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GEVREY SOLUTIONS FOR A VIBRATING BEAM EQUATION

Dedicated to Professor Luigi Rodino on the occasion of his 60th birthday

Abstract. We consider the Cauchy problem for the Euler-Bernoulli equation of the vibrating beam and solve it in Gevrey classes under appropriate Levi conditions on the lower order terms.

1. Introduction and main result

Let us consider the Cauchy problem in $[0,T]\times\mathbb{R}_x$

$$\begin{cases} Lu = 0 \\ u(0,x) = u_0, \quad \partial_t u(0,x) = u_1 \end{cases} \tag{1}$$

for the operator

$$L := D_t^2 - a_4(t)D_x^4 + \sum_{k=0}^{3} a_k(t,x)D_x^k, \tag{2}$$

where $D = -i\partial$ for the sake of the Fourier transform and $a_4(t)$ is a real non-negative function. A motivation to investigate such a problem comes from the Euler-Bernoulli model of the vibrating beam. We admit zeroes of finite order k for $a_4(t)$, and so assume there exists $k\in\mathbb{N}$, $k\geq 2$ such that

$$\sum_{j=0}^{k} |a_4^{(j)}(t)| \neq 0, \quad t\in[0,T]. \tag{3}$$

We assume for the coefficients of L the following regularity conditions:

$$a_4 \in C^k([0,T];\mathbb{R}_+), \quad a_3 \in C^1([0,T];\gamma^s(\mathbb{R})), \quad a_2,a_1,a_0 \in C([0,T];\gamma^s(\mathbb{R})), \tag{4}$$

where $\mathbb{R}_+ = [0,+\infty)$, and $\gamma^s(\mathbb{R})$ is the Gevrey class of index $s\geq 1$ on $\mathbb{R}$, that is the space of all smooth functions f such that

$$|f^{(\alpha)}(x)| \leq CA^\alpha \alpha!^s, \quad C,A>0,\ \alpha\in\mathbb{N}.$$

One can consider L as an anisotropic hyperbolic operator where each derivative with respect to the time variable t has the same weight of two derivatives with respect the space variable x. After that, the two factors $\tau\pm\sqrt{a_4(t)}\xi^2$ of the principal symbol correspond to Schrödinger operators. From the theory of hyperbolic equations, one expects Levi conditions are needed on the lower order terms at the points where the leading coefficient $a_4(t)$ vanishes. On the other hand, from the Schrödinger side, also some decay assumptions as $x \to \infty$ should be taken into account for the imaginary part of these terms, see [4].

Here we assume that the imaginary part of a_3 satisfies the Levi condition

$$|\Im a_3(t,x)| \leq C_0 a_4(t) \langle x \rangle^{-\sigma}, \ \sigma > 1, \tag{5}$$

$\langle x \rangle = (1+x^2)^{1/2}$. Besides the decay rate for $x \to \infty$, (5) says that the order of vanishing of $\Im a_3$ is at least the same of a_4. For the full coefficient a_3, including its real part, for the derivative $\partial_t a_3$, and for the coefficients a_2 and a_1, we require lower orders of zero and not any decay, precisely

$$|\partial_x^\beta a_3(t,x)| \leq CA^\beta \beta!^s a_4(t)^{\eta_1}, \tag{6}$$

$$|\partial_x^\beta \partial_t a_3(t,x)| \leq CA^\beta \beta!^s a_4(t)^{\eta_2}, \tag{7}$$

$$|\partial_x^\beta a_2(t,x)| \leq CA^\beta \beta!^s a_4(t)^{\eta_3}, \tag{8}$$

$$|\partial_x^\beta a_1(t,x)| \leq CA^\beta \beta!^s a_4(t)^{\eta_4}, \tag{9}$$

with $C, A > 0$ and η_i to be specified here below.

We proved in [1] that the problem (1), (2) is well posed in $H^\infty = \cap_{\mu \in \mathbb{R}} H^\mu$ under the assumptions (3), (5) with $\sigma \geq 1$ and

$$\begin{cases} |\partial_x^\beta a_3(t,x)| & \leq & C_\beta a_4(t)^{\eta_1}, & \eta_1 \geq 3/4 - 1/(2k), \\ |\partial_x^\beta \partial_t a_3(t,x)| & \leq & C_\beta a_4(t)^{\eta_2}, & \eta_2 \geq 3/4 - 3/(2k), \\ |\partial_x^\beta a_2(t,x)| & \leq & C_\beta a_4(t)^{\eta_3}, & \eta_3 \geq 1/2 - 1/k, \\ |\partial_x^\beta a_1(t,x)| & \leq & C_\beta a_4(t)^{\eta_4}, & \eta_4 \geq 1/4 - 3/(2k). \end{cases} \tag{10}$$

(H^μ denotes the space of functions f such that $\xi \mapsto \langle \xi \rangle^\mu \hat{f}(\xi)$ is in L^2 where $\hat{}$ is the Fourier transform.) Otherwise, H^∞ well posedness cannot hold; here we are going to prove a result of well posedness in Gevrey classes for (1) in this second case. The main result of this paper is the following:

THEOREM 1. *Let us consider the Cauchy problem (1) for the operator L in (2) under assumptions (3), (4). If the Levi conditions (5)–(9) are fulfilled (but (10) is not necessarily satisfied), then problem (1) is well posed in γ^s for $1 < s < s_0$, where*

