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Preface

This issue of the *Rendiconti del Seminario Matematico* incorporates texts of part of the 30-minute communications delivered at the Second Conference on Pseudo-Differential Operators and Related Topics, held in Växjö, Sweden on 23–27 June 2008.

Topics for the conference included Spectral Theory, Time-Frequency (Gabor) Analysis and Localization Operators, Positivity and Lower Bound Problems, Operators on Singular Manifolds, Fourier Integral Operators, Elliptic and Hyperbolic Problems.

The Växjö Conference was part of the activity of the *International Society for Analysis, its Applications and Computation* (ISAAC) for the year 2008. ISAAC is a non-profit organization established in 1994 to promote and advance analysis, its applications, and its interactions with computation. The president at the time was Professor M. W. Wong (York University, Toronto), whereas the incoming president in 2009 is Professor M. Ruzhansky (Imperial College, London).

During the conference, the participants honoured Professor Luigi Rodino of the University of Turin, on the occasion of his 60th birthday. The event at Växjö was particularly significant in view of the fact that Professor Rodino began his long and productive scientific career in the field of pseudo-differential operators at Lund University and the Mittag-Leffler Institute in 1973–74.

The meeting was organized with 45-minute plenary talks in the morning and three parallel sessions of 30-minute communications in the afternoon. There were about 80 talks altogether in the whole Conference. The texts of the plenary talks were published in the *Rendiconti del Seminario Matematico Università e Politecnico di Torino* (66 no. 4, 2008). Like that issue, the present one is is dedicated to Professor Luigi Rodino. The contributions concerning Elliptic Problems appeared in "Complex Variables and Elliptic Equations" (Taylor & Francis, Oxford, 54/2009). Finally the proceedings in the field of Time Frequency Analysis are appearing in the Journal *CUBO* (Pernambuco University, Brazil).

The editors of this issue were also the scientific organizers of the conference. Accordingly, we wish to thank Växjö University, the Vetenskapsrådet (the Swedish Science Council) and the Mathematics Department "Giuseppe Peano" of the University of Turin for financial support. We are also grateful to Karoline Johansson and Haidar Al-Talibi of Växjö University for their technical support as local organizers.

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Second Conf. Pseudo-Differential Operators

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OPERATOR CALCULUS FOR p-ADIC VALUED SYMBOLS AND QUANTIZATION

Dedicated to Professor Luigi Rodino on the occasion of his 60th birthday

Abstract. The aim of this short review is to attract the attention of the pseudo-differential community to possibilities in the development of operator calculus for symbols (depending on p-adic conjugate variables) taking values in fields of p-adic numbers. Essentials of this calculus were presented in works of the authors of this paper in order to perform p-adic valued quantization. Unfortunately, this calculus still has not attracted a great deal of attention from pure mathematicians, although it opens new and interesting domains for the theory of pseudo-differential operators.

1. Introduction

Quantum formalism with wave functions valued in non-Archimedean fields was developed in a series of papers and books [1]–[13], see also related works of Vladimirov and Volovich [14]–[15] and the book [16] on quantum formalism with p-adic variables but complex-valued wave functions. In this review article, we present the essentials of this theory. We restrict attention to the fields of p-adic numbers. General quantum theory has been developed for an arbitrary non-Archimedean field K, see [11].

The basic objects of this theory are p-adic Hilbert spaces and symmetric operators acting in these spaces. Vectors of a p-adic Hilbert space which are normalized with respect to the inner product represent quantum states. In the p-adic case, the norm is not determined by the inner product. Therefore normalization with respect to the norm and the inner product, which coincides for real and complex Hilbert spaces, is different for p-adic Hilbert spaces. We shall proceed in the following way.

Consider the formal differential expression $\widehat{\mathbf{H}} = H(\partial_{x_j}, x_j)$ of operators of quantum mechanics or quantum field theory. Let us realize this formal expression as a differential operator with variables x_j belonging to the field of p-adic numbers $\mathbb{Q}_p$ and study properties of this operator in a p-adic Hilbert space. Thus we would like to perform a p-adic analogue of Schrödinger's quantization.

We remark that p-adic valued quantum theory suffers from the absence of a "good spectral theorem" for symmetric operators. At the same time, this theory is essentially simpler (mathematically) than ordinary quantum mechanics, since *operators of position and momentum are bounded in the p-adic case*, as was found by Albeverio and Khrennikov [3].

The representation theory of groups in Hilbert spaces forms one of the cornerstones of ordinary quantum mechanics. It is very natural to develop p-adic quantum mechanics in a similar way. We construct a representation of the Weyl–Heisenberg group in a p-adic Hilbert space, namely the space $L_2(\mathbb{Q}_p, \nu_b)$ of L_2-functions with respect to a p-adic valued Gaussian distribution ν_b (the symbol b indicates a p-adic analogue of dispersion), see [3].[1] Here the situation differs very much from that of ordinary quantum mechanics. If we denote by $\widehat{U}(\alpha)$ and $\widehat{V}(\beta)$ the groups of unitary operators corresponding to position and momentum operators, respectively, then these groups are defined only for parameters α and β belonging to balls $U_{R(b)}$ and $U_{r(b)}$, respectively, where $R(b)$ and $r(b)$ depend on the dispersion b of the Gaussian distribution and they are coupled by a kind of Heisenberg uncertainty relation.

We shall also study the representation of the translation group on the space $L_2(\mathbb{Q}_p, \nu_b)$. Here the result also differs from that of ordinary quantum mechanics, and is more similar to one that holds in quantum field theory where Gaussian distributions on infinite dimensional spaces are used.

Let μ be Gaussian measure on the infinite-dimensional real Hilbert space $\mathcal{H}$. It is impossible to construct a representation of translations from all of $\mathcal{H}$ in $L_2(\mathcal{H}, \mu)$, because of the well-known fact that the translation μ^h of a Gaussian measure on $\mathcal{H}$ by a vector $h \in \mathcal{H}$ can be singular with respect to μ. It is well known that μ^h is equivalent to μ if and only if h belongs to a certain proper ("Cameron–Martin") subspace. In a similar way we cannot construct in the space $L_2(\mathbb{Q}_p, \nu_b)$ a representation of translations by all elements h in $\mathbb{Q}_p$; in fact, we have to restrict consideration to translations belonging to some ball (which is an additive subgroup in $\mathbb{Q}_p$) whose radius depends on the dispersion b. This fact is connected with the nonexistence of translation-invariant measures in the p-adic case (similarly for infinite-dimensional spaces over the field of real numbers), see [6].

2. Banach and Hilbert spaces

2.1. p-adic numbers and their quadratic extensions

The field of real numbers $\mathbb{R}$ is constructed as the completion of the field of rational numbers $\mathbb{Q}$ with respect to the metric $\rho_{\mathbb{R}}(x,y) = |x-y|$, where $|\cdot|$ is the usual real valuation (absolute value). The fields of p-adic numbers $\mathbb{Q}_p$ are constructed in a corresponding way, by using other valuations. For any prime number $p > 1$, the p-adic valuation $|\cdot|_p$ is defined in the following way. First we define it for natural numbers. Every natural number n can be represented as the product of prime numbers: $n = 2^{r_2}3^{r_3}\cdots p^{r_p}\cdots$. Then we define $|n|_p = p^{-r_p}$, and in addition set $|0|_p = 0$ and $|-n|_p = |n|_p$. We extend the definition of the p-adic valuation $|\cdot|_p$ to all rational numbers by setting $|n/m|_p = |n|_p/|m|_p$ for $m \neq 0$. The completion of $\mathbb{Q}$ with respect to the metric $\rho_p(x,y) = |x-y|_p$ is the locally compact field of p-adic numbers $\mathbb{Q}_p$. By the well-known *Ostrovsky theorem*, the real valuation (absolute value) $|\cdot|$ and the p-adic valuations $|\cdot|_p$ are the only possible valuations on $\mathbb{Q}$. Thus if one wants to construct a

[1] We remark that ν_b is not a p-adic valued measure, i.e. a bounded linear functional on the space of continuous functions. It is just a distribution, a generalized function, which is primarily defined on the space of analytic test functions. A analogue of the L_2-space can be constructed by completing the space of test functions with respect to a natural norm.

physical model starting with rational numbers, then there are only two possibilities: to proceed to real numbers or to one of the fields of p-adic numbers.[2]

The p-adic valuation satisfies the so-called strong triangle inequality: $|x+y|_p \leq \max[|x|_p, |y|_p]$, which makes ρ_p into an ultrametric. Set $U_r(a) = \{x \in \mathbb{Q}_p : |x-a|_p \leq r\}$ and $U_r^-(a) = \{x \in \mathbb{Q}_p : |x-a|_p < r\}$, with $r = p^n$ and $n = 0, \pm 1, \pm 2, \ldots$; these are ("closed" and "open") balls in $\mathbb{Q}_p$. Set $S_r(a) = \{x \in \mathbb{Q}_p : |x-a|_p = r\}$; these are spheres in $\mathbb{Q}_p$. Any p-adic ball $U_r \equiv U_r(0)$ is an additive subgroup of $\mathbb{Q}_p$. The ball $U_1(0)$ is also a ring, called the *ring of p-adic integers* and denoted by $\mathbb{Z}_p$. For any $x \in \mathbb{Q}_p$, we have a unique canonical expansion (converging in the $|\cdot|_p$-norm) of the form

$$x = \alpha_{-n}/p^n + \cdots \alpha_0 + \cdots + \alpha_k p^k + \cdots, \tag{1}$$

where $\alpha_j = 0, 1, ..., p-1$, are the "digits" of the p-adic expansion. The elements $x \in \mathbb{Z}_p$ have an expansion $x = \alpha_0 + \alpha_1 p + \cdots + \alpha_k p^k + \cdots$, i.e., they are natural generalizations of natural numbers. Moreover, even negative natural numbers can be represented as elements of $\mathbb{Z}_p$, e.g., $-1 = (p-1) + (p-1)p + (p-1)p^2 + \ldots + (p-1)p^n + \ldots$ This is the source of the terminology "p-adic integer".

For $p_1 \neq p_2$, the fields of p-adic numbers $\mathbb{Q}_{p_1}$ and $\mathbb{Q}_{p_2}$ are not isomorphic as topological fields. Thus by moving into the p-adic domain one obtains, in fact, an infinite series of fields for the modeling of, e.g., space geometry. None of these fields is isomorphic to the field of real numbers $\mathbb{R}$. The crucial difference is in the topology.

Fields of p-adic numbers are *disordered.* It is impossible to introduce a linear order on $\mathbb{Q}_p$ (at least in a natural way, e.g., matching algebraic operations). This fact induces interesting departures from the real case. It also plays a fundamental role in the application of p-adic numbers to string theory and cosmology. For a long time, physicists discussed the idea that at Planck distances (which are extremely small) space-time is disordered. In particular, it cannot be described by real numbers. On the other hand, p-adic numbers provide an excellent possibility for the mathematical formulation of this physical idea.

In applications to physics, the following complicated problem arises: "Which p should be used for modeling?" There are various opinions. Igor Volovich proved that some amplitudes used in "ordinary string theory", i.e., based on the real model of space-time, can be reproduced in the limit $p \to \infty$ from the corresponding amplitudes of p-adic string theory [16]. The authors of this paper think that this is not crucial for the new geometry. Therefore the p selected for physical modeling (at least in a theoretical model) does not play an important role. One can switch from one scale to another as one does in the real case by switching in the expansion (1) from one p to another, see [11] for a detailed presentation of this ideology. Of course, each physical phenomenon has its own scale. One can discuss concrete scales, e.g., in the p-adic approach to quantum physics. The authors of this paper proposed selecting $p = [1/\alpha]$: the integer part of the fine structure constant α. However, all such physical discussions have no direct relation to the present paper. For a mathematician, it may be more important to

[2] We remark that experimental data is always rational. It is a consequence of the finite precision of any measurement.

know that typically the case $p=2$ should be treated separately, and proofs obtained for $p>2$ typically do not work for $p=2$.

Let $\tau \in \mathbb{Q}_p$ and suppose that $x^2=\tau$ have no solution in $\mathbb{Q}_p$. The symbol $\mathbb{Q}_p(\sqrt{\tau})$ denotes the corresponding quadratic extension of $\mathbb{Q}_p$. Its elements have the form $z=x+\sqrt{\tau}y$, where $x,y \in \mathbb{Q}_p$. The operation of conjugation is defined by $\bar{z}=x-\sqrt{\tau}y$. We remark that $z\bar{z}=x^2-\tau y^2$ for $z \in \mathbb{Q}_p(\sqrt{\tau})$, and that $z\bar{z} \in Q_p$ for any $z \in \mathbb{Q}_p(\sqrt{\tau})$. The extension of the p-adic valuation from $\mathbb{Q}_p$ onto $\mathbb{Q}_p(\sqrt{\tau})$ is denoted by the same symbol $|\cdot|_p$. We have $|z|_p=\sqrt{|z\bar{z}|_p}$ for $z \in \mathbb{Q}_p(\sqrt{\tau})$. Besides quadratic extensions, we shall also operate with the field of complex p-adic numbers $\mathbb{C}_p$. Its construction is very complicated. Unlike in the real case, we cannot obtain an algebraically closed field by taking a quadratic extension, nor indeed by taking an algebraic extension of any finite order. The algebraic closure $\mathbb{Q}_p^a$ of $\mathbb{Q}_p$ is constructed as an infinite tower of finite extensions. In particular, it is an infinite-dimensional linear space over $\mathbb{Q}_p$ (compare with the real case where the algebraic closure $\mathbb{C}$ is just two dimensional over $\mathbb{R}$). The p-adic valuation is defined on the tower of finite extensions in a consistent way. In this way we obtain the p-adic valuation on $\mathbb{Q}_p^a$. However, this is not the end of the story concerning a p-adic analogue of complex numbers. The field $\mathbb{Q}_p^a$ is not complete with respect to such an extension of the p-adic valuation. Finally, we complete it and obtain that its completion, denoted by $\mathbb{C}_p$, is *algebraically closed!* The latter is a notrivial result, Krasner's theorem. As the reader has seen, the construction of p-adic complex numbers is quite complicated. However, it might be even worse – if Krasner's theorem were not true.

2.2. Banach spaces

Essentials of non-Archimedean functional analysis can be found in, e.g., the book of van Rooji [18].

The symbol K denotes a non-Archimedean field with the valuation (absolute value) $|\cdot|_K$. It is a map from K to $[0,+\infty)$ such that

(1) $|x|_K=0 \Leftrightarrow x=0$;

(2) $|xy|_K=|x|_K\,|y|_K$;

(3) $|x+y|_K \le \max(|x|_K,|y|_K)$.

The latter feature of the valuation is the strong triangle inequality. It plays a fundamental role in the determination of special features of the corresponding non-Archimedean topology. Such terminology is common in so-called non-Archimedean analysis, see e.g. [18]. However, in other domains of mathematics, a non-Archimedean field is a totally (or partially) ordered field containing nonzero infinitesimals, e.g., the field of nonstandard numbers $\mathbb{R}^*$. We emphasize that this paper has nothing to do with the latter case!

Let E be a linear space over a non-Archimedean field K. A *non-Archimedean norm* on E is a mapping $\|\cdot\|: E \to [0,+\infty)$ satisfying the following conditions:

(a) $\|x\|=0 \Leftrightarrow x=0$;

(b) $\|\alpha x\|=|\alpha|_K\,\|x\|,\ \alpha \in K$;

(c) $\|x+y\| \leq \max(\|x\|,\|y\|)$.

As usual, we define non-Archimedean Banach space E as a complete normed space over K. The metric $\rho(x,y) = \|x-y\|$ is ultrametric. Hence every non-Archimedean Banach space is zero-dimensional and totally disconnected. All balls $W_r(a) = \{x \in E : \|x-a\| \leq r\}$ are clopen.

The dual space E' is defined as the space of continuous K-linear functionals $l : E \to K$. Let us introduce the usual norm on E' : $\|l\| = \sup_{x \neq 0} |l(x)|_K / \|x\|$. The space E' endowed with this norm is a Banach space.

The simplest example of a non-Archimedean Banach space is the space $K^n = K \times \cdots \times K$ (n times) with the non-Archimedean norm $\|x\| = \max_{1 \leq j \leq n} |x_j|_K$. More interesting examples are infinite-dimensional non-Archimedean Banach spaces realized as spaces of sequences: set $c_0 \equiv c_0(K) = \{x \in K^\infty : \lim\limits_{n\to\infty} x_n = 0\}$ and $\|x\| = \max_n |x|_K$.

2.3. Hilbert spaces

We take a sequence of p-adic numbers $\lambda = (\lambda_n) \in Q_p^\infty$, $\lambda_n \neq 0$. We set

$$l^2(p,\lambda) = \Big\{ f = (f_n) \in \mathbb{Q}_p^\infty : \text{ the series } \sum f_n^2 \lambda_n \text{ converges in } \mathbb{Q}_p \Big\}.$$

It turns out that $l^2(p,\lambda) = \{ f = (f_n) \in \mathbb{Q}_p^\infty : \lim_{n\to\infty} |f_n|_p \sqrt{|\lambda_n|_p} = 0\}$. In the space $l^2(p,\lambda)$ we introduce the norm $\|f\|_\lambda = \max_n |f_n|_p \sqrt{|\lambda_n|_p}$. The space $l^2(p,\lambda)$ endowed with this norm is non-Archimedean Banach space. On the space $l^2(p,\lambda)$ we also introduce the p-adic valued inner product $(\,\cdot\,,\,\cdot\,)_\lambda$ by setting $(f,g)_\lambda = \sum f_n g_n \lambda_n$.

We remark that $\|f\|_\lambda \in \mathbb{R}$, but $(f,f)_\lambda \in \mathbb{Q}_p$. The norm is not determined by the inner product. Nevertheless, the p-adic inner product $(\,\cdot\,,\,\cdot\,)_\lambda : l^2(p,\lambda) \times l^2(p,\lambda) \to \mathbb{Q}_p$ is continuous and the *Cauchy–Bunyakovsky–Schwarz inequality* holds, namely $|(f,g)_\lambda|_p \leq \|f\|_\lambda \, \|g\|_\lambda$.

DEFINITION 1. *A triplet $(l^2(p,\lambda),\ (\cdot,\cdot)_\lambda,\ \|\cdot\|_\lambda)$ is called a p-adic coordinate Hilbert space.*

More generally, we shall define a p-adic inner product on $\mathbb{Q}_p$-linear space E as an arbitrary non-degenerate symmetric bilinear form $(\cdot,\cdot) : E \times E \to \mathbb{Q}_p$.

REMARK 1. We cannot introduce a p-adic analogue of positive definiteness of a bilinear form. For instance, any element $\gamma \in \mathbb{Q}_p$ can be represented as $\gamma = (x,x)_\lambda$, with $x \in l^2(p,\lambda)$ (this is a simple consequence of properties of bilinear forms over $\mathbb{Q}_p$).

The triplets $(E_j,\ (\cdot,\cdot)_j,\ \|\cdot\|_j)$, $j = 1,2$, where E_j are non-Archimedean Banach spaces, $\|\cdot\|_j$ are norms and $(\cdot,\cdot)_j$ are inner products satisfying the Cauchy–Buniakovski–Schwarz inequality, are isomorphic if the spaces E_1 and E_2 are algebraically isomorphic and the algebraic isomorphism $I : E_1 \to E_2$ is a unitary isometry, i.e., $\|Ix\|_2 = \|x\|_1$ and $(Ix,Iy)_2 = (x,y)_1$.

DEFINITION 2. *The triplet* $(E, (\cdot,\cdot), \|\cdot\|)$ *is a p-adic Hilbert space if it is isomorphic to the coordinate Hilbert space* $(l^2(p,\lambda), (\cdot,\cdot)_\lambda, \|\cdot\|_\lambda)$ *for some sequence of weights* λ.

The isomorphism relation splits the family of p-adic Hilbert spaces into equivalence classes. An equivalence class is characterized by some coordinate representative $l^2(p,\lambda)$. The classification of p-adic Hilbert spaces is an open mathematical problem.

Hilbert spaces over quadratic extensions $\mathbb{Q}_p(\sqrt{\tau})$ of $\mathbb{Q}_p$ can be introduced in the same way. For a given sequence $\lambda = (\lambda_n) \in \mathbb{Q}_p^\infty$, $\lambda_n \neq 0$, we set

$$l^2(p,\lambda,\sqrt{\tau}) = \{f = (f_n) \in \mathbb{Q}_p(\sqrt{\tau})^\infty : \text{ the series} \sum f_n \bar{f}_n \lambda_n \text{ converges}\},$$

with $\|f\|_\lambda = \max_n |f_n|_p \sqrt{|\lambda_n|_p}$ and $(f,g)_\lambda = \sum f_n \bar{g}_n \lambda_n$.

The triplet $(l^2(p,\lambda,\sqrt{\tau}), (\cdot,\cdot)_\lambda, \|\cdot\|_\lambda)$ is the coordinate Hilbert space over the quadratic extension $\mathbb{Q}_p(\sqrt{\tau})$. In general, a Hilbert space $(E, (\cdot,\cdot), \|\cdot\|)$ over the quadratic extension $\mathbb{Q}_p(\sqrt{\tau})$, is by definition isomorphic to some coordinate Hilbert space. We denote a p-adic Hilbert space over $\mathbb{Q}_p(\sqrt{\tau})$ by

$$\mathcal{H}_p \equiv \mathcal{H}_p(\sqrt{\tau}).$$

3. Groups of unitary isometric operators in p-adic Hilbert space

As usual, we introduce unitary operators $\widehat{U} : \mathcal{H}_p \to \mathcal{H}_p$ as operators which preserve the inner product, so $(\widehat{U}x, \widehat{U}y) = (x,y)$ for all $x,y \in \mathcal{H}_p$, with image $\mathrm{Im}\,\widehat{\mathrm{U}} = \widehat{\mathrm{U}}(\mathcal{H}_\mathrm{p}) = \mathcal{H}_\mathrm{p}$. Isometric operators are operators which preserve the norm, so $\|\widehat{U}x\| = \|x\|$, and have $\mathrm{Im}\,\widehat{\mathrm{U}} = \mathcal{H}_\mathrm{p}$. Denote the space of all bounded linear operators $\widehat{A} : \mathcal{H}_p \to \mathcal{H}_p$ by $\mathcal{L}(\mathcal{H}_p)$. It is a Banach space with respect to the operator norm $\|\widehat{A}\| = \sup_{x\neq 0} \|\widehat{A}x\|/\|x\|$. A unitary operator need not be isometric.[3] Indeed, it could even be unbounded. Denote the group of linear isometries of the p-adic Hilbert space $\mathcal{H}_p$ by $IS(\mathcal{H}_p)$, and the group of all bounded unitary operators in $\mathcal{H}_p$ by $UN(\mathcal{H}_p)$. Set $UI(\mathcal{H}_p) = UN(\mathcal{H}_p) \cap UI(\mathcal{H}_p)$.

An operator $\widehat{A} \in \mathcal{L}(\mathcal{H}_p)$ is said to be symmetric if $(\widehat{A}x, y) = (x, \widehat{A}y)$ for all x,y. The following simple fact will be useful later.

THEOREM 1. *The eigenvalue* α *of a symmetric operator* $\widehat{A} : \mathcal{H}_p \to \mathcal{H}_p$ *corresponding to an eigenvector u with nonzero square,* $(u,u) \neq 0$, *belongs to* $\mathbb{Q}_p$. *Eigenvectors corresponding to different eigenvalues of such type are orthogonal.*

The proof is similar to the standard one for complex Hilbert space $\mathcal{H}$.

As usual, we introduce the resolvent set $\mathrm{Res}(\widehat{\mathrm{A}})$ of an operator $\widehat{A} \in \mathcal{L}(\mathcal{H}_p)$; it consists of $\lambda \in \mathbb{Q}_p(\sqrt{\tau})$ such that the operator $(\lambda I - \widehat{A})^{-1}$ exists. The spectrum $\mathrm{Spec}(\widehat{\mathrm{A}})$ of $\widehat{A}$ is the complement of the resolvent set.

[3] Recall that the norm on the p-adic Hilbert space is not determined by the inner product. The only condition of consistency between them is the Cauchy–Bunyakovsky–Schwarz inequality.

Note that every ball U_r in $\mathbb{Q}_p$ is an additive subgroup of $\mathbb{Q}_p$. A map $\widehat{F} : U_r \to \mathcal{L}(\mathcal{H}_p)$ with the properties $\widehat{F}(t+s) = \widehat{F}(t)\widehat{F}(s)$, $t,s \in U_r$, and $\widehat{F}(0) = I$, where I is the unit operator in $\mathcal{H}_p$, is said to be a one-parameter group of operators. If we consider $IS(\mathcal{H}_p), UN(\mathcal{H}_p), UI(\mathcal{H}_p)$ instead of $\mathcal{L}(\mathcal{H}_p)$, we obtain definitions of the parametric groups of isometric, unitary, and isometric unitary operators, respectively. If the map $F : \mathcal{U}_r \to \mathcal{L}(\mathcal{H}_p)$ is analytic the one-parameter group is called analytic.

We recall that any p-adic ball is, in fact, a ball with radius $r = p^k$, with $k = 0, \pm 1, \ldots$ (since the p-adic valuation takes only such values). On the other hand, in a normed space over $\mathbb{Q}_p$ or its quadratic extension, the norm can take any value belonging to $[0, +\infty)$. To match these two ranges of values, we invent the following quantity. Let a be a positive real number. We define

$$[a]_p^- = \sup\{\lambda = p^k,\ k \in \mathbb{Z} : \lambda < a\}. \tag{2}$$

This number approximates (from below) the real number a by numbers from the range of values of the p-adic valuation.

For a bounded operator $\widehat{A}$, we define

$$\gamma(\widehat{A}) = \frac{1}{p^{1/(p-1)}\|\widehat{A}\|} . \tag{3}$$

It is a real number, the reciprocal of the norm $\|\widehat{A}\|$ multiplied by the factor $p^{1/(p-1)}$. The latter appears in connection with convergence of the exponential series in the p-adic case. The series e^y, where in general y belongs to $\mathbb{C}_p$, converges on the ball of radius $r_{\exp} = p^{-1/(p-1)}$.

THEOREM 2. *Let $\widehat{A}$ be a bounded symmetric operator in $\mathcal{H}_p \equiv \mathcal{H}_p(\sqrt{\tau})$. The map*

$$t \mapsto e^{\sqrt{\tau}\, t\widehat{A}}, \quad t \in U_r,\ r = [\gamma(\sqrt{\tau}\widehat{A})]_p^-,$$

is an analytic one-parameter group of isometric unitary operators.

Thus every symmetric operator $\widehat{A} \in \mathcal{L}(\mathcal{H}_p(\sqrt{\tau}))$ generates the one-parameter operator group of isometric unitary operators $t \mapsto \widehat{U}(t) = e^{\sqrt{\tau}\, t\widehat{A}}$. This theorem is a natural generalization of the standard theorem for $\mathbb{C}$-Hilbert space. The following result has no analogue in functional analysis over $\mathbb{C}$.

THEOREM 3. *Suppose that an operator $\widehat{A}$ belongs to $\mathcal{L}(\mathcal{H}_p)$. The map $\alpha \mapsto e^{\alpha\widehat{A}}$, $\alpha \in U_r,\ r = [\gamma(\widehat{A})]_p^-$, is an analytic one-parameter group of isometric operators.*

4. Gaussian integral and spaces of square integrable functions

As already remarked, the mathematical formalism of p-adic quantization does not depend on the choice of a quadratic extension $\mathbb{Q}_p(\sqrt{\tau})$ of $\mathbb{Q}_p$. To make considerations symbolically closer to ordinary complex quantization, we shall proceed for the quadratic extension $\mathbb{Q}_p(i)$. Of course, this choice restricts in an essential way the class of prime numbers under consideration.

To provide the pointwise realization of elements of the p-adic analogue of the L_2-space, we shall consider analytic functions over the field of complex p-adic numbers $\mathbb{C}_p$. In $\mathbb{C}_p$ we denote the ball of radius $s \in \mathbb{R}_+$ with center at $z = 0$ by the symbol $\mathcal{U}_s$. We denote the space of analytic functions $f : \mathcal{U}_s \to \mathbb{C}_p$ by $\mathcal{A}(\mathcal{U}_s)$.

In [2], the general definition of a p-adic valued Gaussian integral was proposed on the basis of distribution theory. In this context, the Gaussian distribution was defined as the distribution having Laplace transform of the form $\exp\{bx^2/2\}$, where $b \in \mathbb{R}$. We recall that in the real case if $b > 0$ then Gaussian distribution is simply a countably additive measure – Gaussian measure with dispersion b. If b is negative or even complex then the Gaussian distribution cannot be realized as a measure.

For our present applications to quantization, we can use a simpler approach based on the definition of Gaussian distribution through the definition of its moments. Roughly speaking, we know moments of Gaussian distribution over the reals. Suppose now that dispersion is a rational number, $b \in \mathbb{Q}$. Then moments can equally well be interpreted as elements of any $\mathbb{Q}_p$. We now can extend by continuity our definition of moments to any “dispersion” $b \in \mathbb{Q}_p$.

Let b be a p-adic number, $b \neq 0$. The p-adic Gaussian distribution ν_b is defined by its moments $(n = 0, 1, \ldots)$:

$$M_{2n} = \int_{\mathbb{Q}_p} x^{2n} \nu_b(dx) \equiv \frac{(2n)!\, b^n}{n!\, 2^n}, \quad M_{2n+1} = \int_{\mathbb{Q}_p} x^{2n+1} \nu_b(dx) \equiv 0.$$

We define the Gaussian integral for polynomial functions by linearity. Then we can define it for some classes of analytic functions. The analytic function $f(x) = \sum_{n=0}^{\infty} c_n x^n$, with $c_n \in \mathbb{C}_p$, is said to be integrable with respect to the Gaussian distribution ν_b if the series

$$\int_{\mathbb{Q}_p} f(x) \nu_b(dx) \equiv \sum_{n=0}^{\infty} c_n M_n = \sum_{n=0}^{\infty} c_{2n} M_{2n} \tag{4}$$

converges. It was shown in [11] that all entire analytic functions on $\mathbb{C}_p$ are integrable. In fact, we do not need analyticity on the whole of $\mathbb{C}_p$ to be able to define the Gaussian integral. The following (real) constant

$$\theta_b \equiv p^{\frac{1}{2(1-p)}} \sqrt{|b/2|_p}$$

will play a fundamental role. If $p \neq 2$, then $\theta_b = p^{\frac{1}{2(1-p)}} \sqrt{|b|_p}$. If $p = 2$, then $\theta_b = \sqrt{|b|_p}$.

PROPOSITION 1. *Let $f(x)$ belong to the class $\mathcal{A}(\mathcal{U}_s)$. If $s > \theta_b$, then the integral (4) converges.*

REMARK 2. There exist functions which are analytic on the ball $\mathcal{U}_{\theta_b}$ but are not integrable, see [11].

In fact, we have proved that the Gaussian distribution is a continuous linear functional on the space of analytic functions $\mathcal{A}(\mathcal{U}_s)$, i.e., it is an analytic generalized function (distribution); for the details see [2]. We shall use the symbol $\int$ to represent the duality between the space of test functions $\mathcal{A}(\mathcal{U}_s)$ and the space of generalized functions $\mathcal{A}'(\mathcal{U}_s)$ by setting $(\mu\prime, f) \equiv \int f(x)\mu(dx)$ for $f \in \mathcal{A}(\mathcal{U}_s)$ and $\mu \in \mathcal{A}'(\mathcal{U}_s)$. As usual, we define the derivative of a generalized function μ by means of the equality $\int f(x)\mu(dx) = -\int f'(x)\mu(dx)$.

It should be remarked that the distribution ν_b is not a bounded measure on any ball of $\mathbb{Q}_p$. (This was proved for the case $p \neq 2$; in the case $p = 2$ the question is still open), see Endo and Khrennikov [19]. Thus we could not integrate continuous functions with respect to the p-adic Gaussian distribution.

We introduce Hermite polynomials over $\mathbb{Q}_p$ by substituting a p-adic variable, in place of a real one, into the ordinary Hermite polynomials over the reals:

$$H_{n,b}(x) = \frac{n!}{b^n}\sum_{k=0}^{[n/2]}\frac{(-1)^k x^{n-2k} b^k}{k!(n-2k)!2^k}.$$

We shall use also the following representation for the Hermite polynomials: $H_{n,b}(x) = (-1)^n e^{x^2/2b}\frac{d^n}{dx^n}e^{-x^2/2b}$. This representation holds on a ball of sufficiently small radius with center at zero. As a consequence, we obtain the following equality in the space of generalized functions $\mathcal{A}'(\mathcal{U}_s)$, with $s > \theta_b$:

$$H_{n,b}(x)\nu_b(dx) = (-1)^n\frac{d^n}{dx^n}\nu_b(dx), \tag{5}$$

i.e., multiplication of the Gaussian distribution by a Hermite polynomial is equivalent to evaluating the corresponding derivative (in the sense of distribution theory).

In the space $\mathcal{P}(\mathbb{Q}_p)$ of polynomials on $\mathbb{Q}_p$ with coefficients belonging to $\mathbb{Q}_p(i)$, we introduce the inner product $(f,g) = \int f(x)\overline{g}(x)\nu_b(dx)$. With respect to this inner product, the polynomials $H_{n,b}$ verify the orthogonal conditions $\int H_{m,b}(x)H_{n,b}(x)\nu_b(dx) = \delta_{nm}\, n!/b^n$.

REMARK 3. In fact, the appearance of such constants $\lambda_n = n!/b^n$ was one of the reasons for introducing p-adic Hilbert spaces that are isomorphic to $l^2(p,\lambda)$.

Any $f \in \mathcal{P}(\mathbb{Q}_p)$ can be written in the following way: $f(x) = \sum_{n=0}^N f_n H_{n,b}(x)$, $N = N(f), f_n \in \mathbb{Q}_p(i)$. We introduce the norm $\|f\|^2 = \max_n |f_n|_p^2(|n!|_p/|b|_p^n)$, and we define $L_2^i(\mathbb{Q}_p,\nu_b)$ as the completion of $\mathcal{P}(\mathbb{Q}_p)$ with respect to $\|\cdot\|$. It is evident that the space $L_2^i(\mathbb{Q}_p,\nu_b)$ is the set

$$\left\{f(x) = \sum_{n=0}^{\infty} f_n H_{n,b}(x),\ f_n \in \mathbb{Q}_p(i):\ \text{the series } \sum_{n=0}^{\infty} f_n \overline{f_n}\frac{n!}{b^n} \text{ converges}\right\}.$$

Let $L_2(\mathbb{Q}_p,\nu_b)$ stand for the subset of $L_2^i(\mathbb{Q}_p,\nu_b)$ consisting of functions that have the Hermite coefficients $f_n \in \mathbb{Q}_p$. This is a Hilbert space over the field $\mathbb{Q}_p$.

For $f(x) \in L_2^i(\mathbb{Q}_p, \nu_b)$ we set

$$\sigma_n^2(f) \equiv \sigma_{n,b}^2(f) = |f_n|_p^2 \left| \frac{n!}{b^n} \right|_p, \tag{6}$$

where

$$f_n = \frac{b^n}{n!} \int f(x) H_{n,b}(x) \nu_{b,p}(dx)$$

are the Hermite coefficients of $f(x)$.

Now we wish to study the relations between $L_2(\mathbb{Q}_p, \nu_b)$-functions and analytic functions. Set $\mathcal{A}_{\mathbb{Q}_p}(\mathcal{U}_r) = \{f \in \mathcal{A}(\mathcal{U}_r) : f : U_r \to \mathbb{Q}_p\}$, i.e., these are functions that have Taylor coefficients belonging to the field $\mathbb{Q}_p$.

THEOREM 4. *Assume* $p \neq 2$. *Then* $L_2(\mathbb{Q}_p, \nu_b) \subset \mathcal{A}_{\mathbb{Q}_p}(\mathcal{U}_{\theta_b})$.

Now we consider the case $p = 2$. In general, L_2-functions are not analytic on the ball $\mathcal{U}_{\theta_b}$.

THEOREM 5. *Let* $s > \theta_b$. *Then* $\mathcal{A}_{\mathbb{Q}_p}(\mathcal{U}_s) \subset L_2(\mathbb{Q}_p, \nu_b)$.

Further we construct the L_2-representation of the translation group. If $|b|_p = p^{2k+1}$ we set $s(b) = p^k$, if $|b|_p = p^{2k}$, we set $s(b) = p^{k-1}$. Set $\widehat{T}_\beta(f)(x) = f(x+\beta), \beta \in \mathbb{Q}_p$. We shall prove that these operators are bounded for $\beta \in U_{s(b)}$. Moreover, these operators are isometries of $L_2(\mathbb{Q}_p, \nu_b)$. Using this fact we shall construct a representation of the translation group in the p-adic Hilbert space $L_2(\mathbb{Q}_p, \nu_b)$.

LEMMA 1. *The formula*

$$\widehat{T}_\beta H_{n,b}(x) = \sum_{j=0}^{n} \binom{n}{j} \left(\frac{\beta}{b}\right)^j H_{n-j,b}(x) \tag{7}$$

holds for the translates of Hermite polynomials.

THEOREM 6. *The operator* $\widehat{T}_\beta$ *belongs to* $IS(L_2(\mathbb{Q}_p, \nu_b))$ *for every* $\beta \in U_{s(b)}$, *and the map* $T : U_{s(b)} \to IS(L_2(\mathbb{Q}_p, \nu_b))$, $\beta \to \widehat{T}_\beta$, *is analytic.*

5. Gaussian representations of position and momentum operators

Just as in ordinary Schrödinger quantum mechanics, let us define the coordinate and momentum operators in $L_2^i(\mathbb{Q}_p, \nu_b)$ by

$$\widehat{\mathbf{q}} f(x) = x f(x),\ \widehat{\mathbf{p}} f(x) = (-i) \left(\frac{d}{dx} - \frac{x}{2b} \right) f(x),$$

where f belongs to the $\mathbb{Q}_p(i)$-linear space $\mathcal{D}$ of linear combinations of Hermite polynomials. The coordinate and momentum operators so defined satisfy on $\mathcal{D}$ the canonical commutation relations

$$[\widehat{\mathbf{q}}, \widehat{\mathbf{p}}] = iI, \tag{8}$$

where I is the unit operator in $L^i_2(\mathbb{Q}_p, \nu_b)$. We shall see that these relations can be extended to the whole of $L^i_2(\mathbb{Q}_p, \nu_b)$.

THEOREM 7 (Albeverio-Khrennikov). *The operators of the coordinate $\widehat{\mathbf{q}}$ and momentum $\widehat{\mathbf{p}}$ are bounded in the Hilbert space $L^i_2(\mathbb{Q}_p, \nu_b)$, with*

$$\|\widehat{\mathbf{q}}\| = \sqrt{|b|_p}, \quad \|\widehat{\mathbf{p}}\| = \frac{1}{\sqrt{|b|_p}}. \tag{9}$$

Moreover $\widehat{\mathbf{q}}$ and $\widehat{\mathbf{p}}$ are symmetric and satisfy (8) on $L^i_2(\mathbb{Q}_p, \nu_b)$.

