^3 = 0\}.$$

Since the principal part P_6 of P is $P_6(\tau,\zeta) = \zeta^6$, it follows that

$$V(P_6) = \{(\tau,\zeta) \in \mathbb{C}^2 :\ \zeta = 0\}.$$

It is therefore trivial that $V(P_6)$ satisfies $PL(\omega)$ for each weight function ω, by Proposition 4.3 of [1]. It is easy to check that

$$V = \{(\lambda^6, \lambda^3 + \lambda^2) :\ \lambda \in \mathbb{C}\}.$$

From this it follows that V has only one irreducible branch near infinity that admits the Puiseux series expansion

$$\zeta(\tau) = \tau^{3/6} + \tau^{2/6}, \quad |\tau| \gg 1,$$

which converts into the expansion

$$\tau(\zeta) = \zeta^2 - 2\zeta^{5/3} + \sum_{\nu=-\infty}^{4} A_\nu \zeta^{\nu/3}, \qquad |\zeta| \gg 1.$$

Since $p = 3$, $q = 6$ and $G_3 = 1 = e^{ik\pi/2}$ for $k = 0$, we are in the last case of Theorem 2 hence we must compute μ^*. Since $A_5 = -2$ and there exists a third root f_3 of 1 such that $\mathrm{Im}(A_5 f_3(1)^5) \neq 0$, we have $\mu(1, f_3) = 5$. Since $5 > 3 = q - p$ we also have $\mu^* = 5$. Consequently, Theorem 2 implies that V satisfies $PL(\omega)$, for $\omega(\tau,\zeta) = \sigma_{\alpha_1}(|\tau|) + \sigma_{\alpha_2}(|\zeta|)$ if and only if

$$\max\{2\alpha_1, \alpha_2\} \geq 1 - \frac{6-5}{3} = \frac{2}{3}.$$

Let us now prove a lemma that is useful in the study of examples.

LEMMA 1. *For $p, q \in \mathbb{N}$, $q > p$, and $a \in \mathbb{C} \setminus \{0\}$ let $P \in \mathbb{C}[\tau, \zeta]$ be defined as*

$$P(\tau,\zeta) := \tau^p - a\zeta^q + \sum_{j=0}^{p-1} b_j \tau^j - \sum_{j=0}^{q-1} a_j \zeta^j.$$

Assume that for $h, s, t \in \mathbb{N}$ we have $p = hs$, $q = ht$, and $(s,t) = 1$ and denote by $\beta_1, \ldots, \beta_h$ the h different h-th roots of a. Then $V(P) := \{(\tau,\zeta) \in \mathbb{C}^2 : P(\tau,\zeta) = 0\}$ has h branches near infinity and for each such branch W there exists $j \in \{1, \ldots, h\}$ such that W admits a Puiseux series expansion which has $\beta_j^{-1/t} \tau^{s/t}$ as leading term.

Proof. Since $p = hs$ and $q = ht$ we have

$$F(\tau,\zeta) := \tau^p - a\zeta^q = \tau^{sh} - a\zeta^{th} = \prod_{j=1}^{h} (\tau^s - \beta_j \zeta^t).$$

Because of $(s,t) = 1$, this shows that $V(F)$ is the union of h irreducible curves, which have the Puiseux series expansions

$$\zeta_j(\tau) = \left(\frac{1}{\beta_j}\right)^{1/t} \tau^{s/t}, \ |\tau| > 0, \ 1 \leq j \leq h.$$

For $1 \leq j \leq h$, $1 \leq k \leq t$, and $\tau \in \mathbb{C}$ with $|\tau| > 0$ denote by $\zeta_{j,k}(\tau)$ the $q = ht$ different roots of $F(\tau, \cdot)$. Then it is easy to check that there exists $\delta > 0$ such that

$$\min\{|\zeta_{i,k}(\tau) - \zeta_{j,m}(\tau)| : \ 1 \leq i, \ j \leq h, \ 1 \leq k, \ m \leq t, (i,k) \neq (j,m)\} \geq \delta |\tau|^{s/t}.$$

Furthermore, there exists $\eta > 0$ such that

$$\min\left\{ |1 - \frac{\beta_\nu}{\beta_j}| : \ 1 \leq j, \ \nu \leq h, \ j \neq \nu \right\} \geq 2\eta.$$