- $\begin{cases} 1/2 & \leq & \eta_1 < 3/4 - 1/(2k) \\ \eta_2 & \geq & 3\eta_1 - 3/2 \\ \eta_3 & \geq & 2\eta_1 - 1 \\ \eta_4 & \geq & 3\eta_1 - 2 \end{cases} \Longrightarrow s_0 = \dfrac{1-\eta_1}{2\,[3/4 - 1/(2k) - \eta_1]}\,,$

- $\begin{cases} \eta_2 & < & 3/4 - 3/(2k) \\ \eta_1 & \geq & \eta_2/3 + 1/2 \\ \eta_3 & \geq & 2\eta_2/3 \\ \eta_4 & \geq & \eta_2 - 1/2 \end{cases} \Longrightarrow s_0 = \dfrac{3/2-\eta_2}{2\,[3/4 - 3/(2k) - \eta_2]}\,,$

- $\begin{cases} \eta_3 & < & 1/2 - 1/k \\ \eta_1 & \geq & \eta_3/2 + 1/2 \\ \eta_2 & \geq & 3\eta_3/2 \\ \eta_4 & \geq & 3\eta_3/2 - 1/2 \end{cases} \Longrightarrow s_0 = \dfrac{1-\eta_3}{2\,[1/2 - \eta_3 - 1/k]}\,,$

- $\begin{cases} \eta_4 & < & 1/4 - 3/(2k) \\ \eta_1 & \geq & \eta_4/3 + 2/3 \\ \eta_2 & \geq & \eta_4 + 1/2 \\ \eta_3 & \geq & 2\eta_4/3 + 1/3 \end{cases} \Longrightarrow s_0 = \dfrac{1-\eta_4}{2\,[1/4 - 3/(2k) - \eta_4]}\,.$

In proving Theorem 1 we need to assume $\sigma > 1$; for a precise explanation of this fact see the final Remark 1.

2. Preliminary results and Schrödinger equations

In what follows, we are going to use pseudo-differential operators $p(x, D_x)$ of order m on $\mathbb{R}$ with symbols $p(x,\xi)$ in the standard class S^m which is the space of all symbols $a(x,\xi)$ satisfying, for any $\alpha, \beta \in \mathbb{Z}_+$,

$$|\partial_\xi^\alpha \partial_x^\beta a(x,\xi)| \leq C_{\alpha,\beta,h} \langle \xi \rangle_h^{m-|\alpha|}, \quad \langle \xi \rangle_h := \sqrt{h^2+\xi^2}, h \geq 1;$$

this is the limit space as $\ell \to \infty$ of the Banach spaces $S^m\ell$ of all symbols such that

$$|a|_{m,\ell} := \sup_{x,\xi} \sup_{\alpha+\beta \leq \ell} |\partial_\xi^\alpha \partial_x^\beta a(x,\xi)| \langle \xi \rangle_h^{-m+|\alpha|} < +\infty.$$

Operators with symbol in S^m are bounded operators from $H^{\mu+m}$ into H^μ for any μ. We shall write $\langle \xi \rangle$ instead of $\langle \xi \rangle_1$.

We are also going to use, given $s \geq 1$, Gevrey-type symbols of class $S^{m,s}$, where $S^{m,s}$ denotes the space of all symbols $a(x,\xi)$ satisfying

$$|\partial_\xi^\alpha \partial_x^\beta a(x,\xi)| \leq C_{\alpha,h} A^\beta \beta!^s \langle \xi \rangle_h^{m-|\alpha|}, \tag{11}$$

which is the limit space

$$S^{m,s} := \varprojlim_{\ell \to +\infty} S_\ell^{m,s}, \quad S_\ell^{m,s} := \varinjlim_{A \to +\infty} S_{\ell,A}^{m,s}$$

of the Banach spaces $S^{m,s}_{\ell,A}$ of all symbols such that

$$|a|_{m,s,A,\ell} := \sup_{\alpha\leq\ell,\beta\in\mathbb{Z}_+} \sup_{x,\xi} |\partial_\xi^\alpha \partial_x^\beta a(x,\xi)| A^{-\beta}\beta!^{-s}\langle\xi\rangle_h^{-m+|\alpha|} < +\infty.$$

Given $\mu\in\mathbb{R}$, $\varepsilon>0$, $s\geq 1$, we deal with the Sobolev–Gevrey spaces

$$H^\mu_{\varepsilon,s}(\mathbb{R}) = e^{-\varepsilon\langle D_x\rangle^{1/s}} H^\mu(\mathbb{R}),$$

where the norm is defined by

$$\|u\|_{\mu,\varepsilon,s} = \|e^{\varepsilon\langle D_x\rangle^{1/s}} u\|_\mu.$$

Operators with symbol in $S^{m,s}$ are bounded from $H^{\mu+m}_{\varepsilon,s}$ to $H^\mu_{\varepsilon,s}$ for $|\varepsilon|<\varepsilon_0$, see [3].

In the present section, following [4], we state some preliminary results concerning Schrödinger equations of the form $Su(t,x)=0$,

$$S = D_t + b_2(t)D_x^2 + b_1(t,x,D_x) + b_0(t,x,D_x), \tag{12}$$

where the function $b_2(t)$ is real valued and does not change sign, say

$$b_2 \in C([0,T];\mathbb{R}_+), \tag{13}$$

the lower order terms are complex valued and such that

$$b_j \in C([0,T];S^j), \quad j=0,1. \tag{14}$$

Let us consider the Cauchy problem

$$\begin{cases} Su=0 \\ u(0,x)=u_0. \end{cases} \tag{15}$$

We say that problem (15) is well posed in H^μ if for any $u_0\in H^\mu$ there is a unique solution $u\in\cap_{j=0}^1 C^j([0,T];H^{\mu-2j})$. We have the following:

THEOREM 2. *Consider the Cauchy problem (15), (12) under the assumptions (13) and (14), and assume moreover that*

$$|\Im b_1(t,x,\xi)| \leq M_0 b_2(t)\langle x\rangle^{-\sigma}|\xi|, \quad |\xi|\geq R,\ \sigma>1. \tag{16}$$

Then (15) is well posed in H^μ.