Proof. Let $f(x) = \sum_{n=0}^{\infty} f_n H_{n,b}(x) \in L^i_2(\mathbb{Q}_p, \nu_b)$. By the recurrence formula

$$H_{n+1,b}(x) = b^{-1}[xH_{n,b}(x) - nH_{n-1,b}(x)], \tag{10}$$

we have

$$\widehat{\mathbf{q}} H_{n,b}(x) = bH_{n+1,b}(x) + nH_{n-1,b}(x), \tag{11}$$

and $\widehat{\mathbf{q}} f(x) = \sum_{n=0}^{\infty} b f_n H_{n+1,b}(x) + \sum_{n=1}^{\infty} n f_n H_{n-1,b}(x)$. Thus, by the strong triangle inequality, we obtain

$$\begin{aligned} \|\widehat{\mathbf{q}} f\|^2 &\le \max\left[\max_n |b|_p^2 |f_n|_p^2 \frac{|(n+1)!|_p}{|b|_p^{n+1}},\ \max_n |n|_p^2 |f_n|_p^2 \frac{|(n-1)!|_p}{|b|_p^{n-1}}\right] \\ &= |b|_p \max\left[\max_n |n+1|_p |f_n|_p^2 \frac{|n!|_p}{|b|_p^n},\ \max_n |n|_p |f_n|_p^2 \frac{|n!|_p}{|b|_p^n}\right] \\ &\le |b|_p \|f\|^2, \end{aligned}$$

(as $|n|_p \le 1$ for all $n \in \mathbf{N}$). Therefore, $\|\widehat{\mathbf{q}}\| \le \sqrt{|b|_p}$. Now we prove that $\|\widehat{\mathbf{q}}\|^2 = |b|_p$. Let $n = p^k$, then

$$D_{k,b} = \|\widehat{\mathbf{q}} H_{p^k,b}\|^2 = \max\left[\frac{|b|_p^2 |(p^k+1)!|_p}{|b|_p^{p^k+1}},\ \frac{|p^k|_p^2 |(p^k-1)!|_p}{|b|_p^{p^k-1}}\right].$$

But $|(p^k+1)!|_p = |p^k!|_p$ and $|p^{2k}(p^k-1)!|_p = p^{-k}|p^k!|_p$. Thus

$$D_{k,b} = |b|_p \frac{|p^k!|_p}{|b|_p^{p^k}} = |b|_p \|H_{p^k,b}\|^2,$$

which proves the first equality in (9).

Further, we have $\frac{d}{dx}H_{n,b}(x) = (x/b)H_{n,b}(x) - H_{n+1,b}(x) = (n/b)H_{n-1,b}(x)$. Set $\widehat{T}_x = (d/dx - (x/2b))$. We have $\widehat{T}_x H_{n,b}(x) = (n/2b)H_{n-1,b}(x) - (1/2)H_{n+1,b}(x)$. To compare this expression with (11), we rewrite it as

$$\widehat{T}_x H_{n,b}(x) = \frac{1}{2b}\big[-bH_{n+1,b}(x) + nH_{n-1,b}(x)\big]. \tag{12}$$

The expression in square brackets is similar to that in (11); the sign does play a role in estimates of max type. Thus we obtain $\|\widehat{T}_x\| = (1/|b|_p)\|\widehat{\mathbf{q}}\|$, which proves the second equality in (9).

Symmetry of the bounded operators $\widehat{\mathbf{q}}, \widehat{\mathbf{p}}$ is easily verified. □

Thus, unlike in the Archimedean case (complex Hilbert space), in the p-adic case the canonical commutation relations (8) are valid not only on a dense subspace, but everywhere on the Hilbert space.

6. One parameter groups generated by position and momentum operators

We shall compute numbers $[\gamma(\widehat{\mathbf{q}})]_p^-$ and $[\gamma(\widehat{\mathbf{p}})]_p^-$, see (2), (3) in section 3.

If $|b|_p = p^{2k+1}$ then $\gamma(\widehat{\mathbf{q}}) = 1/(p^k p^{1/2} p^{1/(p-1)})$. If $p \neq 3$ then $[\gamma(\widehat{\mathbf{q}})]_p^- = 1/p^{k+1}$. If $p = 3$ then $[\gamma(\widehat{\mathbf{q}})]_p^- = 1/p^{k+2}$. If $|b|_p = p^{2k}$ then $\gamma(\widehat{\mathbf{q}}) = 1/(p^k p^{1/(1-p)})$ and $[\gamma(\widehat{\mathbf{q}})]_p^- = 1/p^{k+1}$. Set

$$R(b) = [\gamma(\widehat{\mathbf{q}})]_p^-.$$

If $|b|_p = p^{2k+1}$ then $\gamma(\widehat{\mathbf{p}}) = (p^{1/2}/p^{1/(p-1)})p^k$. If $p \neq 3$ then $[\gamma(\widehat{\mathbf{p}})]_p^- = p^k$. If $p = 3$ then $[\gamma(\widehat{\mathbf{p}})]_p^- = p^{k-1}$. If $|b|_p = p^{2k}$ then $[\gamma(\widehat{\mathbf{p}})]_p^- = p^{k-1}$. Set

$$r(b) = [\gamma(\widehat{\mathbf{p}})]_p^-.$$

THEOREM 8. *(Albeverio–Khrennikov) The maps $\alpha \mapsto \widehat{U}(\alpha) = e^{i\alpha\widehat{\mathbf{q}}}$, $\alpha \in U_{R(b)}$, and $\beta \mapsto \widehat{V}(\beta) = e^{i\beta\widehat{\mathbf{p}}}$, $\beta \in U_{r(b)}$, are analytic one-parameter groups of unitary isometric operators acting on $L_2^i(\mathbb{Q}_p, \nu_b)$. They satisfy the Weyl commutation relations*

$$\widehat{U}(\alpha)\widehat{V}(\beta) = e^{-i\alpha\beta}\widehat{V}(\beta)\widehat{U}(\alpha). \tag{13}$$

We set

$$\widehat{M}_\beta f(x) = e^{-\beta\widehat{\mathbf{q}}/2b} f(x) = \sum_{n=0}^{\infty} \frac{(-\beta\widehat{\mathbf{q}})^n}{n!(2b)^n} f(x), \tag{14}$$

for $f \in L_2(\mathbb{Q}_p, \nu_b)$. By Theorem 7, we easily obtain

PROPOSITION 2. *The map $M : U_{r(b)} \mapsto IS(L_2(\mathbb{Q}_p, \nu_b))$, $\beta \to \widehat{M}_\beta$, is an analytic one-parameter group (indexed by the ball $U_{r(b)}$).*

REMARK 4. The function $x \mapsto e^{-\beta x/2b}$ is not defined on the whole of $\mathbb{Q}_p$ and we cannot consider (14) as a pointwise multiplication operator.

7. Operator calculus

It is well known that in the ordinary $L_2(\mathbb{R}, dx)$ space, the unitary group $\widehat{V}(\beta) = e^{i\beta\widehat{\mathbf{p}}}$, with $\beta \in \mathbb{R}$, can be realized as the translation group, with $\widehat{V}(\beta)\psi(x) = \psi(x+\beta)$ for sufficiently well-behaved functions $\psi(x)$. If we consider the equivalent representation in L_2-space with respect to the Gaussian measure $\nu_b(dx) = (e^{-x^2/2b}/\sqrt{2\pi b})dx$ on $\mathbb{R}$, we obtain

$$\widehat{V}(\beta)\psi(x) = e^{-\beta^2/4b} e^{-\beta x/2b} \psi(x+\beta), \tag{15}$$

or

$$\widehat{V}(\beta) = c_\beta \widehat{M}_\beta \widehat{T}_\beta, \tag{16}$$

where $c_\beta = e^{-\beta^2/4b}$. We shall now prove that (16) is also valid in the p-adic case.

Set $\widehat{S}(\beta) = c_\beta \widehat{M}_\beta \widehat{T}_\beta$, $\beta \in U_{r(b)}$, where the operator $\widehat{M}_\beta$ is defined by (14).

THEOREM 9. *The map $\beta \mapsto \widehat{S}_\beta$, $\beta \in U_{r(b)}$, is a one-parameter analytic group of isometric unitary operators acting in $L_2^i(\mathbb{Q}_p, \nu_b)$.*

LEMMA 2. *The groups $\widehat{S}(\beta)$ and $\widehat{V}(\beta)$ have $\widehat{\mathbf{p}}$ as their common generator.*

As a consequence of this lemma, and the analyticity of the one parameter groups $S(\beta)$ and $V(\beta)$, we easily obtain:

THEOREM 10. *The representation (15), (16) holds for the operator group $\widehat{V}(\beta)$.*

By using one-parameter groups $\widehat{U}(\alpha), \widehat{V}(\beta)$, one can formally define pseudo-differential operators. However, a rigorous mathematical theory is still awaiting development.

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GEVREY SOLUTIONS FOR A VIBRATING BEAM EQUATION

Dedicated to Professor Luigi Rodino on the occasion of his 60th birthday

Abstract. We consider the Cauchy problem for the Euler-Bernoulli equation of the vibrating beam and solve it in Gevrey classes under appropriate Levi conditions on the lower order terms.

1. Introduction and main result

Let us consider the Cauchy problem in $[0,T]\times\mathbb{R}_x$

$$\begin{cases} Lu = 0 \\ u(0,x) = u_0, \quad \partial_t u(0,x) = u_1 \end{cases} \tag{1}$$

for the operator

$$L := D_t^2 - a_4(t)D_x^4 + \sum_{k=0}^{3} a_k(t,x)D_x^k, \tag{2}$$

where $D = -i\partial$ for the sake of the Fourier transform and $a_4(t)$ is a real non-negative function. A motivation to investigate such a problem comes from the Euler-Bernoulli model of the vibrating beam. We admit zeroes of finite order k for $a_4(t)$, and so assume there exists $k\in\mathbb{N}$, $k\geq 2$ such that

$$\sum_{j=0}^{k} |a_4^{(j)}(t)| \neq 0, \quad t\in[0,T]. \tag{3}$$

We assume for the coefficients of L the following regularity conditions:

$$a_4 \in C^k([0,T];\mathbb{R}_+), \quad a_3 \in C^1([0,T];\gamma^s(\mathbb{R})), \quad a_2,a_1,a_0 \in C([0,T];\gamma^s(\mathbb{R})), \tag{4}$$

where $\mathbb{R}_+ = [0,+\infty)$, and $\gamma^s(\mathbb{R})$ is the Gevrey class of index $s\geq 1$ on $\mathbb{R}$, that is the space of all smooth functions f such that

$$|f^{(\alpha)}(x)| \leq CA^\alpha \alpha!^s, \quad C,A>0,\ \alpha\in\mathbb{N}.$$

One can consider L as an anisotropic hyperbolic operator where each derivative with respect to the time variable t has the same weight of two derivatives with respect the space variable x. After that, the two factors $\tau\pm\sqrt{a_4(t)}\xi^2$ of the principal symbol correspond to Schrödinger operators. From the theory of hyperbolic equations, one expects Levi conditions are needed on the lower order terms at the points where the leading coefficient $a_4(t)$ vanishes. On the other hand, from the Schrödinger side, also some decay assumptions as $x \to \infty$ should be taken into account for the imaginary part of these terms, see [4].

Here we assume that the imaginary part of a_3 satisfies the Levi condition

$$|\Im a_3(t,x)| \leq C_0 a_4(t)\langle x\rangle^{-\sigma},\ \sigma > 1, \tag{5}$$

$\langle x\rangle = (1+x^2)^{1/2}$. Besides the decay rate for $x \to \infty$, (5) says that the order of vanishing of $\Im a_3$ is at least the same of a_4. For the full coefficient a_3, including its real part, for the derivative $\partial_t a_3$, and for the coefficients a_2 and a_1, we require lower orders of zero and not any decay, precisely

$$|\partial_x^\beta a_3(t,x)| \leq CA^\beta \beta!^s a_4(t)^{\eta_1}, \tag{6}$$

$$|\partial_x^\beta \partial_t a_3(t,x)| \leq CA^\beta \beta!^s a_4(t)^{\eta_2}, \tag{7}$$

$$|\partial_x^\beta a_2(t,x)| \leq CA^\beta \beta!^s a_4(t)^{\eta_3}, \tag{8}$$

$$|\partial_x^\beta a_1(t,x)| \leq CA^\beta \beta!^s a_4(t)^{\eta_4}, \tag{9}$$

with $C, A > 0$ and η_i to be specified here below.

We proved in [1] that the problem (1), (2) is well posed in $H^\infty = \cap_{\mu\in\mathbb{R}} H^\mu$ under the assumptions (3), (5) with $\sigma \geq 1$ and

$$\begin{cases} |\partial_x^\beta a_3(t,x)| & \leq & C_\beta a_4(t)^{\eta_1}, & \eta_1 \geq 3/4 - 1/(2k), \\ |\partial_x^\beta \partial_t a_3(t,x)| & \leq & C_\beta a_4(t)^{\eta_2}, & \eta_2 \geq 3/4 - 3/(2k), \\ |\partial_x^\beta a_2(t,x)| & \leq & C_\beta a_4(t)^{\eta_3}, & \eta_3 \geq 1/2 - 1/k, \\ |\partial_x^\beta a_1(t,x)| & \leq & C_\beta a_4(t)^{\eta_4}, & \eta_4 \geq 1/4 - 3/(2k). \end{cases} \tag{10}$$

(H^μ denotes the space of functions f such that $\xi \mapsto \langle\xi\rangle^\mu \hat{f}(\xi)$ is in L^2 where $\hat{}$ is the Fourier transform.) Otherwise, H^∞ well posedness cannot hold; here we are going to prove a result of well posedness in Gevrey classes for (1) in this second case. The main result of this paper is the following:

THEOREM 1. *Let us consider the Cauchy problem (1) for the operator L in (2) under assumptions (3), (4). If the Levi conditions (5)–(9) are fulfilled (but (10) is not necessarily satisfied), then problem (1) is well posed in γ^s for $1 < s < s_0$, where*

- $$\begin{cases} 1/2 & \leq & \eta_1 < 3/4 - 1/(2k) \\ \eta_2 & \geq & 3\eta_1 - 3/2 \\ \eta_3 & \geq & 2\eta_1 - 1 \\ \eta_4 & \geq & 3\eta_1 - 2 \end{cases} \implies s_0 = \frac{1-\eta_1}{2\left[3/4 - 1/(2k) - \eta_1\right]},$$

- $$\begin{cases} \eta_2 & < & 3/4 - 3/(2k) \\ \eta_1 & \geq & \eta_2/3 + 1/2 \\ \eta_3 & \geq & 2\eta_2/3 \\ \eta_4 & \geq & \eta_2 - 1/2 \end{cases} \implies s_0 = \frac{3/2-\eta_2}{2\left[3/4 - 3/(2k) - \eta_2\right]},$$

- $$\begin{cases} \eta_3 & < & 1/2 - 1/k \\ \eta_1 & \geq & \eta_3/2 + 1/2 \\ \eta_2 & \geq & 3\eta_3/2 \\ \eta_4 & \geq & 3\eta_3/2 - 1/2 \end{cases} \implies s_0 = \frac{1-\eta_3}{2\left[1/2 - \eta_3 - 1/k\right]},$$

- $$\begin{cases} \eta_4 & < & 1/4 - 3/(2k) \\ \eta_1 & \geq & \eta_4/3 + 2/3 \\ \eta_2 & \geq & \eta_4 + 1/2 \\ \eta_3 & \geq & 2\eta_4/3 + 1/3 \end{cases} \implies s_0 = \frac{1-\eta_4}{2\left[1/4 - 3/(2k) - \eta_4\right]}.$$

In proving Theorem 1 we need to assume $\sigma > 1$; for a precise explanation of this fact see the final Remark 1.

2. Preliminary results and Schrödinger equations

In what follows, we are going to use pseudo-differential operators $p(x, D_x)$ of order m on $\mathbb{R}$ with symbols $p(x,\xi)$ in the standard class S^m which is the space of all symbols $a(x,\xi)$ satisfying, for any $\alpha,\beta \in \mathbb{Z}_+$,

$$|\partial_\xi^\alpha \partial_x^\beta a(x,\xi)| \leq C_{\alpha,\beta,h} \langle\xi\rangle_h^{m-|\alpha|}, \quad \langle\xi\rangle_h := \sqrt{h^2+\xi^2}, h \geq 1;$$

this is the limit space as $\ell \to \infty$ of the Banach spaces $S^m\ell$ of all symbols such that

$$|a|_{m,\ell} := \sup_{x,\xi} \sup_{\alpha+\beta\leq\ell} |\partial_\xi^\alpha \partial_x^\beta a(x,\xi)| \langle\xi\rangle_h^{-m+|\alpha|} < +\infty.$$

Operators with symbol in S^m are bounded operators from $H^{\mu+m}$ into H^μ for any μ. We shall write $\langle\xi\rangle$ instead of $\langle\xi\rangle_1$.

We are also going to use, given $s \geq 1$, Gevrey-type symbols of class $S^{m,s}$, where $S^{m,s}$ denotes the space of all symbols $a(x,\xi)$ satisfying

$$|\partial_\xi^\alpha \partial_x^\beta a(x,\xi)| \leq C_{\alpha,h} A^\beta \beta!^s \langle\xi\rangle_h^{m-|\alpha|}, \tag{11}$$

which is the limit space

$$S^{m,s} := \varprojlim_{\ell\to+\infty} S_\ell^{m,s}, \qquad S_\ell^{m,s} := \varinjlim_{A\to+\infty} S_{\ell,A}^{m,s}$$

of the Banach spaces $S^{m,s}_{\ell,A}$ of all symbols such that

$$|a|_{m,s,A,\ell} := \sup_{\alpha\le\ell,\beta\in\mathbb{Z}_+} \sup_{x,\xi} |\partial_\xi^\alpha \partial_x^\beta a(x,\xi)| A^{-\beta}\beta!^{-s}\langle\xi\rangle_h^{-m+|\alpha|} < +\infty.$$

Given $\mu\in\mathbb{R}$, $\varepsilon>0$, $s\ge 1$, we deal with the Sobolev–Gevrey spaces

$$H^\mu_{\varepsilon,s}(\mathbb{R}) = e^{-\varepsilon\langle D_x\rangle^{1/s}} H^\mu(\mathbb{R}),$$

where the norm is defined by

$$\|u\|_{\mu,\varepsilon,s} = \|e^{\varepsilon\langle D_x\rangle^{1/s}} u\|_\mu.$$

Operators with symbol in $S^{m,s}$ are bounded from $H^{\mu+m}_{\varepsilon,s}$ to $H^\mu_{\varepsilon,s}$ for $|\varepsilon|<\varepsilon_0$, see [3].

In the present section, following [4], we state some preliminary results concerning Schrödinger equations of the form $Su(t,x)=0$,

$$S = D_t + b_2(t)D_x^2 + b_1(t,x,D_x) + b_0(t,x,D_x), \tag{12}$$

where the function $b_2(t)$ is real valued and does not change sign, say

$$b_2 \in C([0,T];\mathbb{R}_+), \tag{13}$$

the lower order terms are complex valued and such that

$$b_j \in C([0,T];S^j), \quad j=0,1. \tag{14}$$

Let us consider the Cauchy problem

$$\begin{cases} Su = 0 \\ u(0,x) = u_0. \end{cases} \tag{15}$$

We say that problem (15) is well posed in H^μ if for any $u_0\in H^\mu$ there is a unique solution $u\in\cap_{j=0}^1 C^j([0,T];H^{\mu-2j})$. We have the following:

THEOREM 2. *Consider the Cauchy problem (15), (12) under the assumptions (13) and (14), and assume moreover that*

$$|\Im b_1(t,x,\xi)| \le M_0 b_2(t)\langle x\rangle^{-\sigma}|\xi|, \quad |\xi|\ge R,\ \sigma>1. \tag{16}$$

Then (15) is well posed in H^μ.

Proof. We define

$$\Lambda(x,\xi) = M_1\omega(\xi/h)\int_0^x \langle y\rangle^{-\sigma} dy, \tag{17}$$

where M_1 is a large constant, $\omega(y)$ a smooth function with $\omega(y) = 0$ for $|y| \leq 1$, $\omega(y) = |y|/y$ for $|y| \geq 2$. For every $\alpha, \beta \in \mathbb{Z}_+$ we have

$$|\partial_\xi^\alpha \partial_x^\beta \Lambda(x,\xi)| \leq \delta_{\alpha,\beta} \langle \xi \rangle_h^{-\alpha}, \tag{18}$$

with constants $\delta_{\alpha,\beta}$ independent on the parameter $h \geq 1$.

Let us now consider the pseudodifferential operators $e^{\pm\Lambda}$ with symbols $e^{\pm\Lambda(x,\xi)}$, and perform the composition $e^{\Lambda}e^{-\Lambda}$. We have:

$$e^{\Lambda}e^{-\Lambda} = I - r(x, D_x),$$

where the principal symbol of r is given by

$$r_{-1}(x,\xi) = D_x \Lambda(x,\xi) \partial_\xi \Lambda(x,\xi). \tag{19}$$

By (18),

$$|r_{(\beta)}^{(\alpha)}(x,\xi)| \leq C_{\alpha,\beta} \langle \xi \rangle_h^{-1-\alpha} \leq C_{\alpha,\beta} h^{-1} \langle \xi \rangle_h^{-\alpha},$$

with $C_{\alpha,\beta}$ independent of h. From this, we can fix a large h in order to have a bounded operator $r(x, D_x)$ on H^μ with norm $\|r(x,D_x)\| < 1$. The operator $I - r(x,D_x)$ is invertible by Neumann series and its inverse operator is given by

$$I + p(x, D_x), \quad p = \sum_{j=1}^{\infty} r^j.$$

This proves that the operator $e^{-\Lambda}(I + p)$ is the right inverse of e^{Λ}. By similar arguments one proves the existence of a left inverse. Thus, the operator e^{Λ} is invertible, the inverse operator is given by

$$\left(e^{\Lambda}\right)^{-1} = e^{-\Lambda}(I + p), \; p(x,\xi) \in S^{-1}, \tag{20}$$

and $p(x,\xi)$ has the principal part (19).

To obtain the well posedness in H^μ of problem (15), we perform the change of variable $v = (e^{\Lambda})^{-1}u$ and we show that the Cauchy problem

$$\begin{cases} S^{\Lambda} v = 0 \\ v(0,x) = (e^{\Lambda})^{-1} u_0 \end{cases} \tag{21}$$

for the operator $S^{\Lambda} := (e^{\Lambda})^{-1} S e^{\Lambda}$ is well posed in H^μ. We have

$$iS = \partial_t + iK(t, x, D_x),$$

where

$$K(t,x,D_x) = b_2(t) D_x^2 + b_1(t,x,D_x) + b_0(t,x,D_x),$$

and

$$iS^{\Lambda} = \partial_t + iK^{\Lambda}, \quad K^{\Lambda} = (e^{\Lambda})^{-1} K e^{\Lambda}.$$

Differentiating with respect to time and taking $\mu = 0$, we have

$$\frac{d}{dt}||v(t)||_0^2 = 2\Re\langle v'(t), v(t)\rangle_0 = -2\Re\langle iK^\Lambda v, v\rangle_0.$$

We write iK as the sum

$$iK = H_K + A_K, \quad H_K = (iK + (iK)^*)/2, \quad A_K = (iK - (iK)^*)/2$$

of its hermitian and anti-hermitian parts. The principal symbol of H_K is given by

$$H_K^0(t,x,\xi) = -\Im b_1(t,x,\xi).$$

The hermitian part H_{K^Λ} of iK^Λ is then

$$H_{K^\Lambda}(t,x,\xi) = 2M_1 b_2(t)|\xi|\langle x\rangle^{-\sigma} - \Im b_1(t,x,\xi) + Q_0(t,x,\xi),$$

with $Q_0(t,x,\xi) \in C([0,T];S^0)$. From (16), taking $M_1 = M_0/2$, we have a positive principal part for $H_{K^\Lambda}(t,x,\xi)$; hence, an application of the sharp Gårding inequality gives

$$2\Re\left(iK^\Lambda u, u\right) \geq -C\|u\|^2, \; u \in H^2. \tag{22}$$

From this, the energy method gives well posedness in L^2 of the Cauchy problem for S^Λ. Well posedness in H^μ immediately follows, since, for any μ, the principal symbol of the hermitian part of $\langle D_x\rangle^\mu iK^\Lambda \langle D_x\rangle^{-\mu}$ is the same of H_{K^Λ}. □

3. Proof of the main result

We approximate the characteristic roots $\pm\sqrt{a_4(t)}\xi^2$ of L by defining

$$\tilde\lambda_2(t,\xi) = \sqrt{a_4(t) + \langle\xi\rangle^{-M}}\xi^2 = \tilde b_2(t,\xi)\xi^2, \tag{23}$$

with $0 \leq M \leq 1/(1-\eta_1)$ to be chosen later on. We immediately notice that

$$\tilde b_2 - b_2 \in C([0,T];S^{-M/2}), \; b_2 = \sqrt{a_4(t)}. \tag{24}$$

Then, we define

$$\tilde b_1(t,x,\xi) = -a_3(t,x)\xi/(2\tilde b_2(t,\xi)), \tag{25}$$

and by (6) with $\eta_1 \geq 1/2$ we have

$$\tilde b_1 \in C([0,T];S^{1,s}). \tag{26}$$

Again, we define the operators

$$\tilde S^\pm = D_t \pm \tilde b_2 D_x^2 \pm \tilde b_1, \tag{27}$$

and compute

$$
\begin{aligned}
\tilde{S}^-\tilde{S}^+ = \; & L - a_2 D_x^2 - a_1 D_x - a_0 - \langle D_x\rangle^{-M} D_x^4 \\
& - \operatorname{op}\left(i\frac{d_\xi \langle\xi\rangle^{-M}\partial_x a_3}{4\tilde{b}_2^2}\xi^3 + i\partial_x a_3 \xi^2 + \cdots \right) - i\frac{a_4'}{2}\tilde{b}_2^{-1} D_x^2 \\
& - i\frac{\partial_t a_3}{2}\tilde{b}_2^{-1} D_x + i\frac{a_4' a_3}{4}\tilde{b}_2^{-3} D_x \\
& + \operatorname{op}\left(-\left(\frac{a_3}{2}\tilde{b}_2^{-1}\xi\right)^2 - \frac{a_3\partial_x a_3}{4}\tilde{b}_2^{-2}\xi + \frac{a_3\partial_x a_3 d_\xi \langle\xi\rangle^{-M}}{8}\tilde{b}_2^{-4}\xi^2 + \cdots \right),
\end{aligned}
$$

where we denote by $\operatorname{op}(p(x,\xi))$ the pseudodifferential operator of symbol $p(x,\xi)$. We have:

LEMMA 1. *Let us consider the operator L given by (2) under the assumptions of Theorem 1 and take $\tilde{S}^{\pm}$ as in* (27)*. Then:*

$$
L = \tilde{S}^-\tilde{S}^+ - (d_0\omega + e_0\omega_0 + f_0\omega_1 + g_0\omega_2 + h_0\omega_3 + l_0\omega_4 + m_0)\,\tilde{b_2}\langle D_x\rangle^2, \tag{28}
$$

where $e_0, d_0, f_0, g_0, h_0, l_0, m_0 \in C([0,T]; S^{0,s}), \omega = \operatorname{op}(\omega(t,\xi))$ *and* $\omega_i = \operatorname{op}\,(\omega_i(t,\xi))$, $i = 0,\dots,4$, *with:*

$$
\omega(t,\xi) = \frac{\langle\xi\rangle^{2-M}}{(a_4(t) + \langle\xi\rangle^{-M})^{1/2}}, \tag{29}
$$

$$
\omega_0(t,\xi) = \frac{a_4'(t)}{a_4(t) + \langle\xi\rangle^{-M}}, \tag{30}
$$

$$
\omega_1(t,\xi) = \frac{1}{(a_4(t) + \langle\xi\rangle^{-M})^{3/2-2\eta_1}}, \tag{31}
$$

$$
\omega_2(t,\xi) = \frac{\langle\xi\rangle^{-1}}{(a_4(t) + \langle\xi\rangle^{-M})^{1-\eta_2}}, \tag{32}
$$

$$
\omega_3(t,\xi) = \frac{1}{(a_4(t) + \langle\xi\rangle^{-M})^{1/2-\eta_3}}, \tag{33}
$$

$$
\omega_4(t,\xi) = \frac{\langle\xi\rangle^{-1}}{(a_4(t) + \langle\xi\rangle^{-M})^{1/2-\eta_4}}. \tag{34}
$$

Proof.

- $i\frac{a_4'}{2}\tilde{b}_2^{-1}D_x^2(\tilde{b}_2\langle D_x\rangle^2)^{-1}$ clearly becomes $d_0\omega_0$.
- $\langle D_x\rangle^{-M}D_x^4(\tilde{b}_2\langle D_x\rangle^2)^{-1}$ clearly becomes $e_0\omega$.
- $\operatorname{op}\left(\left(\frac{a_3}{2}\tilde{b}_2^{-1}\xi\right)^2\right)(\tilde{b}_2\langle D_x\rangle^2)^{-1}$ becomes $f_0\omega_1$ by the Levi condition (6).

- $i\dfrac{\partial_t a_3}{2}\tilde{b}_2^{-1}D_x(\tilde{b}_2\langle D_x\rangle^2)^{-1}$ becomes $g_0\omega_2$ by the Levi condition (7).
- $a_2D_x^2(\tilde{b}_2\langle D_x\rangle^2)^{-1}$ becomes $h_0\omega_3$ by the Levi condition (8).
- $a_1D_x(\tilde{b}_2\langle D_x\rangle^2)^{-1}$ becomes $l_0\omega_4$ by the Levi condition (9).
- $\mathrm{op}\left(i\dfrac{d_\xi\langle\xi\rangle^{-M}\partial_x a_3}{4\tilde{b}_2^2}\xi^3\right)(\tilde{b}_2\langle D_x\rangle^2)^{-1}$ and $\mathrm{op}(i\partial_x a_3\xi^2)(\tilde{b}_2\langle D_x\rangle^2)^{-1}$ have symbols in $C([0,T];S^{0,s})$ by the Levi condition (6) with $\eta_1\geq 1/2$.
- $a_0(\tilde{b}_2D_x^2)^{-1}\in C([0,T];S^{0,s})$.
- $\mathrm{op}\left(\dfrac{a_3\partial_x a_3}{4}\tilde{b}_2^{-2}\xi+\dfrac{a_3\partial_x a_3 d_\xi\langle\xi\rangle^{-M}}{8}\tilde{b}_2^{-4}\xi^2\right)(\tilde{b}_2\langle D_x\rangle^2)^{-1}$ has principal symbol $p_0(t,x,\xi)\langle\xi\rangle^{-1}\omega_1$, with $p_0(t,x,\xi)\in C([0,T];S^{0,s})$, by (6).
- from (6), the principal symbol of $-a_4'a_3\tilde{b}_2^{-3}D_x(\tilde{b}_2\langle D_x\rangle^2)^{-1}$ is dominated by $\omega_0\langle\xi\rangle^{-1}(a_4+\langle\xi\rangle^{-M})^{-(1-\eta_1)}$, and $\langle\xi\rangle^{-1}(a_4+\langle\xi\rangle^{-M})^{-(1-\eta_1)}\in C([0,T];S^{0,s})$ because we are going to choose $M\leq 1/(1-\eta_1)$.

□

LEMMA 2. *The symbols defined by (29)–(34) satisfy:*

$$\left|\partial_\xi^\alpha\int_0^T|\omega_0(t,\xi)|dt\right|\leq\delta_\alpha\langle\xi\rangle^{-\alpha}(1+\log\langle\xi\rangle),\tag{35}$$

$$\left|\partial_\xi^\alpha\int_0^T|\omega(t,\xi)|dt\right|\leq\delta_\alpha\langle\xi\rangle^{2-M(1/2+1/k)-\alpha},\tag{36}$$

$$\left|\partial_\xi^\alpha\partial_x^\beta\int_0^T|\omega_1(t,x,\xi)|dt\right|\leq\begin{cases}\delta_{\alpha,\beta}\langle\xi\rangle^{-\alpha}(1+\log\langle\xi\rangle)\text{ if }\eta_1\geq 3/4-1/(2k)\\ \delta_{\alpha,\beta}\langle\xi\rangle^{M(3/2-1/k-2\eta_1)-\alpha}\text{ if }\eta_1<3/4-1/(2k),\end{cases}\tag{37}$$

$$\left|\partial_\xi^\alpha\partial_x^\beta\int_0^T|\omega_2(t,x,\xi)|dt\right|\leq\begin{cases}\delta_{\alpha,\beta}\langle\xi\rangle^{-\alpha}\text{ if }\eta_2\geq 1-1/k\\ \delta_{\alpha,\beta}\langle\xi\rangle^{-1+M(1-1/k-\eta_2)-\alpha}\text{ if }\eta_2<1-1/k,\end{cases}\tag{38}$$

$$\left|\partial_\xi^\alpha\partial_x^\beta\int_0^T|\omega_3(t,x,\xi)|dt\right|\leq\begin{cases}\delta_{\alpha,\beta}\langle\xi\rangle^{-\alpha}(1+\log\langle\xi\rangle)\text{ if }\eta_3\geq 1/2-1/k\\ \delta_{\alpha,\beta}\langle\xi\rangle^{M(1/2-1/k-\eta_3)-\alpha}\text{ if }\eta_3<1/2-1/k,\end{cases}\tag{39}$$

$$\left|\partial_\xi^\alpha\partial_x^\beta\int_0^T|\omega_4(t,x,\xi)|dt\right|\leq\begin{cases}\delta_{\alpha,\beta}\langle\xi\rangle^{-\alpha}\text{ if }\eta_4\geq 1/2-1/k\\ \delta_{\alpha,\beta}\langle\xi\rangle^{-1+M(1/2-1/k-\eta_4)-\alpha}\text{ if }\eta_4<1/2-1/k.\end{cases}\tag{40}$$

Proof. The proof is a simple application of Lemma 1 and Lemma 2 of [2]. □

The next step of the proof is to reduce L to a first order system of a suitable form by using factorization (28).

LEMMA 3. *Let us consider the operator L in (2) under the assumptions of Theorem 1. Let us denote*

$$\tilde{K}_1 = \tilde{b}_2 D_x^2 + \tilde{b}_1 = b_2 D_x^2 + b_1, \quad b_1 = \tilde{b}_1 + (\tilde{b}_2 - b_2) D_x^2 \tag{41}$$

where $b_1 \in C([0,T]; S^{1,s})$ *and, in view of (24) and (26),* $\Im b_1 = \Im \tilde{b_1}$. *Then, the scalar equation* $Lu = 0$ *is equivalent to the* 2×2 *system* $\mathcal{W} U = 0$,

$$\mathcal{W} = D_t + \tilde{K} + D_0 \omega + E_0 \omega_0 + F_0 \omega_1 + G_0 \omega_2 + H_0 \omega_3 + L_0 \omega_4 + M_0, \tag{42}$$

where

$$\tilde{K} = \begin{pmatrix} \tilde{K}_1 & 0 \\ 0 & -\tilde{K}_1 \end{pmatrix}, \tag{43}$$

$D_0, \dots, M_0 \in C([0,T]; S^{0,s})$, ω, ω_i *(*$i = 0, \dots, 4$*) as in (29)–(34).*

Proof. For a scalar unknown u we define the vector $U_0 = {}^t(u_0, u_1)$ by

$$\begin{cases} u_0 = \tilde{b_2} \langle D_x \rangle^2 u \\ u_1 = \tilde{S}^+ u \end{cases}$$

so that, from (28), the scalar equation $Lu = 0$ is equivalent to the system $\mathcal{W}_0 U_0 = 0$ with

$$\begin{aligned} \mathcal{W}_0 = D_t \; &+ \begin{pmatrix} \tilde{K}_1 & -\tilde{b_2} \langle D_x \rangle^2 \\ 0 & -\tilde{K}_1 \end{pmatrix} \\ &+ \begin{pmatrix} -i\omega_0/2 & 0 \\ d_0 \omega + e_0 \omega_0 + f_0 \omega_1 + g_0 \omega_2 + h_0 \omega_3 + l_0 \omega_4 & 0 \end{pmatrix} \\ &+ \begin{pmatrix} \left[\tilde{b_2} \langle D_x \rangle^2, \tilde{K}_1\right] \left(\tilde{b_2} \langle D_x \rangle^2\right)^{-1} & 0 \\ m_0 & 0 \end{pmatrix}, \end{aligned} \tag{44}$$

where we use $(\partial_t \tilde{b_2}) \langle D_x \rangle^2 u = (\omega_0/2) u_0$. The term $\left[\tilde{b_2} \langle D_x \rangle^2, \tilde{K}_1\right] \cdot \left(\tilde{b_2} \langle D_x \rangle^2\right)^{-1}$ is of order 0 because $\tilde{b_2}$ does not depend on x and $\partial_\xi^\alpha \tilde{b_2} = p_{-\alpha} \tilde{b_2}$ with $p_{-\alpha}$ of order $-\alpha$.