Then we have for $\lambda \in \mathbb{C}$ with $|\lambda| = \varepsilon|\tau|^{s/t}$:

$$(11) \qquad |F(\tau, \zeta_{j,k}(\tau)+\lambda)| = |\tau^s - \beta_j(\zeta_{j,k}(\tau)+\lambda)^t| \prod_{\substack{\nu=1 \\ \nu\neq j}}^{h} |\tau^s - \beta_\nu(\zeta_{j,k}(\tau)+\lambda)^t|.$$

Now note that $\zeta_{j,k}(\tau)^t = \tau^s/\beta_j$ and the choice of η imply the existence of $0 < \varepsilon_1 < \delta$ such that for each ε with $0 < \varepsilon \leq \varepsilon_1$ we have

$$(12) \qquad |\tau^s - \beta_\nu(\zeta_{j,k}(\tau)+\lambda)^t| = |\tau|^s \left|1 - \frac{\beta_\nu}{\beta_j}\left(1 + \beta_j^{1/t}\frac{\lambda}{\tau^{s/t}}\right)^t\right| \geq \eta|\tau|^s.$$

Similary, we get

$$\tau^s - \beta_j(\zeta_{j,k}(\tau)+\lambda)^t = \tau^s - \beta_j \sum_{l=1}^{t} \binom{t}{l} \zeta_{j,k}(\tau)^{t-l}\lambda^l.$$

This shows that we can choose $\varepsilon_1 > 0$ so small that for each ε with $0 < \varepsilon < \varepsilon_1$ we have

$$(13) \qquad |\tau^s - \beta_j(\zeta_{j,k}(\tau)+\lambda)^t| \geq |\beta_j|^{1/t} t \frac{\varepsilon}{2} |\tau|^s.$$

From (11), (12), and (13) we now get

$$(14) \qquad |F(\tau, \zeta_{j,k}(\tau)+\lambda)| \geq \eta^{h-1}|\beta_j|^{1/t} t \frac{\varepsilon}{2}|\tau|^{sh}, \ |\lambda| = \varepsilon|\tau|^{s/t}.$$

To apply the Theorem of Rouché to F and P on the circles $\partial B(\zeta_{j,k}(\tau), \varepsilon|\tau|^{s/t})$, we note that there exists $C > 1$ such that

$$|P(\tau,\zeta) - F(\tau,\zeta)| \leq C(|\tau|^{p-1} + |\zeta|^{q-1}), \ |\tau| \geq 1.$$

Since for $\lambda \in \mathbb{C}$ with $|\lambda| = \varepsilon|\tau|^{s/t}$ we have

$$|(\zeta_{j,k}(\tau)+\lambda)^{q-1}| \leq |\tau|^{s(q-1)/t} \left(\left|\frac{1}{\beta_j^{1/t}}\right| + \varepsilon\right)^{q-1},$$

and since $p < q$ there exists $D > 1$ such that, for $|\lambda| = \varepsilon|\tau|^{s/t}$,

$$(15) \qquad |P(\tau, \zeta_{j,k}(\tau)+\lambda) - F(\tau, \zeta_{j,k}(\tau)+\lambda)| \leq CD|\tau|^{s(q-1)/t} = CD|\tau|^{sh-s/t}.$$

From (14) and (15) we now get that for each $0 < \varepsilon < \varepsilon_1$ there exists $\tau_0 > 1$ such that for each $\tau \in \mathbb{C}$ with $|\tau| \geq \tau_0$ we have

$$|F(\tau, \zeta_{j,k}(\tau)+\lambda) - P(\tau, \zeta_{j,k}(\tau)+\lambda)| < |F(\tau, \zeta_{j,k}(\tau)+\lambda)|, \ |\lambda| = \varepsilon|\tau|^{s/t}.$$

By the Theorem of Rouché, it follows that for each $\tau \in \mathbb{C}$, $|\tau| \geq \tau_0$, the function $\zeta \mapsto P(\tau,\zeta)$ has a zero $\xi_{j,k}(\tau)$ which satisfies $|\xi_{j,k}(\tau) - \zeta_{j,k}(\tau)| \leq \varepsilon|\tau|^{s/t}$ for each ε with

$0 < \varepsilon < \varepsilon_1$. By the choice of ε_1, the disks $B(\zeta_{j,k}(\tau), \varepsilon|\tau|^{s/t})$, $1 \le j \le h$, $1 \le k \le t$, are pairwise disjoint. Since each branch W of $V(P)$ near infinity admits a Puiseux series expansion of the form

$$\zeta(\tau) = \sum_{\nu=-\infty}^{w} A_\nu \tau^{\nu/r}$$

it now follows that the leading term of such an expansion has the given form. □

In the following example we use Lemma 1 and Theorem 2 to give a correct proof of [1], Example 5.3. The proof that was given in [1] is based on that part of [1], Theorem 4.16, in which we have a flaw. Nevertheless, the assertions of [1], Example 5.3, are right, as the new proof shows.