Proof. We define

$$\Lambda(x,\xi) = M_1\omega(\xi/h)\int_0^x \langle y\rangle^{-\sigma} dy, \tag{17}$$

where M_1 is a large constant, $\omega(y)$ a smooth function with $\omega(y) = 0$ for $|y| \leq 1$, $\omega(y) = |y|/y$ for $|y| \geq 2$. For every $\alpha, \beta \in \mathbb{Z}_+$ we have

$$|\partial_\xi^\alpha \partial_x^\beta \Lambda(x,\xi)| \leq \delta_{\alpha,\beta} \langle\xi\rangle_h^{-\alpha}, \tag{18}$$

with constants $\delta_{\alpha,\beta}$ independent on the parameter $h \geq 1$.

Let us now consider the pseudodifferential operators $e^{\pm\Lambda}$ with symbols $e^{\pm\Lambda(x,\xi)}$, and perform the composition $e^\Lambda e^{-\Lambda}$. We have:

$$e^\Lambda e^{-\Lambda} = I - r(x, D_x),$$

where the principal symbol of r is given by

$$r_{-1}(x,\xi) = D_x \Lambda(x,\xi) \partial_\xi \Lambda(x,\xi). \tag{19}$$

By (18),

$$|r_{(\beta)}^{(\alpha)}(x,\xi)| \leq C_{\alpha,\beta} \langle\xi\rangle_h^{-1-\alpha} \leq C_{\alpha,\beta} h^{-1} \langle\xi\rangle_h^{-\alpha},$$

with $C_{\alpha,\beta}$ independent of h. From this, we can fix a large h in order to have a bounded operator $r(x,D_x)$ on H^μ with norm $\|r(x,D_x)\| < 1$. The operator $I - r(x,D_x)$ is invertible by Neumann series and its inverse operator is given by

$$I + p(x,D_x), \quad p = \sum_{j=1}^{\infty} r^j.$$

This proves that the operator $e^{-\Lambda}(I+p)$ is the right inverse of e^Λ. By similar arguments one proves the existence of a left inverse. Thus, the operator e^Λ is invertible, the inverse operator is given by

$$\left(e^\Lambda\right)^{-1} = e^{-\Lambda}(I+p), \; p(x,\xi) \in S^{-1}, \tag{20}$$

and $p(x,\xi)$ has the principal part (19).

To obtain the well posedness in H^μ of problem (15), we perform the change of variable $v = (e^\Lambda)^{-1} u$ and we show that the Cauchy problem

$$\begin{cases} S^\Lambda v = 0 \\ v(0,x) = (e^\Lambda)^{-1} u_0 \end{cases} \tag{21}$$

for the operator $S^\Lambda := (e^\Lambda)^{-1} S e^\Lambda$ is well posed in H^μ. We have

$$iS = \partial_t + iK(t,x,D_x),$$

where

$$K(t,x,D_x) = b_2(t) D_x^2 + b_1(t,x,D_x) + b_0(t,x,D_x),$$

and

$$iS^\Lambda = \partial_t + iK^\Lambda, \quad K^\Lambda = (e^\Lambda)^{-1} K e^\Lambda.$$

Differentiating with respect to time and taking $\mu = 0$, we have

$$\frac{d}{dt}||v(t)||_0^2 = 2\Re\langle v'(t), v(t)\rangle_0 = -2\Re\langle iK^\Lambda v, v\rangle_0.$$

We write iK as the sum

$$iK = H_K + A_K, \quad H_K = (iK + (iK)^*)/2, \quad A_K = (iK - (iK)^*)/2$$

of its hermitian and anti-hermitian parts. The principal symbol of H_K is given by

$$H_K^0(t,x,\xi) = -\Im b_1(t,x,\xi).$$

The hermitian part H_{K^Λ} of iK^Λ is then

$$H_{K^\Lambda}(t,x,\xi) = 2M_1 b_2(t)|\xi|\langle x\rangle^{-\sigma} - \Im b_1(t,x,\xi) + Q_0(t,x,\xi),$$

with $Q_0(t,x,\xi) \in C([0,T];S^0)$. From (16), taking $M_1 = M_0/2$, we have a positive principal part for $H_{K^\Lambda}(t,x,\xi)$; hence, an application of the sharp Gårding inequality gives

$$2\Re\left(iK^\Lambda u, u\right) \geq -C\|u\|^2, \; u \in H^2. \tag{22}$$

From this, the energy method gives well posedness in L^2 of the Cauchy problem for S^Λ. Well posedness in H^μ immediately follows, since, for any μ, the principal symbol of the hermitian part of $\langle D_x\rangle^\mu iK^\Lambda \langle D_x\rangle^{-\mu}$ is the same of H_{K^Λ}. □