We begin to diagonalize the matrix

$$\begin{pmatrix} \tilde{K}_1 & -\tilde{b_2} \langle \xi \rangle^2 \\ 0 & -\tilde{K}_1 \end{pmatrix}$$

by means of

$$\mathcal{D}_0(\xi) = \begin{pmatrix} 1 & \langle\xi\rangle^2/2\xi^2 \\ 0 & 1 \end{pmatrix}, \quad |\xi| \geq R > 0, \tag{45}$$

which is in S^0. At the operator level, for the system $\mathcal{W}_0$ in (44) we have

$$\mathcal{D}_0^{-1}\mathcal{W}_0\mathcal{D}_0 = \mathcal{W}_1$$

with $\mathcal{W}_1$ equal to $\mathcal{W}$ in (42) modulo a term of the form

$$\begin{pmatrix} 0 & \tilde{z}_1 \\ 0 & 0 \end{pmatrix},$$

where

$$\tilde{z}_1(t,x,\xi) = \langle\xi\rangle^2\xi^{-2}\tilde{b}_1(t,x,\xi), \quad |\xi| \geq R > 0.$$

We perform a second step of diagonalization by means of the operator with symbol

$$\mathcal{D}_1 = \begin{pmatrix} 1 & \tilde{d}_1 \\ 0 & 1 \end{pmatrix}, \quad \tilde{d}_1 = -\tilde{z}_1/2\tilde{b}_2(t)\xi^2, \ |\xi| \geq R. \tag{46}$$

By (6), we have

$$\tilde{d}_1 \in C([0,T]; S^{-1+M(1-\eta_1),s}) \subseteq C([0,T]; S^{0,s}).$$

Moreover, from (6) and (7),

$$\partial_t\tilde{d}_1 = p_0\omega_0 + q_0\omega_1 + r_0\omega_2, \quad p_0, q_0, r_0 \in C([0,T]; S^{0,s}).$$

Thus, $\mathcal{D}_1^{-1}\mathcal{W}_1\mathcal{D}_1 = \mathcal{W}$, with $\mathcal{W}$ in (42). □

Proof of Theorem 1. To prove the well posedness in Gevrey classes of the Cauchy problem (1) for the scalar operator L, we are going to prove the well posedness in Sobolev–Gevrey spaces of the equivalent problem

$$\begin{cases} \mathcal{W}U(t,x) = 0 \\ U(0,x) = G(x), \end{cases} \tag{47}$$

for the system $\mathcal{W}$ in (42). We notice that under the assumptions of Theorem 1, recalling also (41), (25) and the Levi condition (5), the diagonal part $D_t + \tilde{K}$ of $\mathcal{W}$ satisfies the hypotheses of Theorem 2. Thus we can apply Theorem 2 to $D_t + \tilde{K}$. We take the operator Λ in (17) and consider the transformed system

$$\mathcal{W}^\Lambda := \begin{pmatrix} e^\Lambda & 0 \\ 0 & e^{-\Lambda} \end{pmatrix}^{-1} \mathcal{W} \begin{pmatrix} e^\Lambda & 0 \\ 0 & e^{-\Lambda} \end{pmatrix}. \tag{48}$$

We know that, taking sufficiently large C_0 in (5) and h in (17), we have

$$iW^\Lambda = \partial_t + i\tilde{K}_\Lambda + D_1\omega + E_1\omega_0 + F_1\omega_1 + G_1\omega_2 + H_1\omega_3 + L_1\omega_4 + M_1,$$

where

$$2\Re\left(i\tilde{K}_\Lambda U, U\right) \geq -C\|U\|^2,\ U \in H^2;$$

moreover, since both the transformations $e^\Lambda, (e^\Lambda)^{-1}$ are of order zero, and

$$\partial_\xi^\alpha \omega(t,\xi) = q_{-\alpha}(t,\xi)\omega(t,\xi), \quad q_{-\alpha} \in C([0,T]; S^{-\alpha}),$$
$$\partial_\xi^\alpha \omega_j(t,\xi) = q_{-\alpha,j}(t,\xi)\omega_j(t,\xi), \quad q_{-\alpha,j} \in C([0,T]; S^{-\alpha}),\ j = 0,\dots,4,$$

we have

$$\begin{aligned} &i(D_0\omega + E_0\omega_0 + F_0\omega_1 + G_0\omega_2 + H_0\omega_3 + L_0\omega_4 + M_0)^\Lambda \\ &\quad = D_1\omega + E_1\omega_0 + F_1\omega_1 + G_1\omega_2 + H_1\omega_3 + L_1\omega_4 + M_1 \end{aligned}$$

with

$$D_1, E_1, F_1, G_1, H_1, L_1, M_1 \in C([0,T]; S^{0,s}).$$

The next step in the proof consists in the transformation also of $D_1\omega + E_1\omega_0 + F_1\omega_1 + G_1\omega_2 + H_1\omega_3 + L_1\omega_4 + M_1$ into a positive operator, modulo a remainder of order zero. There a loss of derivatives will appear. We perform the change of variable given by $e^{\phi(t,D_x)}$, where

$$\phi(t,\xi) = C\int_0^t \left(|\omega_0(\tau,\xi)| + \omega(\tau,\xi) + \sum_{i=1}^4 \omega_i(\tau,\xi)\right) d\tau,$$

C a large enough constant to be chosen. The change of variable carries a loss, see Lemma 2; the loss becomes greater as the order $\mathrm{ord}(\phi)$ of the symbol $\phi(t,\xi)$ increases. Thus we choose the parameter M that minimizes $\mathrm{ord}(\phi)$, which is the maximum between the orders of $\int_0^t \omega(\tau)d\tau$, $\int_0^t |\omega_0(\tau,\xi)|d\tau$, $\int_0^t \omega_i(\tau,\xi)d\tau$, $i = 1,\dots,4$. In a comparison between (35)–(40), we notice that the following cases can occur:

- $\begin{cases} 1/2 \leq \eta_1 < 3/4 - 1/(2k) \\ \eta_2 \geq 3\eta_1 - 3/2 \\ \eta_3 \geq 2\eta_1 - 1 \\ \eta_4 \geq 3\eta_1 - 2 \end{cases} \Longrightarrow \begin{array}{l} M = 1/(1-\eta_1), \\ \mathrm{ord}(\phi) = \dfrac{2\,[3/4 - 1/(2k) - \eta_1]}{1-\eta_1}, \end{array}$

- $\begin{cases} \eta_2 < 3/4 - 3/(2k) \\ \eta_1 \geq \eta_2/3 + 1/2 \\ \eta_3 \geq 2\eta_2/3 \\ \eta_4 \geq \eta_2 - 1/2 \end{cases} \Longrightarrow \begin{array}{l} M = 3/(3/2-\eta_2), \\ \mathrm{ord}(\phi) = \dfrac{2\,[3/4 - 3/(2k) - \eta_2]}{3/2-\eta_2}, \end{array}$

- $\begin{cases} \eta_3 < 1/2 - 1/k \\ \eta_1 \geq \eta_3/2 + 1/2 \\ \eta_2 \geq 3\eta_3/2 \\ \eta_4 \geq 3\eta_3/2 - 1/2 \end{cases} \Longrightarrow \begin{array}{l} M = 2/(1-\eta_3), \\ \mathrm{ord}(\phi) = \dfrac{2\,[1/2 - \eta_3 - 1/k]}{1-\eta_3}, \end{array}$

- $\begin{cases} \eta_4 < 1/4 - 3/(2k) \\ \eta_1 \geq \eta_4/3 + 2/3 \\ \eta_2 \geq \eta_4 + 1/2 \\ \eta_3 \geq 2\eta_4/3 + 1/3 \end{cases} \Longrightarrow \begin{array}{l} M = 3/(1-\eta_4), \\ \mathrm{ord}(\phi) = \dfrac{2\,[1/4 - 3/(2k) - \eta_4]}{1-\eta_4}. \end{array}$

In each case, we have that $\mathrm{ord}(\phi) = 1/s_0$, s_0 as in the statement of Theorem 1; in what follows we use the notation

$$\phi \in C([0,T]; S^{1/s_0})$$

which covers all the four cases that can occur. The change of variable can be considered only if $\phi(t,\xi)\langle\xi\rangle^{-1/s}$ is small enough (see [3]), and it is

$$\begin{aligned} i\mathcal{W}^{\Lambda,\phi} &:= e^{-\phi} i\mathcal{W}^{\Lambda} e^{\phi} \\ &= \partial_t + \partial_t\phi(t,D_x)\,I + i\tilde{K}^{\Lambda} + R(t,x,D_x) \\ &\quad + D_2\omega + E_2\omega_0 + F_2\omega_1 + G_2\omega_2 + H_2\omega_3 + L_2\omega_4 + M_2 \\ &= \partial_t + i\tilde{K}^{\Lambda} + (D_2\omega + C\omega I) + (E_2\omega_0 + C|\omega_0| I) + (F_2\omega_1 + C\omega_1 I) \\ &\quad + (G_2\omega_2 + C\omega_2 I) + (H_2\omega_3 + C\omega_3 I) + (L_2\omega_4 + C\omega_4 I) \\ &\quad + M_2 + R(t,x,D_x), \end{aligned} \tag{49}$$

where I is the 2×2 identity matrix,

$$D_2, E_2, F_2, G_2, H_2, L_2, M_2 \in C([0,T]; S^{0,s}),$$

and $R \in C([0,T]; S^{1/s,s})$.

Taking now C sufficiently large, from the sharp Gårding inequality for matrix operators, see [5, Theorem 4.4, page 134], we immediately get that $D_2\omega + C\omega I$, $E_2\omega_0 + C|\omega_0| I$, $F_2\omega_1 + C\omega_1 I$, $G_2\omega_2 + C\omega_2 I$, $H_2\omega_3 + C\omega_3 I$ and $L_2\omega_4 + C\omega_4 I$ in (49) are positive modulo terms with symbol in $C([0,T]; S^{0,s})$.

It only remains to make R a positive operator. To this aim, we take $\mu = 0$ and, for a function $r(t) \in C^1[0,T]$ to be chosen, we perform the last change of variable given by $e^{r(t)\langle D_x\rangle^{1/s} - \varepsilon\langle D_x\rangle^{1/s}}$, $\varepsilon > 0$, and consider the final operator

$$i\tilde{\mathcal{W}} := e^{-\left(\phi(t,D_x) + r(t)\langle D_x\rangle^{1/s} - \varepsilon\langle D_x\rangle^{1/s}\right)} i\mathcal{W}^{\Lambda} e^{\phi(t,D_x) + r(t)\langle D_x\rangle^{1/s} - \varepsilon\langle D_x\rangle^{1/s}}. \tag{50}$$

By [3] we know that there exists an $\varepsilon_0 > 0$ such that if

$$\phi(x,D_x) + r(t)\langle D_x\rangle^{1/s} \leq \varepsilon\langle D_x\rangle^{1/s}, \quad 0 \leq \varepsilon \leq \varepsilon_0,$$

then

$$\begin{aligned} i\tilde{\mathcal{W}} &= \partial_t + i\tilde{K}^{\Lambda} + (D_2\omega + C\omega I) + (E_2\omega_0 + C|\omega_0| I) + (F_2\omega_1 + C\omega_1 I) \\ &\quad + (G_2\omega_2 + C\omega_2 I) + (H_2\omega_3 + C\omega_3 I) + (L_2\omega_4 + C\omega_4 I) \\ &\quad + M_2 + \tilde{R}(t,x,D_x) + r'(t)\langle D_x\rangle^{1/s} I, \end{aligned} \tag{51}$$

where $\tilde{R} \in C([0,T]; S^{1/s})$ has seminorms such that $|\tilde{R}(t)|_\ell \le r_\ell(t)$ for some functions $r_\ell \in C[0,T]$ not depending on $r(t)$. An application of Calderón–Vaillancourt's Theorem to the operator $\tilde{R}$ gives the existence of a positive constant ℓ_0 only depending on the space dimension n such that

$$|\langle \tilde{R}v, v\rangle_{L^2}| \le r_{\ell_0}(t)\langle\langle D_x\rangle^{1/s}v, v\rangle_{L^2}.$$

Thus, taking $r(t)$ such that $r'(t) = r_{\ell_0}(t)$, we have that also $\tilde{R}(t,x,D_x) + r'(t)\langle D_x\rangle^{1/s} I$ becomes a positive operator modulo terms of order zero. So, from (51), we obtain by Gronwall's method

$$\|U(t)\|_0^2 \le C_0\|U(0)\|_0^2.$$

This procedure can be generalized to the case $\mu \neq 0$, since for each μ we have

$$\langle D_x\rangle^{\mu}\big(i\tilde{\mathcal{W}}\big)\langle D_x\rangle^{-\mu} = i\tilde{\mathcal{W}} + R_\mu,$$

with R_μ of order zero. From this, the energy method gives well posedness in H^μ of the Cauchy problem for $i\tilde{\mathcal{W}}$, which corresponds to well posedness of (47) in $H^\mu_{\varepsilon,s}$, $0 < \varepsilon \le \varepsilon_0$. □

REMARK 1. If, with the assumptions of Theorem 2, we take $\sigma = 1$ or $\sigma \in (0,1)$, then the Cauchy problem (15) is not well posed in H^μ, but it is well posed respectively in H^∞ or in γ^s for $s < 1/(1-\sigma)$, see [4]. This is because the symbol Λ in (17) has positive order under a decay at infinity condition of type $\Im b_1 \sim \langle x\rangle^{-\sigma}$ with $\sigma \in (0,1]$. Regarding second order equations, in the statement of Theorem 1 we only admit $\sigma > 1$, see (5). This is because in the proof of Theorem 1 we need Λ of order zero; otherwise, the transformation (48) carries a loss of derivatives, and as now we cannot simultaneously control the two losses coming from the decay condition at infinity (transformation (48)) and from the Levi conditions (transformation (50)). The problem of giving an analogue of Theorem 1 in the case $\sigma \in (0,1]$ is still open.

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EVOLUTION FOR OVERDETERMINED SYSTEMS IN (SMALL) GEVREY CLASSES

Dedicated to Professor Luigi Rodino on the occasion of his 60th birthday

Abstract. Given a system of linear partial differential operators with constant coefficients whose affine algebraic varieties $V(\check{\wp})$ have dimension 1, we establish in which classes of (small) Gevrey functions the associated Cauchy problem admits at least one solution, looking at the Puiseux series expansions on the branches at infinity of the algebraic curves $V(\check{\wp})$. We focus, in particular, on the case of two variables, giving some examples.

1. Introduction and main theorems

Let $A_0(D)$ be an $a_1 \times a_0$ matrix of linear partial differential operators with constant coefficients in the N indeterminates $z_1, \dots, z_N$.

To allow different scales of regularity in the time-variables t and in the space-variables x, we split $\mathbb{R}^N_z \simeq \mathbb{R}^k_t \times \mathbb{R}^n_x$ and consider then the spaces of (ultra-)differentiable functions of Beurling type

$$\begin{aligned}\mathcal{E}_\omega(\mathbb{R}^N) = \ & \{ f \in \mathcal{E}(\mathbb{R}^N) : \ \forall K \subset\subset \mathbb{R}^N \ \forall \varepsilon > 0 \ \exists c > 0 : \\ & \sup_K |D_t^\beta D_x^\gamma f(t,x)| \leq c \varepsilon^{|\gamma|+|\beta|} (\beta!)^{1/\alpha_1} (\gamma!)^{1/\alpha_2} \ \forall \gamma \in \mathbb{N}_0^n, \beta \in \mathbb{N}_0^k \},\end{aligned}$$

where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $0 \leq \alpha_j < 1$. If $\alpha_1 = \alpha_2 = 1/s$ with $s > 1$ this space coincides with the space of (small) Gevrey functions of order s. If $\alpha_1 = \alpha_2 = 0$ it is identified with the space $\mathcal{E}(\mathbb{R}^N)$ of smooth functions. We assume in the following that $\alpha_1 = 0$ if $\alpha_2 = 0$, so that we allow ultradifferentiability in all variables or only in the space-variables, but not only in the time-variables.

We want to consider the Cauchy problem for $A_0(D)$ in these classes of (ultra-) differentiable functions with initial data on $\{(t,x) \in \mathbb{R}^k \times \mathbb{R}^n : t = 0\}$. In order to avoid the problem of formal coherence of the initial data, which can be particularly intricate if the system is overdetermined, we allow Whitney functions as initial data, which means that we give functions with all their normal derivatives on $\{t = 0\}$. By Whitney's extension theorem it is not restrictive to give zero initial-data, so that we are concerned with the following (overdetermined) Cauchy problem:

$$(1) \qquad \begin{cases} \text{given } f \in \mathcal{E}_\omega(\mathbb{R}^N)^{a_1} \\ \text{find } \varphi \in \mathcal{E}_\omega(\mathbb{R}^N)^{a_0} \text{ such that} \\ A_0(D)\varphi = f \\ D_t^\alpha \varphi(0,x) = 0 \quad \forall \alpha \in \mathbb{N}_0^k, \forall x \in \mathbb{R}^n. \end{cases}$$

Let $\theta = (\tau, \zeta) \in \mathbb{C}^k \times \mathbb{C}^n$ be the dual coordinates of $z = (t, x)$ and denote by $\mathcal{P} = \mathbb{C}[\theta_1, \dots, \theta_N]$ the ring of complex polynomials in the N indeterminates $\theta_1, \dots, \theta_N$. By the formal substitution $\theta_j \leftrightarrow D_j = \frac{1}{i}\frac{\partial}{\partial z_j}$ we can associate to the operator $A_0(D)$ the $\mathcal{P}$-homomorphism $A_0(\theta)$ and insert it into a Hilbert resolution of the $\mathcal{P}$-module $\mathcal{M} = \operatorname{coker}{}^t A_0(\theta)$:

$$0 \to \mathcal{P}^{a_d} \xrightarrow{{}^tA_{d-1}(\theta)} \mathcal{P}^{a_{d-1}} \longrightarrow \dots \longrightarrow \mathcal{P}^{a_2} \xrightarrow{{}^tA_1(\theta)} \mathcal{P}^{a_1} \xrightarrow{{}^tA_0(\theta)} \mathcal{P}^{a_0} \longrightarrow \mathcal{M} \to 0.$$

When the map $A_1(\theta)$ is not trivial the system (1) is overdetermined and in order to be solvable f must satisfy the compatibility conditions

$$\begin{cases} A_1(D) f = 0 \\ D_t^\alpha f(0,x) = 0 \quad \forall \alpha \in \mathbb{N}_0^k,\ \forall x \in \mathbb{R}^n. \end{cases} \tag{2}$$

We say that the pair $(\mathbb{R}^n_x, \mathbb{R}^k_t \times \mathbb{R}^n_x)$ is *of evolution* for $A_0(D)$ (or for $\mathcal{M}$) in the class $\mathcal{E}_\omega$ if the Cauchy problem (1) admits at least one solution φ for each datum f satisfying the compatibility conditions (2).

Let us denote by $V = V(\check{\wp})$, for $\wp \in \operatorname{Ass}(\mathcal{M})$, the algebraic variety

$$V(\check{\wp}) = \{\theta \in \mathbb{C}^N :\ p(-\theta) = 0\ \forall p \in \wp\}.$$

It was proved in [3], [4] that evolution is equivalent to the validity of the following Phragmén-Lindelöf principle for every $V = V(\check{\wp})$ with $\wp \in \operatorname{Ass}(\mathcal{M})$:

$$PL(\omega) \quad \begin{cases} \exists A > 0 \text{ such that } \forall v \in \mathrm{PSH}(V) \text{ satisfying, for some } \alpha_v > 0, \\ \begin{cases} (\alpha) \quad v(\tau,\zeta) \le |\operatorname{Im}\tau| + |\operatorname{Im}\zeta| + \omega(\tau,\zeta) \quad \forall (\tau,\zeta) \in V \\ (\beta) \quad v(\tau,\zeta) \le \alpha_v(|\operatorname{Im}\zeta| + \omega(\tau,\zeta) + 1) \quad \forall (\tau,\zeta) \in V \end{cases} \\ \text{then } v \text{ must also satisfy} \\ (\gamma) \quad v(\tau,\zeta) \le A(|\operatorname{Im}\zeta| + \omega(\tau,\zeta) + 1) \quad \forall (\tau,\zeta) \in V, \end{cases}$$

where $\mathrm{PSH}(V)$ is the set of plurisubharmonic functions on V (cf. [2]), and $\omega(\tau,\zeta) = \sigma_{\alpha_1}(|\tau|) + \sigma_{\alpha_2}(|\zeta|)$ is the *weight function* defined, for $0 \le \alpha_1, \alpha_2 < 1$ and $t \ge 0$, by

$$\sigma_\alpha(t) = \begin{cases} t^\alpha & \text{if } 0 < \alpha < 1 \\ \log(1+t) & \text{if } \alpha = 0. \end{cases}$$

When the algebraic variety V has dimension one, i.e. is an algebraic curve, we can describe its branches at infinity by means of Puiseux series expansions. It turns out that the orders α_1, α_2 for which V satisfies $PL(\omega)$ are strictly related to the coefficients and the exponents of the Puiseux series expansions on its branches at infinity. This seems particularly useful since Puiseux series expansions can be computed by several programs, such as MAPLE, for instance.

Given an algebraic curve $V \subset \mathbb{C}^N \simeq \mathbb{C}^k_\tau \times \mathbb{C}^n_\zeta$ with cone of limiting directions

$$V^h = \bigcup_{j=1}^{\ell} V_j = \bigcup_{j=1}^{\ell} \mathbb{C} \cdot v_j$$

for $v_j = (\tau_j^o, \zeta_j^o) \in (\mathbb{C}^k \times \mathbb{C}^n) \setminus \{(0,0)\}$, there are two kinds of Puiseux series expansions on the branches of V near infinity, depending on whether their cone of limiting directions V_j is contained in $\mathbb{C}^k \times \{0\}$ or not. More precisely (cf. Lemma 3.6 of [2]):

1) If $V_j \not\subset \mathbb{C}^k \times \{0\}$ and, for instance, the first component of ζ_j^o is not zero, then on the branches W of V with cone of limiting directions V_j we have a Puiseux series expansion of the form

$$(3) \qquad (\tau, \zeta_1, \zeta') = (\tau_j^o, 1, a)\zeta_1 + \sum_{\nu=-\infty}^{\kappa} (D_\nu, 0, E_\nu)\zeta_1^{\nu/m}, \qquad |\zeta_1| \gg 1$$

where $\zeta' = (\zeta_2, \ldots, \zeta_n)$, $m \in \mathbb{N}$, $\kappa \in \mathbb{Z} \cup \{-\infty\}$, $\kappa < m$, $D_\nu \in \mathbb{C}^k$, $a, E_\nu \in \mathbb{C}^{n-1}$ for all $\nu \leq \kappa$.

2) If $V_j \subset (\mathbb{C}^k \setminus \{0\}) \times \{0\}$ and, for instance, the first component of τ_j^o is not zero, then on the branches W of V with cone of limiting directions V_j we have a Puiseux series expansion of the form

$$(4) \qquad (\tau_1, \tau', \zeta) = (1, 0, 0)\tau_1 + \sum_{\nu=-\infty}^{p'} (0, F_\nu, G_\nu)\tau_1^{\nu/q}, \qquad |\tau_1| \gg 1$$

where $\tau' = (\tau_2, \ldots, \tau_k)$, $q \in \mathbb{N}$, $p' \in \mathbb{Z} \cup \{-\infty\}$, $p' < q$, $F_\nu \in \mathbb{C}^{k-1}$, $G_\nu \in \mathbb{C}^n$ for all $\nu \leq p'$.

Note that all the indices and the coefficients in (3) and (4) depend on the branches W (cf. [2]), so that we should write $\kappa = \kappa(W)$, $p' = p'(W)$, etc.

Moreover, we can multiply the coefficients D_ν, E_ν in (3) by ω_m^ν and the coefficients F_ν, G_ν in (4) by ω_q^ν (where ω_m and ω_q are, respectively, any m-th root and any q-th root of unity), obtaining an equivalent representation of W.

On each of these branches we have several necessary and/or sufficient conditions for $PL(\omega)$ to be valid (cf. [2]). In the case of one time-variable (and one or more space-variables) these necessary and sufficient conditions perfectly fit, so that we have a complete characterization of systems which are of evolution in $\mathcal{E}_\omega$. In this case the Puiseux series expansion (3) is of the form

$$(5) \qquad \begin{cases} \tau(\zeta_1) = \tau_j^o \zeta_1 + \sum_{\nu=-\infty}^{s} D_\nu \zeta_1^{\nu/m} \\ \zeta'(\zeta_1) = a\zeta_1 + \sum_{\nu=-\infty}^{t} E_\nu \zeta_1^{\nu/m} \end{cases} \qquad |\zeta_1| \gg 1$$

where $s = \max\{\nu \leq \kappa : \, D_\nu \neq 0\}$, $t = \max\{\nu \leq \kappa : \, E_\nu \neq 0\}$ (and the maximum of the empty set is defined, here and in the following, as $-\infty$).

The Puiseux expansion (4) is of the form

$$(6) \qquad \zeta(\tau) = \sum_{\nu=-\infty}^{p} G_\nu \tau^{\nu/q}, \qquad |\tau| \gg 1$$

with $p = \max\{\nu \leq p' : G_\nu \neq 0\}$. If $G_p \in \mathbb{C}\mathbb{R}^n$ then we can assume, up to a real linear change of variables, that $G_p = (G_{p,1}, 0, \ldots, 0)$ and hence obtain from (6) (cf. [1], [2]):

$$\tau(\zeta_1) = \sum_{\nu=-\infty}^{q} A_\nu \zeta_1^{\nu/p} = \sum_{\nu=-\infty}^{q} A_\nu \zeta_1^{a\nu/r}, \qquad |\zeta_1| \gg 1,\ A_q = \left(\frac{1}{G_{p,1}^{1/p}}\right)^q \tag{7}$$

and, for $n \geq 2$,

$$\zeta'(\zeta_1) = \sum_{\nu=-\infty}^{P} B_\nu \zeta_1^{\nu/Q} = \sum_{\nu=-\infty}^{P} B_\nu \zeta_1^{b\nu/r} \qquad |\zeta_1| \gg 1$$

for $P \in \mathbb{Z}$, $Q \in \mathbb{N}$, $P < Q$, $r = ap = bQ$ the least common multiple of p and Q ($r := p$ if $n = 1$).

Let us define, for $\zeta_o \in \{-1, 1\}$, for any branch f_m of the m-th root and for any branch f_r of the r-th root:

$$\begin{aligned}
&u(\zeta_o, f_m) = \max\{\nu \leq t : \operatorname{Im}(E_\nu f_m(\zeta_o)^\nu) \neq 0\} \\
&w(\zeta_o, f_m) = \max\{\nu \leq s : \operatorname{Im}(D_\nu f_m(\zeta_o)^\nu) \neq 0\} \\
&w_0 = \max\{w(\zeta_o, f_m) : \zeta_o \in \{-1,1\}, f_m \text{ a branch of the } m\text{-th root}, \\
&\qquad\qquad\qquad\qquad w(\zeta_o, f_m) > \max\{0, u(\zeta_o, f_m)\}\} \\
&\mu(\zeta_o, f_r) = \max\{\nu < q : \operatorname{Im}(A_\nu f_r(\zeta_o)^{a\nu}) \neq 0\} \\
&\mu^* = \max\Big\{\mu(\zeta_o, f_r) :\ \zeta_o \in \{-1,1\},\ f_r \text{ a branch of the } r\text{-th root}, \\
&\qquad\qquad \mu(\zeta_o, f_r) > q - p, \text{ and} \\
&\qquad\qquad \operatorname{Im}(B_\nu f_r(\zeta_o)^{b\nu}) = 0 \quad \forall \nu \geq Q\left(1 - \frac{q - \mu(\zeta_o, f_r)}{p}\right)\Big\},
\end{aligned}$$

where we mean, in the definition of μ^*, that we do not place any requirement on the B_ν if $n = 1$. Here again everything depends on the branch W of V that has V_j as cone of limiting directions (cf. [2]), so that we should write $w_0 = w_0(W)$, $\mu^* = \mu^*(W)$, etc.

We can then state the following theorem (cf. Theorem 5.16 of [2]):

THEOREM 1. *Let V be an algebraic curve in $\mathbb{C}_\tau \times \mathbb{C}^n_\zeta$ with cone of limiting directions*

$$V^h = \bigcup_{j=1}^{\ell} V_j = \bigcup_{j=1}^{\ell} \mathbb{C} \cdot v_j$$

for $v_j = (\tau_j^o, \zeta_j^o) \in (\mathbb{C} \times \mathbb{C}^n) \setminus \{(0,0)\}$, and let $\omega(\tau, \zeta) = \sigma_{\alpha_1}(|\tau|) + \sigma_{\alpha_2}(|\zeta|)$ be a given weight function. Then the following conditions are equivalent:

(1) V satisfies $PL(\omega)$.

(2) For each $j \in \{1, \ldots, \ell\}$ and for each branch W of V with cone of limiting directions V_j, one of the following conditions holds (where we write p, q, etc. instead of $p(W), q(W)$, etc.):

(i) $\zeta_j^o \notin \mathbb{C}\mathbb{R}^n$;

(ii) $v_j = (\tau_j^o, \zeta_j^o) \in (\mathbb{R} \setminus \{0\}) \times (\mathbb{R}^n \setminus \{0\})$ *and*

$$\max\{\alpha_1, \alpha_2\} \geq w_0/m;$$

(iii) $v_j = (0, \zeta_j^o) \in \{0\} \times (\mathbb{R}^n \setminus \{0\})$ *and*

$$\max\left\{\frac{s}{m}\alpha_1, \alpha_2\right\} \geq \frac{w_0}{m};$$

(iv) $v_j = (\tau_j^o, 0) \in (\mathbb{R} \setminus \{0\}) \times \{0\}$, $p \leq 0$ *or* $G_p \notin \mathbb{C}\mathbb{R}^n$;

(v) $v_j = (\tau_j^o, 0) \in (\mathbb{R} \setminus \{0\}) \times \{0\}$, $p > 0$, $G_p \in \lambda\mathbb{R}^n$ *for some* $\lambda \in \mathbb{C}$, $q/p \notin \mathbb{N}$, $\alpha_1 \geq p/q$;

(vi) $v_j = (\tau_j^o, 0) \in (\mathbb{R} \setminus \{0\}) \times \{0\}$, $p > 0$, $G_p \in \lambda\mathbb{R}^n$, $q/p \in \mathbb{N}$, $\lambda/|\lambda| \notin \{e^{ik\pi p/q} : k \in \mathbb{Z}\}$, $\alpha_1 \geq p/q$;

(vii) $v_j = (\tau_j^o, 0) \in (\mathbb{R} \setminus \{0\}) \times \{0\}$, $p > 0$, $G_p \in \lambda\mathbb{R}^n$, $q/p \in \mathbb{N}$, $\lambda/|\lambda| \in \{e^{ik\pi p/q} : k \in \mathbb{Z}\}$,

$$\max\left\{\frac{q}{p}\alpha_1, \alpha_2\right\} \geq 1 - \frac{q - \mu^*}{p}.$$

We now want to find a more explicit formulation of this theorem in the case of two variables, i.e. $k = n = 1$. In this case there exists a polynomial $P \in \mathbb{C}[\tau, \zeta]$ of degree $m' > 0$ such that

$$V = V(P) = \{(\tau, \zeta) \in \mathbb{C}^2 : \ P(\tau, \zeta) = 0\},$$
$$V^h = V(P_{m'}) = \{(\tau, \zeta) \in \mathbb{C}^2 : \ P_{m'}(\tau, \zeta) = 0\},$$

where $P_{m'}$ is the principal part of P and is of the form

$$P_{m'}(\tau, \zeta) = b\tau^\nu \zeta^\mu \prod_{j=1}^{\sigma} (\tau - a_j\zeta)^{m_j}, \qquad (\tau, \zeta) \in \mathbb{C}^2$$

for some $\mu, \nu, \sigma \in \mathbb{N}_0$, $b \in \mathbb{C} \setminus \{0\}$, and $m_j \in \mathbb{N}_0$, $a_j \in \mathbb{C} \setminus \{0\}$ for $1 \leq j \leq \sigma$.

Therefore the Puiseux series expansions (5) reduce to

$$\tau(\zeta) = A\zeta + \sum_{\nu=-\infty}^{s} D_\nu \zeta^{\nu/m}, \qquad |\zeta| \gg 1, \tag{8}$$

with $A = 0$ or $A = a_j$ for some $j \in \{1, \ldots, \sigma\}$.

The series expansions (6) and (7) are of the form:

$$\zeta(\tau) = \sum_{\nu=-\infty}^{p} G_\nu \tau^{\nu/q}, \qquad |\tau| \gg 1, \tag{9}$$

$$\tau(\zeta) = \sum_{\nu=-\infty}^{q} A_\nu \zeta^{\nu/p}, \qquad |\zeta| \gg 1, \tag{10}$$

for $G_\nu \in \mathbb{C}$ and $A_q = (1/G_p^{1/p})^q$.

Now we check what this specialization means for the conditions $(i)-(vii)$ in *(2)* of Theorem 1. Obviously, the condition (i) is empty when $n = 1$.

Let us look at the conditions *(2)(ii)* and *(2)(iii)* for $n = 1$. We first prove that if $s > 0$ then $w_0 = s$. To this aim we choose the branch $g(\rho e^{i\varphi}) = \rho^{1/m}\exp(i\varphi/m)$ of the m-th root. Then, for $D_s = re^{i\psi}$, we have that

$$D_s g(1)^s = re^{i\psi} \in \mathbb{R} \qquad \text{iff} \qquad \psi = h\pi,\ h \in \mathbb{Z};$$

in this case

$$D_s g(-1)^s = re^{i(\psi+\pi s/m)} = re^{i(h\pi+\pi s/m)} = \pm re^{i\frac{s}{m}\pi} \notin \mathbb{R}$$

since $s/m \notin \mathbb{Z}$ for $0 < s < m$. This means that we can find $\zeta_o \in \{-1,1\}$ and a branch $f_m = g$ of the m-th root such that $w(\zeta_o, f_m) = s > 0$. Since $n = 1$ we have $u(\zeta_o, f_m) = -\infty$ and hence $w_0 = s$. Therefore the conditions *(2)(ii)* and *(2)(iii)* of Theorem 1 become, respectively:

(ii)′ $v_j = (\tau_j^o, \zeta_j^o) \in (\mathbb{R}\setminus\{0\}) \times (\mathbb{R}\setminus\{0\})$ and

$$\max\{\alpha_1, \alpha_2\} \geq s/m;$$

(iii)′ $v_j = (0, \zeta_j^o) \in \{0\} \times (\mathbb{R}\setminus\{0\})$ and $\alpha_2 \geq s/m$.

If, on the contrary, $s \leq 0$, then $w_0 = -\infty$ and the conditions *(2)(ii)* and *(2)(ii)’*, *(2)(iii)* and *(2)(iii)’* are empty, and hence coincide again.

In case of *(2)(iv)* we have only the condition $p \leq 0$, since $G_p \in \mathbb{C}\mathbb{R}$ is always satisfied.

Let us now take $p = 1$ and look at the conditions *(2)(v)–(vii)*. We have $q/p = q \in \mathbb{N}$ (hence the condition *(2)(v)* is empty) and

$$\frac{G_1}{|G_1|} := e^{i\phi} \in \{e^{ik\frac{p}{q}\pi} : k \in \mathbb{Z}\} = \{e^{i\frac{k\pi}{q}} : k \in \mathbb{Z}\}$$

if and only if $\phi q = k\pi$ for some $k \in \mathbb{Z}$, i.e. if and only if $G_1^q \in \mathbb{R}$. In this case $\mu^* = -\infty$, since the condition

$$q - 1 = q - p < \mu(\zeta_o, f_p) < q$$

cannot be satisfied for any integer $\mu(\zeta_o, f_p)$. Therefore the condition *(2)(vii)* is empty.

If, on the contrary, $G_1^q \notin \mathbb{R}$ then we have the condition $\alpha_1 \geq p/q$ from *(2)(vi)*.

Let us now take $p = 2$. If q is odd then $q/p \notin \mathbb{N}$ and we have the condition $\alpha_1 \geq p/q$ from *(2)(v)*.

If q is even then $q/p \in \mathbb{N}$ and

$$\frac{G_2}{|G_2|} := e^{i\phi} \in \{e^{ik\pi\frac{p}{q}} : k \in \mathbb{Z}\} = \{e^{i\frac{2k\pi}{q}} : k \in \mathbb{Z}\}$$

if and only if $\phi q = 2k\pi$ for some $k \in \mathbb{Z}$, i.e. if and only if $G_2^q > 0$. Let us now investigate μ^* in this case. If $A_{q-1} \neq 0$, then there exist $\zeta_o \in \{-1,1\}$ and a branch f_2 of the square root such that

$$\mathrm{Im}(A_{q-1} f_2(\zeta_o)^{q-1}) \neq 0$$

since $q-1$ is odd. In this case $\mu^* = q-1 > q-2 = q-p$, and the condition *(2)(vii)* becomes

$$\max\left\{\frac{q}{2}\alpha_1, \alpha_2\right\} \geq 1 - \frac{q-(q-1)}{2} = \frac{1}{2}.$$

If, on the contrary, $A_{q-1} = 0$ then for any $\zeta_o \in \{-1,1\}$ and any branch f_2 of the square root we have that $\mu(\zeta_o, f_2) < q-1$ and hence $\mu^* = -\infty$, because the condition

$$q-2 = q-p < \mu(\zeta_o, f_2) < q-1$$

cannot be satisfied for any integer $\mu(\zeta_o, f_2)$. In this case the condition *(2)(vii)* is therefore empty.

If we assume that $G_2^q \in \mathbb{C} \setminus \mathbb{R}$ or $G_2^q < 0$ then $G_2/|G_2| \notin \{e^{i\frac{2k\pi}{q}} : k \in \mathbb{Z}\}$. In this case we have the condition $\alpha_1 \geq p/q$ from *(2)(vi)*.

Let us finally remark that if $V(P)$ satisfies $PL(\omega)$, then also $V(P_{m'})$ satisfies $PL(\omega)$ because of Theorem 5.3 of [2]. Vice versa, if $V(P_{m'}) = V_1 \cup \ldots \cup V_\ell$ satisfies $PL(\omega)$, then every V_j, for $j \in \{1, \ldots, \ell\}$, admits a real generator $v_j = (\tau_j^o, \zeta_j^o) \in \mathbb{R}^2 \setminus \{0\}$ by Theorem 3.3 of [2].

All the above considerations allow us to reformulate Theorem 1 in the case of two variables as follows:

THEOREM 2. *For $P \in \mathbb{C}[\tau, \zeta] \setminus \mathbb{C}$ with principal part $P_{m'}$ and a weight function $\omega(\tau, \zeta) = \sigma_{\alpha_1}(|\tau|) + \sigma_{\alpha_2}(|\zeta|)$ the algebraic curve $V(P)$ satisfies $PL(\omega)$ if and only if the following two conditions are satisfied:*

(1) $V(P_{m'})$ satisfies $PL(\omega)$.

(2) For each $j \in \{1, \ldots, \ell\}$ and for each branch W of V with cone of limiting directions V_j, one of the following conditions holds:

(i) $v_j = (\tau_j^o, \zeta_j^o) \in \mathbb{R} \times (\mathbb{R} \setminus \{0\})$ and

$$\begin{cases} \max\{\alpha_1, \alpha_2\} \geq \frac{s}{m} & \text{if } \tau_j^o \neq 0 \\ \alpha_2 \geq \frac{s}{m} & \text{if } \tau_j^o = 0; \end{cases}$$

(ii) $v_j = (\tau_j^o, 0) \in (\mathbb{R} \setminus \{0\}) \times \{0\}$, $p \leq 0$, or $p = 1$ and $G_1^q \in \mathbb{R}$, or $p = 2$, $G_2^q > 0$, q is even and $A_{q-1} = 0$;

(iii) $v_j = (\tau_j^o, 0) \in (\mathbb{R}\setminus\{0\}) \times \{0\}$, $p > 0$ *and*

$$\begin{cases} \alpha_1 \geq \frac{p}{q} & \text{if } p \in \{1,2\} \text{ and } G_p^q \in \mathbb{C}\setminus\mathbb{R} \\ & \text{or if } p = 2 \text{ and } G_2^q < 0 \\ & \text{or if } p = 2,\ G_2^q > 0,\ q \text{ odd} \\ & \text{or if } p \geq 3 \text{ and } \frac{q}{p} \notin \mathbb{N} \\ & \text{or if } p \geq 3,\ \frac{q}{p} \in \mathbb{N}, \text{ and} \\ & \quad \frac{G_p}{|G_p|} \notin \left\{ e^{ik\pi\frac{p}{q}} : k \in \mathbb{Z} \right\} \\ \max\left\{\frac{q}{2}\alpha_1, \alpha_2\right\} \geq \frac{1}{2} & \text{if } p = 2,\ G_2^q > 0, \\ & \quad q \text{ even, and } A_{q-1} \neq 0 \\ \max\left\{\frac{q}{p}\alpha_1, \alpha_2\right\} \geq 1 - \frac{q-\mu^*}{p} & \text{if } p \geq 3,\ \frac{q}{p} \in \mathbb{N}, \text{ and} \\ & \quad \frac{G_p}{|G_p|} \in \left\{ e^{ik\pi\frac{p}{q}} : k \in \mathbb{Z} \right\}. \end{cases}$$

REMARK 1. Theorem 2 corrects [1], Theorem 4.16, which is not correct, due to a mistake in the proof of part (1) of Lemma 4.10 in [1]. However, the arguments for this part of Lemma 4.10 are right whenever $(p,q) = 1$. Therefore Theorem 2 concides with Theorem 4.16 of [1] if $(p,q) = 1$ on every branch W of $V(P)$. Note that [1], Theorem 4.16, is also correct if $V(P)$ has no branches W for which $p \geq 3$, $q/p \in \mathbb{N}$ and $G_p/|G_p| \in \{e^{ik\pi\frac{p}{q}} : k \in \mathbb{Z}\}$.