EXAMPLE 2. For $p, q \in \mathbb{N}$, $p, q, \ge 2$, and $a \in \mathbb{C} \setminus \{0\}$ let $P \in \mathbb{C}[\tau, \zeta]$ be defined as

$$P(\tau, \zeta) := \tau^p - a\zeta^q + \sum_{j=0}^{p-1} b_j \tau^j - \sum_{j=0}^{q-1} a_j \zeta^j.$$

Then for $\omega(\tau, \zeta) := \sigma_{\alpha_1}(|\tau|) + \sigma_{\alpha_2}(|\zeta|)$ the following assertions hold for

$$V = V(P) = \{(\tau, \zeta) \in \mathbb{C} \times \mathbb{C} : P(\tau, \zeta) = 0\}.$$

(1) If $p > q \ge 2$ then V satisfies $PL(\omega)$ if and only if $\alpha_2 \ge q/p$.

(2) If $q > p \ge 3$ then V satisfies $PL(\omega)$ if and only if $\alpha_1 \ge p/q$.

(3) If $q > p = 2$ with $a \in \mathbb{C} \setminus \mathbb{R}$ or $a < 0$ or $a > 0$ and q odd, then V satisfies $PL(\omega)$ if and only if $\alpha_1 \ge p/q$.

(4) If $q > p = 2$, $a \in \mathbb{R}$, q even and $a > 0$ then V satisfies $PL(\omega)$ for all $0 \le \alpha_1, \alpha_2 < 1$.

(5) If $p = q \ge 3$ or $p = q = 2$ and $a \in \mathbb{C} \setminus \, [0, \infty[$ then V does not satisfy $PL(\omega)$ for any $(\alpha_1, \alpha_2) \in \, [0, 1[\times [0, 1[$.

(6) If $p = q = 2$, $a \in \mathbb{R}$ and $a > 0$ then V satisfies $PL(\omega)$ for all $0 \le \alpha_1, \alpha_2 < 1$.

To prove these assertions we argue as follows.

(1) In this case the principal part P_p of P is given by $P_p(\tau, \zeta) = \tau^p$. Hence $V(P_p)$ satisfies $PL(\kappa)$ for each weight function κ by [1], Proposition 4.3. Now fix any branch W of V near infinity. By [1], Lemma 4.4, W admits a Puiseux series expansion of the form (8). The present hypothesis $q < p$ implies $A = 0$ so that (8) gives

$$\tau(\zeta) = \sum_{\nu=-\infty}^{s} D_\nu \zeta^{\nu/m}.$$

Since $P(\tau(\zeta), \zeta) = 0$, we have $D_s^p \zeta^{sp/m} - a\zeta^q = 0$ and consequently $s/m = q/p$. Hence we get from Theorem 2, part *(2)(i)* that $PL(\omega)$ holds on W if and only if $\alpha_2 \ge s/m = q/p$. Since W was an arbitrary branch of V the proof of (1) is complete.

(2) In this case the principal part P_q of P is given by $P_q(\tau,\zeta) = -a\zeta^q$. Hence $V(P_q)$ satisfies $PL(\kappa)$ for each weight function κ by [1], Proposition 4.3. Next assume that there are $h,s,t \in \mathbb{N}$ with $p = hs$, $q = ht$, and $(s,t) = 1$. If we denote by $\beta_1,\ldots,\beta_h$ the h different roots of a, then we get from Lemma 1 that for each branch W of V near infinity, there exists $1 \le j \le h$ such that W admits a Puiseux series expansion of the form