3. Proof of the main result

We approximate the characteristic roots $\pm\sqrt{a_4(t)}\xi^2$ of L by defining

$$\tilde\lambda_2(t,\xi) = \sqrt{a_4(t) + \langle\xi\rangle^{-M}}\xi^2 = \tilde b_2(t,\xi)\xi^2, \tag{23}$$

with $0 \leq M \leq 1/(1-\eta_1)$ to be chosen later on. We immediately notice that

$$\tilde b_2 - b_2 \in C([0,T];S^{-M/2}), \; b_2 = \sqrt{a_4(t)}. \tag{24}$$

Then, we define

$$\tilde b_1(t,x,\xi) = -a_3(t,x)\xi/(2\tilde b_2(t,\xi)), \tag{25}$$

and by (6) with $\eta_1 \geq 1/2$ we have

$$\tilde b_1 \in C([0,T];S^{1,s}). \tag{26}$$

Again, we define the operators

$$\tilde S^\pm = D_t \pm \tilde b_2 D_x^2 \pm \tilde b_1, \tag{27}$$

and compute

$$\begin{aligned}\tilde{S}^{-}\tilde{S}^{+} = \;& L - a_2 D_x^2 - a_1 D_x - a_0 - \langle D_x\rangle^{-M} D_x^4 \\ & - \operatorname{op}\left(i\frac{d_\xi \langle\xi\rangle^{-M}\partial_x a_3}{4\tilde{b}_2^2}\xi^3 + i\partial_x a_3 \xi^2 + \cdots \right) - i\frac{a_4'}{2}\tilde{b}_2^{-1} D_x^2 \\ & - i\frac{\partial_t a_3}{2}\tilde{b}_2^{-1} D_x + i\frac{a_4' a_3}{4}\tilde{b}_2^{-3} D_x \\ & + \operatorname{op}\left(-\left(\frac{a_3}{2}\tilde{b}_2^{-1}\xi\right)^2 - \frac{a_3\partial_x a_3}{4}\tilde{b}_2^{-2}\xi + \frac{a_3\partial_x a_3 d_\xi\langle\xi\rangle^{-M}}{8}\tilde{b}_2^{-4}\xi^2 + \cdots \right),\end{aligned}$$

where we denote by $\operatorname{op}(p(x,\xi))$ the pseudodifferential operator of symbol $p(x,\xi)$. We have:

LEMMA 1. *Let us consider the operator L given by (2) under the assumptions of Theorem 1 and take* $\tilde{S}^{\pm}$ *as in* (27)*. Then:*

$$L = \tilde{S}^{-}\tilde{S}^{+} - (d_0\omega + e_0\omega_0 + f_0\omega_1 + g_0\omega_2 + h_0\omega_3 + l_0\omega_4 + m_0)\,\tilde{b_2}\langle D_x\rangle^2, \tag{28}$$

where $e_0, d_0, f_0, g_0, h_0, l_0, m_0 \in C([0,T]; S^{0,s}), \omega = \operatorname{op}(\omega(t,\xi))$ *and* $\omega_i = \operatorname{op}\,(\omega_i(t,\xi))$, $i = 0,\ldots,4$, *with:*

$$\omega(t,\xi) = \frac{\langle\xi\rangle^{2-M}}{(a_4(t)+\langle\xi\rangle^{-M})^{1/2}}, \tag{29}$$

$$\omega_0(t,\xi) = \frac{a_4'(t)}{a_4(t)+\langle\xi\rangle^{-M}}, \tag{30}$$

$$\omega_1(t,\xi) = \frac{1}{(a_4(t)+\langle\xi\rangle^{-M})^{3/2-2\eta_1}}, \tag{31}$$

$$\omega_2(t,\xi) = \frac{\langle\xi\rangle^{-1}}{(a_4(t)+\langle\xi\rangle^{-M})^{1-\eta_2}}, \tag{32}$$

$$\omega_3(t,\xi) = \frac{1}{(a_4(t)+\langle\xi\rangle^{-M})^{1/2-\eta_3}}, \tag{33}$$

$$\omega_4(t,\xi) = \frac{\langle\xi\rangle^{-1}}{(a_4(t)+\langle\xi\rangle^{-M})^{1/2-\eta_4}}. \tag{34}$$

Proof.

- $i\frac{a_4'}{2}\tilde{b}_2^{-1}D_x^2(\tilde{b}_2\langle D_x\rangle^2)^{-1}$ clearly becomes $d_0\omega_0$.
- $\langle D_x\rangle^{-M}D_x^4(\tilde{b}_2\langle D_x\rangle^2)^{-1}$ clearly becomes $e_0\omega$.
- $\operatorname{op}\left(\left(\frac{a_3}{2}\tilde{b}_2^{-1}\xi\right)^2\right)(\tilde{b}_2\langle D_x\rangle^2)^{-1}$ becomes $f_0\omega_1$ by the Levi condition (6).