2. Examples

EXAMPLE 1. Let us consider the algebraic curve

$$V = \{(\tau,\zeta) \in \mathbb{C}^2 : P(\tau,\zeta) = \zeta^6 + 3\zeta^2\tau^2 + \tau^2 - 3\zeta^4\tau - 6\zeta\tau^2 - 2\zeta^3\tau - \tau^3 = 0\}.$$

Since the principal part P_6 of P is $P_6(\tau,\zeta) = \zeta^6$, it follows that

$$V(P_6) = \{(\tau,\zeta) \in \mathbb{C}^2 :\ \zeta = 0\}.$$

It is therefore trivial that $V(P_6)$ satisfies $PL(\omega)$ for each weight function ω, by Proposition 4.3 of [1]. It is easy to check that

$$V = \{(\lambda^6, \lambda^3 + \lambda^2) :\ \lambda \in \mathbb{C}\}.$$

From this it follows that V has only one irreducible branch near infinity that admits the Puiseux series expansion

$$\zeta(\tau) = \tau^{3/6} + \tau^{2/6}, \quad |\tau| \gg 1,$$

which converts into the expansion

$$\tau(\zeta) = \zeta^2 - 2\zeta^{5/3} + \sum_{\nu=-\infty}^{4} A_\nu \zeta^{\nu/3}, \qquad |\zeta| \gg 1.$$

Since $p = 3$, $q = 6$ and $G_3 = 1 = e^{ik\pi/2}$ for $k = 0$, we are in the last case of Theorem 2 hence we must compute μ^*. Since $A_5 = -2$ and there exists a third root f_3 of 1 such that $\mathrm{Im}(A_5 f_3(1)^5) \neq 0$, we have $\mu(1, f_3) = 5$. Since $5 > 3 = q - p$ we also have $\mu^* = 5$. Consequently, Theorem 2 implies that V satisfies $PL(\omega)$, for $\omega(\tau,\zeta) = \sigma_{\alpha_1}(|\tau|) + \sigma_{\alpha_2}(|\zeta|)$ if and only if

$$\max\{2\alpha_1, \alpha_2\} \geq 1 - \frac{6-5}{3} = \frac{2}{3}.$$

Let us now prove a lemma that is useful in the study of examples.

LEMMA 1. *For $p, q \in \mathbb{N}$, $q > p$, and $a \in \mathbb{C} \setminus \{0\}$ let $P \in \mathbb{C}[\tau,\zeta]$ be defined as*

$$P(\tau,\zeta) := \tau^p - a\zeta^q + \sum_{j=0}^{p-1} b_j \tau^j - \sum_{j=0}^{q-1} a_j \zeta^j.$$

Assume that for $h, s, t \in \mathbb{N}$ we have $p = hs$, $q = ht$, and $(s,t) = 1$ and denote by $\beta_1, \ldots, \beta_h$ the h different h-th roots of a. Then $V(P) := \{(\tau,\zeta) \in \mathbb{C}^2 : P(\tau,\zeta) = 0\}$ has h branches near infinity and for each such branch W there exists $j \in \{1,\ldots,h\}$ such that W admits a Puiseux series expansion which has $\beta_j^{-1/t}\tau^{s/t}$ as leading term.

Proof. Since $p = hs$ and $q = ht$ we have

$$F(\tau,\zeta) := \tau^p - a\zeta^q = \tau^{sh} - a\zeta^{th} = \prod_{j=1}^{h} (\tau^s - \beta_j \zeta^t).$$

Because of $(s,t) = 1$, this shows that $V(F)$ is the union of h irreducible curves, which have the Puiseux series expansions

$$\zeta_j(\tau) = \left(\frac{1}{\beta_j}\right)^{1/t} \tau^{s/t},\ |\tau| > 0,\ 1 \leq j \leq h.$$

For $1 \leq j \leq h$, $1 \leq k \leq t$, and $\tau \in \mathbb{C}$ with $|\tau| > 0$ denote by $\zeta_{j,k}(\tau)$ the $q = ht$ different roots of $F(\tau,\cdot)$. Then it is easy to check that there exists $\delta > 0$ such that

$$\min\{|\zeta_{i,k}(\tau) - \zeta_{j,m}(\tau)| :\ 1 \leq i,\ j \leq h,\ 1 \leq k,\ m \leq t, (i,k) \neq (j,m)\} \geq \delta|\tau|^{s/t}.$$

Furthermore, there exists $\eta > 0$ such that

$$\min\left\{|1 - \frac{\beta_\nu}{\beta_j}| :\ 1 \leq j,\ \nu \leq h,\ j \neq \nu\right\} \geq 2\eta.$$

Then we have for $\lambda \in \mathbb{C}$ with $|\lambda| = \varepsilon|\tau|^{s/t}$:

$$(11) \qquad |F(\tau,\zeta_{j,k}(\tau)+\lambda)| = |\tau^s - \beta_j(\zeta_{j,k}(\tau)+\lambda)^t| \prod_{\substack{\nu=1\\ \nu\neq j}}^{h} |\tau^s - \beta_\nu(\zeta_{j,k}(\tau)+\lambda)^t|.$$

Now note that $\zeta_{j,k}(\tau)^t = \tau^s/\beta_j$ and the choice of η imply the existence of $0 < \varepsilon_1 < \delta$ such that for each ε with $0 < \varepsilon \leq \varepsilon_1$ we have

$$(12) \qquad |\tau^s - \beta_\nu(\zeta_{j,k}(\tau)+\lambda)^t| = |\tau|^s \left|1 - \frac{\beta_\nu}{\beta_j}\left(1+\beta_j^{1/t}\frac{\lambda}{\tau^{s/t}}\right)^t\right| \geq \eta|\tau|^s.$$

Similary, we get

$$\tau^s - \beta_j(\zeta_{j,k}(\tau)+\lambda)^t = \tau^s - \beta_j \sum_{l=1}^{t} \binom{t}{l} \zeta_{j,k}(\tau)^{t-l}\lambda^l.$$

This shows that we can choose $\varepsilon_1 > 0$ so small that for each ε with $0 < \varepsilon < \varepsilon_1$ we have

$$(13) \qquad |\tau^s - \beta_j(\zeta_{j,k}(\tau)+\lambda)^t| \geq |\beta_j|^{1/t} t \frac{\varepsilon}{2}|\tau|^s.$$

From (11), (12), and (13) we now get

$$(14) \qquad |F(\tau,\zeta_{j,k}(\tau)+\lambda)| \geq \eta^{h-1}|\beta_j|^{1/t} t\frac{\varepsilon}{2}|\tau|^{sh},\ |\lambda| = \varepsilon|\tau|^{s/t}.$$

To apply the Theorem of Rouché to F and P on the circles $\partial B(\zeta_{j,k}(\tau),\varepsilon|\tau|^{s/t})$, we note that there exists $C > 1$ such that

$$|P(\tau,\zeta) - F(\tau,\zeta)| \leq C(|\tau|^{p-1} + |\zeta|^{q-1}),\ |\tau| \geq 1.$$

Since for $\lambda \in \mathbb{C}$ with $|\lambda| = \varepsilon|\tau|^{s/t}$ we have

$$|(\zeta_{j,k}(\tau)+\lambda)^{q-1}| \leq |\tau|^{s(q-1)/t} \left(\left|\frac{1}{\beta_j^{1/t}}\right| + \varepsilon\right)^{q-1},$$

and since $p < q$ there exists $D > 1$ such that, for $|\lambda| = \varepsilon|\tau|^{s/t}$,

$$(15) \qquad |P(\tau,\zeta_{j,k}(\tau)+\lambda) - F(\tau,\zeta_{j,k}(\tau)+\lambda)| \leq CD|\tau|^{s(q-1)/t} = CD|\tau|^{sh-s/t}.$$

From (14) and (15) we now get that for each $0 < \varepsilon < \varepsilon_1$ there exists $\tau_0 > 1$ such that for each $\tau \in \mathbb{C}$ with $|\tau| \geq \tau_0$ we have

$$|F(\tau,\zeta_{j,k}(\tau)+\lambda) - P(\tau,\zeta_{j,k}(\tau)+\lambda)| < |F(\tau,\zeta_{j,k}(\tau)+\lambda)|,\ |\lambda| = \varepsilon|\tau|^{s/t}.$$

By the Theorem of Rouché, it follows that for each $\tau \in \mathbb{C}$, $|\tau| \geq \tau_0$, the function $\zeta \mapsto P(\tau,\zeta)$ has a zero $\xi_{j,k}(\tau)$ which satisfies $|\xi_{j,k}(\tau) - \zeta_{j,k}(\tau)| \leq \varepsilon|\tau|^{s/t}$ for each ε with

$0 < \varepsilon < \varepsilon_1$. By the choice of ε_1, the disks $B(\zeta_{j,k}(\tau), \varepsilon|\tau|^{s/t})$, $1 \le j \le h$, $1 \le k \le t$, are pairwise disjoint. Since each branch W of $V(P)$ near infinity admits a Puiseux series expansion of the form

$$\zeta(\tau) = \sum_{\nu=-\infty}^{w} A_\nu \tau^{\nu/r}$$

it now follows that the leading term of such an expansion has the given form. □

In the following example we use Lemma 1 and Theorem 2 to give a correct proof of [1], Example 5.3. The proof that was given in [1] is based on that part of [1], Theorem 4.16, in which we have a flaw. Nevertheless, the assertions of [1], Example 5.3, are right, as the new proof shows.

EXAMPLE 2. For $p, q \in \mathbb{N}$, $p, q, \ge 2$, and $a \in \mathbb{C} \setminus \{0\}$ let $P \in \mathbb{C}[\tau, \zeta]$ be defined as

$$P(\tau, \zeta) := \tau^p - a\zeta^q + \sum_{j=0}^{p-1} b_j \tau^j - \sum_{j=0}^{q-1} a_j \zeta^j.$$

Then for $\omega(\tau, \zeta) := \sigma_{\alpha_1}(|\tau|) + \sigma_{\alpha_2}(|\zeta|)$ the following assertions hold for

$$V = V(P) = \{(\tau, \zeta) \in \mathbb{C} \times \mathbb{C} : \ P(\tau, \zeta) = 0\}.$$

(1) If $p > q \ge 2$ then V satisfies $PL(\omega)$ if and only if $\alpha_2 \ge q/p$.

(2) If $q > p \ge 3$ then V satisfies $PL(\omega)$ if and only if $\alpha_1 \ge p/q$.

(3) If $q > p = 2$ with $a \in \mathbb{C} \setminus \mathbb{R}$ or $a < 0$ or $a > 0$ and q odd, then V satisfies $PL(\omega)$ if and only if $\alpha_1 \ge p/q$.

(4) If $q > p = 2$, $a \in \mathbb{R}$, q even and $a > 0$ then V satisfies $PL(\omega)$ for all $0 \le \alpha_1, \alpha_2 < 1$.

(5) If $p = q \ge 3$ or $p = q = 2$ and $a \in \mathbb{C} \setminus \ [0, \infty[$ then V does not satisfy $PL(\omega)$ for any $(\alpha_1, \alpha_2) \in \ [0, 1[\times [0, 1[$.

(6) If $p = q = 2$, $a \in \mathbb{R}$ and $a > 0$ then V satisfies $PL(\omega)$ for all $0 \le \alpha_1, \alpha_2 < 1$.

To prove these assertions we argue as follows.

(1) In this case the principal part P_p of P is given by $P_p(\tau, \zeta) = \tau^p$. Hence $V(P_p)$ satisfies $PL(\kappa)$ for each weight function κ by [1], Proposition 4.3. Now fix any branch W of V near infinity. By [1], Lemma 4.4, W admits a Puiseux series expansion of the form (8). The present hypothesis $q < p$ implies $A = 0$ so that (8) gives

$$\tau(\zeta) = \sum_{\nu=-\infty}^{s} D_\nu \zeta^{\nu/m}.$$

Since $P(\tau(\zeta), \zeta) = 0$, we have $D_s^p \zeta^{sp/m} - a\zeta^q = 0$ and consequently $s/m = q/p$. Hence we get from Theorem 2, part *(2)(i)* that $PL(\omega)$ holds on W if and only if $\alpha_2 \ge s/m = q/p$. Since W was an arbitrary branch of V the proof of (1) is complete.

(2) In this case the principal part P_q of P is given by $P_q(\tau,\zeta) = -a\zeta^q$. Hence $V(P_q)$ satisfies $PL(\kappa)$ for each weight function κ by [1], Proposition 4.3. Next assume that there are $h,s,t \in \mathbb{N}$ with $p = hs$, $q = ht$, and $(s,t) = 1$. If we denote by $\beta_1,\dots,\beta_h$ the h different roots of a, then we get from Lemma 1 that for each branch W of V near infinity, there exists $1 \le j \le h$ such that W admits a Puiseux series expansion of the form

$$\zeta(\tau) = \frac{1}{\beta_j^{1/t}}(\tau^s)^{1/t} + \ l.o.t.$$

This shows that $G_s^t = 1/\beta_j$. Now we distinguish the following cases:
(i) $s = 1$.
This means that $p = h$ and $q/p \in \mathbb{N}$. Since $h = p \ge 3$ by the present hypotheses, at least one of the numbers $\beta_1,\dots,\beta_p$ is not real. If $\beta_j \in \mathbb{C}\setminus\mathbb{R}$ then $G_s^t \in \mathbb{C}\setminus\mathbb{R}$. Therefore, it follows from Theorem 2 $(2)(iii)$ (and Theorem 2 $(2)(ii)$ for $\beta_j \in \mathbb{R}$) that V satisfies $PL(\omega)$ if and only if $\alpha_1 \ge p/q$.
(ii) $s \ge 2$, $h = 1$.
Then $s = p \ge 3$ and $(p,q) = 1$. Hence $p/q \notin \mathbb{N}$ and it follows from Theorem 2 $(2)(iii)$ that V satisfies $PL(\omega)$ if and only if $\alpha_1 \ge p/q$.
(iii) $s = 2$, $h \ge 2$.
Then $(s,t) = 1$ implies that t must be odd. Hence it follows from Theorem 2 $(2)(iii)$ that, no matter whether $\frac{1}{\beta_j} \in \mathbb{C}\setminus\mathbb{R}$ or $\frac{1}{\beta_j} \in \mathbb{R}\setminus\{0\}$, V satisfies $PL(\omega)$ if and only if $\alpha_1 \ge s/t = p/q$.
(iv) $s \ge 3$, $h \ge 2$.
Then $s/t \notin \mathbb{N}$ together with Theorem 2 $(2)(iii)$ implies also in this case that V satisfies $PL(\omega)$ if and only if $\alpha_1 \ge s/t = p/q$.

(3) As in part (2) we get that $V(P_q)$ satisfies $PL(\kappa)$ for each weight function κ. If $p = 2$, $a > 0$, and q is odd then $\tau^2 - a\zeta^q$ is irreducible and hence V has a Puiseux series expansion of the form $\zeta(\tau) = \left(\frac{\tau^2}{a}\right)^{1/q} + l.o.t.$ This shows that $G_2 = \left(\frac{1}{a}\right)^{1/q}$ and hence $G_2^q = \frac{1}{a} > 0$. Therefore, it follows from Theorem 2 $(2)(iii)$ that V satisfies $PL(\omega)$ if and only if $\alpha_1 \ge p/q$.
If $p = 2$ and q is odd the same argument as above shows that $G_2^q = \frac{1}{a}$ is negative if $a < 0$ or is not real if a is not real. Hence we get the same conclusion as before.
If $p = 2$ and q is even, then $q = 2m$ and $\tau^p - a\zeta^q$ factors as $(\tau - \sqrt{a}\zeta^m)(\tau + \sqrt{a}\zeta^m)$. By Lemma 1, the two branches of V near infinity are then given by

$$\zeta(\tau) = \left(\frac{\pm 1}{\sqrt{a}}\right)^{1/m} + \ l.o.t.$$

Hence $G_1^m = \frac{\pm 1}{\sqrt{a}}$. This number is not real if $a \in \mathbb{C}\setminus\ [0,\infty[$. Therefore, it follows from Theorem 2 $(2)(iii)$ that (3) holds.

(4) The same arguments as in (3) show that now $G_1^m = \frac{\pm 1}{\sqrt{a}}$ is real since $a > 0$. Hence (4) follows from Theorem 2 $(2)(ii)$.

(5) In both cases the principal part of P is given by $P_p(\tau,\zeta) = \tau^p - a\zeta^p$ and we can find $\alpha \in \mathbb{C}\setminus\mathbb{R}$ such that $\alpha^p = a$. Hence P_p admits a factor $\tau - \alpha\zeta$ with $\alpha \in \mathbb{C}\setminus\mathbb{R}$,

which implies that $V(P_p)$ does not satisfy $PL(\omega)$ for any weight function ω because of [1], Proposition 4.3. By [1], Corollary 4.9, also $V(P)$ cannot satisfy $PL(\omega)$ for any weight function ω. Hence (5) holds.

(6) In this case the principal part P_2 of P is given by $P_2(\tau,\zeta) = \tau^2 - a\zeta^2 = (\tau - \sqrt{a}\zeta)(\tau + \sqrt{a}\zeta)$. Since a is positive by the present hypothesis, $V(P_2)$ satisfies $PL(\kappa)$ for each weight function κ, by [1], Proposition 4.3. Since $V(P_2)$ has two irreducible components, it follows similarly as in the proof of Lemma 1 that V has two branches near infinity and that these can be described as

$$\tau = -\frac{b_1}{2} \pm \sqrt{a}\zeta\Big(1 + \frac{a_1}{a}\,\frac{1}{\zeta} + \frac{a_0'}{a}\,\frac{1}{\zeta^2}\Big)^{1/2}.$$

This implies the existence of $C > 0$ such that

$$|\operatorname{Im}\tau| \le C|\operatorname{Im}\zeta| + C, \qquad (\tau,\zeta) \in V.$$

Hence condition (γ) of $PL(\omega)$ follows from condition (α) of $PL(\omega)$ for each weight function ω.

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THE KERNEL THEOREM IN ULTRADISTRIBUTIONS: MICROLOCAL REGULARITY OF THE KERNEL

Dedicated to Professor Luigi Rodino on the occasion of his 60th birthday

Abstract. In this paper we study kernels associated with continuous operators between spaces of Gevrey ultradistributions. The existence of such kernels has been established, in analogy with the kernel theorem of L. Schwartz for classical distributions, by H. Komatsu, and our aim here is to study these kernels from a microlocal point of view. The main results, which are the theorems 2, 3 below, show that there is a significant difference between the results which hold true in the case of Beurling ultradistributions and the results valid for Roumieu ultradistributions.

1. Introduction

The Schwartz kernel theorem states that the linear continuous operators T mapping $\mathcal{D}(U)$ to $\mathcal{D}'(V)$ are precisely the operators for which there is $\mathcal{K} \in \mathcal{D}'(V \times U)$ such that

$$(1) \qquad Tu(\varphi) = \mathcal{K}(\varphi \otimes u), \quad u \in \mathcal{D}(U),\ \varphi \in \mathcal{D}(V).$$

(Cf. L. Schwartz, [17].) $\mathcal{K}$ is called the "kernel" of T and in this situation we write $Tu(x) = \int_U \mathcal{K}(x,y)u(y)dy$. Here U and V are open sets in $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively, $\mathcal{D}(U)$ is the space of $C_0^\infty(U)$ functions endowed with the Schwartz topology and $\mathcal{D}'(W)$ the space of distributions on W, with $W = V$ or $W = V \times U$. The Schwartz theorem has been extended to the case of ultradistributions by H. Komatsu and both L. Schwartz and H. Komatsu have also studied linear continuous operators defined on compactly supported distributions, respectively ultradistributions, to distributions or ultradistributions. We shall consider for the moment only the distribution case. The problem is then to consider a linear continuous operator $T : \mathcal{E}'(U) \to \mathcal{D}'(V)$, where $\mathcal{E}'(U)$ is the space of compactly supported distributions on U. T induces a linear continuous operator on $\mathcal{D}(U)$ and therefore it has a distributional kernel $\mathcal{K} \in \mathcal{D}'(V \times U)$. The relation (1) associates a separately continuous bilinear form $(\varphi, u) \mapsto \mathcal{K}(\varphi \otimes u)$ on $\mathcal{D}(V) \times \mathcal{D}(U)$ with T whereas the initial operator defined on $\mathcal{E}'(U)$ is associated with the bilinear form $(\varphi, u) \mapsto T(u)(\varphi)$ defined on $\mathcal{D}(V) \times \mathcal{E}'(U)$. If we want to understand the class of kernels $\mathcal{K} \in \mathcal{D}'(V \times U)$ which correspond to linear continuous operators $\mathcal{E}'(U) \to \mathcal{D}'(V)$, we may then just study the bilinear form $(\varphi, u) \mapsto \mathcal{K}(\varphi \otimes u)$ as a form on $\mathcal{D}(V) \times \mathcal{E}'(U)$. This has led to a sophisticated theory of tensor products of topological vector spaces in which the notion of "nuclear" spaces (introduced by

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A. Grothendieck) plays a central role. It turns out that most common spaces of distributions or ultradistributions are nuclear and the central result concerning the kernel theorem in distributions is that the operator $T : \mathcal{D}(U) \to \mathcal{D}'(V)$ associated with some $\mathcal{K} \in \mathcal{D}'(V \times U)$ can be extended to a linear continuous operator $\mathcal{E}'(U) \to \mathcal{D}'(V)$ if and only if $\mathcal{K}$ can be identified in a natural way with an element in $[\mathcal{D}(V)\hat{\otimes}\mathcal{E}'(U)]'$, where $\mathcal{D}(V)\hat{\otimes}\mathcal{E}'(U)$ is, say, the ε topological tensor product of $\mathcal{D}(V)$ with $\mathcal{E}'(U)$. Since the spaces under consideration are nuclear, we may as well work with the π tensor product. For definitions and details we refer to [2] and [19]. There is also an interpretation of this in terms of C^∞ functions with distributional values.

The theory of tensor products of topological vector spaces is very powerful and it explains, among other things, why kernel theorems in Banach spaces of (possibly generalized) functions must typically be more complicated than those in distributions (see e.g., [1] for some examples of kernel theorems in Lebesgue spaces): infinite dimensional Banach spaces are never nuclear. On the other hand, when one wants to consider kernel theorems in hyperfunctions, this kind of approach is not usable in practice since hyperfunctions have no reasonable topology. One may then try another approach, which has been worked out in microlocal analysis. The central notion is this time the "wave front set" of a distribution, ultradistribution, or hyperfunction (introduced in 1969 by M. Sato for hyperfunction, [15] and in 1970 by L. Hörmander for distributions, [3]). The main condition is then

$$(2) \qquad \{(x,y,0,\eta); x \in V, y \in U, \eta \neq 0\} \cap \mathrm{WF}(\mathcal{K}) = \emptyset.$$

When $\mathcal{K}$ is a distribution, $\mathrm{WF}(\mathcal{K})$ stands for the C^∞ wave front set and if (2) holds then microlocal analysis gives a natural meaning to $\int_U \mathcal{K}(x,y)u(y)dy$ when $u \in \mathcal{E}'(U)$. (See [3], [20].) The same is true also in hyperfunctions if WF denotes the analytic wave front set: there is a natural meaning for $\int_U \mathcal{K}(x,y)u(y)dy$ when u is a real-analytic functional on U. Integration is then defined in terms of "integration along fibers" and $\int_U \mathcal{K}(x,y)u(y)dy$ has a meaning in hyperfunctions: see e.g., [16], [5] for details.

There is now however a fundamental difference between the two main cases contemplated by microlocal analysis, the distributional and the hyperfunctional one.

It is in fact not difficult to see that the condition (2) is not equivalent to the fact that $\mathcal{K} \in [\mathcal{D}(V)\hat{\otimes}\mathcal{E}'(U)]'$. This means that (2) is not a necessary condition when we want $\mathcal{K}$ to define a continuous operator from $\mathcal{E}'(U)$ to $\mathcal{D}'(V)$. On the other hand, it is part of the results described in [10], [11], that for hyperfunctions a reasonable operator acting from some space of analytic functionals to the space of hyperfunctions can only be defined in presence of condition (2). It seemed then natural to the present authors to look into the case of Gevrey ultradistributions and to study if microlocal conditions of type (2) are necessary for reasonable operators in ultradistributions to exist. It came, at least at first, as a surprise, that the answer depends on which type of ultradistributions one is considering: for ultradistributions of Beurling type, one may work with weaker conditions than the ones corresponding to (2), whereas for ultradistributions of Roumieu type such conditions are also necessary: see section 2 for the terminology and the theorems 2, 3 for the precise statements.

2. Definitions and main results

For the convenience of the reader, we shall now recall some of the definitions related to Gevrey-ultradistributions. (For most of the notions considered here, cf. e.g., Lions–Magenes, vol.3, section 1.3, or [14].)

Consider $s > 1$, $L > 0$, U open in $\mathbb{R}^n$ and let K be a compact set in U. We shall denote by $f \mapsto |f|_{s,L,K}$ the quasinorm

$$|f|_{s,L,K} = \sup_{\alpha \in \mathbb{N}^n} \sup_{x \in K} \frac{|(\partial/\partial x)^\alpha f(x)|}{L^{|\alpha|}(\alpha!)^s}, \tag{3}$$

defined on $\mathcal{C}^\infty(U)$. We further denote by

- $\mathcal{D}^{s,L}(K)$ the space of $\mathcal{C}^\infty$ functions f on $\mathbb{R}^n$ which vanish outside K such that for them $|f|_{s,L,K} < \infty$,
- $\mathcal{D}^{(s)}(K) = \bigcap_{L>0} \mathcal{D}^{s,L}(K), \mathcal{D}^{\{s\}}(K) = \bigcup_{L>0} \mathcal{D}^{s,L}(K)$,
- $\mathcal{D}^{\{s\}}(U) = \bigcup_{K \subset U} \mathcal{D}^{\{s\}}(K)$, respectively $\mathcal{D}^{(s)}(U) = \bigcup_{K \subset U} \mathcal{D}^{(s)}(K)$,
- $\mathcal{E}^{(s)}(U) = \{f \in \mathcal{C}^\infty(U); \forall K \Subset U, \forall L > 0, |f|_{s,L,K} < \infty\}$, respectively $\mathcal{E}^{\{s\}}(U) = \{f \in \mathcal{C}^\infty(U); \forall K \Subset U, \exists L > 0, |f|_{s,L,K} < \infty\}$.

The functions in $\mathcal{E}^{\{s\}}(U)$, are called "ultradifferentiable" of Roumieu type, and those in $\mathcal{E}^{(s)}(U)$, ultradifferentiable of Beurling type, with Gevrey index s. Since we shall often encounter statements for the two types of classes which are quite similar, we now introduce the convention that we shall write $\mathcal{D}^*(U)$ when we give a statement which refers to both the case $* = (s)$ and the case $* = \{s\}$. The same convention also applies for other spaces associated with the two cases.

All the spaces mentioned above carry natural topologies:

- $\mathcal{D}^{s,L}(K)$ is a Banach space when endowed with $|\cdot|_{s,L,K}$ as a norm,
- $\mathcal{D}^{(s)}(K)$ is the projective limit (for "$L \to 0+$") of the spaces $\mathcal{D}^{s,L}(K)$, whereas $\mathcal{D}^{\{s\}}(K)$ is the inductive limit (for "$L \to \infty$") of the same spaces. The spaces $\mathcal{D}^{(s)}(K)$ are FS (i.e., Fréchet-Schwartz), whereas the spaces $\mathcal{D}^{\{s\}}(K)$ are DFS (duals of Fréchet–Schwartz). (The topological properties of these spaces are studied in [6].)
- $\mathcal{D}^{\{s\}}(U)$ is the inductive limit (for $K \subset U$) of the spaces $\mathcal{D}^{\{s\}}(K)$, whereas $\mathcal{D}^{(s)}(U)$ is the inductive limit (again for $K \subset U$) of the spaces $\mathcal{D}^{(s)}(K)$.
- We shall define topologies on $\mathcal{E}^{(s)}(U)$ and $\mathcal{E}^{\{s\}}(U)$ as follows. At first we define for $K \Subset U$ and $L > 0$ the space $Y_{K,L}$ of restrictions to K of functions in $\mathcal{C}^\infty(U)$, which satisfy $|f|_{s,L,K} < \infty$, endowed with the topology given by the semi-norm $|\cdot|_{s,L,K}$. Then,

$$\mathcal{E}^{(s)}(U) = \varprojlim_{K \Subset U} \varprojlim_{L>0} Y_{K,L}, \quad \mathcal{E}^{\{s\}}(U) = \varprojlim_{K \Subset U} \varinjlim_{L>0} Y_{K,L}.$$

We have continuous inclusions $\mathcal{D}^*(U) \subset \mathcal{E}^*(U)$ with dense image and $\mathcal{D}^*(K)$ is the subspace of $\mathcal{E}^*(U)$ $(K \subset U)$ consisting of the functions with compact support lying in K.

For a systematic study of the topological properties of these spaces we refer to [13], [6]. We shall however strive to use only a minimum of results on the topological structure of the spaces we shall consider. On the other hand, we shall consider later a new class of spaces in which we can state results which can serve as a common background for both the Roumieu and the Beurling case.

Finally, we shall denote by $\mathcal{D}^{\{s\}\prime}(U)$, $\mathcal{D}^{(s)\prime}(U)$, $\mathcal{E}^{\{s\}\prime}(U)$, $\mathcal{E}^{(s)\prime}(U)$, the strong dual spaces (called Gevrey-ultradistributions of Roumieu, respectively Beurling type) of the spaces $\mathcal{D}^{\{s\}}(U)$, $\mathcal{D}^{(s)}(U)$, $\mathcal{E}^{\{s\}}(U)$, $\mathcal{E}^{(s)}(U)$.

We then also have by duality continuous inclusions

$$\mathcal{E}^{*\prime}(U) \subset \mathcal{D}^{*\prime}(U). \tag{4}$$

As for integral operators, the following remark is easy to check (cf. [6]):

- assume $\mathcal{K} \in \mathcal{D}^{*\prime}(V \times U)$. Then the prescription $T(\varphi)(\psi) = \mathcal{K}(\psi \otimes \varphi)$ defines a linear continuous operator T from $\mathcal{D}^*(U)$ to $\mathcal{D}^{*\prime}(V)$.

We shall write this as

$$(T\varphi)(x) = \int_V \mathcal{K}(x,y)\varphi(y)dy, \ \varphi \in \mathcal{D}^*(U).$$

It is part of the results proved in [6], [7], [8], that also the converse is true:

THEOREM 1 (Komatsu). *a) Any linear continuous operator* $T : \mathcal{D}^*(U) \to \mathcal{D}^{*\prime}(V)$ *is of form* $T(\varphi)(\psi) = \mathcal{K}(\psi \otimes \varphi)$ *for some* $\mathcal{K} \in \mathcal{D}^{*\prime}(V \times U)$.

b) (See [8], page 655.) Any linear continuous operator $T : \mathcal{E}^{*\prime}(U) \to \mathcal{D}^{*\prime}(V)$ *is of form* $T(\varphi)(\psi) = \mathcal{K}(\psi \otimes \varphi)$ *for some* $\mathcal{K} \in \mathcal{D}^{*\prime}(V) \otimes_\varepsilon \mathcal{E}^*(U)$. *("$\otimes_\varepsilon$" is the ε-tensor product.)*

Before we can state our own results, we must still introduce the notions of Gevrey wave front sets. In order to justify them, we start from the following straightforward (and standard) result, which is in fact also central in the calculations:

REMARK 1 (See e.g., [6]). Let B be a closed ball in $\mathbb{R}^n$ (or $\mathbb{R}^m$; in the case $\mathbb{R}^m$, notation should be changed slightly).

There are constants $c > 0, c' > 0$, such that for $f \in C_0^\infty(B)$ we have

$$\sup_{\xi \in \mathbb{R}^n} |\hat{f}(\xi)| \exp[c'(|\xi|/L)^{1/s}] \le c|f|_{s,L,B}, \ |f|_{s,c'L,B} \le c \sup_{\xi \in \mathbb{R}^n} |\hat{f}(\xi)| \exp[(|\xi|/L)^{1/s}]. \tag{5}$$

"Hats" will denote the Fourier transform, which we define by

$$\hat{f}(\xi) = \mathcal{F} f(\xi) = \int_{\mathbb{R}^n} f(x) \exp[-i\langle x, \xi \rangle] dx.$$

The relation (5) is based on the following elementary inequality, which is valid for $d|\xi| \geq 1$:

(6) $$|\xi|^{|\alpha|} \exp[-d|\xi|^{1/s}] \leq |\xi|^{|\alpha|} \inf_{\beta} |\beta|^{|\beta|}/(d|\xi|^{1/s})^{|\beta|} \leq (4s)^{s|\alpha|} d^{-s|\alpha|} |\alpha|^{s|\alpha|};$$

the last inequality is obtained by evaluating the function $F(\beta) = |\beta|^{|\beta|} \times (d|\xi|^{1/s})^{-|\beta|}$ for $|\beta| = [s|\alpha|] + 1$, where $[s|\alpha|]$ is the integer part of $s|\alpha|$. (The factor "$4^{s|\alpha|}$" appears because of the "integer part".)
We have the following relations:

- A function $f \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ lies in $\mathcal{D}^{\{s\}}(\mathbb{R}^n)$ precisely if there are constants $c, d > 0$, such that $|\hat{f}(\xi)| \leq c \exp[-d|\xi|^{1/s}]$.
- A function $f \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ lies in $\mathcal{D}^{(s)}(\mathbb{R}^n)$ precisely if there is c and a function $\ell : \mathbb{R}^n \to \mathbb{R}_+$ such that

  (7) $$\forall d > 0, \exists c' \quad \text{s.t.} \quad d|\xi|^{1/s} \leq \ell(\xi) + c', \forall \xi \in \mathbb{R}^n,$$

  and such that

  (8) $$|\hat{f}(\xi)| \leq c \exp[-\ell(\xi)], \forall \xi \in \mathbb{R}^n.$$

- A real analytic functional u lies in $\mathcal{E}^{(s)\prime}(\mathbb{R}^n)$ if there are constants $c, d > 0$ such that $|\hat{u}(\xi)| \leq c \exp[d|\xi|^{1/s}]$.
- A real analytic functional u lies in $\mathcal{E}^{\{s\}\prime}(\mathbb{R}^n)$ if for every $d > 0$ there is a constant c such that $|\hat{u}(\xi)| \leq c \exp[d|\xi|^{1/s}]$.

A useful remark is the sub-additivity of the function $\xi \mapsto |\xi|^{1/s}$ for $s \geq 1$, that is,

(9) $$|\xi + \theta|^{1/s} \leq |\xi|^{1/s} + |\theta|^{1/s}, \quad \forall \xi, \theta \in \mathbb{R}^n.$$

We now introduce the wave front sets corresponding to the ultradistribution spaces considered above. (Cf., e.g., [4], [9], [14].)

DEFINITION 1. *a) Let $u \in \mathcal{D}^{(s)\prime}(U)$ and consider $(x^0, \xi^0) \in U \times \dot{\mathbb{R}}^n$. We shall say that $(x^0, \xi^0) \notin \mathrm{WF}_{(s)}(u)$, if we can find $\varepsilon > 0$, $v \in \mathcal{E}^{(s)\prime}(\mathbb{R}^n)$, an open convex cone Γ which contains ξ^0, $c > 0$ and a function ℓ as in (7) with the following properties:*

(10) $$u \equiv v \ \text{ on } \ |x - x^0| < \varepsilon, \quad |\hat{v}(\xi)| \leq c \exp[-\ell(\xi)] \ \text{ for } \ \xi \in \Gamma.$$

b) Let $u \in \mathcal{D}^{\{s\}\prime}(U)$. We shall say that $(x^0, \xi^0) \notin \mathrm{WF}_{\{s\}}(u)$, if we can find $\varepsilon > 0$, $v \in \mathcal{E}^{\{s\}\prime}(\mathbb{R}^n)$, an open convex cone Γ which contains ξ^0 and $c, d > 0$ such that

(11) $$u \equiv v \ \text{ on } \ |x - x^0| < \varepsilon, \quad |\hat{v}(\xi)| \leq c \exp[-d|\xi|^{1/s}] \ \text{ for } \ \xi \in \Gamma.$$

The $\mathrm{WF}_{\{s\}}(u)$, $\mathrm{WF}_{(s)}(u)$ are the Gevrey wave front sets of u of Roumieu, respectively Beurling, type with Gevrey index s.

We now state the main results.

THEOREM 2. *Let $V \times U$ be an open set in $\mathbb{R}^n \times \mathbb{R}^m$ and consider a linear continuous map $T : \mathcal{D}^{\{s\}}(U) \to \mathcal{D}^{\{s\}\prime}(V)$ given by some kernel $\mathcal{K} \in \mathcal{D}^{\{s\}\prime}(V \times U)$. Then the following statements are equivalent:*

i) T can be extended to a continuous and linear map $T : \mathcal{E}^{\{s\}\prime}(U) \to \mathcal{D}^{\{s\}\prime}(V)$.

ii) $\mathcal{K}$ satisfies the Gevrey wave front set condition of Roumieu type:

$$\mathrm{WF}_{\{s\}}(\mathcal{K}) \cap \{(x,y,0,\eta); \eta \neq 0\} = \emptyset.$$

THEOREM 3. *With V and U as before, consider a linear continuous map T : $\mathcal{D}^{(s)}(U) \to \mathcal{D}^{(s)\prime}(V)$ given by some kernel $\mathcal{K} \in \mathcal{D}^{(s)\prime}(V \times U)$. Then the following statements are equivalent:*

a) T can be extended to a continuous and linear map $T : \mathcal{E}^{(s)\prime}(U) \to \mathcal{D}^{(s)\prime}(V)$.

b) For every $(x^0,y^0) \in V \times U$ and for all $d > 0$, $\exists \varepsilon > 0$, $\exists c, \exists c_1$, and $\exists \mathcal{K}' \in \mathcal{E}^{(s)\prime}(V \times U)$ such that $\mathcal{K}' = \mathcal{K}$ on $|(x,y)-(x^0,y^0)| < \varepsilon$ and

$$|(\mathcal{F}\mathcal{K}')(\xi,\eta)| \leq c_1 \exp[-d|\eta|^{1/s}] \text{ for } |\xi| \leq c|\eta|. \tag{12}$$

REMARK 2. A comparison of condition b) in Theorem 3 with part a) of Definition 1 shows that $\mathrm{WF}_{(s)}(\mathcal{K}) \cap \{(x,y,0,\eta); \eta \neq 0\} = \emptyset$ implies b) in the theorem. We shall see later on that the converse is not true: there are kernels which satisfy condition b), but do not satisfy the wave front set condition $\mathrm{WF}_{(s)}(\mathcal{K}) \cap \{(x,y,0,\eta); \eta \neq 0\} = \emptyset$.

REMARK 3. Note that, taking into account Theorem 1, the conditions ii) and b) in the preceding theorems may be regarded as characterizations of the respective spaces $\mathcal{D}^{\{s\}\prime}(V) \otimes_\varepsilon \mathcal{E}^{\{s\}}(U)$ and $\mathcal{D}^{(s)\prime}(V) \otimes_\varepsilon \mathcal{E}^{(s)}(U)$, as subspaces of $\mathcal{D}^{\{s\}\prime}(V \times U)$ and $\mathcal{D}^{(s)\prime}(V \times U)$.