$$\zeta(\tau) = \frac{1}{\beta_j^{1/t}}(\tau^s)^{1/t} + \; l.o.t.$$

This shows that $G_s^t = 1/\beta_j$. Now we distinguish the following cases:
(i) $s = 1$.
This means that $p = h$ and $q/p \in \mathbb{N}$. Since $h = p \ge 3$ by the present hypotheses, at least one of the numbers $\beta_1,\ldots,\beta_p$ is not real. If $\beta_j \in \mathbb{C}\setminus\mathbb{R}$ then $G_s^t \in \mathbb{C}\setminus\mathbb{R}$. Therefore, it follows from Theorem 2 $(2)(iii)$ (and Theorem 2 $(2)(ii)$ for $\beta_j \in \mathbb{R}$) that V satisfies $PL(\omega)$ if and only if $\alpha_1 \ge p/q$.
(ii) $s \ge 2$, $h = 1$.
Then $s = p \ge 3$ and $(p,q) = 1$. Hence $p/q \notin \mathbb{N}$ and it follows from Theorem 2 $(2)(iii)$ that V satisfies $PL(\omega)$ if and only if $\alpha_1 \ge p/q$.
(iii) $s = 2$, $h \ge 2$.
Then $(s,t) = 1$ implies that t must be odd. Hence it follows from Theorem 2 $(2)(iii)$ that, no matter whether $\frac{1}{\beta_j} \in \mathbb{C}\setminus\mathbb{R}$ or $\frac{1}{\beta_j} \in \mathbb{R}\setminus\{0\}$, V satisfies $PL(\omega)$ if and only if $\alpha_1 \ge s/t = p/q$.
(iv) $s \ge 3$, $h \ge 2$.
Then $s/t \notin \mathbb{N}$ together with Theorem 2 $(2)(iii)$ implies also in this case that V satisfies $PL(\omega)$ if and only if $\alpha_1 \ge s/t = p/q$.

(3) As in part (2) we get that $V(P_q)$ satisfies $PL(\kappa)$ for each weight function κ. If $p = 2$, $a > 0$, and q is odd then $\tau^2 - a\zeta^q$ is irreducible and hence V has a Puiseux series expansion of the form $\zeta(\tau) = \left(\frac{\tau^2}{a}\right)^{1/q} + l.o.t.$ This shows that $G_2 = \left(\frac{1}{a}\right)^{1/q}$ and hence $G_2^q = \frac{1}{a} > 0$. Therefore, it follows from Theorem 2 $(2)(iii)$ that V satisfies $PL(\omega)$ if and only if $\alpha_1 \ge p/q$.
If $p = 2$ and q is odd the same argument as above shows that $G_2^q = \frac{1}{a}$ is negative if $a < 0$ or is not real if a is not real. Hence we get the same conclusion as before.
If $p = 2$ and q is even, then $q = 2m$ and $\tau^p - a\zeta^q$ factors as $(\tau - \sqrt{a}\zeta^m)(\tau + \sqrt{a}\zeta^m)$. By Lemma 1, the two branches of V near infinity are then given by

$$\zeta(\tau) = \left(\frac{\pm 1}{\sqrt{a}}\right)^{1/m} + \; l.o.t.$$

Hence $G_1^m = \frac{\pm 1}{\sqrt{a}}$. This number is not real if $a \in \mathbb{C}\setminus\ [0,\infty[$. Therefore, it follows from Theorem 2 $(2)(iii)$ that (3) holds.

(4) The same arguments as in (3) show that now $G_1^m = \frac{\pm 1}{\sqrt{a}}$ is real since $a > 0$. Hence (4) follows from Theorem 2 $(2)(ii)$.

(5) In both cases the principal part of P is given by $P_p(\tau,\zeta) = \tau^p - a\zeta^p$ and we can find $\alpha \in \mathbb{C}\setminus\mathbb{R}$ such that $\alpha^p = a$. Hence P_p admits a factor $\tau - \alpha\zeta$ with $\alpha \in \mathbb{C}\setminus\mathbb{R}$,

which implies that $V(P_p)$ does not satisfy $PL(\omega)$ for any weight function ω because of [1], Proposition 4.3. By [1], Corollary 4.9, also $V(P)$ cannot satisfy $PL(\omega)$ for any weight function ω. Hence (5) holds.

(6) In this case the principal part P_2 of P is given by $P_2(\tau,\zeta)=\tau^2-a\zeta^2=(\tau-\sqrt{a}\zeta)(\tau+\sqrt{a}\zeta)$. Since a is positive by the present hypothesis, $V(P_2)$ satisfies $PL(\kappa)$ for each weight function κ, by [1], Proposition 4.3. Since $V(P_2)$ has two irreducible components, it follows similarly as in the proof of Lemma 1 that V has two branches near infinity and that these can be described as

$$\tau=-\frac{b_1}{2}\pm\sqrt{a}\zeta\Big(1+\frac{a_1}{a}\,\frac{1}{\zeta}+\frac{a_0'}{a}\,\frac{1}{\zeta^2}\Big)^{1/2}.$$

This implies the existence of $C>0$ such that

$$|\operatorname{Im}\tau|\le C|\operatorname{Im}\zeta|+C,\qquad (\tau,\zeta)\in V.$$

Hence condition (γ) of $PL(\omega)$ follows from condition (α) of $PL(\omega)$ for each weight function ω.

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