- $i\dfrac{\partial_t a_3}{2}\tilde{b}_2^{-1}D_x(\tilde{b}_2\langle D_x\rangle^2)^{-1}$ becomes $g_0\omega_2$ by the Levi condition (7).
- $a_2D_x^2(\tilde{b}_2\langle D_x\rangle^2)^{-1}$ becomes $h_0\omega_3$ by the Levi condition (8).
- $a_1D_x(\tilde{b}_2\langle D_x\rangle^2)^{-1}$ becomes $l_0\omega_4$ by the Levi condition (9).
- $\mathrm{op}\left(i\dfrac{d_\xi\langle\xi\rangle^{-M}\partial_x a_3}{4\tilde{b}_2^2}\xi^3\right)(\tilde{b}_2\langle D_x\rangle^2)^{-1}$ and $\mathrm{op}(i\partial_x a_3\xi^2)(\tilde{b}_2\langle D_x\rangle^2)^{-1}$ have symbols in $C([0,T];S^{0,s})$ by the Levi condition (6) with $\eta_1\geq 1/2$.
- $a_0(\tilde{b}_2D_x^2)^{-1}\in C([0,T];S^{0,s})$.
- $\mathrm{op}\left(\dfrac{a_3\partial_x a_3}{4}\tilde{b}_2^{-2}\xi+\dfrac{a_3\partial_x a_3 d_\xi\langle\xi\rangle^{-M}}{8}\tilde{b}_2^{-4}\xi^2\right)(\tilde{b}_2\langle D_x\rangle^2)^{-1}$ has principal symbol $p_0(t,x,\xi)\langle\xi\rangle^{-1}\omega_1$, with $p_0(t,x,\xi)\in C([0,T];S^{0,s})$, by (6).
- from (6), the principal symbol of $-a_4'a_3\tilde{b}_2^{-3}D_x(\tilde{b}_2\langle D_x\rangle^2)^{-1}$ is dominated by $\omega_0\langle\xi\rangle^{-1}(a_4+\langle\xi\rangle^{-M})^{-(1-\eta_1)}$, and $\langle\xi\rangle^{-1}(a_4+\langle\xi\rangle^{-M})^{-(1-\eta_1)}\in C([0,T];S^{0,s})$ because we are going to choose $M\leq 1/(1-\eta_1)$.

$\square$

LEMMA 2. *The symbols defined by (29)–(34) satisfy:*

$$\left|\partial_\xi^\alpha\int_0^T|\omega_0(t,\xi)|dt\right|\leq\delta_\alpha\langle\xi\rangle^{-\alpha}(1+\log\langle\xi\rangle),\tag{35}$$

$$\left|\partial_\xi^\alpha\int_0^T|\omega(t,\xi)|dt\right|\leq\delta_\alpha\langle\xi\rangle^{2-M(1/2+1/k)-\alpha},\tag{36}$$

$$\left|\partial_\xi^\alpha\partial_x^\beta\int_0^T|\omega_1(t,x,\xi)|dt\right|\leq\begin{cases}\delta_{\alpha,\beta}\langle\xi\rangle^{-\alpha}(1+\log\langle\xi\rangle)\text{ if }\eta_1\geq 3/4-1/(2k)\\[2ex]\delta_{\alpha,\beta}\langle\xi\rangle^{M(3/2-1/k-2\eta_1)-\alpha}\text{ if }\eta_1<3/4-1/(2k),\end{cases}\tag{37}$$

$$\left|\partial_\xi^\alpha\partial_x^\beta\int_0^T|\omega_2(t,x,\xi)|dt\right|\leq\begin{cases}\delta_{\alpha,\beta}\langle\xi\rangle^{-\alpha}\text{ if }\eta_2\geq 1-1/k\\[2ex]\delta_{\alpha,\beta}\langle\xi\rangle^{-1+M(1-1/k-\eta_2)-\alpha}\text{ if }\eta_2<1-1/k,\end{cases}\tag{38}$$

$$\left|\partial_\xi^\alpha\partial_x^\beta\int_0^T|\omega_3(t,x,\xi)|dt\right|\leq\begin{cases}\delta_{\alpha,\beta}\langle\xi\rangle^{-\alpha}(1+\log\langle\xi\rangle)\text{ if }\eta_3\geq 1/2-1/k\\[2ex]\delta_{\alpha,\beta}\langle\xi\rangle^{M(1/2-1/k-\eta_3)-\alpha}\text{ if }\eta_3<1/2-1/k,\end{cases}\tag{39}$$

$$\left|\partial_\xi^\alpha\partial_x^\beta\int_0^T|\omega_4(t,x,\xi)|dt\right|\leq\begin{cases}\delta_{\alpha,\beta}\langle\xi\rangle^{-\alpha}\text{ if }\eta_4\geq 1/2-1/k\\[2ex]\delta_{\alpha,\beta}\langle\xi\rangle^{-1+M(1/2-1/k-\eta_4)-\alpha}\text{ if }\eta_4<1/2-1/k.\end{cases}\tag{40}$$

Proof. The proof is a simple application of Lemma 1 and Lemma 2 of [2]. □

The next step of the proof is to reduce L to a first order system of a suitable form by using factorization (28).

LEMMA 3. *Let us consider the operator L in (2) under the assumptions of Theorem 1. Let us denote*

$$\tilde{K}_1 = \tilde{b}_2 D_x^2 + \tilde{b}_1 = b_2 D_x^2 + b_1, \quad b_1 = \tilde{b}_1 + (\tilde{b}_2 - b_2) D_x^2 \tag{41}$$

where $b_1 \in C([0,T]; S^{1,s})$ *and, in view of (24) and (26),* $\Im b_1 = \Im \tilde{b_1}$*. Then, the scalar equation* $Lu = 0$ *is equivalent to the* 2×2 *system* $\mathcal{W} U = 0$,

$$\mathcal{W} = D_t + \tilde{K} + D_0 \omega + E_0 \omega_0 + F_0 \omega_1 + G_0 \omega_2 + H_0 \omega_3 + L_0 \omega_4 + M_0, \tag{42}$$

where

$$\tilde{K} = \begin{pmatrix} \tilde{K}_1 & 0 \\ 0 & -\tilde{K}_1 \end{pmatrix}, \tag{43}$$