The following remark is immediate.

REMARK 4. Let $*$ denote (s) or $\{s\}$ with $s > 1$, and consider $\chi_1 \in \mathcal{D}^*(V)$, $\chi_2 \in \mathcal{D}^*(U)$. We denote by B_1 the support of χ_1 and by B_2 the support of χ_2. If $T : \mathcal{E}^{*\prime}(U) \to \mathcal{D}^{*\prime}(V)$ is a linear continuous operator, then so is $T_1 : \mathcal{E}^{*\prime}(B_2) \to \mathcal{E}^{*\prime}(B_1)$ defined by $T_1 u = \chi_1 T(\chi_2 u)$. Conversely, if all operators obtained in this way are continuous for some linear operator $T : \mathcal{E}^{*\prime}(U) \to \mathcal{D}^{*\prime}(V)$, then T is continuous. Note that if T corresponds to a kernel $\mathcal{K}(x,y)$, then T_1 corresponds to the kernel $\chi_1(x)\chi_2(y)\mathcal{K}(x,y)$. In view of this remark we may assume henceforth without loss of generality that $U = \mathbb{R}^m$, respectively that $V = \mathbb{R}^n$, and that

$$\operatorname{supp} \mathcal{K} \subset B' \times B, \tag{13}$$

for some closed balls $B \subset \mathbb{R}^m$, $B' \subset \mathbb{R}^n$.

REMARK 5. If $\mathcal{K} \in \mathcal{D}^{*\prime}(\mathbb{R}^{n+m})$ satisfies (13), then $\operatorname{supp} Tg \subset B'$ for every $g \in \mathcal{D}^*(\mathbb{R}^m)$. Conversely, if $\operatorname{supp} Tg \subset B'$ for every $g \in \mathcal{D}^*(\mathbb{R}^m)$, then $\operatorname{supp} \mathcal{K} \subset B' \times \mathbb{R}^m$.

3. Intermediate spaces and weight functions

In this section we define spaces which are intermediate between Roumieu and Beurling ultradistributions. We fix a closed ball B and consider for $f \in \mathcal{C}_0^\infty(\mathbb{R}^n)$, $u \in \mathcal{A}'(\mathbb{R}^n)$, a new set of quasinorms

$$\|f\|_{s,d} = \sup_{\xi\in\mathbb{R}^n} |\hat{f}(\xi)| \exp[d|\xi|^{1/s}],$$

$$\|u\|^{s,d} = \sup_{\xi\in\mathbb{R}^n} |\hat{u}(\xi)| / \exp[d|\xi|^{1/s}]. \tag{14}$$

(Here $\mathcal{A}'(\mathbb{R}^n)$ denotes the real-analytic functionals on $\mathbb{R}^n$.) Thus formally, $\|u\|^{s,d} = \|u\|_{s,-d}$, but the two quasinorms refer to different situations, so we wanted to make the difference visible also notationally.

DEFINITION 2. *We denote by $\mathcal{G}^{s,d}(B)$ the space of $\mathcal{C}^\infty$ functions u with support in B such that $\|u\|_{s,d} < \infty$, endowed with the norm $\|u\|_{s,d}$. In a similar way, we consider the space $\mathcal{G}_d^{s,\prime}(B)$ of ultradistributions u with support in B for which $\|u\|^{s,d} < \infty$, endowed with the norm $\|u\|^{s,d}$.*

Also note that, using the estimates (5), we have for suitable constants c', c'', the following continuous inclusions:

$$\mathcal{G}^{s,1/L^{1/s}}(B) \subset \mathcal{D}^{s,c'L}(B) \subset \mathcal{G}^{s,c''/L^{1/s}}(B), \text{ if } L > 0. \tag{15}$$

Thus (for fixed s) the spaces $\mathcal{G}^{s,d}(B)$ form a scale (indexed by $d > 0$) of function spaces which is essentially equivalent with the scale $\mathcal{D}^{s,L}(B)$. For example, we have

$$\mathcal{D}^{\{s\}}(B) = \varinjlim_{d>0} \mathcal{G}^{s,d}(B) \tag{16}$$

as locally convex spaces. (Also see [6].)

REMARK 6. When $f \in \mathcal{D}^{*\prime}(\mathbb{R}^n)$ has compact support and $g \in \mathcal{D}^*(\mathbb{R}^n)$, we can calculate $f(g)$ by $f(g) = (2\pi)^{-n} \int_{\mathbb{R}^n} \hat{f}(\xi)\hat{g}(-\xi)d\xi$, where the integral is the standard Lebesgue integral. (See [7].)

We now mention that $\mathcal{G}_d^{s,\prime}(B)$ is not defined as a dual space and, in some sense, the norms $\|u\|^{s,d}$ are not optimal for duality arguments. We now state a lemma that will help us to bypass this shortcoming. This is typically used for the cut-off multiplier $\chi \in \mathcal{D}^{(s)}(B'')$ for balls $B' \Subset B''$, satisfying $\chi \equiv 1$ on B'.

LEMMA 1. *Consider $\chi \in \mathcal{D}^{(s)}(\mathbb{R}^n)$, $d > 0$. Then the constants $c_1 := \|\chi\|_{s,d}$ and $c_2 := \|\hat{\chi}(\xi)\exp[d|\xi|^{1/s}]\|_{\mathcal{L}^1(\mathbb{R}^n)}$ are finite.*
a) Moreover, we have

$$\|\chi f\|_{s,d} \leq (2\pi)^{-n} c_1 \|\hat{f}(\xi)\exp[d|\xi|^{1/s}]\|_{\mathcal{L}^1(\mathbb{R}^n)}. \tag{17}$$

b) In a similar vein, we also have

$$\|\chi f\|_{s,d} \le (2\pi)^{-n} c_2 \|f\|_{s,d}.$$

c) Finally, if h is measurable,

$$|(\hat{\chi} * h)(\xi)| \le c_2 \|h(\xi)\exp[-d|\xi|^{1/s}]\|_{\mathcal{L}^\infty(\mathbb{R}^n)} \cdot \exp[d|\xi|^{1/s}].$$

Proof. The finiteness of the constants comes from (8). For a), we have

$$\begin{aligned}
&(2\pi)^n|\mathcal{F}(\chi f)(\xi)| \cdot \exp[d|\xi|^{1/s}] = \left|\int_{\mathbb{R}^n} \hat{\chi}(\xi-\theta)\hat{f}(\theta)d\theta\right| \cdot \exp[d|\xi|^{1/s}] \\
&\le \Big|\int_{\mathbb{R}^n} \hat{\chi}(\xi-\theta)\exp[d|\xi-\theta|^{1/s}] \cdot \hat{f}(\theta)\exp[d|\theta|^{1/s}] \\
&\qquad\qquad \times \exp[d|\xi|^{1/s} - d|\theta|^{1/s} - d|\xi-\theta|^{1/s}]d\theta\Big| \\
&\le \|\chi\|_{s,d} \cdot \|\hat{f}(\theta)\exp[d|\theta|^{1/s}]\|_{\mathcal{L}^1(\mathbb{R}^n)}.
\end{aligned}$$

Here we used the inequality $|\xi|^{1/s} \le |\xi-\theta|^{1/s} + |\theta|^{1/s}$. See (9).

Parts b) and c) are proved with a similar argument. □

A measurable and non-negative valued function on $\mathbb{R}^n$ is called a weight function. A weight function $\varphi(\xi)$ is said to be sub-linear if it satisfies

$$\sup_{\xi\in\mathbb{R}^n} (\varphi(\xi) - \varepsilon|\xi|) < +\infty, \quad \text{for any } \varepsilon > 0.$$

In this article, we only consider radial weight functions, and we say, by abuse of notation, that a weight function is increasing when it is an increasing function of $|\xi|$.

Now consider two sub-linear weight functions $\varphi, \psi : \mathbb{R}^n \to \mathbb{R}_+$ and assume that $\psi(\theta) - |\xi-\theta|^{1/s} \le \varphi(\xi) + c$, $\forall\xi$, $\forall\theta$, in $\mathbb{R}^n$. If $\chi \in \mathcal{D}^{(s)}(\mathbb{R}^n)$, then there exists a constant c' such that

$$\|(\hat{\chi} * h)e^{\psi}\|_{\mathcal{L}^1(\mathbb{R}^n)} \le c'\|he^{\varphi}\|_{\mathcal{L}^1(\mathbb{R}^n)} \tag{18}$$

holds for any measurable function h. Indeed, the left hand side of (18) is estimated from above by

$$\begin{aligned}
&\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |\hat{\chi}(\theta-\xi)h(\xi)|\exp[\psi(\theta)]d\xi d\theta \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{\psi(\theta)-|\xi-\theta|^{1/s}-\varphi(\xi)} \cdot |\hat{\chi}(\theta-\xi)|e^{|\xi-\theta|^{1/s}} \cdot |h(\xi)|e^{\varphi(\xi)}d\xi d\theta \\
&\le e^c\|\hat{\chi}(\theta)e^{|\theta|^{1/s}}\|_{\mathcal{L}^1(\mathbb{R}^n)} \cdot \|he^{\varphi}\|_{\mathcal{L}^1(\mathbb{R}^n)}.
\end{aligned}$$

REMARK 7. Our next lemma is similar to Lemma 1, c), but is more abstract and therefore less precise. We also mention that in the proof of the lemma we consider Lebesgue-spaces associated with weights. We briefly recall the terminology. We assume that we are given a continuous weight function $\varphi : \mathbb{R}^n \to \mathbb{R}_+$ and say that two measurable functions on $\mathbb{R}^n$ are equivalent if they are equal except on a set of Lebesgue measure zero. Then we denote by $\mathcal{L}^1(\mathbb{R}^n, \varphi)$ the space of equivalence classes of measurable functions on $\mathbb{R}^n$ for which the integral $\int_{\mathbb{R}^n} |f(\xi)| \exp[\varphi(\xi)] d\xi$ is finite. The norm on this space is of course

$$f \mapsto \|f\|_{\mathcal{L}^1,\varphi} = \int_{\mathbb{R}^n} |f(\xi)| \exp[\varphi(\xi)] d\xi. \tag{19}$$

If $L : \mathcal{L}^1(\mathbb{R}^n, \varphi) \to \mathbb{C}$ is a linear continuous map, then there is a measurable function h defined on $\mathbb{R}^n$ such that $L(f) = \int_{\mathbb{R}^n} f(\xi)h(\xi)d\xi$, $\forall f \in \mathcal{L}^1(\mathbb{R}^n, \varphi)$ and we have $|h(\xi)| \le \|L\|_1 \exp[\varphi(\xi)]$, for almost all $\xi \in \mathbb{R}^n$, where $\|L\|_1$ is the norm of L as a functional on $\mathcal{L}^1(\mathbb{R}^n, \varphi)$.

LEMMA 2. *Let $B' \Subset B''$ be two balls in $\mathbb{R}^n$, χ a function in $\mathcal{D}^{(s)}(B'')$ satisfying $\chi \equiv 1$ on B', and $d > 0$. Consider two sub-linear weight functions $\varphi, \psi : \mathbb{R}^n \to \mathbb{R}_+$. Assume that*

$$\int_{\mathbb{R}^n} |\mathcal{F}(\chi f)(\xi)| \exp[\psi(\xi)] d\xi \le c_1 \int_{\mathbb{R}^n} |\hat{f}(\xi)| \exp[\varphi(\xi)] d\xi, \forall f \in \mathcal{L}^2(\mathbb{R}^n), \tag{20}$$

for some constant c_1, provided the right hand side in (20) is finite. Also denote by $\mathcal{N}(B'', \psi)$ the set

$$\mathcal{N}(B'', \psi) := \{g \in \mathcal{D}^{(s)}(B''); \int_{\mathbb{R}^n} |\hat{g}(\xi)| \exp[\psi(\xi)] d\xi \le 1\}.$$

Then there is a constant c_2 such that for any $v \in \mathcal{D}^{(s)\prime}(\mathbb{R}^n)$ with $\operatorname{supp} v \subset B'$ *we have that*

$$|\hat{v}(\xi)| \le c_2 \exp[\varphi(-\xi)] \sup_{g \in \mathcal{N}(B'', \psi)} |v(g)|.$$

Proof. We define the spaces Z and Y, $Y \subset Z$, by

$$Z = \{f \in \mathcal{L}^2(\mathbb{R}^n); \int_{\mathbb{R}^n} |\hat{f}(\xi)| \exp[\varphi(\xi)] d\xi < \infty\},$$
$$Y = \{f \in Z; \|f\|_{s,d'} < \infty \text{ for all } d'\}.$$

It is easy to see that Y is dense in Z if the latter is endowed with the norm defined by $f \mapsto \|\hat{f}\|_{\mathcal{L}^1,\varphi}$: if f is given in Z, then $k \mapsto f_k = \mathcal{F}^{-1}(\exp[-(1/k)|\xi|]\hat{f})$, $k = 1, 2, \ldots$ is a sequence of functions in Y which approximates f. Now, $Y \subset \mathcal{E}^{(s)}(\mathbb{R}^n)$ and we also observe that if $\mu \in C_0^\infty(\mathbb{R}^n)$, then $\mathcal{F}^{-1}\mu \in Z$.

It suffices to construct c_2 such that

$$|\hat{v}(\xi)| \le c_2 \exp[\varphi(-\xi)]$$

holds for any $v \in \mathcal{D}^{(s)\prime}(\mathbb{R}^n)$ satisfying $\operatorname{supp} v \subset B'$ and

$$\sup_{g \in \mathcal{N}(B'', \psi)} |v(g)| \le 1. \tag{21}$$

Now we fix such a v and consider the functional $f \mapsto v(\chi f)$, which is initially defined on $\mathcal{E}^{(s)}(\mathbb{R}^n)$. For $f \in Y$, we have

$$|v(\chi f)| \le \int_{\mathbb{R}^n} |\mathcal{F}(\chi f)(\xi)| \exp[\psi(\xi)] d\xi \le c_1 \|\hat{f}\|_{\mathcal{L}^1,\varphi},$$

where the first inequality follows from (21), and the second from (20). Therefore, this functional can be extended, by continuity, to a linear continuous functional L on Z. Next we introduce the space $\hat{Z} = \{f \in \mathcal{L}^2(\mathbb{R}^n); \int_{\mathbb{R}^n} |f(\xi)| \exp[\varphi(\xi)] d\xi < \infty\}$, which is the image of Z under the Fourier transform. We endow $\hat{Z}$ with the norm $f \mapsto \|f\|_{\mathcal{L}^1,\varphi}$; this is of course the norm induced by the norm of Z if we use the Fourier transform to identify Z and $\hat{Z}$. The map L gives rise in this way to a linear continuous map $\hat{L}$ on $\hat{Z}$ defined by $\hat{L}(f) = L(\mathcal{F}^{-1} f)$.

Finally, we can apply the Hahn-Banach theorem to extend $\hat{L}$ to a linear continuous map defined on the space $\mathcal{L}^1(\mathbb{R}^n, \varphi)$ introduced in Remark 7, with the norm not greater than c_1. (Instead of applying the Hahn-Banach theorem, we can also use the density of Z in $\mathcal{L}^1(\mathbb{R}^n, \varphi)$.) It follows therefore from Remark 7, that $\hat{L}$ is of form $\hat{L}(f) = \int_{\mathbb{R}^n} \hat{f}(\xi) h(\xi) d\xi$, for some suitable measurable function h on $\mathbb{R}^n$ which satisfies $|h(\xi)| \le c_1 \exp[\varphi(\xi)]$ for almost all ξ. The proof of the lemma will come to an end if we can show that $\hat{v}(\xi) = (2\pi)^n h(-\xi)$. This is the case, since

$$\begin{aligned} \int_{\mathbb{R}^n} \mu(\xi) h(\xi) d\xi &= \hat{L}(\mu) = L(\mathcal{F}^{-1}\mu) = v(\chi \mathcal{F}^{-1}\mu) = v(\mathcal{F}^{-1}\mu) \\ &= (2\pi)^{-n} \int_{\mathbb{R}^n} \hat{v}(-\xi) \mu(\xi) d\xi \end{aligned}$$

for $\mu \in C_0^\infty(\mathbb{R}^n)$, which means that $h(\xi)$ and $(2\pi)^{-n}\hat{v}(-\xi)$ coincide as distributions. Here we have used the fact that $\operatorname{supp} v \subset B'$ and that $\chi \equiv 1$ on B''. □

COROLLARY 1. *There is a constant c' for which we have the following implication for $v \in \mathcal{D}^{(s)\prime}(\mathbb{R}^n)$ satisfying* $\operatorname{supp} v \subset B'$*:*

(22) $\quad |v(f)| \le 1$ *for all* $f \in \mathcal{D}^{(s)}(B'')$ *with* $\|f\|_{s,d} \le 1$, *implies* $\|v\|^{s,2d} \le c'$.

In other words, the quasinorm $v \mapsto \|v\|^{s,2d}$ can be estimated from above by the inequality

$$\|v\|^{s,2d} \le c' \sup_{f \in \mathcal{M}} |v(f)|$$

using the bounded set $\mathcal{M} = \{f \in \mathcal{D}^{(s)}(B''); \|f\|_{s,d} \le 1\}$ in $\mathcal{D}^{(s)}(B'')$, and a constant c' depending only on B', B'', and d. Since, in the opposite direction, we have

$$\sup_{f \in \mathcal{M}} |v(f)| \le c'' \|v\|^{s,d/2}$$

for some constant c'' independent of v, it is clear that the topology induced on $\mathcal{E}^{*\prime}(B)$ as a subspace of $\mathcal{D}^{*\prime}(\mathbb{R}^n)$ is given as the inductive/projective limit of the spaces $\mathcal{G}_d^{s,\prime}(B)$.

The corollary follows from Lemma 2, if we also take into account Lemma 1.

REMARK 8. The statement in the corollary is meaningful also for $v \in \mathcal{D}^{\{s\}\prime}(\mathbb{R}^n)$. In this case, we know from the very beginning that there is a constant c', which may depend on v, with $\|v\|^{s,2d} \le c'$, and the lemma just gives an estimate by duality of the norm $\|v\|^{s,2d}$.

We now consider a sequence of numbers C_j which satisfies the condition

$$j^2 \le C_j, \tag{23}$$

(other conditions on the constants C_j will be introduced in a moment) and denote by ℓ the (increasing) function

$$\ell(\xi) = \sup_j (j|\xi|^{1/s} - C_j). \tag{24}$$

REMARK 9. a) The function ℓ is well-defined since $j|\xi|^{1/s} - C_j$ is negative for $|\xi| < j$. (This implies that the "sup" is finite for every ξ.) Somewhat more specifically, $j|\xi|^{1/s} - C_j \le -j(j - |\xi|^{1/s})$ tends to $-\infty$ for $j \to \infty$ when ξ is fixed, and therefore we also see that actually, $\ell(\xi) = \max_j (j|\xi|^{1/s} - C_j)$, i.e., the "sup" is actually a "max".

b) The function ℓ clearly satisfies (7).

c) Assume now that C_j also satisfies

$$C_j \ge 4C_{[j/2]+1}, [j/2] \text{ the integer part of } j/2. \tag{25}$$

Since $k|\xi|^{1/s} - C_k \le 4\left(([k/2]+1)|\xi/2|^{1/s} - C_{[k/2]+1}\right)$, we then also have

$$\ell(\xi) \le 4\ell(\xi/2). \tag{26}$$

We recall here the fact that when one defines function spaces by inequalities of type $|\hat{f}(\xi)| \le \exp[\varphi(\xi)]$, then conditions of type $\varphi(\xi) \le c\varphi(\xi/2)$ are used (for increasing weight functions) in relation to the requirement that the function space be stable under multiplication. (When the weight functions are not increasing, the formulation of the corresponding condition is somewhat more involved. We shall not use c) in this paper. Also cf. the "ring condition" in [12].)

The condition (23) is needed to show that the function ℓ is finite. We now put further conditions on the constants C_j to show that we can make ℓ sub-linear and Lipschitz-continuous. We should mention that while the fact that ℓ is sub-linear is essential, the fact that it is Lipschitz continuous is not strictly needed in this paper. Lipschitzianity is however needed as soon as one wants to develop a theory of pseudodifferential and Fourier integral operators in spaces related to weight functions and therefore we show also in this paper that we can choose the functions ℓ with this property. (See [12].)

LEMMA 3. *a) Consider a sequence of constants $C_j \ge j^2$, and define a function $\tilde{\rho} : \mathbb{R}_+ \to \mathbb{R}_+$ by*

$$\tilde{\rho}(\tau) = \begin{cases} \sup_j (j\tau^{1/s} - C_j), & (\tau \ge 1) \\ \sup_j (j - C_j), & (\tau < 1). \end{cases} \tag{27}$$

Then $\tilde{\rho}$ is finite. If C_j tends to infinity quick enough and is suitably chosen, then $\tilde{\rho}$ is sub-linear and Lipschitz. Moreover, we may assume that if $s' > s$ is fixed, then

$$\lim_{\tau\to\infty} \tilde{\rho}(\tau)/\tau^{1/s'} = 0. \tag{28}$$

b) Let $\tilde{\rho}$ be as in the conclusion of part a) and denote by $\rho : \mathbb{R}^n \to \mathbb{R}_+$ the function $\rho(\xi) = \tilde{\rho}(|\xi|)$. Then ρ is sub-linear and Lipschitz.

Proof. We choose a sequence of positive numbers $M_j \searrow 0$, with $M_1 = 1$. Further, we iteratively define numbers τ_j, $j \geq 0, C_j \geq j^2$, $j \geq 1$, and functions ρ_j with the following properties:

- $\tau_0 = 0, C_1 = 0, \rho_1(\tau) = \tau^{1/s}$,
- $\rho_j(\tau) = j\tau^{1/s} - C_j$,
- the sequence $j \mapsto (C_{j+1} - C_j)$ is strictly increasing,
- $\tau_j^{1/s} = C_{j+1} - C_j$, and therefore also the sequence $j \mapsto \tau_j$ is strictly increasing,
- $j(1/s)\tau^{-1+1/s} = \rho_j'(\tau) \leq M_j$ on $[\tau_{j-1}, \infty)$,
- $\rho_j(\tau) \geq \rho_{j-1}(\tau)$ for $\tau \geq \tau_{j-1}$, $\rho_j(\tau) \leq \rho_{j-1}(\tau)$ for $\tau \leq \tau_{j-1}$.

As a preparation for this, we notice that, independently of the way we choose the constants C_j, we shall have $\rho_j'(\tau) \geq \rho_{j+1}'(\tau)$, $\forall\tau$. Therefore, if τ_j is chosen with $\rho_j(\tau_j) = \rho_{j+1}(\tau_j)$, then we have $\rho_j(\tau) \geq \rho_{j+1}(\tau)$ for $\tau \leq \tau_j$, respectively $\rho_j(\tau) \leq \rho_{j+1}(\tau)$ for $\tau \geq \tau_j$. We now return to the construction of the C_j, τ_j. Note that, by our requirements, we have to set $\tau_0 = 0, C_1 = 0$. We next note that the functions $\rho_j(\tau)$ are concave and $\rho_2(\tau) = 2\tau^{1/s} - C_2$ is negative for $\tau > 0$ small, whatever the value of $C_2 > 0$ may be, whereas ρ_1 is positive. Moreover, when C_2 increases so does τ_1 given by $\tau_1^{1/s} = C_2 - C_1 = C_2$ and we fix some $C_2 \geq 2^2$ so that $2(1/s)\tau_1^{-1+1/s} \leq M_2$. This already defines ρ_2 by $\rho_2(\tau) = 2\tau^{1/s} - C_2$, and it is automatic that $\rho_2'(\tau) \leq M_2$ for every $\tau \geq \tau_1$. We may now assume that we have found C_j, τ_{j-1} and have set $\rho_j = j\tau^{1/s} - C_j$. In particular, $\rho_j(\tau) \geq \rho_{j-1}(\tau)$ for $\tau \geq \tau_{j-1}$ and $\rho_j'(\tau) \leq M_j$ for $\tau \geq \tau_j$. Next we fix $C_{j+1} \geq (j+1)^2$, large enough such that for $\tau_j^{1/s} = C_{j+1} - C_j$ we have $j(1/s)\tau_j^{-1+1/s} \leq M_j$ and set $\rho_{j+1}(\tau) = (j+1)\tau^{1/s} - C_{j+1}$.

This concludes the construction of the numbers τ_j, C_j, ρ_j by iteration. If we also want to have (28), then it suffices to choose τ_{j-1} so that $j(1/s)\tau^{1/s-1} \leq M_j\tau^{1/s'-1}$ on $[\tau_{j-1}, \infty)$.

It follows for these choices that

$$\sup_k \rho_k(\tau) = \rho_j(\tau) \text{ on } [\tau_j, \tau_{j+1}] \text{ for } \tau \geq 1, \tag{29}$$

and we set $\tilde{\rho}(\tau) = \sup_k \rho_k(\tau)$ for such τ.

This shows that $\tilde{\rho}'(\tau) \to 0$ there where the derivative is defined (which is except the points $\tau = \tau_j$) when $\tau \to \infty$. The sub-linearity and the Lipschitz-continuity of ρ is then clear, so part a) of the lemma is proved. Part b) is an immediate consequence. □

LEMMA 4. *Let $\tilde{\ell} : \mathbb{R}^n \to \mathbb{R}_+$ be a function which satisfies (7) and denote $\mathcal{M} = \{f \in \mathcal{D}^{(s)}(B_2); \int_{\mathbb{R}^n} |\hat{f}(\xi)| \exp[\tilde{\ell}(\xi)] d\xi \leq 1\}$. Then $\mathcal{M}$ is a bounded set in $\mathcal{D}^{(s)}(\mathbb{R}^n)$.*

We apply this for "$\tilde{\ell} = \ell'/2$", where ℓ' will be constructed later on.

Proof. In view of the support condition in the definition of $\mathcal{M}$, we only need to estimate the derivatives of the elements in $\mathcal{M}$, and in fact show that for every j there is a constant $\tilde{c}_j$ such that $|(\partial/\partial x)^\alpha f(x)| \leq \tilde{c}_j j^{-s|\alpha|} |\alpha|^{s|\alpha|}$, for $f \in \mathcal{M}$. We write for this purpose for fixed j, α,

$$
\begin{aligned}
j^{s|\alpha|} \left| \frac{\partial^\alpha}{\partial x^\alpha} f(x) \right| &\leq j^{s|\alpha|} \sup_{\xi \in \mathbb{R}^n} |\xi^\alpha| \exp[-sj|\xi|^{1/s}] \cdot \int_{\mathbb{R}^n} |\hat{f}(\xi)| \exp[sj|\xi|^{1/s}] d\xi \\
&\leq |\alpha|^{s|\alpha|} \int_{\mathbb{R}^n} |\hat{f}(\xi)| \exp[\ell(\xi) + \ln c_j] d\xi \\
&\leq \tilde{c}_j |\alpha|^{s|\alpha|},
\end{aligned}
$$

since $|\xi^\alpha| \exp[-sj|\xi|^{1/s}] \leq c' j^{-s|\alpha|} |\alpha|^{s|\alpha|}$, $\forall \alpha \in \mathbb{N}^n$. (See, e.g., the argument for studying (6). The point is that by analogy, $\exp[sj|\xi|^{1/s}] \geq (sj|\xi|^{1/s})^{s|\alpha|}/(s|\alpha|)^{|\alpha|}$. In the second inequality we have used $sj|\xi|^{1/s} \leq \ell(\xi) + \ln c_j$ for some constants c_j.) □

PROPOSITION 1. *Fix $\chi \in \mathcal{D}^{(s)}(B)$, and consider sequences of constants C_j, C'_j. Assume that $\int_{\mathbb{R}^n} |\hat{f}(\xi)| \exp[2j|\xi|^{1/s}] d\xi \leq C_{2j}$ implies $\|\chi f\|_{s,j} \leq C'_j$. (See Lemma 1.) Assume further that both sequences satisfy the condition $\ln C_j \geq j^2, \ln C'_j \geq j^2$.*

We now denote by $\ell, \ell' : \mathbb{R}^n \to \mathbb{R}_+$ the functions $\ell(\xi) = \sup_j (j|\xi|^{1/s} - \ln C_j)$, $\ell'(\xi) = \sup_j (j|\xi|^{1/s} - \ln C'_j)$. Then we have that

$$
|\mathcal{F}(\chi f)(\xi)| \leq \exp[-\ell'(\xi)] \cdot \|\hat{f}(\xi) \exp \ell(\xi)\|_{L^1(\mathbb{R}^n)} \tag{30}
$$

and

$$
\int_{\mathbb{R}^n} |\mathcal{F}(\chi f)(\xi)| \exp[\ell'(\xi)/2] d\xi \leq \|\hat{f}(\xi) \exp \ell(\xi)\|_{L^1(\mathbb{R}^n)} \cdot \int_{\mathbb{R}^n} \exp[-\ell'(\xi)/2] d\xi.
$$

Proof. It suffices to argue for the case $\|\hat{f}(\xi) \exp \ell(\xi)\|_{L^1(\mathbb{R}^n)} = 1$. Thus, f satisfies $\|\hat{f}(\xi) \exp[j|\xi|^{1/s} - \ln C_j]\|_{L^1(\mathbb{R}^n)} \leq 1$, so it follows from the assumption on C_j, that $\|\mathcal{F}(\chi f)(\xi)\|_{s,j} \leq C'_j$ for every j. This shows that

$$
|\mathcal{F}(\chi f)(\xi)| \leq \inf_j \exp[-j|\xi|^{1/s} + \ln C'_j] = \exp[-\ell'(\xi)].
$$

Since $\exp[-\ell'(\xi)/2]$ is integrable, we also obtain the last inequality. □

4. Kernels and the spaces $\mathcal{G}^s$

It seems natural to study the integral operator $Tu(x) = \int_{\mathbb{R}^m} \mathcal{K}(x,y)u(y)dy$ in the frame of the spaces $\mathcal{G}^s$. The conditions which we use for $\mathcal{K}$ in this section are motivated by the following considerations:

- let $\mathcal{K} \in \mathcal{D}^{(s)\prime}(\mathbb{R}^{n+m})$ have compact support. Then there is $d > 0$ and $c > 0$ such that

  (31) $$|\hat{\mathcal{K}}(\xi,\eta)| \le c\exp[d(|\xi|^{1/s} + |\eta|^{1/s})], \forall(\xi,\eta) \in \mathbb{R}^{n+m}.$$

- From (31), condition b) in Theorem 3 is equivalent to the following:

  (32) $$\forall d'', \exists d' > 0, \exists c, \text{ s.t. } |\hat{\mathcal{K}}(\xi,\eta)| \le c\exp[d'|\xi|^{1/s} - d''|\eta|^{1/s}].$$

Most of our arguments are based on the following simple relation:

(33) $$\mathcal{K}(\psi) = (2\pi)^{-n-m}\int_{\mathbb{R}^{n+m}} \hat{\mathcal{K}}(\xi,\eta)\hat{\psi}(-\xi,-\eta)d\xi d\eta, \psi \in \mathcal{D}^{(s)}(\mathbb{R}^{n+m}),$$

the integral being the Lebesgue integral as above. (See Remark 6.) It follows that

(34) $$\mathcal{F}(Tg)(\xi) = (2\pi)^{-m}\int_{\mathbb{R}^m} \hat{\mathcal{K}}(\xi,\eta)\hat{g}(-\eta)d\eta.$$

PROPOSITION 2. *a) Let $\mathcal{K} \in \mathcal{D}^{(s)\prime}(\mathbb{R}^{n+m})$ satisfy (13) and assume that (31) holds for some $d > 0$. Also consider $\tilde{d} > d$. Then*

$$(\varphi,u) \mapsto (Tu)(\varphi) := \int_{\mathbb{R}^{n+m}} \hat{\mathcal{K}}(\xi,\eta)\hat{\varphi}(-\xi)\hat{u}(-\eta)d\xi d\eta,$$

for $\varphi \in \mathcal{D}^{(s)}(\mathbb{R}^n)$ defines a continuous operator $T : \mathcal{G}^{s,\tilde{d}}(\mathbb{R}^m) \to \mathcal{G}_d^{s,\prime}(B')$.

b) Let $\mathcal{K}$ be a ultradistribution with support in $B' \times B$ which satisfies the estimate

(35) $$|\hat{\mathcal{K}}(\xi,\eta)| \le \exp[d_1|\xi|^{1/s} - d_2|\eta|^{1/s}], \text{ for some constants } d_1 > 0, d_2 > 0.$$

Also fix $d_3 < d_2$, $B_1 \Supset B$. Then the correspondence

$$g \mapsto Tg(x) := \int_U \mathcal{K}(x,y)g(y)dy,$$

for $g \in \mathcal{D}^{(s)}(B_1)$, can be extended to a continuous operator $\mathcal{G}_{d_3}^{s,\prime}(B) \to \mathcal{G}_{d_1}^{s,\prime}(B')$.

Proof. We only prove b). (Part a) is proved by similar arguments but is even simpler.) We have already observed in Remark 5 that $\operatorname{supp} Tg \subset B'$. When $g \in \mathcal{D}^{(s)}(B_1)$, then $\mathcal{F}(Tg)(\xi)$ is given (34). We claim that we have for some $c > 0$ the estimate

(36) $$\|Tg\|^{s,d_1} \le c\|g\|^{s,d_3}, \forall g \in \mathcal{D}^{(s)}(B_1).$$

To prove this, we just argue as follows:

$$\Big|\int_{\mathbb{R}^m} \hat{\mathcal{K}}(\xi,\eta)\hat{g}(-\eta)d\eta\Big|$$
$$\leq \exp[d_1|\xi|^{1/s}]\int_{\mathbb{R}^m} \exp[-d_1|\xi|^{1/s}+d_2|\eta|^{1/s}]|\hat{\mathcal{K}}(\xi,\eta)|\exp[-d_2|\eta|^{1/s}]|\hat{g}(-\eta)|d\eta$$

and notice that

$$\int_{\mathbb{R}^m} \exp[-d_2|\eta|^{1/s}]|\hat{g}(-\eta)|d\eta \leq \|g\|^{s,d_3}\int_{\mathbb{R}^m} \exp[(-d_2+d_3)|\eta|^{1/s}]d\eta.$$

We have now proved (36) and can conclude the argument by observing that we can approximate elements in $\mathcal{G}^{s,\prime}_{d_3}(B)$ with functions in $\mathcal{D}^{(s)}(B_1)$ by convolution: we fix $\kappa \in \mathcal{D}^{(s)}(y;|y|\leq 1)$ with $\hat{\kappa}(0)=1$ and approximate $\hat{u}$ by $\hat{\kappa}(\eta/j)\hat{u}$. We have then for j large that $\mathcal{F}^{-1}(\hat{\kappa}(\cdot/j))*u \in \mathcal{D}^{(s)}(B_1)$ and that $\sup_\eta \exp[-d_3|\eta|^{1/s}]|(1-\hat{\kappa}(\eta/j))\hat{u}(\eta)| \to 0$ as $j\to\infty$. □

REMARK 10. The proposition gives in particular the implications ii)⇒i) in Theorem 2 and b)⇒a) in Theorem 3. See Remark 4 and Corollary 1.

To establish the remaining implications in the theorems 2, 3, we first prove a lemma (part of which will be used only in section 6):

LEMMA 5. *Let $\chi \in \mathcal{D}^{(s)}(B')$, $\kappa \in \mathcal{D}^{(s)}(B)$ and fix L, d. Then there is $c>0$ such that*

$$|\exp[-i\langle x,\xi\rangle]|_{s,L,B} = \sup_\alpha \frac{|\xi^\alpha|}{L^{|\alpha|}(\alpha!)^s} \leq \exp[c|\xi|^{1/s}/L^{1/s}], \tag{37}$$

$$\|\chi(x)\exp[-i\langle x,\xi\rangle]\|_{s,d} \leq \|\chi\|_{s,d}\exp[d|\xi|^{1/s}], \tag{38}$$

and

$$\|\kappa(y)\exp[-i\langle y,\eta\rangle]\|^{s,d} \leq \|\kappa\|_{s,d}\exp[-d|\eta|^{1/s}]. \tag{39}$$

Note that (39) is an estimate referring to the spaces $\mathcal{G}^{s,\prime}_d$, although the function $y\mapsto \kappa(y)\exp[-i\langle y,\eta\rangle]$ lies in $\mathcal{D}^{(s)}(\mathbb{R}^m)$.

Proof. (37) is a direct calculation.

For (38) we have to calculate $\sup_\theta |\mathcal{F}(\chi\exp[-i\langle x,\xi\rangle])(\theta)|\exp[d|\theta|^{1/s}]$. Since $\mathcal{F}(\chi\exp[-i\langle x,\xi\rangle])(\theta) = \hat{\chi}(\theta+\xi)$, it suffices to observe that

$$\sup_\theta |\hat{\chi}(\theta+\xi)|\exp[d|\theta|^{1/s}] \leq \sup_\theta \|\chi\|_{s,d}\exp[d|\theta|^{1/s}-d|\theta+\xi|^{1/s}]$$
$$\leq \|\chi\|_{s,d}\exp[d|\xi|^{1/s}],$$

where we used $|\theta|^{1/s} \le |\theta+\xi|^{1/s} + |\xi|^{1/s}$. As for (39), we can argue similarly as

$$\begin{aligned}\|\kappa(y)\exp[-i\langle y,\eta\rangle]\|^{s,d} &= \sup_\theta |\hat{\kappa}(\theta+\eta)|\exp[-d|\theta|^{1/s}] \\ &\le \sup_\theta \|\kappa\|_{s,d}\exp[-d|\theta+\eta|^{1/s} - d|\theta|^{1/s}] \\ &\le \|\kappa\|_{s,d}\exp[-d|\eta|^{1/s}].\end{aligned}$$

Here we again used (9). □

We can now prove the following converse to part b) in Proposition 2:

PROPOSITION 3. *Let $\mathcal{K}$ be a ultradistribution with support in $B' \times B$. Denote by T the operator $Tu(x) = \int_{\mathbb{R}^m} \mathcal{K}(x,y)u(y)dy$. Assume that there are constants c, d_1, d_2 and balls B_1, B_2, $B \Subset B_1, B' \Subset B_2$, such that T can be extended to a continuous operator $\mathcal{G}^{s,\prime}_{d_1}(B_1) \to \mathcal{G}^{s,\prime}_{d_2}(B_2)$. Then the Fourier transform of $\mathcal{K}$ satisfies the estimate*

$$|\hat{\mathcal{K}}(\xi,\eta)| \le c_1 \exp[d_2|\xi|^{1/s} - d_1|\eta|^{1/s}]. \tag{40}$$

In particular, we have $|\hat{\mathcal{K}}(\xi,\eta)| \le c_1 \exp[-d_1|\eta|^{1/s}/2]$ if $|\xi| \le d_1|\eta|/(2d_2)$.

Proof. We shall obtain (40) starting from the estimate

$$\|T(\kappa(y)\exp[-i\langle y,\eta\rangle])\|^{s,d_2} \le c_2\|\kappa(y)\exp[-i\langle y,\eta\rangle]\|^{s,d_1},$$

where $\kappa \in \mathcal{D}^{(s)}(B_1)$ is identically 1 on B. On the other hand, by fixing $\chi \in \mathcal{D}^{(s)}(B_2)$ identically one on B', we have that

$$\begin{aligned}&\|T(\kappa(y)\exp[-i\langle y,\eta\rangle])\|^{s,d_2} \\ &= \sup_\xi \exp[-d_2|\xi|^{1/s}]|\mathcal{F}_{x\to\xi}(T(\kappa(y)\exp[-i\langle y,\eta\rangle]))(\xi)| \\ &= \sup_\xi \exp[-d_2|\xi|^{1/s}]|\mathcal{K}(\chi(x)\kappa(y)\exp[-i\langle x,\xi\rangle - i\langle y,\eta\rangle])| \\ &= \sup_\xi \exp[-d_2|\xi|^{1/s}]|\hat{\mathcal{K}}(\xi,\eta)|.\end{aligned}$$

The last equality follows from the fact that $\chi(x)\kappa(y)$ is identically one on the support of $\mathcal{K}$. By applying (39), we now obtain that

$$\sup_\xi \exp[-d_2|\xi|^{1/s}]|\hat{\mathcal{K}}(\xi,\eta)| \le c_3 \exp[-d_1|\eta|^{1/s}],$$

which is the estimate we wanted to prove. □

There is a result dual to Proposition 3 which we now consider.

PROPOSITION 4. *Let $\mathcal{K}$ be as in the previous proposition and assume that the map $S : \mathcal{G}^{s,d_1}(B_2) \to \mathcal{G}^{s,\prime}_{d_2}(B_1)$ such that*

$$\mathcal{F}(S\varphi)(\eta) = \int \hat{\mathcal{K}}(-\xi,\eta)\hat{\varphi}(\xi)d\xi$$

maps $\mathcal{G}^{s,d_1}(B_2)$ to $\mathcal{G}^{s,d_3}(B_1)$ and is continuous as a map $\mathcal{G}^{s,d_1}(B_2) \to \mathcal{G}^{s,d_3}(B_1)$. Then there is c such that

$$|\hat{\mathcal{K}}(-\xi,\eta)| \leq c \exp[d_1|\xi|^{1/s} - d_3|\eta|^{1/s}]. \tag{41}$$

REMARK 11. Proposition 4 can be reduced to Proposition 3 by tricks, but the proof is rather simple and does not seem worth the effort this would require.

Proof of Proposition 4. Continuity of S means that there is a constant c' such that

$$\|S\varphi\|_{s,d_3} \leq c'\|\varphi\|_{s,d_1}, \forall\varphi \in \mathcal{G}^{s,d_1}(B_2). \tag{42}$$

We shall apply this for the family of functions $\varphi_{\tilde{\xi}}$ defined by

$$\varphi_{\tilde{\xi}}(x) := \chi(x)e^{-i\langle x,\tilde{\xi}\rangle},$$

where $\chi \in \mathcal{D}^{(s)}(V)$ is a fixed function with the property that $\chi \equiv 1$ on B'. Note that then $\hat{\varphi}_{\tilde{\xi}}(\xi) = \hat{\chi}(\xi+\tilde{\xi})$, so we also have $\mathcal{F}(S\varphi_{\tilde{\xi}})(\eta) = \int \hat{\mathcal{K}}(-\xi,\eta)\hat{\varphi}_{\tilde{\xi}}(\xi)d\xi = \int \hat{\mathcal{K}}(-\xi,\eta)\hat{\chi}(\xi+\tilde{\xi})d\xi$. Now, since $\chi \equiv 1$ on B', $\mathcal{F}(S\varphi_{\tilde{\xi}})(\eta)$ is just $\hat{\mathcal{K}}(\tilde{\xi},\eta)$. It follows from the continuity of S that

$$\sup_\eta |\hat{\mathcal{K}}(\tilde{\xi},\eta)| \exp[d_3|\eta|^{1/s}] \leq c'\|\varphi_{\tilde{\xi}}\|_{s,d_1}. \tag{43}$$

We can also write this as

$$|\hat{\mathcal{K}}(\tilde{\xi},\eta)| \leq c \exp[d_1|\tilde{\xi}|^{1/s} - d_3|\eta|^{1/s}], \tag{44}$$

if we also use (38) for $\|\varphi_{\tilde{\xi}}\|_{s,d_1}$. □

5. Proof of Theorem 3

In this section we apply Proposition 3 to prove a)⇒b) in Theorem 3. For the implication b)⇒a), see Remark 10.

As a preparation, we choose balls $B_2 \Supset B_1 \Supset B'$ in $\mathbb{R}^n$ and consider the spaces X, Y_d, where X is the space $\{v \in \mathcal{D}^{(s)\prime}(\mathbb{R}^n); \operatorname{supp} v \subset B'\}$ and $Y_d = \mathcal{G}^{s,\prime}_d(B_1) = \{v; \operatorname{supp} v \subset B_1, \|v\|^{s,d} < \infty\}$. The spaces Y_d are clearly Banach spaces with the natural norm and the inclusions $Y_d \subset Y_{d'}$ are continuous for $d < d'$. Moreover, $X \subset Y := \bigcup_d Y_d$. We endow X with the topology induced by $\mathcal{D}^{(s)\prime}(\mathbb{R}^n)$ and also Y with the inductive limit topology by $Y = \varinjlim_d Y_d$. It is then, in the terminology of [2], a LF-space.

We have the following result:

PROPOSITION 5. *a) The inclusion $Y \subset \mathcal{D}^{(s)\prime}(\mathbb{R}^n)$ is continuous.*
b) The inclusion $X \subset Y$ is continuous.

Proof. In all the argument we fix some $\chi \in \mathcal{D}^{(s)}(B_1)$ which is identically one on B'. Whenever we refer in the argument which follows to some result obtained in a previous section in which a cut-off function is used, it will be this one.

a) Let us first show that the inclusions $Y_d \subset \mathcal{D}^{(s)\prime}(\mathbb{R}^n)$ are continuous. Assume then that $v \mapsto \|v\|_q$ is a continuous semi-norm on $\mathcal{D}^{(s)\prime}(\mathbb{R}^n)$. There is no loss of generality to assume that it has the form $\|v\|_q = \sup_{f\in\mathcal{M}} |v(f)|$ for some bounded set $\mathcal{M} \subset \mathcal{D}^{(s)}(\mathbb{R}^n)$. It follows that there exists a ball $\tilde{B}$ such that $\mathcal{M} \subset \mathcal{D}^{(s)}(\tilde{B})$ and such that $\mathcal{M}$ is bounded in the space $\mathcal{G}^{s,d}(\tilde{B})$ for every $d > 0$.

Then from Lemma 1 b), we can see that the set $\mathcal{N} = \{\chi f; f \in \mathcal{M}\}$ is bounded in $\mathcal{G}^{s,d}(B_2)$ for every $d > 0$, and for $v \in Y_d$ we have

$$\|v\|_q = \sup_{f\in\mathcal{M}} |v(f)| = \sup_{f\in\mathcal{M}} |v(\chi f)| = \sup_{g\in\mathcal{N}} |v(g)|$$
$$\leq \|v\|^{s,d} \cdot \sup_{g\in\mathcal{N}} \|g\|_{s,2d} \cdot \int_{\mathbb{R}^n} \exp[-d|\xi|^{1/s}]d\xi.$$

Here we used Remark 6 for the last inequality. Since the second and the last factor in the right hand side are bounded, the inclusion $Y_d \to \mathcal{D}^{(s)\prime}(\mathbb{R}^n)$ is continuous, as claimed.

b) Now let $\mathcal{U} \subset Y$ be a convex set such that its intersection with the space Y_d is a neighborhood of the origin for every $d > 0$. This means in particular that for every j we can find a constant $c''_j > 0$ such that $\{v \in Y_j; \|v\|^{s,j} \leq c''_j\} \subset \mathcal{U}$. (The constants c''_j will have to be, in general, small.) We now choose constants c_j such that $|h(\xi)| \leq c_j \exp[j|\xi|^{1/s}]$ implies that $|(\hat{\chi} * h)(\xi)| \leq 2^{-j} c''_{2j} \exp[2j|\xi|^{1/s}]$. (See Lemma 1.) Note that c_j must be small compared with c''_{2j}.

By using Corollary 1 we also see that there are constants C_j such that if $v \in \mathcal{G}^{s,\prime}_d(B_1)$ and if $f \in \mathcal{L}^2(\mathbb{R}^n)$, $\|f\|_{s,j} \leq C_j$ implies $|v(f)| \leq 1$, then $\|v\|^{s,2j} \leq c''_{2j}$ and hence $v \in \mathcal{U}$. The constants C_j will typically be large and once we have found such constants, we may increase them still further. We then assume that they are larger than $\max(1/c_j, \exp[j^2])$.

Next, we now consider an increasing sequence of positive constants C'_j for which the numbers $\ln C'_j$ satisfy (23) and for which we also have that for the sequence C_j chosen above, it follows from $\int_{\mathbb{R}^n} |\hat{f}(\xi)| \exp[2j|\xi|^{1/s}]d\xi \leq C_{2j}$ that $|\mathcal{F}(\chi f)(\xi)| \leq C'_j \exp[-j|\xi|^{1/s}]$. Again this can be obtained using Lemma 1. (In all this argument we denote "large constants" by capital letters and "small" ones, by small letters.)

We now denote $\ell(\xi) = \sup_j [j|\xi|^{1/s} - \ln C_j]$, $\ell'(\xi) = \sup_j [j|\xi|^{1/s} - \ln C'_j]$ and consider $\mathcal{M} = \{f \in \mathcal{D}^{(s)}(B_2); \int_{\mathbb{R}^n} |\hat{f}(\xi)| \exp[\ell'(\xi)/2]d\xi \leq 1\}$. $\mathcal{M}$ is then a bounded set in $\mathcal{D}^{(s)}(\mathbb{R}^n)$: see Lemma 4.

For a fixed positive constant $\tilde{c}$, it follows that the set

$$W = \{v \in X; |v(f)| \leq \tilde{c}, \forall f \in \mathcal{M}\}$$

is a neighborhood of the origin in X. To conclude the argument it will therefore suffice to show that $W \subset \mathcal{U}$ if $\tilde{c}$ is chosen suitably.

Assume then that $v \in W$, which means in particular that $v \in \mathcal{G}_d^{s,\prime}(B')$ for some d, since $X = \bigcup_d \mathcal{G}_d^{s,\prime}(B')$ as vector spaces.

Since $|v(f)| \leq \tilde{c}$ for all $f \in \mathcal{M}$ it follows combining Proposition 1 with Lemma 2 that $|\hat{v}(\xi)| \leq c''\tilde{c}\exp[\ell(\xi)]$ for some constant c'' which depends only on ℓ and ℓ'. We now put on $\tilde{c}$ the condition $c''\tilde{c} \leq 1$. Since we also know that $|\hat{v}(\xi)| \leq C\exp[d|\xi|^{1/s}]$ for some C and d, we conclude that

$$|\hat{v}(\xi)| \leq \exp[\min(\ell(\xi), d|\xi|^{1/s} + \ln C)], \forall \xi \in \mathbb{R}^n. \tag{45}$$

Note that the constants C and d depend on v. Now we choose a natural number $k > d+1$. If $|\xi|^{1/s}$ is large enough, say, larger than $\ln C + \ln C_k$, it follows that

$$d|\xi|^{1/s} + \ln C \leq k|\xi|^{1/s} - |\xi|^{1/s} - \ln C_k + \ln C_k + \ln C \leq k|\xi|^{1/s} - \ln C_k.$$

This shows that there is σ, which also depends on v, such that

$$|\hat{v}(\xi)| \leq \max_{j=1,\ldots,\sigma} \exp[j|\xi|^{1/s} - \ln C_j].$$

Indeed, for $|\xi|^{1/s} \geq \ln C + \ln C_k$, this is true by what we saw before if we assume $\sigma \geq k$, and for $|\xi|^{1/s} \leq \ln C + \ln C_k$, we have that $j|\xi|^{1/s} - \ln C_j \leq j(\ln C + \ln C_k) - \ln C_j \to -\infty$, with $j \to \infty$ (uniformly for the vectors ξ under consideration), such that $\ell(\xi) \leq \sup_{j \leq j^0}(j|\xi|^{1/s} - \ln C_j)$, for some j^0.

We can now find measurable functions h_j, $j = 1, \ldots, \sigma$, such that $\hat{v} = \sum_{j=1}^{\sigma} h_j$ and such that $|h_j(\xi)| \leq c_j \exp[j|\xi|^{1/s}]$. Multiplying $w_j = \mathcal{F}^{-1}h_j$ with the cut-off function χ, we obtain in this way ultradistributions $v_j = \chi w_j$, $j = 1, \ldots, \sigma$, such that $|\hat{v}_j(\xi)| \leq 2^{-j}c''_{2j}\exp[2j|\xi|^{1/s}]$ and such that $v = \sum_{j=1}^{\sigma} v_j$. Since the ultradistributions $2^j v_j$ lie in $\mathcal{U}$ and $\mathcal{U}$ is convex and contains the origin, it follows that $v \in \mathcal{U}$. This concludes the proof. □

We have now proved Proposition 5 and turn to the proof of Theorem 3. Recall that we may assume that $\operatorname{supp} \mathcal{K} \subset B' \times B$, with B and B' closed balls in $\mathbb{R}^m$, respectively $\mathbb{R}^n$. (See Remark 4.) Let us then assume that $T : \mathcal{D}^{(s)\prime}(\mathbb{R}^m) \to \mathcal{D}^{(s)\prime}(\mathbb{R}^n)$ is a continuous operator such that the restriction to $\mathcal{D}^{(s)}(\mathbb{R}^m)$ is given by the kernel $\mathcal{K}$. Since the inclusions $\mathcal{G}_d^{s,\prime}(B) \to \mathcal{D}^{(s)\prime}(\mathbb{R}^n)$ are continuous we obtain for every $d > 0$ a continuous map (denoted again T) $T : \mathcal{G}_d^{s,\prime}(B) \to \mathcal{D}^{(s)\prime}(\mathbb{R}^n)$ and consider $\chi \in \mathcal{D}^{(s)}(B_2)$ which is identically one on B'. On $\mathcal{G}_d^{s,\prime}(B)$ the operator T coincides with χT, so in particular it is trivial that T defines a continuous operator $T : \mathcal{G}_d^{s,\prime}(B) \to X$. By part b) of Proposition 5, it also defines a continuous operator $T : \mathcal{G}_d^{s,\prime}(B) \to Y$. It follows therefore from Grothendieck's theorem which we recall in a moment, that there is d' with $T(\mathcal{G}_d^{s,\prime}(B)) \subset Y_{d'}$ and such that the map $T : \mathcal{G}_d^{s,\prime}(B) \to Y_{d'}$ is continuous. At this moment we can essentially apply Proposition 3 to conclude the argument.

THEOREM 4 (Grothendieck, [2]). *Let* $\cdots \to X_i \to X_{i+1} \to \cdots$ *be a sequence of Fréchet spaces and continuous maps. Denote by* X *the inductive limit of the spaces* X_i, *by* $f_i : X_i \to X$ *the natural maps and consider a continuous linear map* $T : F \to X$ *where* F *is a Fréchet space. Assume that* X *is Hausdorff. Then there is an index* i^0 *such that* $T(F) \subset f_{i^0}(X_{i^0})$. *Moreover if* f_{i^0} *is injective, then there is a continuous map* $T^0 : F \to X_{i^0}$ *such that* T *is factorized into* $F \xrightarrow{T^0} X_{i^0} \xrightarrow{f_{i^0}} X$.

6. Proof of Theorem 2

In this section we prove the implication i)⇒ii) in Theorem 2. For the implication ii)⇒i), see Remark 10.

PROPOSITION 6. *Let* $S : \mathcal{D}^{\{s\}}(B) \to \mathcal{D}^{\{s\}}(B')$ *be a continuous integral operator associated with a kernel* $\mathcal{K}$ *with support in* $B' \times B$, B, B', *balls in* $\mathbb{R}^m$, *respectively* $\mathbb{R}^n$, *and fix* $d > 0$. *Then there is* $d' > 0$ *such that* S *induces a continuous operator* $\mathcal{G}^{s,d}(B) \to \mathcal{G}^{s,d'}(B')$.

Proof. Using (16), we have a continuous operator from a Banach space to a countable inductive limit of Banach spaces:

$$\mathcal{G}^{s,d}(B) \to \varinjlim_{d>0} \mathcal{G}^{s,d}(B) \xrightarrow{S} \varinjlim_{j\in\mathbb{N}} \mathcal{G}^{s,j}(B'),$$

where the first map is the standard inclusion given by the definition of an inductive limit. Then the conclusion follows from Theorem 4. □

Proof of Theorem 2. The assumption is that $Tu(x) = \int \mathcal{K}(x,y)u(y)dy$ is a linear continuous operator $\mathcal{E}^{\{s\}\prime}(U) \to \mathcal{D}^{\{s\}\prime}(V)$. Since we can multiply with cut-off functions in the x and in the y variables, there is again no loss of generality to assume that $U = \mathbb{R}^m$, $V = \mathbb{R}^n$ and that $\operatorname{supp} \mathcal{K} \subset B' \times B$ for two balls $B' \subset \mathbb{R}^n$, $B \subset \mathbb{R}^m$. By duality, we obtain then a continuous operator $S : \mathcal{D}^{\{s\}}(\mathbb{R}^n) \to \mathcal{E}^{\{s\}}(\mathbb{R}^m)$ defined by

$$\varphi(x) \mapsto (S\varphi)(y) = \int \mathcal{K}(x,y)\varphi(x)dx.$$

From the support condition, the image of S is included in $\mathcal{D}^{\{s\}}(B)$, and S becomes a continuous operator

$$S : \mathcal{D}^{\{s\}}(B') \to \mathcal{D}^{\{s\}}(B),$$

since the topology of $\mathcal{D}^{\{s\}}(B)$ is equal to the one induced by the inclusion $\mathcal{D}^{\{s\}}(B) \subset \mathcal{E}^{\{s\}}(\mathbb{R}^m)$. It follows therefore from Proposition 6 that if we fix $d' > 0$, then there is $d > 0$ such that S induces a continuous operator $\mathcal{G}^{s,d'}(B') \to \mathcal{G}^{s,d}(B)$. The conclusion in the theorem is then a consequence of Proposition 4. □

7. An example and some comments

In this section we give an example of a distribution which satisfies condition b) in theorem 3, but does not satisfy a wave front set condition of form $\mathrm{WF}_{(s)}(\mathcal{K})\cap\{(x,y,0,\eta);\ \eta\neq 0\}=\emptyset$.

We shall work for $n=m=1$, on $V\times U=T^2=T\times T$, T the one-dimensional torus. Since we are dealing with a non-quasianalytic setup, there is no real loss of generality in doing so. (We say something about this in Remark 12 below.) On the other hand, working on the torus makes the example a little bit simpler.

We denote $\exp[-k^{1+1/s}/j]$, for $j\in\mathbb{N}$, $k\in\mathbb{N}$, by a_{jk} and define the distribution $\mathcal{K}$ on T^2 by

$$\mathcal{K}(x,y)=(2\pi)^{-2}\sum_{j,k=1}^{\infty}a_{jk}\exp[i(jx+ky)]. \tag{46}$$

(The numbers a_{jk} are thus the Fourier coefficients of $\mathcal{K}$ and convergence in (46) is in the space of classical distributions.) It is immediate that $\mathcal{K}$ defines a continuous operator $L:\mathcal{D}^{(s)\prime}(T)\to\mathcal{D}^{(s)\prime}(T)$ by

$$Lu=(2\pi)^{-1}\sum_{j=1}^{\infty}b_j\exp[ijx],\quad b_j=\sum_{k=1}^{\infty}a_{jk}\hat{u}(-k) \tag{47}$$

where $\hat{u}(k)=u(\exp[-iyk])$ are the Fourier coefficients of u and and convergence in the first part of (47) is in the space of ultradistributions.

We claim that we have

PROPOSITION 7. *Let $\mathcal{K}$ be the kernel defined by (46). Then there is $(x^0,y^0)\in T^2$ such that $((x^0,y^0),(0,1))\in\mathrm{WF}_{(s)}(\mathcal{K})$. (Also see Remark 13 below.)*

Thus $\mathcal{K}$ defines a continuous operator $\mathcal{D}^{(s)\prime}(T)\to\mathcal{D}^{(s)\prime}(T)$, but we do not have $\mathrm{WF}_{(s)}(\mathcal{K})\cap\{(x,y,0,\eta);x\in T,y\in T,\eta\neq 0\}=\emptyset$.

To prove Proposition 7, we first state

PROPOSITION 8. *Consider $w\in\mathcal{D}^{(s)\prime}(T^2)$ and suppose that for some (x^0,y^0), $((x^0,y^0),(0,1))\notin\mathrm{WF}_{(s)}(w)$. Then there is $\varepsilon>0$ such that if $\chi\in\mathcal{D}^{(s)}(\mathbb{R}^2)$ is supported in an ε-neighborhood of (x^0,y^0), then $|\mathcal{F}(\chi w)(\xi,\eta)|\leq\exp[-\ell(\xi,\eta)]$ for some sublinear function ℓ as in (7) when (ξ,η) is in a suitably small conic neighborhood of $(0,1)$.*

The proof of this proposition is straightforward and is similar e.g., to the proof of lemma 1.7.3 in [14]. We omit details.

We can now prove Proposition 7. In fact, arguing by contradiction and using the preceding proposition, we can find a partition of unity formed of functions χ_i, $i=1,\dots,\sigma$, in $\mathcal{D}^{(s)}(T^2)$ such that for some conic neighborhood Γ of $(0,1)$ in $\mathbb{R}^2$ and some function ℓ as in (7) we have $|\mathcal{F}(\chi_i\mathcal{K})(\xi,\eta)|\leq\exp[-\ell(\xi,\eta)]$ for $(\xi,\eta)\in\Gamma$ and

$i = 1, \ldots, \sigma$. Since $a_{jk} = \sum_{i=1}^{\sigma} \mathcal{F}(\chi_i \mathcal{K})(j,k)$ it would follow that $|a_{jk}| \leq \sigma \exp[-\ell(j,k)]$ when $(j,k) \in \Gamma$, which is false.

REMARK 12. We have argued on the torus but we can now also immediately obtain from this an example of a kernel $\mathcal{K}'$ defined on $\mathbb{R} \times \mathbb{R}$ which satisfies condition b), but not the wave front set relation $\mathrm{WF}_{(s)}(\mathcal{K}) \cap \{(x,y,0,\eta); x \in \mathbb{R}, y \in \mathbb{R}, \eta \neq 0\} = \emptyset$. To simplify notations, we first observe that after a translation on the torus, it follows from above that there are kernels which define linear continuous maps $\mathcal{D}^{(s)\prime}(T) \to \mathcal{D}^{(s)\prime}(T)$, but with $((0,0),(0,1)) \in \mathrm{WF}_{(s)}(\mathcal{K})$. Next, pick $\psi \in \mathcal{D}^{(s)}(\mathbb{R}^2)$ which has support in a small neighborhood of $0 \in \mathbb{R}^2$ with $\psi \equiv 1$ in a still smaller neighborhood of 0. If $\mathcal{K} \in \mathcal{D}'(T^2)$ is the one just introduced above, then $\mathcal{K}' = \psi\mathcal{K}$ has a natural interpretation as a distribution on $\mathbb{R}^2$. Since $\mathcal{K}$ gave rise to a linear continuous operator $\mathcal{D}^{(s)\prime}(T) \to \mathcal{D}^{(s)\prime}(T)$, $\mathcal{K}'$ defines in a natural way a linear continuous operator $\mathcal{D}^{(s)\prime}(\mathbb{R}) \to \mathcal{D}^{(s)\prime}(\mathbb{R})$. It clearly does not satisfy the wave front set condition we would like to have.

REMARK 13. With a small extra effort, we can show that actually $((0,0),(0,1)) \in \mathrm{WF}_{(s)}(\mathcal{K})$, $\mathcal{K}$ the one defined in (46). To prove this it is essential that the coefficients a_{jk} are positive. We leave the details to the reader.

REMARK 14. The arguments in this paper can in principle be extended to more general classes of non-quasianalytic ultradistributions but we have not tried to work out such cases.

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$L^p(\mathbb{R})$ BOUNDEDNESS AND COMPACTNESS OF LOCALIZATION OPERATORS ASSOCIATED WITH THE STOCKWELL TRANSFORM

Dedicated to Professor Luigi Rodino on the occasion of his 60th birthday

Abstract. In this article, we prove the boundedness and compactness of localization operators associated with Stockwell transforms, which depend on a symbol and two windows, on $L^p(\mathbb{R})$, $1 \le p \le \infty$.

1. Introduction

1.1. The Stockwell transform

The Stockwell transform, which was defined in [13], is a hybrid of the Gabor transform and the wavelet transform. For a signal $f \in L^2(\mathbb{R})$, the Stockwell transform $S_\varphi f$ with respect to the window $\varphi \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ is given by

$$(1) \qquad S_\varphi f(b,\xi) = (2\pi)^{-1/2}|\xi| \int_{-\infty}^{\infty} e^{-ix\xi} f(x)\overline{\varphi(\xi(x-b))}\,dx, \quad b \in \mathbb{R},\ \xi \in \mathbb{R}.$$

More precisely,

$$S_\varphi f(b,\xi) = (f, \varphi^{b,\xi}),$$

where

$$(2) \qquad \varphi^{b,\xi} = (2\pi)^{-1/2}|\xi| e^{ix\xi}\varphi(\xi(x-b)),$$

or

$$\varphi^{b,\xi} = (2\pi)^{-1/2} M_\xi T_{-b} D_\xi \varphi,$$

and $(\ ,\ )$ is the inner product in $L^2(\mathbb{R})$. Here, M_ξ, T_{-b} and D_ξ are the modulation operator, the translation operator and the dilation operator, defined by

$$(M_\xi h)(x) = e^{ix\xi} h(x),$$

$$(T_{-b} h)(x) = h(x-b),$$

$$(D_\xi h)(x) = |\xi| h(\xi x),$$

for all $x \in \mathbb{R}$ and all measurable function h on $\mathbb{R}$.

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A great amount of articles use the Stockwell transform to study applied problems, covering areas as geophysics, engineering or biomedicine (see the references list in the papers [9] and [14]). Some mathematical aspects of such a transform are studied or expanded in the papers [2, 8, 9, 10, 11, 14].

1.2. Reconstruction formula

In an attempt to reconstruct a signal f from its Stockwell spectrum $\{S_\varphi f(b,\xi) : b,\xi \in \mathbb{R}\}$, we have the following result in [8].

THEOREM 1. *Let* $\varphi \in L^2(\mathbb{R})$ *be such that* $\|\varphi\|_{L^2(\mathbb{R})} = 1$ *and*

$$\int_{-\infty}^{\infty} \frac{|\hat{\varphi}(\xi-1)|^2}{|\xi|} d\xi < \infty. \tag{3}$$

Then for all signals f *and* g *in* $L^2(\mathbb{R})$*,*

$$(f,g)_{L^2(\mathbb{R})} = \frac{1}{c_\varphi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} S_\varphi f(b,\xi) \overline{S_\varphi g(b,\xi)} \frac{db\, d\xi}{|\xi|}, \tag{4}$$

where

$$c_\varphi = \int_{-\infty}^{\infty} \frac{|\hat{\varphi}(\xi-1)|^2}{|\xi|} d\xi, \tag{5}$$

and $\hat{}$ *denotes the Fourier transform defined by*

$$\hat{F}(\zeta) = (2\pi)^{-N/2} \int_{\mathbb{R}^N} e^{-ix\cdot\zeta} F(x)\, dx$$

for all F *in* $L^1(\mathbb{R}^N)$.

REMARK 1. Theorem 1 is known as the Plancherel formula or the resolution of the identity formula for the one-dimensional Stockwell transform. The integrability condition (3) is the admissibility condition for a function φ in $L^2(\mathbb{R})$ to be a window. An important corollary of Theorem 1 is that every signal f can be reconstructed from its Stockwell spectrum by means of the inversion formula

$$f = \frac{1}{c_\varphi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (f, \varphi^{b,\xi})_{L^2(\mathbb{R})} \varphi^{b,\xi} \frac{db\, d\xi}{|\xi|}. \tag{6}$$

That the admissibility condition (3) is a necessary condition for the inversion formula for the Stockwell transform can be seen by letting $f = g = \varphi$ in (4). Details can be found in [7].

1.3. Localization operators

Let φ, ψ be measurable functions on $\mathbb{R}$, σ be measurable function on $\mathbb{R}^2$, then for all functions $f \in L^p(\mathbb{R})$, we define the localization operator $L_{\sigma,\varphi,\psi} f$, by

$$L_{\sigma,\varphi,\psi} f = \int_{\mathbb{R}} \int_{\mathbb{R}} \sigma(b,\xi)(S_\varphi f)(b,\xi)\psi^{b,\xi} \frac{dbd\xi}{|\xi|} = \int_{\mathbb{R}^2} \sigma(b,\xi)(f,\varphi^{b,\xi})\psi^{b,\xi} \frac{dbd\xi}{|\xi|}. \tag{7}$$

REMARK 2. The symbol can be understood as a filter of the Stockwell spectrum. Formula (6) reconstructs the signal using the Stockwell spectrum $\{S_\varphi f(b,\xi) : b,\xi \in \mathbb{R}\}$ with respect to the window component $\varphi^{b,\xi}$. The localization operator using the filtered Stockwell spectrum $\{\sigma(b,\xi)S_\varphi f(b,\xi) : b,\xi \in \mathbb{R}\}$ may be defined by

$$T_{\sigma,\varphi} f = f = \frac{1}{c_\varphi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sigma(b,\xi)(f,\varphi^{b,\xi})_{L^2(\mathbb{R})} \varphi^{b,\xi} \frac{db\,d\xi}{|\xi|}.$$

However, in order to allow some linearity properties with respect to the windows, we consider the localization operator designed in the original way (7).

In accordance with the different choices ot the symbols $\sigma(b,\xi)$ and the different continuities required, we need to impose different conditions on φ and ψ. And then we obtain an operator on $L^p(\mathbb{R})$.

In the paper [15] by Wong, the L^p-boundedness of localization operators associated to left regular representations is studied for $1 \leq p \leq \infty$. L^p-boundedness and L^p-compactness of two-wavelet localization operators on the Weyl-Heisenberg group can be found in the papers [4] by Boggiatto and Wong, and [3] by Boggiatto, Oliaro and Wong. The aim of this paper is to give another set of results on the L^p-boundedness and also L^p-compactness of the localization operators defined by (7).

In Section 2, we prove that the localization operator associated with the Stockwell transform, with symbols in $L^1(\mathbb{R})$ and windows $\varphi \in L^{p'}(\mathbb{R})$ and $\psi \in L^p(\mathbb{R})$ are bounded linear operators on $L^p(\mathbb{R})$, $1 \leq p \leq \infty$. Herein, p' is the conjugate of p, such that

$$\frac{1}{p} + \frac{1}{p'} = 1. \tag{8}$$

If the symbols are in $L^r(\mathbb{R}^2)$, $1 \leq r \leq 2$, and the admissible windows φ, ψ are in $L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, then the localization operators are proved in Section 3 to be bounded linear operators on $L^p(\mathbb{R})$, $r \leq p \leq r'$. Section 4 deals with the compactness for symbols in $L^1(\mathbb{R}^2)$. The last section treats the localization operators associated to the generalized Stockwell transform defined in [10] and [11]. Due to the close relation between the Stockwell transform and generalized Stockwell transform, all our conclusions obtained in Section 2, Section 3 and Section 4 can be applied to these localization operators.

2. Symbols in $L^1(\mathbb{R}^n)$

For $1 \le p \le \infty$, let $\sigma \in L^1(\mathbb{R}^2)$, $\varphi \in L^{p'}(\mathbb{R})$ and $\psi \in L^p(\mathbb{R})$. We are going to show that $L_{\sigma,\varphi,\psi}$ is a bounded linear operator on $L^p(\mathbb{R})$.

Let us start with the following estimates:

PROPOSITION 1. *For $1 \le p \le \infty$, let $\psi \in L^p(\mathbb{R})$ and $f \in L^{p'}(\mathbb{R})$, where p' is the conjugate of p. Then*

$$\|\psi^{b,\xi}\|_p = (2\pi)^{-1/2}|\xi|^{1/p'}\|\psi\|_p, \tag{9}$$

and

$$|S_\psi f(b,\xi)| \le (2\pi)^{-1/2}|\xi|^{1/p'}\|\psi\|_p\|f\|_{p'}. \tag{10}$$

Proof. For $p = \infty$, the first equality is trivial. For $p \ne \infty$, by Fubini's theorem, we have

$$\begin{aligned}
\|\psi^{b,\xi}\|_p &= \left\{\int \left|(2\pi)^{-1/2}|\xi|e^{ix\xi}\psi(\xi(x-b))\right|^p dx\right\}^{1/p} \\
&= (2\pi)^{-1/2}|\xi|\left\{\int |\psi(\xi(x-b))|^p\, dx\right\}^{1/p} \\
&= (2\pi)^{-1/2}|\xi|^{1/p'}\|\psi\|_p.
\end{aligned}$$

Applying Hölder's inequality and (9), we have

$$|S_\psi f(b,\xi)| = |(f,\psi^{b\xi})| \le \|f\|_{p'}\|\psi^{b,\xi}\|_p = (2\pi)^{-1/2}|\xi|^{1/p'}\|f\|_{p'}\|\psi\|_p.$$

□

In the following we denote with $\|\cdot\|_{B(L^p(\mathbb{R}))}$ the operator norm in the Banach space $B(L^p)$ of bounded linear operators on $L^p, 1 \le p \le \infty$.

We start with the result about the boundedness of $L_{\sigma,\varphi,\psi}$ on $L^1(\mathbb{R})$.

PROPOSITION 2. *Let $\sigma \in L^1(\mathbb{R}^2)$ and $\varphi \in L^\infty(\mathbb{R}), \psi \in L^1(\mathbb{R})$. Then $L_{\sigma,\varphi,\psi}$: $L^1(\mathbb{R}) \to L^1(\mathbb{R})$ is a bounded linear operator and*

$$\|L_{\sigma,\varphi,\psi}\|_{B(L^1(\mathbb{R}))} \le \frac{1}{2\pi}\|\sigma\|_1\|\varphi\|_\infty\|\psi\|_1.$$

Proof. For any $f \in L^1(\mathbb{R})$, by (7), (2), and (10), we have

$$\begin{aligned}
&\|L_{\sigma,\varphi,\psi} f\|_1 = \int \Big| \iint \sigma(b,\xi) S_\varphi f(b,\xi) \psi^{b,\xi}(x) \frac{db\,d\xi}{|\xi|} \Big| dx \\
\leq\ & \iiint |\sigma(b,\xi)| \Big((2\pi)^{-1/2} |\xi| \|f\|_1 \|\varphi\|_\infty \Big) \Big((2\pi)^{-1/2} |\xi| |\psi(\xi(x-b))| \Big) \frac{db\,d\xi}{|\xi|}\, dx \\
\leq\ & \frac{1}{2\pi} \|f\|_1 \|\varphi\|_\infty \iiint |\sigma(b,\xi)| |\psi(\xi(x-b))| |\xi|\, db\,d\xi\,dx \\
=\ & \frac{1}{2\pi} \|f\|_1 \|\varphi\|_\infty \iint |\sigma(b,\xi)| \Big(\int |\xi| |\psi(\xi(x-b))|\, dx \Big) db\,d\xi \\
=\ & \Big(\frac{1}{2\pi} \|\varphi\|_\infty \|\sigma\|_1 \|\psi\|_1 \Big) \|f\|_1,
\end{aligned}$$

which completes our proof. ☐

For $p \neq 1$, we have the following conclusion about the boundedness of $L_{\sigma,\varphi,\psi}$.

PROPOSITION 3. *Let* $\sigma \in L^1(\mathbb{R}^2)$, $\varphi \in L^{p'}(\mathbb{R})$ *and* $\psi \in L^p(\mathbb{R})$. *Then* $L_{\sigma,\varphi,\psi} : L^p(\mathbb{R}) \to L^p(\mathbb{R})$ *is a bounded linear operator for* $1 \leq p < \infty$ *and*

$$\|L_{\sigma,\varphi,\psi}\|_{B(L^p(\mathbb{R}))} \leq \frac{1}{2\pi} \|\sigma\|_1 \|\varphi\|_{p'} \|\psi\|_p.$$

Proof. For any $f \in L^p(\mathbb{R})$, consider the linear functional

$$T_f : L^{p'}(\mathbb{R}) \to \mathbb{C}, \quad g \mapsto (g, L_{\sigma,\varphi,\psi} f).$$

By (7), we have

$$\begin{aligned}
|(g, L_{\sigma,\varphi,\psi} f)| &= |(L_{\sigma,\varphi,\psi} f, g)| \\
&= \Big| \int \sigma(b,\xi) S_\varphi f(b,\xi) \overline{S_\psi g(b,\xi)} \frac{db\,d\xi}{|\xi|} \Big| \\
&= \int |\sigma| |S_\varphi f(b,\xi)| |S_\psi g(b,\xi)| \frac{db\,d\xi}{|\xi|}.
\end{aligned}$$

Applying Proposition 1, we have

$$\begin{aligned}
&|(g, L_{\sigma,\varphi,\psi} f)| \\
\leq\ & \int |\sigma(b,\xi)| \Big((2\pi)^{-1/2} |\xi|^{1/p} \|f\|_p \|\varphi\|_{p'} \Big) \Big((2\pi)^{-1/2} |\xi|^{1/p'} \|g\|_{p'} \|\psi\|_p \Big) \frac{db\,d\xi}{|\xi|} \\
=\ & \Big(\frac{1}{2\pi} \|\sigma(b,\xi)\|_1 \|\varphi\|_{p'} \|\psi\|_p \|f\|_p \Big) \|g\|_{p'}
\end{aligned}$$

which implies that T_f is a continuous linear functional on $L^{p'}(\mathbb{R})$, and the operator norm

$$\|T_f\|_{B(L^{p'}(\mathbb{R}))} \leq \frac{1}{2\pi} \|\sigma\|_1 \|\varphi\|_{p'} \|\psi\|_p \|f\|_p.$$

Since $T_f g = (g, L_{\sigma,\varphi,\psi} f)$, by the Riesz representation theorem, we have

$$\|L_{\sigma,\varphi,\psi} f\|_p = \|T_f\|_{B(L^{p'}(\mathbb{R}))} \leq \frac{1}{2\pi}\|\sigma\|_1\|\varphi\|_{p'}\|\psi\|_p\|f\|_p,$$

which establishes the proposition. □

To sum up the two propositions above, we have the following theorem.

THEOREM 2. *Let $\sigma \in L^1(\mathbb{R}^2)$, $\varphi \in L^{p'}(\mathbb{R})$, $\psi \in L^p(\mathbb{R})$. Then $L_{\sigma,\varphi,\psi} : L^p(\mathbb{R}) \to L^p(\mathbb{R})$ is bounded linear operator for $1 \leq p \leq \infty$ and*

$$\|L_{\sigma,\varphi,\psi}\|_{B(L^p(\mathbb{R}))} \leq \frac{1}{2\pi}\|\sigma\|_1\|\varphi\|_{p'}\|\psi\|_p.$$

3. Symbols in $L^r(\mathbb{R})$, $1 \leq r \leq 2$

In this section, we study the localization operators $L_{\sigma,\varphi,\psi}$ for symbols $\sigma \in L^r(\mathbb{R})$, $1 \leq r \leq 2$.