$D_0, \ldots, M_0 \in C([0,T]; S^{0,s})$, ω, ω_i *(*$i = 0, \ldots, 4$*) as in (29)–(34).*

Proof. For a scalar unknown u we define the vector $U_0 = {}^t(u_0, u_1)$ by

$$\begin{cases} u_0 = \tilde{b_2} \langle D_x \rangle^2 u \\ u_1 = \tilde{S}^+ u \end{cases}$$

so that, from (28), the scalar equation $Lu = 0$ is equivalent to the system $\mathcal{W}_0 U_0 = 0$ with

$$\begin{aligned} \mathcal{W}_0 = D_t \ &+ \begin{pmatrix} \tilde{K}_1 & -\tilde{b_2} \langle D_x \rangle^2 \\ 0 & -\tilde{K}_1 \end{pmatrix} \\ &+ \begin{pmatrix} -i\omega_0/2 & 0 \\ d_0 \omega + e_0 \omega_0 + f_0 \omega_1 + g_0 \omega_2 + h_0 \omega_3 + l_0 \omega_4 & 0 \end{pmatrix} \\ &+ \begin{pmatrix} \left[\tilde{b_2} \langle D_x \rangle^2, \tilde{K}_1\right] \left(\tilde{b_2} \langle D_x \rangle^2\right)^{-1} & 0 \\ m_0 & 0 \end{pmatrix}, \end{aligned} \tag{44}$$

where we use $(\partial_t \tilde{b_2}) \langle D_x \rangle^2 u = (\omega_0/2) u_0$. The term $\left[\tilde{b_2} \langle D_x \rangle^2, \tilde{K}_1\right] \cdot \left(\tilde{b_2} \langle D_x \rangle^2\right)^{-1}$ is of order 0 because $\tilde{b_2}$ does not depend on x and $\partial_\xi^\alpha \tilde{b_2} = p_{-\alpha} \tilde{b_2}$ with $p_{-\alpha}$ of order $-\alpha$.

We begin to diagonalize the matrix

$$\begin{pmatrix} \tilde{K}_1 & -\tilde{b_2} \langle \xi \rangle^2 \\ 0 & -\tilde{K}_1 \end{pmatrix}$$

by means of

$$\mathcal{D}_0(\xi) = \begin{pmatrix} 1 & \langle\xi\rangle^2/2\xi^2 \\ 0 & 1 \end{pmatrix}, \ |\xi| \geq R > 0, \tag{45}$$

which is in S^0. At the operator level, for the system $\mathcal{W}_0$ in (44) we have

$$\mathcal{D}_0^{-1}\mathcal{W}_0\mathcal{D}_0 = \mathcal{W}_1$$

with $\mathcal{W}_1$ equal to $\mathcal{W}$ in (42) modulo a term of the form

$$\begin{pmatrix} 0 & \tilde{z}_1 \\ 0 & 0 \end{pmatrix},$$

where

$$\tilde{z}_1(t,x,\xi) = \langle\xi\rangle^2\xi^{-2}\tilde{b}_1(t,x,\xi), \ \ |\xi| \geq R > 0.$$

We perform a second step of diagonalization by means of the operator with symbol

$$\mathcal{D}_1 = \begin{pmatrix} 1 & \tilde{d}_1 \\ 0 & 1 \end{pmatrix}, \ \tilde{d}_1 = -\tilde{z}_1/2\tilde{b}_2(t)\xi^2, \ |\xi| \geq R. \tag{46}$$

By (6), we have

$$\tilde{d}_1 \in C([0,T]; S^{-1+M(1-\eta_1),s}) \subseteq C([0,T]; S^{0,s}).$$

Moreover, from (6) and (7),

$$\partial_t\tilde{d}_1 = p_0\omega_0 + q_0\omega_1 + r_0\omega_2, \ \ p_0, q_0, r_0 \in C([0,T]; S^{0,s}).$$

Thus, $\mathcal{D}_1^{-1}\mathcal{W}_1\mathcal{D}_1 = \mathcal{W}$, with $\mathcal{W}$ in (42). $\square$

Proof of Theorem 1. To prove the well posedness in Gevrey classes of the Cauchy problem (1) for the scalar operator L, we are going to prove the well posedness in Sobolev–Gevrey spaces of the equivalent problem

$$\begin{cases} \mathcal{W}U(t,x) = 0 \\ U(0,x) = G(x), \end{cases} \tag{47}$$

for the system $\mathcal{W}$ in (42). We notice that under the assumptions of Theorem 1, recalling also (41), (25) and the Levi condition (5), the diagonal part $D_t + \tilde{K}$ of $\mathcal{W}$ satisfies the hypotheses of Theorem 2. Thus we can apply Theorem 2 to $D_t + \tilde{K}$. We take the operator Λ in (17) and consider the transformed system

$$\mathcal{W}^\Lambda := \begin{pmatrix} e^\Lambda & 0 \\ 0 & e^{-\Lambda} \end{pmatrix}^{-1} \mathcal{W} \begin{pmatrix} e^\Lambda & 0 \\ 0 & e^{-\Lambda} \end{pmatrix}. \tag{48}$$