PROPOSITION 4. *Let ψ and φ be admissible windows, $\psi \in L^2(\mathbb{R})$ and $\varphi \in L^2(\mathbb{R})$, $\sigma \in L^2(\mathbb{R}^2)$. Then $L_{\sigma,\varphi,\psi} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is a bounded linear operator and*

$$\|L_{\sigma,\varphi,\psi}\|_{B(L^2(\mathbb{R}))} \leq \left(\frac{\sqrt{c_\varphi c_\psi}}{2\pi}\|\varphi\|_2\|\psi\|_2\right)^{1/2}\|\sigma\|_2.$$

To prove the proposition, let us start with the following lemma.

LEMMA 1. *Let ψ and φ be admissible windows, $\psi \in L^2(\mathbb{R})$ and $\varphi \in L^2(\mathbb{R})$, $\sigma \in L^\infty(\mathbb{R}^2)$. Then $L_{\sigma,\varphi,\psi} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is a bounded linear operator and*

$$\|L_{\sigma,\varphi,\psi}\|_{B(L^2(\mathbb{R}))} \leq \sqrt{c_\varphi c_\psi}\,\|\sigma\|_\infty.$$

Proof. For any $f, g \in L^2(\mathbb{R})$, by (7) and Hölder's inequality, we have

$$\begin{aligned} |(L_{\sigma,\varphi,\psi} f, g)| &= \left|\int_{\mathbb{R}^2}\sigma(b,\xi)S_\varphi f(b,\xi)\overline{S_\psi g(b,\xi)}\frac{db\,d\xi}{|\xi|}\right| \\ &\leq \|\sigma\|_\infty\int_{\mathbb{R}^2}|S_\varphi f(b,\xi)||\overline{S_\psi g(b,\xi)}|\frac{db\,d\xi}{|\xi|} \\ &\leq \|\sigma\|_\infty\left(\int_{\mathbb{R}^2}|S_\varphi f(b,\xi)|^2\frac{db\,d\xi}{|\xi|}\right)^{1/2}\left(\int_{\mathbb{R}^2}|\overline{S_\psi g(b,\xi)}|^2\frac{db\,d\xi}{|\xi|}\right)^{1/2}. \end{aligned}$$

By Theorem 1, we have

$$\begin{aligned} |(L_{\sigma,\varphi,\psi} f, g)| &\leq \|\sigma\|_\infty(c_\varphi)^{1/2}(c_\psi)^{1/2}\|f\|_2\|g\|_2 \\ &= \sqrt{c_\varphi c_\psi}\,\|\sigma\|_\infty\|f\|_2\|g\|_2, \end{aligned}$$

which completes the proof. □

Proof of Proposition 4. For any fixed $f \in L^2(\mathbb{R})$, admissible windows $\varphi, \psi \in L^2(\mathbb{R})$, we define a linear map from $L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ to $L^2(\mathbb{R})$ by

$$T(\sigma) = L_{\sigma,\varphi,\psi} f.$$

From the above lemma we have

$$\|T(\sigma)\|_2 \leq \sqrt{c_\varphi c_\psi} \|f\|_2 \|\sigma\|_\infty, \tag{11}$$

and let $p = 2$ in Theorem 2, we have

$$\|T(\sigma)\|_2 \leq \left(\frac{1}{2\pi} \|f\|_2 \|\varphi\|_2 \|\psi\|_2 \right) \|\sigma\|_1. \tag{12}$$

Applying interpolation theory, see [1] for instance, we have

$$\begin{aligned} \|T(\sigma)\|_2 &\leq (\sqrt{c_\varphi c_\psi} \|f\|_2)^{1/2} \left(\frac{1}{2\pi} \|f\|_2 \|\varphi\|_2 \|\psi\|_2 \right)^{1/2} \|\sigma\|_2 \\ &= \left(\frac{\sqrt{c_\varphi c_\psi}}{2\pi} \|\varphi\|_2 \|\psi\|_2 \right)^{1/2} \|f\|_2 \|\sigma\|_2. \end{aligned}$$

By the definition of $T(\sigma)$, we have

$$\|L_{\sigma,\varphi,\psi} f\|_2 \leq \left(\frac{\sqrt{c_\varphi c_\psi}}{2\pi} \|\varphi\|_2 \|\psi\|_2 \right)^{1/2} \|f\|_2 \|\sigma\|_2.$$

Thus the proof is complete. □

THEOREM 3. *Let ψ and φ be admissible windows, $\psi \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and $\varphi \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$. Let $\sigma \in L^r(\mathbb{R}^2)$, $1 \leq r \leq 2$. Then there exists a unique bounded linear operator $L_{\sigma,\varphi,\psi} : L^p(\mathbb{R}) \to L^p(\mathbb{R})$ for all $p \in [r, r']$ such that*

$$\|L_{\sigma,\varphi,\psi}\|_{B(L^p(\mathbb{R}))} \leq M_1^{1-\theta} M_2^{\theta} \|\sigma\|_p, \tag{13}$$

where

$$M_1 = \left(\frac{1}{2\pi} \|\varphi\|_\infty \|\psi\|_1 \right)^{\frac{2}{r}-1} \left(\frac{\sqrt{c_\varphi c_\psi}}{2\pi} \|\varphi\|_2 \|\psi\|_2 \right)^{\frac{1}{r'}},$$

$$M_2 = \left(\frac{1}{2\pi} \|\varphi\|_1 \|\psi\|_\infty \right)^{\frac{2}{r}-1} \left(\frac{\sqrt{c_\varphi c_\psi}}{2\pi} \|\varphi\|_2 \|\psi\|_2 \right)^{\frac{1}{r'}}.$$

Proof. Let T be the bilinear mapping from $\{L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)\} \times \{L^1(\mathbb{R}) \cap L^2(\mathbb{R})\}$ to $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, defined by

$$T(\sigma, f) = L_{\sigma,\varphi,\psi} f. \tag{14}$$

By Proposition 2 and Propostion 3.1, we have

$$\|T(\sigma, f)\|_1 \leq \frac{1}{2\pi} \|\varphi\|_\infty \|\psi\|_1 \|\sigma\|_1 \|f\|_1,$$

$$\|T(\sigma, f)\|_2 \leq \frac{\sqrt{c_\varphi c_\psi}}{2\pi} \|\varphi\|_2 \|\psi\|_2 \|\sigma\|_2 \|f\|_2.$$

By the multi-linear interpolation theory, see Section 10.1 in [5] for reference, we get a unique bounded linear operator $T(\sigma, f) : L^r(\mathbb{R}^2) \times L^r(\mathbb{R}) \to L^r(\mathbb{R})$ such that

$$\|T(\sigma, f)\|_r \leq M_1 \|\sigma\|_r \|f\|_r, \tag{15}$$

where

$$M_1 = \Big(\frac{1}{2\pi}\|\varphi\|_\infty\|\psi\|_1\Big)^{1-\alpha}\Big(\frac{\sqrt{c_\varphi c_\psi}}{2\pi}\|\varphi\|_2\|\psi\|_2\Big)^{\alpha/2},$$

with

$$\frac{1-\alpha}{1}+\frac{\alpha}{2}=\frac{1}{r} \qquad or \qquad \alpha = 2-\frac{2}{r}.$$

By the definition of T in (14), we have

$$\|L_{\sigma,\varphi,\psi}\|_{B(L^r(\mathbb{R}))} \leq \Big(\frac{1}{2\pi}\|\varphi\|_\infty\|\psi\|_1\Big)^{\frac{2}{r}-1}\Big(\frac{\sqrt{c_\varphi c_\psi}}{2\pi}\|\varphi\|_2\|\psi\|_2\Big)^{\frac{1}{r'}}\|\sigma\|_r. \tag{16}$$

Since the adjoint of $L_{\sigma,\varphi,\psi}$ is $L_{\overline{\sigma},\overline{\psi},\overline{\varphi}}$, so $L_{\sigma,\varphi,\psi}$ is a bounded linear map on $L^{r'}(\mathbb{R})$, with its operator norm

$$\begin{aligned}\|L_{\sigma,\varphi,\psi}\|_{B(L^{r'}(\mathbb{R}))} &= \|L_{\overline{\sigma},\overline{\psi},\overline{\varphi}}\|_{B(L^r(\mathbb{R}))}\\ &\leq \Big(\frac{1}{2\pi}\|\varphi\|_1\|\psi\|_\infty\Big)^{\frac{2}{r}-1}\Big(\frac{\sqrt{c_\varphi c_\psi}}{2\pi}\|\varphi\|_2\|\psi\|_2\Big)^{\frac{1}{r'}}\|\sigma\|_r. \end{aligned}\tag{17}$$

Using an interpolation of (16) and (17), we have that, for any $p \in [r, r']$,

$$\|L_{\sigma,\varphi,\psi}\|_{B(L^p(\mathbb{R}))} \leq M_1^{1-\theta}M_2^{\theta}\|\sigma\|_p,$$

with

$$\frac{1-\theta}{r}+\frac{\theta}{r'}=\frac{1}{p} \qquad or \qquad \theta = \Big(\frac{1}{r}-\frac{1}{p}\Big)\Big/\Big(\frac{1}{r}-\frac{1}{r'}\Big).$$

□

4. Compact operators

In this section, we study the compactness of the localization operators $L_{\sigma,\varphi,\psi} : L^p(\mathbb{R}) \to L^p(\mathbb{R})$. We start with a simple case:

LEMMA 2. *For $1 \leq p < \infty$, let $\varphi \in L^{p'}(\mathbb{R})$, σ and ψ be compactly supported and continuous. Then $L_{\sigma,\varphi,\psi} : L^p(\mathbb{R}) \to L^p(\mathbb{R})$ is compact.*

Proof. To prove that $L_{\sigma,\varphi,\psi}$ is compact, it is enough to show that the image of any bounded sequence has a convergent subsequence. Let $\{f_j\}_{j=1}^\infty$ be a sequence of functions in $L^p(\mathbb{R})$ such that

$$\|f_j\|_p \leq 1, \qquad j = 1, 2, \ldots.$$

Because σ is compactly supported, we may assume that

$$\sigma(b,\xi) = 0, \qquad \text{for all } (b,\xi) \text{ such that } (|b|^2+|\xi|^2)^{1/2} > M.$$

From Proposition 1 and the fact that ψ is continuous, we have

$$|\psi^{b,\xi}(x)| \le (2\pi)^{-1/2}|\xi|\|\psi\|_\infty,$$

$$|(S_\varphi f_j)(b,\xi)| \le (2\pi)^{-1/2}\|f_j\|_p\||\xi|^{1/p}\|\varphi\|_{p'} \le (2\pi)^{-1/2}|\xi|^{1/p}\|\varphi\|_{p'}.$$

Therefore

$$\begin{aligned}
&|L_{\sigma,\varphi,\psi} f_j(x)| \\
&= \left| \iint_{\mathbb{R}^2} \sigma(b,\xi)(S_\varphi f_j)(b,\xi)\psi^{b,\xi}(x)\frac{db\,d\xi}{|\xi|} \right| \\
&\le \iint_{\substack{b\in\mathbb{R}\\ |\xi|\le M}} |\sigma(b,\xi)|((2\pi)^{-1/2}|\xi|^{1/p}\|\varphi\|_{p'})((2\pi)^{-1/2}|\xi|\|\psi\|_\infty)\frac{db\,d\xi}{|\xi|} \\
&\le \frac{1}{2\pi}\|\varphi\|_{p'}\|\psi\|_\infty \iint_{\substack{b\in\mathbb{R}\\ |\xi|\le M}} |\sigma(b,\xi)||\xi|^{1/p}db\,d\xi \\
&\le \frac{1}{2\pi}M^{1/p}\|\varphi\|_{p'}\|\psi\|_\infty\|\sigma\|_1,
\end{aligned}$$

for all $j = 1,2,\ldots$. Thus the sequence $\{L_{\sigma,\varphi,\psi} f_j\}_{j=1}^\infty$ is uniformly bounded.

Let ε be any positive number. Since ψ is compactly supported and continuous, it is therefore uniformly continuous. So there exists $\delta_1 > 0$, such that

$$|\psi(x)-\psi(y)| \le \varepsilon, \qquad \text{for any } |x-y| < \delta_1.$$

Let $\delta = \min\left\{\frac{\delta_1}{1+M}, \frac{\varepsilon}{1+M}\right\}$. Then for any $|x-y| < \delta$, $|\xi| \le M$,

$$\begin{aligned}
&|\psi^{b,\xi}(x)-\psi^{b,\xi}(y)| \\
&= (2\pi)^{-1/2}|\xi|\left|e^{ix\xi}\psi(\xi(x-b)) - e^{iy\xi}\psi(\xi(y-b))\right| \\
&\le (2\pi)^{-1/2}|\xi|\left(|e^{ix\xi}|\left|\psi(\xi(x-b))-\psi(\xi(y-b))\right| + \left|e^{ix\xi}-e^{iy\xi}\right||\psi(\xi(y-b))|\right) \\
&\le (2\pi)^{-1/2}|\xi|(|\psi(\xi(x-b))-\psi(\xi(y-b))| + |x-y||\xi|\|\psi\|_\infty) \\
&\le (2\pi)^{-1/2}|\xi|(\varepsilon + \|\psi\|_\infty\varepsilon),
\end{aligned}$$

and thus for any $x,y \in \mathbb{R}$ such that $|x-y| < \delta$,

$$\begin{aligned}
&|(L_{\sigma,\varphi,\psi} f_j)(x) - (L_{\sigma,\varphi,\psi} f_j)(y)| \\
&\le \frac{1}{c_{\varphi,\psi}} \iint_{\substack{b\in\mathbb{R}\\ |\xi|\le M}} |\sigma(b,\xi)||(S_\varphi f_j)(b,\xi)|\left|\psi^{b,\xi}(x)-\psi^{b,\xi}(y)\right|\frac{db\,d\xi}{|\xi|} \\
&\le \frac{1}{c_{\varphi,\psi}} \iint_{\substack{b\in\mathbb{R}|\\ \xi|\le M}} |\sigma(b,\xi)|\left((2\pi)^{-1/2}|\xi|^{1/p}\|\varphi\|_{p'}\right)\left((2\pi)^{-1/2}|\xi|(\varepsilon+\|\psi\|_\infty\varepsilon)\right)\frac{db\,d\xi}{|\xi|} \\
&\le \frac{1}{2\pi c_{\varphi,\psi}}\|\sigma\|_1\|\varphi\|_{p'}M^{1/p}(1+\|\psi\|_\infty)\varepsilon.
\end{aligned}$$

So $\{L_{\sigma,\varphi,\psi} f_j\}_{j=1}^{\infty}$ is equicontinuous on $\mathbb{R}$. Therefore for every compact subset K of $\mathbb{R}$, the Ascoli–Arzelà theorem ensures that $\{L_{\sigma,\varphi,\psi} f_j\}_{j=1}^{\infty}$ has a subsequence that converges uniformly on K. Thus by the Cantor diagonal procedure, we can find a subsequence $\{L_{\sigma,\varphi,\psi} f_{j_k}\}_{k=1}^{\infty}$ converging pointwise to a function g on $\mathbb{R}$. By (7) and (2), and the inequality (10), we have

$$|(L_{\sigma,\varphi,\psi} f_j)(x)|^p \le ((2\pi)^{-1}\|\varphi\|_{p'})^p \left(\iint |\sigma(b,\xi)||\xi|^{1/p}|\psi(\xi(x-b))|\, db\, d\xi\right)^p.$$

Denote the function on the left hand side of the above inequality by h. By Hölder's inequality, we have

$$\begin{aligned}
&\int |h(x)|\, dx \\
&= C\int \left(\iint |\sigma(b,\xi)||\xi|^{1/p}|\psi(\xi(x-b))|\, db\, d\xi\right)^p dx \\
&= C\int \left(\iint_{|b|^2+|\xi|^2\le M2} |\sigma(b,\xi)||\xi|^{1/p}|\psi(\xi(x-b))|\, db\, d\xi\right)^p dx \\
&\le C\int \left(\iint \left(|\sigma(b,\xi)||\xi|^{1/p}|\psi(\xi(x-b))|\right)^p db\, d\xi\right)\cdot\left(\iint_{|b|^2+|\xi|^2\le M2} 1^{p'}\, db\, d\xi\right)^{p/p'} dx \\
&= C(2\pi M^2)^{p/p'}(\|\sigma(b,\xi)\|_p\|\psi\|_p)^p < \infty,
\end{aligned}$$

where C is the constant $((2\pi)^{-1}\|\varphi\|_{p'})^p$. So by Lebesgue's dominated convergence theorem, the sequence $\{|L_{\sigma,\varphi,\psi} f_{j_k}|^p\}_{k=1}^{\infty}$ converges to $|g|^p$ in $L^1(\mathbb{R})$ as $k\to\infty$. And thus,

$$|L_{\sigma,\varphi,\psi} f_{j_k}(x) - g(x)|^p \le 2^p(|L_{\sigma,\varphi,\psi} f_{j_k}(x)|^p + |g(x)|^p) \le 2^{p+1}h(x),$$

and $|L_{\sigma,\varphi,\psi} f_{j_k} - g|^p$ converges to 0 pointwise, so by the Lebesgue's dominated convergence theorem, $\int |L_{\sigma,\varphi,\psi} f_{j_k}(x) - g(x)|^p dx$ converges to 0. Thus $\{L_{\sigma,\varphi,\psi} f_{j_k}\}_{k=1}^{\infty}$ converges to g in $L^p(\mathbb{R})$. Therefore $L_{\sigma,\varphi,\psi}$ is compact. □

PROPOSITION 5. *For* $1\le p<\infty$, *let* $\sigma\in L^1(\mathbb{R}^2)$ *and* $\psi\in L^p(\mathbb{R}), \varphi\in L^{p'}(\mathbb{R})$. *Then* $L_{\sigma,\varphi,\psi}: L^p(\mathbb{R})\to L^p(\mathbb{R})$ *is compact.*

Proof. For any $\sigma,\tau\in L^1(\mathbb{R}^2)$, $\varphi\in L^{p'}(\mathbb{R})$ and $\psi,\phi\in L^p(\mathbb{R})$, by (7) and Theorem 2, we have

$$\begin{aligned}
\|L_{\sigma,\varphi,\psi} - L_{\tau,\varphi,\psi}\|_{B(L^p(\mathbb{R}))} &= \|L_{\sigma-\tau,\varphi,\psi}\|_{B(L^p(\mathbb{R}))} \\
&\le (2\pi)^{-1}\|\sigma-\tau\|_1\|\varphi\|_{p'}\|\psi\|_p,
\end{aligned}$$

and

$$\begin{aligned}
\|L_{\sigma,\varphi,\psi} - L_{\sigma,\varphi,\phi}\|_{B(L^p(\mathbb{R}))} &= \|L_{\sigma,\varphi,\psi-\phi}\|_{B(L^p(\mathbb{R}))} \\
&\le (2\pi)^{-1}\|\sigma\|_1\|\varphi\|_{p'}\|\psi-\phi\|_p.
\end{aligned}$$

By the above lemma, and the fact that $C_0(\mathbb{R}^2)$ is dense in $L^1(\mathbb{R}^2)$, and $C_0(\mathbb{R})$ is dense in $L^p(\mathbb{R})$ for $1 \le p < \infty$, and the fact that the set of compact operators is closed in $B(L^p(\mathbb{R}))$, the proposition holds. □

THEOREM 4. *Under the same hypotheses on* σ, φ, ψ *as Theorem 2, the bounded linear operator* $L_{\sigma,\varphi,\psi} : L^p(\mathbb{R}) \to L^p(\mathbb{R})$ *is compact for* $1 \le p \le \infty$.

Proof. From the previous proposition, we only need to show that the conclusion holds for $p = \infty$. In fact, the operator $L_{\sigma,\varphi,\psi} : L^\infty(\mathbb{R}) \to L^\infty(\mathbb{R})$ is the adjoint of the operator $L_{\bar{\sigma},\bar{\psi},\bar{\varphi}} : L^1(\mathbb{R}) \to L^1(\mathbb{R})$, which is compact by Proposition 5. Thus by the duality property, $L_{\sigma,\varphi,\psi} : L^\infty(\mathbb{R}) \to L^\infty(\mathbb{R})$ is compact. □

5. Localization operators associated with the modified Stockwell transform

In the papers [10, 11], the modified Stockwell transform is defined by

$$\begin{aligned} (S^s_\varphi f)(b,\xi) &= (2\pi)^{-1} \int f(x)\, e^{-ix\xi} |\xi|^{1/s} \overline{\varphi(\xi(x-b))}\, dx \\ &= (f, \varphi_s^{b,\xi}), \end{aligned} \tag{18}$$

where

$$\varphi_s^{b,\xi}(x) = e^{ix\xi} |\xi|^{1/s} \varphi(\xi(x-b)) = |\xi|^{1/s-1} \varphi^{b,\xi}(b,\xi)(x).$$

The connection between the modified Stockwell transform and Stockwell transform is

$$S^s_\varphi f = |\xi|^{1/s-1} S_\varphi f(b,\xi).$$

And so the localization operators associated with the modified Stockwell transform can be expressed by

$$\begin{aligned} L^s_{\sigma,\varphi,\psi} f &= \int_{\mathbb{R}} \int_{\mathbb{R}} \sigma(b,\xi)(S^s_\varphi f)(b,\xi) \varphi_s^{b,\xi} \frac{db\, d\xi}{|\xi|^{(2/s)-1}} \\ &= \iint_{\mathbb{R}^2} \sigma(b,\xi)(|\xi|^{1/s-1} S_\varphi f(b,\xi)(|\xi|^{1/s-1} \varphi^{b,\xi}) \frac{db d\xi}{|\xi|^{(2/s)-1}} \\ &= \iint \sigma(b,\xi) S_\varphi f(b,\xi) \psi^{b,\xi} \frac{db\, d\xi}{|\xi|} \\ &= L_{\sigma,\varphi,\psi} f. \end{aligned}$$

So our results in this paper can be extended to the localization operators associated with the modified Stockwell transform.

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SQUARE-INTEGRABLE GROUP REPRESENTATIONS AND LOCALIZATION OPERATORS FOR MODIFIED STOCKWELL TRANSFORMS

Dedicated to Professor Luigi Rodino on the occasion of his 60th birthday

Abstract. Recently discovered square-integrable group representations are used to study localization operators for the modified Stockwell transforms. The Schatten–von Neumann properties of these localization operators are established in this paper, and for trace class localization operators, the traces and the trace class norm inequalities are presented.

1. Introduction

Let $\varphi \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Then as a hybrid of the Gabor transform and the wavelet transform, the Stockwell transform $S_\varphi f$ of a signal f in $L^2(\mathbb{R})$ with respect to the window φ is defined by

$$(S_\varphi f)(b,\xi) = (2\pi)^{-1/2}|\xi| \int_{-\infty}^{\infty} e^{-ix\xi} f(x)\overline{\varphi(\xi(x-b))}\, dx$$

for all b in $\mathbb{R}$ and ξ in $\mathbb{R}\setminus\{0\}$. Alternatively, we can write for all f in $L^2(\mathbb{R})$, b in $\mathbb{R}$ and ξ in $\mathbb{R}\setminus\{0\}$,

$$(S_\varphi f)(b,\xi) = (f,\varphi^{b,\xi})_{L^2(\mathbb{R})},$$

where

$$\varphi^{b,\xi} = (2\pi)^{-1/2} M_\xi T_{-b} D_\xi^1 \varphi,$$

the modulation operator M_ξ, the translation operator T_{-b} and the dilation operator D_ξ^1 are defined by

$$(M_\xi h)(x) = e^{ix\xi} h(x),$$

$$(T_{-b} h)(x) = h(x-b),$$

$$(D_\xi^1 h)(x) = |\xi| h(\xi x),$$

for all x in $\mathbb{R}$ and all measurable functions h on $\mathbb{R}$.

The Stockwell transform is a versatile tool first introduced in [11]. More recent results on the Stockwell transform in the contexts of applications can be found in [6, 10]. The mathematical underpinnings of Stockwell transforms are developed in [4, 5, 7, 8, 9, 13].

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Prompted by applications in time-frequency analysis, the Stockwell transform S_φ has been extended in [6, 7] to a family $\{S_\varphi^s : 0 < s \le \infty\}$ of modified Stockwell transforms, which include the classical Stockwell transform when $s = 1$ and a variant of the wavelet transform when $s = 2$. To wit, for all functions in $L^2(\mathbb{R})$, the modified Stockwell transform $S_\varphi^s f$ of f for $0 < s \le \infty$ is defined by

$$(S_\varphi^s f)(b,\xi) = (f, \varphi_s^{b,\xi})_{L^2(\mathbb{R})}, \quad b \in \mathbb{R}, \xi \in \mathbb{R}\setminus\{0\},$$

where

$$\varphi_s^{b,\xi} = (2\pi)^{-1/2} M_\xi T_{-b} D_\xi^s \varphi,$$

and for all t in $(0,\infty]$ the dilation operator D_ξ^t is defined by

$$(D_\xi^t h)(x) = |\xi|^{1/t} h(\xi x)$$

for all x in $\mathbb{R}$ and all measurable functions h on $\mathbb{R}$. More explicitly,

$$(S_\varphi^s f)(b,\xi) = (2\pi)^{-1/2} |\xi|^{1/s} \int_{-\infty}^{\infty} e^{-ix\xi} f(x) \overline{\varphi(\xi(x-b))}\, dx$$

for all b in $\mathbb{R}$ and ξ in $\mathbb{R}\setminus\{0\}$. For a comparison with the classical Stockwell transform, we note that

$$(S_\varphi^s f)(b,\xi) = |\xi|^{-1/s'} (S_\varphi f)(b,\xi), \quad b \in \mathbb{R}, \xi \in \mathbb{R}\setminus\{0\}, \tag{1}$$

where s' is the conjugate index of s given by $1/s + 1/s' = 1$. An important property of the modified Stockwell transform is the following resolution of the identity formula in [6, 7].

THEOREM 1. *Let $\varphi \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ be such that*

$$\int_{-\infty}^{\infty} \varphi(x)\, dx = 1$$

and

$$\int_{-\infty}^{\infty} \frac{|\hat{\varphi}(\xi-1)|^2}{|\xi|}\, d\xi < \infty,$$

where $\hat{\varphi}$ is the Fourier transform of φ defined by

$$\hat{\varphi}(\xi) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} e^{-ix\xi} \varphi(x)\, dx, \quad \xi \in \mathbb{R}.$$

Then for all f and g in $L^2(\mathbb{R})$, we get for $0 < s \le \infty$,

$$(f,g)_{L^2(\mathbb{R})} = \frac{1}{c_\varphi} \int_{-\infty}^{\infty} (S_\varphi^s f)(b,\xi) \overline{(S_\varphi^s g)(b,\xi)} \frac{db\, d\xi}{|\xi|^{1-(2/s')}},$$

where

$$c_\varphi = \int_{-\infty}^{\infty} \frac{|\hat{\varphi}(\xi-1)|^2}{|\xi|} d\xi.$$

Based on the resolution of the identity formula in Theorem 1, localization operators can be introduced and their Schatten–von Neumann properties investigated. Results in this direction are announced in [8].

The aim of this paper is to use square-integrable group representations found by Boggiatto, Fernández and Galbis [4] to construct localization operators. These localization operators turn out to be the same as the localization operators based on the resolution of the identity formulas for the modified Stockwell transforms. From this fact follow the Schatten–von Neumann properties, a trace formula and trace class norm inequalities for the localization operators defined using the resolution of the identity formulas for the Stockwell transforms.

In Section 2, we give a brief recapitulation of Schatten–von Neumann classes and localization operators corresponding to square-integrable representations of locally compact and Hausdorff groups. Square-integrable group representations suggested by the one in [4] are summarized in Section 3. In Section 4, localization operators arising from these square-integrable representations are introduced. They are shown to coincide with localization operators defined using the resolution of the identity formulas for the modified Stockwell transforms. The Schatten–von Neumann properties of these localization operators are established, and the traces of trace class localization operators are computed. We give in Section 5 the trace class norm inequalities for the trace class localization operators studied in Section 4 and give an explicit formula for the function that occurs in the lower bound for the trace norm of such a trace class localization operator.

2. Schatten–von Neumann classes and localization operators

Let X be an infinite-dimensional, complex and separable Hilbert space in which the inner product and norm are denoted, respectively, by $(\cdot,\cdot)$ and $\|\cdot\|$. Let $A: X \to X$ be a compact operator. Then the operator $|A|: X \to X$ defined by

$$|A| = \sqrt{A^*A}$$

is positive and compact. So, using the spectral theorem, there exists for X an orthonormal basis $\{\varphi_k : k = 1,2,\ldots\}$ consisting of eigenvectors of $|A|$. For $k = 1,2,\ldots$, let s_k be the eigenvalue of $|A|: X \to X$ corresponding to the eigenvector φ_k. We say that the compact operator $A: X \to X$ is in the Schatten–von Neumann class S_p, $1 \le p < \infty$, if

$$\sum_{k=1}^{\infty} s_k^p < \infty.$$

If a compact operator $A: X \to X$ is in S_p, $1 \le p < \infty$, then we define the norm $\|A\|_{S_p}$ of A by

$$\|A\|_{S_p} = \left\{ \sum_{k=1}^{\infty} s_k^p \right\}^{1/p}.$$

By convention, the Schatten–von Neumann class S_∞ is taken to be simply the C^*-algebra $B(X)$ of all bounded linear operators on X and the norm $\|\cdot\|_{S_\infty}$ in S_∞ is simply the norm in $B(X)$.

Of particular interest is the Schatten–von Neumann class S_1, which is also known as the trace class. If a compact operator $A : X \to X$ is in the trace class S_1, then we can define the trace $\mathrm{tr}(A)$ of A by

$$\mathrm{tr}(A) = \sum_{k=1}^{\infty} (A\varphi_k, \varphi_k),$$

where $\{\varphi_k : k = 1, 2, \ldots\}$ is any orthonormal basis for X.

Let G be a locally compact and Hausdorff group on which the left Haar measure is denoted by $d\mu$. Let $U(X)$ be the group of all unitary operators on X and let $\pi : G \to U(X)$ be an irreducible and unitary representation of G on X. Suppose that the representation π is square-integrable in the sense that there exists a nonzero vector φ in X such that

$$\int_G |(\varphi, \pi(g)\varphi)|^2 d\mu(g) < \infty. \tag{2}$$

The condition (2) is known as the admissibility condition for the square-integrable representation of G on X. We call any vector φ for which $\|\varphi\| = 1$ and the admissibility condition (2) is fulfilled an admissible wavelet for the square-integrable representation of G on X. For any admissible wavelet φ, we define the constant c_φ by

$$c_\varphi = \int_G |(\varphi, \pi(g)\varphi)|^2 d\mu(g).$$

We need the following result, which is Theorem 14.5 in [12].

THEOREM 2. *Let φ be an admissible wavelet for a square-integrable representation $\pi : G \to U(X)$ of G on X. Let $F \in L^p(G)$, $1 \le p \le \infty$. For every x in X, we define $L_{F,\varphi}x$ in X by*

$$(L_{F,\varphi}x, y) = \frac{1}{c_\varphi} \int_G F(g)(x, \pi(g)\varphi)(\pi(g)\varphi, y)\, d\mu(g)$$

for all y in X. Then $L_{F,\varphi} : X \to X$ is in the Schatten–von Neumann class S_p and

$$\|L_{F,\varphi}\|_{S_p} \le c_\varphi^{-1/p} \|F\|_{L^p(G)}.$$

REMARK 1. The linear operator $L_{F,\varphi} : X \to X$ is called the localization operator for the transform $X \ni x \mapsto (x, \pi(\cdot)\varphi) \in L^2(G)$.

The following trace formula is given in Theorem 13.6 in [12].

THEOREM 3. *Let φ be an admissible wavelet for a square-integrable representation $\pi : G \to U(X)$ of G on X. Let $F \in L^1(G)$. Then the trace $\mathrm{tr}(L_{F,\varphi})$ of the trace* *class localization operator* $L_{F,\varphi}: X \to X$ *is given by*

$$\operatorname{tr}(L_{F,\varphi}) = \frac{1}{c_\varphi} \int_G F(g)\, d\mu(g).$$

A lower bound for the norm $\|L_{F,\varphi}\|_{S_1}$ of the trace class localization operator $L_{F,\varphi}: X \to X$ can be given in terms of the function F_φ on G defined by

$$F_\varphi(g) = (L_{F,\varphi}\pi(g)\varphi, \pi(g)\varphi), \quad g \in G.$$

Indeed, we have the following result, which is Theorem 14.1 in [12].

THEOREM 4. *Let* φ *be an admissible wavelet for a square-integrable representation* $\pi: G \to U(X)$ *of* G *on* X. *Let* $F \in L^1(G)$. *Then*

$$\frac{1}{c_\varphi}\|F_\varphi\|_{L^1(G)} \le \|L_{F,\varphi}\|_{S_1} \le \frac{1}{c_\varphi}\|F\|_{L^1(G)}.$$

The function F_φ is the expectation value of the observable $L_{F,\varphi}: X \to X$ in the coherent states $\pi(g)\varphi$, $g \in G$. Information about coherent states and related topics can be found in [1, 2, 3].

3. Square-integrable representations

Let $\mathbb{G}$ be the set $\mathbb{R} \times (\mathbb{R} \setminus \{0\}) \times \mathbb{S}^1$, where $\mathbb{S}^1$ is the unit circle centered at the origin. If we identify $\mathbb{S}^1$ with the interval $[-\pi, \pi]$, then $\mathbb{G}$ becomes a group with respect to the multiplication $\cdot$ given by

$$(b_1, \xi_1, \theta_1) \cdot (b_2, \xi_2, \theta_2) = \left(b_1 + \frac{b_2}{\xi_1}, \xi_1\xi_2, \theta_1 + \theta_2 + b_1\xi_1(1 - \xi_2) \right)$$

for all (b_1, ξ_1, θ_1) and (b_2, ξ_2, θ_2) in $\mathbb{G}$. In fact, $\mathbb{G}$ is a Lie group on which the left Haar measure is just the Lebesgue measure. For future reference, let us note that $(0, 1, 0)$ is the identity element in $\mathbb{G}$ and

$$(b, \xi, \theta)^{-1} = (-b\xi, 1/\xi, -\theta + b(1 - \xi))$$

for all (b, ξ, θ) in $\mathbb{G}$. For $\alpha \in (-\infty, 1)$, we let H_α be the set defined by

$$H_\alpha = \left\{ f \in \mathcal{S}'(\mathbb{R}) : \int_{-\infty}^{\infty} |\hat{f}(u)|^2 |u|^\alpha du < \infty \right\}.$$

Then H_α becomes a Hilbert space in which the inner product $(\cdot, \cdot)_{H_\alpha}$ and the norm $\|\cdot\|_{H_\alpha}$ are given by

$$(f, g)_{H_\alpha} = \int_{-\infty}^{\infty} \hat{f}(u)\overline{\hat{g}(u)}|u|^\alpha du$$

and

$$\|f\|^2_{H_\alpha} = \int_{-\infty}^{\infty} |\hat{f}(u)|^2 |u|^\alpha du$$

for all f and g in H_α. We assume throughout this paper that $\alpha \in (-\infty, 1)$.

Let $U(H_\alpha)$ be the group of all unitary operators on H_α. Then we define the mapping $\rho_\alpha : \mathbb{G} \to U(H_\alpha)$ by

$$\rho_\alpha(b,\xi,\theta) f = e^{i(\theta + b\xi)} |\xi|^{-(\alpha+1)/2} \pi_{b,\xi} f \tag{3}$$

for all (b,ξ,θ) in $\mathbb{G}$ and all f in H_α, where

$$(\pi_{b,\xi} f)(x) = |\xi| f(\xi(x-b)), \quad x \in \mathbb{R}.$$

THEOREM 5. *$\rho_\alpha : \mathbb{G} \to U(H_\alpha)$ is an irreducible and unitary representation of $\mathbb{G}$ on H_α.*

In fact, the following theorem tells us much more about the representation $\rho_\alpha : \mathbb{G} \to U(H_\alpha)$.

THEOREM 6. *The representation $\rho_\alpha : \mathbb{G} \to U(H_\alpha)$ of the group $\mathbb{G}$ on H_α is square-integrable.*

THEOREM 7. *Let $\varphi \in H_{-\alpha,1} \cap H_{-\alpha-1,1}$, where*

$$H_{\beta,1} = \left\{ f \in \mathcal{S}' : \int_{-\infty}^{\infty} |\hat{f}(u)|^2 |u+1|^\beta du < \infty \right\}, \quad \beta \in (-\infty, 1).$$

Let ψ be the function on $\mathbb{R}$ defined by

$$\psi(t) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} e^{itw} |w|^{-\alpha} \hat{\varphi}(w-1)\, dw, \quad t \in \mathbb{R}.$$

Then for all f in H_α, the modified Stockwell transform $S^s_\varphi f$ of f for $0 < s \le \infty$, is given by

$$(f, \rho_\alpha(b,\xi,\theta)\psi)_{H_\alpha} = (2\pi)^{1/2} e^{-i\theta} |\xi|^{(\alpha+1)/2-(1/s)} (S^s_\varphi f)(b,\xi), \quad (b,\xi,\theta) \in \mathbb{G}.$$

REMARK 2. In fact,

$$\psi = \mathcal{F}^{-1}(|\bullet|^{-\alpha} \hat{\varphi}(\bullet - 1)),$$

where $\mathcal{F}^{-1}$ denotes the inverse Fourier transform.

THEOREM 8. *Let f and g be in H_α. Then for $0 < s \le \infty$ and for all $\varphi \in H_{-\alpha-1,1}$,*

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (S^s_\varphi f)(b,\xi) \overline{(S^s_\varphi g)(b,\xi)} \frac{db\, d\xi}{|\xi|^{(2/s)-(\alpha+1)}} = \|\varphi\|^2_{H_{-\alpha-1,1}} (f,g)_{H_\alpha}.$$

Theorem 8 can be seen as another set of resolution of the identity formulas for the modified Stockwell transforms S^s_φ, $0 < s \le \infty$, and is the basis for the localization operators studied in the following section.