We know that, taking sufficiently large C_0 in (5) and h in (17), we have

$$iW^\Lambda = \partial_t + i\tilde{K}_\Lambda + D_1\omega + E_1\omega_0 + F_1\omega_1 + G_1\omega_2 + H_1\omega_3 + L_1\omega_4 + M_1,$$

where

$$2\Re\left(i\tilde{K}_\Lambda U,U\right) \geq -C\|U\|^2,\ U \in H^2;$$

moreover, since both the transformations $e^\Lambda, (e^\Lambda)^{-1}$ are of order zero, and

$$\begin{aligned}
&\partial_\xi^\alpha \omega(t,\xi) = q_{-\alpha}(t,\xi)\omega(t,\xi), \quad q_{-\alpha} \in C([0,T];S^{-\alpha}),\\
&\partial_\xi^\alpha \omega_j(t,\xi) = q_{-\alpha,j}(t,\xi)\omega_j(t,\xi), \quad q_{-\alpha,j} \in C([0,T];S^{-\alpha}),\ j = 0,\dots,4,
\end{aligned}$$

we have

$$\begin{aligned}
&i(D_0\omega + E_0\omega_0 + F_0\omega_1 + G_0\omega_2 + H_0\omega_3 + L_0\omega_4 + M_0)^\Lambda\\
&\quad = D_1\omega + E_1\omega_0 + F_1\omega_1 + G_1\omega_2 + H_1\omega_3 + L_1\omega_4 + M_1
\end{aligned}$$

with

$$D_1, E_1, F_1, G_1, H_1, L_1, M_1 \in C([0,T];S^{0,s}).$$

The next step in the proof consists in the transformation also of $D_1\omega + E_1\omega_0 + F_1\omega_1 + G_1\omega_2 + H_1\omega_3 + L_1\omega_4 + M_1$ into a positive operator, modulo a remainder of order zero. There a loss of derivatives will appear. We perform the change of variable given by $e^{\phi(t,D_x)}$, where

$$\phi(t,\xi) = C\int_0^t \left(|\omega_0(\tau,\xi)| + \omega(\tau,\xi) + \sum_{i=1}^4 \omega_i(\tau,\xi)\right) d\tau,$$

C a large enough constant to be chosen. The change of variable carries a loss, see Lemma 2; the loss becomes greater as the order $\mathrm{ord}(\phi)$ of the symbol $\phi(t,\xi)$ increases. Thus we choose the parameter M that minimizes $\mathrm{ord}(\phi)$, which is the maximum between the orders of $\int_0^t \omega(\tau)d\tau$, $\int_0^t |\omega_0(\tau,\xi)|d\tau$, $\int_0^t \omega_i(\tau,\xi)d\tau$, $i = 1,\dots,4$. In a comparison between (35)–(40), we notice that the following cases can occur:

- $$\begin{cases} 1/2 \leq \eta_1 < 3/4 - 1/(2k)\\ \eta_2 \geq 3\eta_1 - 3/2\\ \eta_3 \geq 2\eta_1 - 1\\ \eta_4 \geq 3\eta_1 - 2 \end{cases} \Longrightarrow \quad \begin{aligned} &M = 1/(1-\eta_1),\\ &\mathrm{ord}(\phi) = \frac{2\,[3/4 - 1/(2k) - \eta_1]}{1-\eta_1}, \end{aligned}$$

- $$\begin{cases} \eta_2 < 3/4 - 3/(2k)\\ \eta_1 \geq \eta_2/3 + 1/2\\ \eta_3 \geq 2\eta_2/3\\ \eta_4 \geq \eta_2 - 1/2 \end{cases} \Longrightarrow \quad \begin{aligned} &M = 3/(3/2-\eta_2),\\ &\mathrm{ord}(\phi) = \frac{2\,[3/4 - 3/(2k) - \eta_2]}{3/2-\eta_2}, \end{aligned}$$

- $$\begin{cases} \eta_3 < 1/2 - 1/k\\ \eta_1 \geq \eta_3/2 + 1/2\\ \eta_2 \geq 3\eta_3/2\\ \eta_4 \geq 3\eta_3/2 - 1/2 \end{cases} \Longrightarrow \quad \begin{aligned} &M = 2/(1-\eta_3),\\ &\mathrm{ord}(\phi) = \frac{2\,[1/2 - \eta_3 - 1/k]}{1-\eta_3}, \end{aligned}$$

- $\begin{cases} \eta_4 < 1/4 - 3/(2k) \\ \eta_1 \geq \eta_4/3 + 2/3 \\ \eta_2 \geq \eta_4 + 1/2 \\ \eta_3 \geq 2\eta_4/3 + 1/3 \end{cases} \qquad \Longrightarrow \qquad \begin{array}{l} M = 3/(1-\eta_4), \\ \mathrm{ord}(\phi) = \dfrac{2\,[1/4 - 3/(2k) - \eta_4]}{1-\eta_4}. \end{array}$

In each case, we have that $\mathrm{ord}(\phi) = 1/s_0$, s_0 as in the statement of Theorem 1; in what follows we use the notation

$$\phi \in C([0,T]; S^{1/s_0})$$

which covers all the four cases that can occur. The change of variable can be considered only if $\phi(t,\xi)\langle\xi\rangle^{-1/s}$ is small enough (see [3]), and it is

$$\begin{aligned}
i\mathcal{W}^{\Lambda,\phi} &:= e^{-\phi} i\mathcal{W}^{\Lambda} e^{\phi} \\
&= \partial_t + \partial_t\phi(t,D_x)\,I + i\tilde{K}^{\Lambda} + R(t,x,D_x) \\
&\quad + D_2\omega + E_2\omega_0 + F_2\omega_1 + G_2\omega_2 + H_2\omega_3 + L_2\omega_4 + M_2 \\
&= \partial_t + i\tilde{K}^{\Lambda} + (D_2\omega + C\omega I) + (E_2\omega_0 + C|\omega_0| I) + (F_2\omega_1 + C\omega_1 I) \\
&\quad + (G_2\omega_2 + C\omega_2 I) + (H_2\omega_3 + C\omega_3 I) + (L_2\omega_4 + C\omega_4 I) \\
&\quad + M_2 + R(t,x,D_x)\,,
\end{aligned} \tag{49}$$

where I is the 2×2 identity matrix,

$$D_2, E_2, F_2, G_2, H_2, L_2, M_2 \in C([0,T]; S^{0,s}),$$

and $R \in C([0,T]; S^{1/s,s})$.

Taking now C sufficiently large, from the sharp Gårding inequality for matrix operators, see [5, Theorem 4.4, page 134], we immediately get that $D_2\omega + C\omega I$, $E_2\omega_0 + C|\omega_0| I$, $F_2\omega_1 + C\omega_1 I$, $G_2\omega_2 + C\omega_2 I$, $H_2\omega_3 + C\omega_3 I$ and $L_2\omega_4 + C\omega_4 I$ in (49) are positive modulo terms with symbol in $C([0,T]; S^{0,s})$.

It only remains to make R a positive operator. To this aim, we take $\mu = 0$ and, for a function $r(t) \in C^1[0,T]$ to be chosen, we perform the last change of variable given by $e^{r(t)\langle D_x\rangle^{1/s} - \varepsilon\langle D_x\rangle^{1/s}}$, $\varepsilon > 0$, and consider the final operator

$$i\tilde{\mathcal{W}} := e^{-\left(\phi(t,D_x) + r(t)\langle D_x\rangle^{1/s} - \varepsilon\langle D_x\rangle^{1/s}\right)} i\mathcal{W}^{\Lambda} e^{\phi(t,D_x) + r(t)\langle D_x\rangle^{1/s} - \varepsilon\langle D_x\rangle^{1/s}}. \tag{50}$$

By [3] we know that there exists an $\varepsilon_0 > 0$ such that if

$$\phi(x,D_x) + r(t)\langle D_x\rangle^{1/s} \leq \varepsilon\langle D_x\rangle^{1/s}, \quad 0 \leq \varepsilon \leq \varepsilon_0,$$

then

$$\begin{aligned}
i\tilde{\mathcal{W}} &= \partial_t + i\tilde{K}^{\Lambda} + (D_2\omega + C\omega I) + (E_2\omega_0 + C|\omega_0| I) + (F_2\omega_1 + C\omega_1 I) \\
&\quad + (G_2\omega_2 + C\omega_2 I) + (H_2\omega_3 + C\omega_3 I) + (L_2\omega_4 + C\omega_4 I) \\
&\quad + M_2 + \tilde{R}(t,x,D_x) + r'(t)\langle D_x\rangle^{1/s} I\,,
\end{aligned} \tag{51}$$

where $\tilde{R} \in C([0,T]; S^{1/s})$ has seminorms such that $|\tilde{R}(t)|_\ell \leq r_\ell(t)$ for some functions $r_\ell \in C[0,T]$ not depending on $r(t)$. An application of Caldéron–Vaillancourt's Theorem to the operator $\tilde{R}$ gives the existence of a positive constant ℓ_0 only depending on the space dimension n such that

$$|\langle \tilde{R}v, v\rangle_{L^2}| \leq r_{\ell_0}(t)\langle\langle D_x\rangle^{1/s}v, v\rangle_{L^2}.$$

Thus, taking $r(t)$ such that $r'(t) = r_{\ell_0}(t)$, we have that also $\tilde{R}(t,x,D_x) + r'(t)\langle D_x\rangle^{1/s} I$ becomes a positive operator modulo terms of order zero. So, from (51), we obtain by Gronwall's method

$$\|U(t)\|_0^2 \leq C_0\|U(0)\|_0^2.$$

This procedure can be generalized to the case $\mu \neq 0$, since for each μ we have

$$\langle D_x\rangle^\mu \big(i\tilde{\mathcal{W}}\big)\langle D_x\rangle^{-\mu} = i\tilde{\mathcal{W}} + R_\mu,$$

with R_μ of order zero. From this, the energy method gives well posedness in H^μ of the Cauchy problem for $i\tilde{\mathcal{W}}$, which corresponds to well posedness of (47) in $H^\mu_{\varepsilon,s}$, $0 < \varepsilon \leq \varepsilon_0$. $\square$

REMARK 1. If, with the assumptions of Theorem 2, we take $\sigma = 1$ or $\sigma \in (0,1)$, then the Cauchy problem (15) is not well posed in H^μ, but it is well posed respectively in H^∞ or in γ^s for $s < 1/(1-\sigma)$, see [4]. This is because the symbol Λ in (17) has positive order under a decay at infinity condition of type $\Im b_1 \sim \langle x\rangle^{-\sigma}$ with $\sigma \in (0,1]$. Regarding second order equations, in the statement of Theorem 1 we only admit $\sigma > 1$, see (5). This is because in the proof of Theorem 1 we need Λ of order zero; otherwise, the transformation (48) carries a loss of derivatives, and as now we cannot simultaneously control the two losses coming from the decay condition at infinity (transformation (48)) and from the Levi conditions (transformation (50)). The problem of giving an analogue of Theorem 1 in the case $\sigma \in (0,1]$ is still open.

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