4. Localization operators for modified Stockwell transforms

Let φ be a nonzero function in $H_{-\alpha,1}\cap H_{-\alpha-1,1}$ and let ψ be the function on $\mathbb{R}$ defined by

$$\psi(t)=(2\pi)^{-1/2}\int_{-\infty}^{\infty}e^{itw}|w|^{-\alpha}\hat{\varphi}(w-1)\,dw,\quad t\in\mathbb{R}. \tag{4}$$

Without loss of generality, we can choose φ in such a way that

$$\|\psi\|_{H_\alpha}=1. \tag{5}$$

Let $F\in L^p(\mathbb{G})$, $1\le p\le\infty$. Then for all f in H_α, we define $\tilde{L}_{F,\psi}f$ by

$$\begin{aligned}&(\tilde{L}_{F,\psi}f,g)_{H_\alpha}\\ &= \frac{1}{c_\psi}\int_{-\pi}^{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}F(b,\xi,\theta)(f,\rho_\alpha(b,\xi,\theta)\psi)_{H_\alpha}(\rho_\alpha(b,\xi,\theta)\psi,g)_{H_\alpha}db\,d\xi\,d\theta\end{aligned}$$

for all g in H_α, where

$$c_\psi=\int_{-\pi}^{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|(\psi,\rho_\alpha(b,\xi,\theta)\psi)_{H_\alpha}|^2db\,d\xi\,d\theta.$$

LEMMA 1. *Let φ and ψ be as in* (4) *and* (5)*. Then*

$$c_\psi=4\pi^2\|\varphi\|^2_{H_{-\alpha-1,1}}.$$

Proof. We note that

$$\begin{aligned}&\int_{-\pi}^{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|(\psi,\rho_\alpha(b,\xi,\theta)\psi)_{H_\alpha}|^2db\,d\xi\,d\theta\\ &= \int_{-\pi}^{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|(\psi,e^{i(\theta+b\xi)}|\xi|^{-(\alpha+1)/2}\pi_{b,\xi}\psi)_{H_\alpha}|^2db\,d\xi\,d\theta\\ &= \int_{-\pi}^{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|(\psi,|\xi|^{-(\alpha+1)/2}\pi_{b,\xi}\psi)_{H_\alpha}|^2db\,d\xi\,d\theta.\end{aligned}$$

By Theorem 7,

$$(\psi,e^{ib\xi}\pi_{b,\xi}\psi)_{H_\alpha}=(2\pi)^{1/2}|\xi|^{\alpha}(S_\varphi\psi)(b,\xi),\quad b\in\mathbb{R},\quad \xi\in\mathbb{R}\setminus\{0\},$$

and hence

$$(\psi,\pi_{b,\xi}\psi)_{H_\alpha}=(2\pi)^{1/2}e^{ib\xi}|\xi|^{\alpha}(S_\varphi\psi)(b,\xi),\quad b\in\mathbb{R},\quad \xi\in\mathbb{R}\setminus\{0\}.$$

So,

$$\begin{aligned}&\int_{-\pi}^{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|(\psi,\rho_\alpha(b,\xi,\theta)\psi)_{H_\alpha}|^2db\,d\xi\,d\theta\\ &= 4\pi^2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|(S_\varphi\psi)(b,\xi)|^2\frac{db\,d\xi}{|\xi|^{1-\alpha}}=4\pi^2\|\varphi\|^2_{H_{-\alpha-1,1}}\|\psi\|^2_{H_\alpha}.\end{aligned}$$

Since $\|\psi\|_{H_\alpha}=1$, the lemma is proved. □

Now, by Theorem 2, we can conclude that $\tilde{L}_{F,\psi}: H_\alpha \to H_\alpha$ is in the Schatten–von Neumann class S_p. This fact can be used to prove the following result.

THEOREM 9. *Let* φ *be as given in* (4) *and* (5), $F \in L^p(\mathbb{R}\times\mathbb{R})$, $1 \le p \le \infty$. *If for* $0 < s \le \infty$, *we define* $L^s_{F,\varphi} f$ *for all* f *in* H_α *by*

$$\begin{aligned}&(L^s_{F,\varphi} f, g)_{H_\alpha}\\ &= \frac{1}{\|\varphi\|^2_{H_{-\alpha-1,1}}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(b,\xi)\,(S^s_\varphi f)(b,\xi)\overline{(S^s_\varphi g)(b,\xi)}\,\frac{db\,d\xi}{|\xi|^{(2/s)-(\alpha+1)}}\end{aligned}$$

for all g *in* H_α, *then* $L^s_{F,\varphi}: H_\alpha \to H_\alpha$ *is in the Schatten–von Neumann class* S_p. *Moreover,*

$$\|L^s_{F,\varphi}\|_{S_p} \le (2\pi\|\varphi\|^2_{H_{-\alpha-1,1}})^{-1/p}\|F\|_{L^p(\mathbb{R}\times\mathbb{R})}.$$

Proof. If we define the function $\tilde{F}$ on $\mathbb{G}$ by

$$\tilde{F}(b,\xi,\theta) = F(b,\xi), \quad (b,\xi,\theta)\in\mathbb{G},$$

then $\tilde{F} \in L^p(\mathbb{G})$. But for all f and g in H_α, we get by (3)

$$\begin{aligned}&(\tilde{L}_{\tilde{F},\psi} f, g)_{H_\alpha}\\ (6)\quad &= \frac{1}{c_\psi}\int_{-\pi}^{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\tilde{F}(b,\xi,\theta)(f,\rho_\alpha(b,\xi,\theta)\psi)_{H_\alpha}(\rho_\alpha(b,\xi,\theta)\psi,g)_{H_\alpha}\,db\,d\xi\,d\theta\\ &= \frac{2\pi}{c_\psi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}F(b,\xi)(f,\pi_{b,\xi}\psi)_{H_\alpha}(\pi_{b,\xi}\psi,g)_{H_\alpha}|\xi|^{-(\alpha+1)}\,db\,d\xi.\end{aligned}$$

By (1), (3) and Theorem 7, we have

$$(7)\qquad (f,\pi_{b,\xi}\psi)_{H_\alpha} = (2\pi)^{1/2}e^{-ib\xi}|\xi|^{\alpha+1-(1/s)}(S^s_\varphi f)(b,\xi)$$

and

$$(8)\qquad (\pi_{b,\xi}\psi,g)_{H_\alpha} = (2\pi)^{1/2}e^{ib\xi}|\xi|^{\alpha+1-(1/s)}\overline{(S^s_\varphi g)(b,\xi)}$$

for all b in $\mathbb{R}$ and ξ in $\mathbb{R}\setminus\{0\}$. Putting (7) and (8) in (6) and using Lemma 1, we get for all f and g in H_α,

$$\begin{aligned}&(\tilde{L}_{\tilde{F},\psi} f, g)_{H_\alpha}\\ (9)\quad &= \frac{4\pi^2}{c_\psi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}F(b,\xi)\,(S^s_\varphi f)(b,\xi)\,\overline{(S^s_\varphi g)(b,\xi)}\,\frac{db\,d\xi}{|\xi|^{(2/s)-(\alpha+1)}}\\ &= (L^s_{F,\varphi} f,g)_{H_\alpha}.\end{aligned}$$

So, $L^s_{F,\varphi}: H_\alpha \to H_\alpha$ is the same as $\tilde{L}_{\tilde{F},\varphi}: H_\alpha \to H_\alpha$ and is hence in the Schatten–von Neumann class S_p. Finally, using the inequality in Theorem 2 and Lemma 1, we get

$$\|L^s_{F,\varphi}\|_{S_p} = \|\tilde{L}_{\tilde{F},\psi}\|_{S_p} \le \frac{1}{c_\psi}\|\tilde{F}\|_{L^p(\mathbb{G})} = (2\pi\|\varphi\|^2_{H_{-\alpha-1,1}})^{-1/p}\|F\|_{L^p(\mathbb{R}\times\mathbb{R})}.$$

□

REMARK 3. It is important to bring out the fact that by (9), the localization operators $L^s_{F,\varphi}$ are all equal to $\tilde{L}_{\tilde{F},\psi}$ for all s in $(0,\infty]$. This fact can also be seen from the formula (1). Notwithstanding the variety of Hilbert spaces H_α offered by Theorem 9, it is to be noted, however, that in view of applications to signal analysis and imaging, the Hilbert space H_0, i.e., $L^2(\mathbb{R})$, is most commonly used.

A formula for traces of localization operators for the modified Stockwell transforms is given in the following theorem.

THEOREM 10. *Let φ be as given in* (4) *and* (5). *Then for all functions F in $L^1(\mathbb{R}\times\mathbb{R})$, the trace* $\mathrm{tr}(L^s_{F,\varphi})$ *of the trace class localization operator $L^s_{F,\varphi}: H_\alpha \to H_\alpha$, $0 < s \le \infty$, is given by*

$$\mathrm{tr}(L^s_{F,\varphi}) = \frac{1}{2\pi\|\varphi\|^2_{H_{-\alpha-1,1}}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(b,\xi)\,db\,d\xi.$$

Proof. By Theorem 3, Lemma 1 and (9), we get

$$\begin{aligned}\mathrm{tr}(L^s_{F,\varphi}) &= \mathrm{tr}(\tilde{L}_{\tilde{F},\psi})\\ &= \frac{1}{c_\psi}\int_{-\pi}^{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\tilde{F}(b,\xi,\theta)\,db\,d\xi\,d\theta\\ &= \frac{1}{2\pi\|\varphi\|^2_{H_{-\alpha-1,1}}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}F(b,\xi)\,db\,d\xi,\end{aligned}$$

as required. □

5. Trace class norm inequalities

We give in this section a result on the trace class norm inequalities for the localization operators $L^s_{F,\varphi}: H_\alpha \to H_\alpha$, $0 < s \le \infty$.

THEOREM 11. *Let φ be as given in* (4) *and* (5). *Then for all functions F in $L^1(\mathbb{R}\times\mathbb{R})$, we get for $0 < s \le \infty$*

$$\frac{1}{2\pi\|\varphi\|^2_{H_{-\alpha-1,1}}}\|F_\varphi\|_{L^1(\mathbb{R}\times\mathbb{R})} \le \|L^s_{F,\varphi}\|_{S_1} \le \frac{1}{2\pi\|\varphi\|^2_{H_{-\alpha-1,1}}}\|F\|_{L^1(\mathbb{R}\times\mathbb{R})},$$

where

$$F_\varphi(b,\xi) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}F(b',\xi')\left|\left(\varphi, T_{\xi(b-b')}M_{(\xi'/\xi)-1}D^{2/(\alpha+1)}_{\xi'/\xi}\varphi\right)_{H_{-\alpha,1}}\right|^2 db'\,d\xi'$$

for all b in $\mathbb{R}$ and ξ in $\mathbb{R}\setminus\{0\}$.

To prove Theorem 11, we note that by Theorem 9, we only need to establish the lower bound for $\|L^s_{F,\varphi}\|_{S_1}$. To that end, let us recall that by Remark 3,

$$L^s_{F,\varphi} = \tilde{L}_{\tilde{F},\psi},$$

where ψ is given in (4) and

$$\tilde{F}(b,\xi,\theta) = F(b,\xi), \quad (b,\xi,\theta) \in \mathbb{G}.$$

So, by Theorem 4,

$$\frac{1}{c_\psi}\|\tilde{F}_\psi\|_{L^1(\mathbb{G})} \le \|L^s_{F,\varphi}\|_{S_1}.$$

By Lemma 1, Theorem 11 is proved if we show that

$$\|\tilde{F}_\psi\|_{L^1(G)} = 2\pi\|F_\varphi\|_{L^1(\mathbb{R}\times\mathbb{R})},$$

where

$$\tilde{F}_\psi(b,\xi,\theta) = \left(\tilde{L}_{\tilde{F},\psi}\rho_\alpha(b,\xi,\theta)\psi, \rho_\alpha(b,\xi,\theta)\psi\right)_{H_\alpha}, \quad (b,\xi,\theta) \in \mathbb{G}.$$

This follows from the next formula.

THEOREM 12. *Under the hypotheses of Theorem* 11,

$$\begin{aligned}&\tilde{F}_\psi(b,\xi,\theta)\\ &= \frac{2\pi}{c_\psi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(b',\xi')\left|\left(\varphi, T_{\xi(b-b')}M_{(\xi'/\xi)-1}D^{2/(\alpha+1)}_{\xi'/\xi}\varphi\right)_{H_{-\alpha,1}}\right|^2 db'\,d\xi'.\end{aligned}$$

We give two proofs of Theorem 12. The first proof is based on the explicit Fourier transform of $\rho_\alpha(b,\xi,\theta)\psi$ for all (b,ξ,θ) in $\mathbb{G}$ and the second one, in which the same Fourier transform is still a key ingredient, explicates the use of the underlying group structure.

First proof of Theorem 12. The starting point is the formula

$$(10)\qquad (\rho_\alpha(b,\xi,\theta)\psi)^\wedge(u) = e^{i(\theta+b\xi)}|\xi|^{(\alpha-1)/2}e^{-ibu}\hat{\varphi}\left(\frac{u}{\xi}-1\right)|u|^{-\alpha}, \quad u \in \mathbb{R},$$

for all (b,ξ,θ) in $\mathbb{G}$. So

$$\begin{aligned}&\left|\left(\rho_\alpha(b,\xi,\theta)\psi, \rho_\alpha(b',\xi',\theta')\psi\right)_{H_\alpha}\right|\\ &= \left|\int_{-\infty}^{\infty}(\rho_\alpha(b,\xi,\theta)\psi)^\wedge(u)\overline{(\rho_\alpha(b',\xi',\theta')\psi)^\wedge(u)}|u|^\alpha du\right|\\ &= |\xi\xi'|^{(\alpha-1)/2}\left|\int_{-\infty}^{\infty}\hat{\varphi}\left(\frac{u}{\xi}-1\right)\overline{\hat{\varphi}\left(\frac{u}{\xi'}-1\right)}e^{-i(b-b')u}|u|^{-\alpha}du\right|\end{aligned}$$

$$\begin{aligned}
&= |\xi\xi'|^{(\alpha-1)/2}\left|\int_{-\infty}^{\infty}\hat{\varphi}(v)\overline{\hat{\varphi}\left(\frac{\xi(v+1)-\xi'}{\xi'}\right)}e^{-i(b-b')\xi v}|v+1|^{-\alpha}|\xi|^{1-\alpha}dv\right|\\
&= \left|\frac{\xi'}{\xi}\right|^{(\alpha-1)/2}\left|\int_{-\infty}^{\infty}\hat{\varphi}(u)\overline{e^{i(b-b')\xi v}\hat{\varphi}\left(\frac{v+1-(\xi'/\xi)}{\xi'/\xi}\right)}|v+1|^{-\alpha}dv\right|\\
&= \left|\frac{\xi'}{\xi}\right|^{(\alpha-1)/2}\left|\int_{-\infty}^{\infty}\hat{\varphi}(u)\left|\frac{\xi'}{\xi}\right|\overline{(T_{(b-b')\xi}M_{(\xi'/\xi)-1}D_{\xi'/\xi}\varphi)^{\wedge}(v)}|v+1|^{-\alpha}dv\right|\\
&= \left|\int_{-\infty}^{\infty}\hat{\varphi}(v)\overline{(T_{(b-b')\xi}M_{(\xi'/\xi)-1}D_{\xi'/\xi}^{2/(\alpha+1)}\varphi)^{\wedge}(v)}|v+1|^{-\alpha}dv\right|\\
&= \left|\left(\varphi,T_{(b-b')\xi}M_{(\xi'/\xi)-1}D_{\xi'/\xi}^{2/(\alpha+1)}\varphi\right)_{H_{-\alpha,1}}\right|
\end{aligned}$$

for all (b,ξ,θ) in $\mathbb{G}$. Hence

$$\begin{aligned}
&\tilde{F}_{\psi}(b,\xi,\theta)\\
&= \frac{1}{c_{\psi}}\int_{-\pi}^{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}F(b',\xi')|(\rho_{\alpha}(b,\xi,\theta)\psi,\rho_{\alpha}(b',\xi',\theta')\psi)_{H_{\alpha}}|^{2}db'\,d\xi'\,d\theta'\\
&= \frac{2\pi}{c_{\psi}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}F(b',\xi')\left|\left(\varphi,T_{\xi(b-b')}M_{(\xi'/\xi)-1}D_{\xi'/\xi}^{2/(\alpha+1)}\varphi\right)_{H_{-\alpha,1}}\right|^{2}db'\,d\xi'
\end{aligned}$$

for all (b,ξ,θ) in $\mathbb{G}$, as claimed. □

Second proof of Theorem 12. Since $\rho_{\alpha}:\mathbb{G}\to U(H_{\alpha})$ is a unitary representation of $\mathbb{G}$ on H_{α}, it follows that

$$\begin{aligned}
&\left(\rho_{\alpha}(b,\xi,\theta)\psi,\rho_{\alpha}(b',\xi',\theta')\psi\right)_{H_{\alpha}}\\
&= (\psi,\rho_{\alpha}((b,\xi,\theta)^{-1}\cdot(b',\xi',\theta'))\psi)_{H_{\alpha}}\\
&= (\psi,\rho_{\alpha}(\xi(b'-b),\xi'/\xi,(\theta'-\theta+b(\xi'-\xi))\psi)_{H_{\alpha}}.
\end{aligned}$$

To simplify notation, we let $\tilde{b}=\xi(b'-b)$, $\tilde{\xi}=\xi'/\xi$ and $\tilde{\theta}=(\theta'-\theta)+b(\xi'-\xi)$. Then by (10),

$$\begin{aligned}
&\left(\rho_{\alpha}(b,\xi,\theta)\psi,\rho_{\alpha}(b',\xi',\theta')\psi\right)_{H_{\alpha}}\\
&= \left(\psi,\rho_{\alpha}(\tilde{b},\tilde{\xi},\tilde{\theta})\psi\right)_{H_{\alpha}}\\
&= \int_{-\infty}^{\infty}\hat{\psi}(u)\overline{(\rho_{\alpha}(\tilde{b},\tilde{\xi},\tilde{\theta})\psi)^{\wedge}(u)}|u|^{\alpha}du\\
&= \int_{-\infty}^{\infty}|u|^{-\alpha}\hat{\varphi}(u-1)e^{-i(\tilde{\theta}+\tilde{b}\tilde{\xi})}e^{i\tilde{b}u}|\tilde{\xi}|^{(\alpha-1)/2}\overline{\hat{\varphi}\left(\frac{u}{\tilde{\xi}}-1\right)}|u|^{-\alpha}|u|^{\alpha}du
\end{aligned}$$

$$
\begin{aligned}
&= e^{-i(\tilde\theta+\tilde b\tilde\xi)}|\tilde\xi|^{(\alpha-1)/2}\int_{-\infty}^{\infty}\hat\varphi(u-1)e^{i\tilde b u}\overline{\hat\varphi\left(\frac{u}{\tilde\xi}-1\right)}|u|^{-\alpha}du\\
&= e^{i\tilde b}e^{-i(\tilde\theta+\tilde b\tilde\xi)}|\tilde\xi|^{(\alpha-1)/2}\int_{-\infty}^{\infty}\hat\varphi(v)e^{i\tilde b v}\overline{\hat\varphi\left(\frac{v+1-\tilde\xi}{\tilde\xi}\right)}|v+1|^{-\alpha}dv\\
&= e^{i\tilde b}e^{-i(\tilde\theta+\tilde b\tilde\xi)}|\tilde\xi|^{(\alpha-1)/2}\int_{-\infty}^{\infty}\hat\varphi(v)\overline{(M_{-\tilde b}T_{1-\tilde\xi}D_{1/\tilde\xi}\hat\varphi)(v)}|v+1|^{-\alpha}dv\\
&= e^{i\tilde b}e^{-i(\tilde\theta+\tilde b\tilde\xi)}|\tilde\xi|^{(\alpha+1)/2}\int_{-\infty}^{\infty}\hat\varphi(v)\overline{(T_{-\tilde b}M_{\tilde\xi-1}D_{\tilde\xi}\varphi)^\wedge(v)}|v+1|^{-\alpha}dv\\
&= e^{i\tilde b}e^{-i(\tilde\theta+\tilde b\tilde\xi)}\left(\varphi, T_{-\tilde b}M_{\tilde\xi-1}D_{\tilde\xi}^{2/(\alpha+1)}\varphi\right)_{H_{-\alpha,1}}
\end{aligned}
$$

and then we can proceed as in the first proof. □

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CHARACTERISTIC INITIAL BOUNDARY VALUE PROBLEMS FOR SYMMETRIZABLE SYSTEMS

Dedicated to Professor Luigi Rodino on the occasion of his 60th birthday

Abstract. We consider an initial-boundary value problem for a linear Friedrichs symmetrizable system, with characteristic boundary of constant rank. Assuming that the problem is L^2 well posed, we show the regularity of the L^2 solution, for sufficiently smooth data, in the framework of anisotropic Sobolev spaces.

1. Introduction

We consider an initial boundary value problem for a linear Friedrichs symmetrizable system, with characteristic boundary of constant rank. It is well-known that for solutions of symmetric or symmetrizable hyperbolic systems with characteristic boundary full regularity (i.e. solvability in the usual Sobolev spaces H^m) cannot be expected generally because of the possible loss of derivatives in the normal direction to the boundary, see [23, 12].

The natural space is the anisotropic Sobolev space H^m_*, which comes from the observation that the one-order gain of normal differentiation should be compensated by two-order loss of tangential differentiation (cf. [4]). The theory has been developed mostly for characteristic boundaries of *constant multiplicity* (see the definition in assumption (B)) and *maximally nonnegative* boundary conditions, see [4, 5, 11, 16, 17, 18, 19, 21].

However, there are important characteristic problems of physical interest where boundary conditions are not maximally nonnegative. Under the more general *Kreiss-Lopatinski condition* (KL), the theory has been developed for problems satisfying the *uniform* KL condition with *uniformly* characteristic boundaries (when the boundary matrix has constant rank in a neighborhood of the boundary), see [8, 1] and references therein.

In this paper we are interested in the problem of the regularity. We assume the existence of the strong L^2 solution, satisfying a suitable energy estimate, without assuming any structural assumption sufficient for existence, such as the fact that the boundary conditions are maximally dissipative or satisfy the Kreiss–Lopatinski condition. We show that this is enough in order to get the regularity of solutions, in the natural framework of weighted anisotropic Sobolev spaces H^m_*, provided the data are sufficiently smooth. Obviously, the present results contain in particular what has been previously obtained for maximally nonnegative boundary conditions.

Let Ω be an open bounded subset of $\mathbb{R}^n$ (for a fixed integer $n \geq 2$), lying locally on one side of its smooth, connected boundary $\Gamma := \partial\Omega$. For any real $T > 0$, we set $Q_T := \Omega \times]0,T[$ and $\Sigma_T := \Gamma \times]0,T[$; in addition we define $Q_\infty := \Omega \times [0,+\infty[$, $\Sigma_\infty := \partial\Omega \times [0,+\infty[$, $Q := \Omega \times \mathbb{R}$ and $\Sigma := \partial\Omega \times \mathbb{R}$. We are interested in the following initial boundary value problem (written in the sequel IBVP)

$$Lu = F, \quad \text{in } Q_T \tag{1}$$

$$Mu = G, \quad \text{on } \Sigma_T \tag{2}$$

$$u_{|t=0} = f, \quad \text{in } \Omega, \tag{3}$$

where L is the first order linear partial differential operator

$$L = \partial_t + \sum_{i=1}^{n} A_i(x,t)\partial_i + B(x,t), \tag{4}$$

$\partial_t := \frac{\partial}{\partial t}$, $\partial_i := \frac{\partial}{\partial x_i}$, $i = 1,\dots,n$ and $A_i(x,t), B(x,t)$ are $N \times N$ real matrix-valued functions of (x,t), for a given integer size $N \geq 1$, defined over Q_∞. The unknown $u = u(x,t)$ and the data $F = F(x,t)$, $f = f(x)$ are real vector-valued functions with N components, defined on $\overline{Q}_T$ and $\overline{\Omega}$ respectively. In the boundary conditions (2), M is a smooth $d \times N$ matrix-valued function of (x,t), defined on Σ_∞, with maximal constant rank d. The boundary datum $G = G(x,t)$ is a d-vector valued function, defined on $\overline{\Sigma}_T$.

Let us denote by $\nu(x) := (\nu_1(x),\dots,\nu_n(x))$ the unit outward normal to Γ at the point $x \in \Gamma$; then

$$A_\nu(x,t) = \sum_{i=1}^{n} A_i(x,t)\nu_i(x), \quad (x,t) \in \Sigma_\infty, \tag{5}$$

is the *boundary matrix*. Let $P(x,t)$ be the orthogonal projection onto the orthogonal complement of $\ker A_\nu(x,t)$, denoted $\ker A_\nu(x,t)^\perp$; it is defined by

$$P(x,t) = \frac{1}{2\pi i}\int_{C(x,t)} (\lambda - A_\nu(x,t))^{-1}\, d\lambda, \quad (x,t) \in \Sigma_\infty, \tag{6}$$

where $C(x,t)$ is a closed rectifiable Jordan curve with positive orientation in the complex plane, enclosing all and only all non-zero eingenvalues of $A_\nu(x,t)$. Denoting again by P an arbitrary smooth extension on $\overline{Q}_\infty$ of the above projection, Pu and $(I-P)u$ are called respectively the *non characteristic* and the *characteristic* components of the vector field $u = u(x,t)$.

We study the problem (1)–(3) under the following assumptions:

(A) The operator L is Friedrichs symmetrizable, meaning that for all $(x,t) \in \overline{Q}_\infty$ there exists a symmetric positive definite matrix $S_0(x,t)$ such that the matrices $S_0(x,t)A_i(x,t)$, $i = 1,\cdots,n$, are also real symmetric; this implies, in particular, that the *symbol* $A(x,t,\xi) = \sum_{i=1}^{n} A_i(x,t)\xi_i$ is diagonalizable with real eigenvalues, whenever $(x,t,\xi) \in \overline{Q}_\infty \times \mathbb{R}^n$.

(B) The boundary is *characteristic, with constant rank*, namely the boundary matrix A_ν is singular on Σ_∞ and has constant rank $0 < r := \mathrm{rank} A_\nu(x,t) < N$ for all $(x,t) \in \Sigma_\infty$; this assumption, together with the symmetrizability of L and that Γ is connected, yields that the number of negative eigenvalues of A_ν (the so-called *incoming modes*) remains constant on Σ_∞.

(C) $\ker A_\nu(x,t) \subseteq \ker M(x,t)$, for all $(x,t) \in \Sigma_\infty$; moreover $d = \mathrm{rank} M(x,t)$ must equal the number of negative eigenvalues of $A_\nu(x,t)$.

(D) The orthogonal projection $P(x,t)$ onto $\ker A_\nu(x,t)^\perp$, $(x,t) \in \Sigma_\infty$, can be extended as a matrix-valued C^∞ function over $\overline{Q}_\infty$.

Concerning the solvability of the IBVP (1)–(3), we state the following well-posedness assumption:

(E) *Existence of the L^2 weak solution.* Assume that $S_0,\ A_i \in \mathrm{Lip}(\overline{Q}_\infty)$ for $i = 1,\dots,n$. For all $T > 0$ and all matrices $B \in L^\infty(\overline{Q}_T)$, there exist constants $\gamma_0 \geq 1$ and $C_0 > 0$ such that for all $F \in L^2(Q_T)$, $G \in L^2(\Sigma_T)$, $f \in L^2(\Omega)$ there exists a unique solution $u \in L^2(Q_T)$ of (1)–(3), with data (F, G, f), satisfying the following properties:

i. $u \in C([0,T]; L^2(\Omega))$;

ii. $Pu_{|\Sigma_T} \in L^2(\Sigma_T)$;

iii. for all $\gamma \geq \gamma_0$ and $0 < \tau \leq T$ the solution u enjoys the following a priori estimate

$$
\begin{aligned}
& e^{-2\gamma\tau}\|u(\tau)\|^2_{L^2(\Omega)} + \gamma \int_0^\tau e^{-2\gamma t}\|u(t)\|^2_{L^2(\Omega)}\,dt \\
& \qquad + \int_0^\tau e^{-2\gamma t}\|Pu_{|\partial\Omega}(t)\|^2_{L^2(\partial\Omega)}\,dt \\
& \leq C_0\left(\|f\|^2_{L^2(\Omega)} + \int_0^\tau e^{-2\gamma t}\left(\frac{1}{\gamma}\|F(t)\|^2_{L^2(\Omega)} + \|G(t)\|^2_{L^2(\partial\Omega)}\right)dt\right).
\end{aligned} \tag{7}
$$

When the IBVP (1)–(3) admits an a priori estimate of type (7), with $F = Lu$, $G = Mu$, for all $\tau > 0$ and all sufficiently smooth functions u, one says that the problem is *strongly L^2 well posed*, see e.g. [1]. A necessary condition for (7) is the validity of the *uniform Kreiss-Lopatinski condition* (UKL) (an estimate of type (7) has been obtained by Rauch [13]). On the other hand, UKL is not sufficient for the well posedness and other structural assumptions have to be taken into account, see [1].

Finally, we require the following technical assumption that for C^∞ approximations of problem (1)–(3) one still has the existence of L^2 solutions. This stability property holds true for maximally nonnegative boundary conditions and for uniform KL conditions.

(F) Given matrices $(S_0, A_i, B) \in \mathcal{C}_T(H^\sigma_*) \times \mathcal{C}_T(H^\sigma_*) \times \mathcal{C}_T(H^{\sigma-2}_*)$, where $\sigma \geq [\frac{n+1}{2}]+4$, enjoying properties (A)–(E), let $(S_0^{(k)}, A_i^{(k)}, B^{(k)})$ be C^∞ matrix-valued functions converging to (S_0, A_i, B) in $\mathcal{C}_T(H^\sigma_*) \times \mathcal{C}_T(H^\sigma_*) \times \mathcal{C}_T(H^{\sigma-2}_*)$ as $k \to \infty$, and satisfying properties (A)–(D). Then, for k sufficiently large, property (E) holds also for the approximating problems with coefficients $(S_0^{(k)}, A_i^{(k)}, B^{(k)})$.

The solution of (1)–(3), considered in the statements (E), (F), must be intended in the sense of Rauch [15]. This means that for all $v \in H^1(Q_T)$ such that $v_{|\Sigma_T} \in (A_\nu(\ker M))^\perp$ and $v(T,\cdot) = 0$ in Ω, there holds:

$$\int_0^T \langle u(t), L^* v(t)\rangle\, dt = \int_0^T \langle F(t), v(t)\rangle\, dt - \int_{\Sigma_T} \langle A_\nu g, v\rangle\, d\sigma_x\, dt + \int_\Omega \langle f, v(0)\rangle\, dx,$$

where L^* is the adjoint operator of L and g is a function defined on Σ_T such that $Mg = G$. Notice also that for such a weak solution to (1)–(3), the boundary condition (2) makes sense. Indeed, in [15, Theorem 1] it is shown that for any $u \in L^2(Q_T)$, with $Lu \in L^2(Q_T)$, the trace of $A_\nu u$ on Σ_T exists in $H^{-1/2}(\Sigma_T)$. Moreover, for a given boundary matrix $M(x,t)$ satisfying assumption (C), there exists another matrix $M_0(x,t)$ such that $M(x,t) = M_0(x,t)A_\nu(x,t)$ for all $(x,t) \in \Sigma_\infty$. Therefore, for L^2 solutions of (1) one has

$$Mu = G \quad \text{on } \Sigma_T \iff M_0 A_\nu u_{|\Sigma_T} = G \quad \text{on } \Sigma_T. \tag{8}$$

In order to study the regularity of solutions to the IBVP (1)–(3), the data F, G, f need to satisfy some compatibility conditions. The compatibility conditions are defined in the usual way (see [14]). Given the IBVP (1)–(3), we recursively define $f^{(h)}$ by formally taking $h-1$ time derivatives of $Lu = F$, solving for $\partial_t^h u$ and evaluating it at $t = 0$; for $h = 0$ we set $f^{(0)} := f$. The *compatibility condition* of order $k \geq 0$ for the IBVP reads as

$$\sum_{h=0}^{p} \binom{p}{h} (\partial_t^{p-h} M)_{|t=0} f^{(h)} = \partial_t^h G_{|t=0}\,, \quad \text{on } \Gamma,\ p = 0, \dots, k. \tag{9}$$

In the framework of the preceding assumptions, we are able to prove the following theorem.

THEOREM 1. *Let $m \in \mathbb{N}$ and $s = \max\{m, [\frac{n+1}{2}] + 5\}$. Assume that $S_0, A_i \in \mathcal{C}_T(H^s_*)$, for $i = 1, \dots, n$, and that $B \in \mathcal{C}_T(H^{s-1}_*)$ (or $B \in \mathcal{C}_T(H^s_*)$ if $m = s$). Assume also that problem* (1)–(3) *obeys the assumptions* (A)–(F). *Then for all $F \in H^m_*(Q_T)$, $G \in H^m(\Sigma_T)$, $f \in H^m_*(\Omega)$, with $f^{(h)} \in H^{m-h}_*(\Omega)$ for $h = 1, \dots, m$, satisfying the compatibility condition* (9) *of order $m-1$, the unique solution u to* (1)–(3)*, with data (F, G, f), belongs to $\mathcal{C}_T(H^m_*)$ and $Pu_{|\Sigma_T} \in H^m(\Sigma_T)$. Moreover u satisfies the a priori estimate*

$$\|u\|_{\mathcal{C}_T(H^m_*)} + \|Pu_{|\Sigma_T}\|_{H^m(\Sigma_T)} \leq C_m \big(|||f|||_{m,*} + \|F\|_{H^m_*(Q_T)} + \|G\|_{H^m(\Sigma_T)} \big), \tag{10}$$

with a constant $C_m > 0$ depending only on A_i, B.

The function spaces involved in the statement above (cf. also the assumption (F)), and the norms appearing in the energy estimate (10) are introduced in the next section.

2. Function spaces

For every integer $m \geq 1$, $H^m(\Omega)$, $H^m(Q_T)$ denote the usual Sobolev spaces of order m over Ω and Q_T respectively.

In order to define the anisotropic Sobolev spaces, first we need to introduce the differential operators in *tangential direction*. Throughout the paper, for every $j = 1,2,\dots,n$, the differential operator Z_j is defined by

$$Z_1 := x_1\partial_1\,, \quad Z_j := \partial_j\,, \text{ for } j = 2,\dots,n\,.$$

Then, for every multi-index $\alpha = (\alpha_1,\dots,\alpha_n) \in \mathbb{N}^n$, the tangential differential operator Z^α of order $|\alpha| = \alpha_1 + \cdots + \alpha_n$ is defined by setting

$$Z^\alpha := Z_1^{\alpha_1}\dots Z_n^{\alpha_n}$$

(we also write, with the standard multi-index notation, $\partial^\alpha = \partial_1^{\alpha_1}\dots\partial_n^{\alpha_n}$).

We denote by $\mathbb{R}^n_+$ the n-dimensional positive half-space $\mathbb{R}^n_+ := \{x = (x_1,x') \in \mathbb{R}^n : x_1 > 0\ x' := (x_2,\dots,x_n) \in \mathbb{R}^{n-1}\}$. For every positive integer m, the *tangential* (or *conormal*) Sobolev space $H^m_{tan}(\mathbb{R}^n_+)$ and the *anisotropic* Sobolev space $H^m_*(\mathbb{R}^n_+)$ are defined respectively by:

(11) $$H^m_{tan}(\mathbb{R}^n_+) := \{w \in L^2(\mathbb{R}^n_+) : Z^\alpha w \in L^2(\mathbb{R}^n_+)\,,\ |\alpha| \leq m\},$$

(12) $$H^m_*(\mathbb{R}^n_+) := \{w \in L^2(\mathbb{R}^n_+) : Z^\alpha\partial_1^k w \in L^2(\mathbb{R}^n_+)\,,\ |\alpha| + 2k \leq m\},$$

and equipped respectively with norms

(13) $$\|w\|^2_{H^m_{tan}(\mathbb{R}^n_+)} := \sum_{|\alpha|\leq m} \|Z^\alpha w\|^2_{L^2(\mathbb{R}^n_+)}\,,$$

(14) $$\|w\|^2_{H^m_*(\mathbb{R}^n_+)} := \sum_{|\alpha|+2k\leq m} \|Z^\alpha\partial_1^k w\|^2_{L^2(\mathbb{R}^n_+)}\,.$$

To extend the definition of the above spaces to an open bounded subset Ω of $\mathbb{R}^n$ (fulfilling the assumptions made at the beginning of the previous section), we proceed as follows. First, we take an open covering $\{U_j\}_{j=0}^l$ of $\overline{\Omega}$ such that $U_j \cap \overline{\Omega}$, $j = 1,\dots,l$, are diffeomorphic to $\overline{\mathbb{B}}_+ := \{x_1 \geq 0,\, |x| < 1\}$, with Γ corresponding to $\partial\mathbb{B}_+ := \{x_1 = 0,\, |x| < 1\}$, and $U_0 \subset\subset \Omega$. Next we choose a smooth partition of unity $\{\psi_j\}_{j=0}^l$ subordinate to the covering $\{U_j\}_{j=0}^l$. We say that a distribution u belongs to $H^m_{tan}(\Omega)$, if and only if $\psi_0 u \in H^m(\mathbb{R}^n)$ and, for all $j = 1,\dots,l$, $\psi_j u \in H^m_{tan}(\mathbb{R}^n_+)$, in local coordinates in U_j. The space $H^m_{tan}(\Omega)$ is provided with the norm

(15) $$\|u\|^2_{H^m_{tan}(\Omega)} := \|\psi_0 u\|^2_{H^m(\mathbb{R}^n)} + \sum_{j=1}^{l} \|\psi_j u\|^2_{H^m_{tan}(\mathbb{R}^n_+)}\,.$$

The anisotropic Sobolev space $H^m_*(\Omega)$ is defined in a completely similar way as the set of distributions u in Ω such that $\psi_0 u \in H^m(\mathbb{R}^n)$ and $\psi_j u \in H^m_*(\mathbb{R}^n_+)$, in local coordinates in U_j, for all $j = 1, \dots, l$; it is provided with the norm

$$\|u\|^2_{H^m_*(\Omega)} := \|\psi_0 u\|^2_{H^m(\mathbb{R}^n)} + \sum_{j=1}^{l} \|\psi_j u\|^2_{H^m_*(\mathbb{R}^n_+)} \,. \tag{16}$$

The definitions of $H^m_{tan}(\Omega)$ and $H^m_*(\Omega)$ do not depend on the choice of the coordinate patches $\{U_j\}_{j=0}^l$ and the corresponding partition of unity $\{\psi_j\}_{j=0}^l$, and the norms arising from different choices of U_j, ψ_j are equivalent.

For an extensive study of the anisotropic Sobolev spaces, we refer the reader to [24], [20]; here we just remark that the continuous imbeddings

$$\begin{aligned} &H^m_{tan}(\Omega) \hookrightarrow H^p_{tan}(\Omega)\,, \quad H^m_*(\Omega) \hookrightarrow H^p_*(\Omega)\,, \quad \forall m \geq p \geq 1\,,\\ &H^m(\Omega) \hookrightarrow H^m_*(\Omega) \hookrightarrow H^m_{tan}(\Omega)\,, \quad \forall m \geq 1\,,\\ &H^m_*(\Omega) \hookrightarrow H^{[m/2]}(\Omega)\,, \quad H^1_*(\Omega) = H^1_{tan}(\Omega) \end{aligned} \tag{17}$$

hold true. For the sake of convenience, we also set $H^0_*(\Omega) = H^0_{tan}(\Omega) = L^2(\Omega)$. The spaces $H^m_{tan}(\Omega)$, $H^m_*(\Omega)$, endowed with their norms (15), (16), become Hilbert spaces. Analogously, we define the spaces $H^m_{tan}(Q_T)$ and $H^m_*(Q_T)$.

Let $C^m([0,T];X)$ denote the set of all m-times continuously differentiable functions over $[0,T]$, taking values in a Banach space X. We define the spaces

$$\mathcal{C}_T(H^m_{tan}) := \bigcap_{j=0}^{m} C^j([0,T]; H^{m-j}_{tan}(\Omega))\,, \quad \mathcal{C}_T(H^m_*) := \bigcap_{j=0}^{m} C^j([0,T]; H^{m-j}_*(\Omega))\,,$$

equipped respectively with the norms

$$\begin{aligned} \|u\|^2_{\mathcal{C}_T(H^m_{tan})} &:= \sum_{j=0}^{m} \sup_{t\in[0,T]} \|\partial_t^j u(t)\|^2_{H^{m-j}_{tan}(\Omega)}\,,\\ \|u\|^2_{\mathcal{C}_T(H^m_*)} &:= \sum_{j=0}^{m} \sup_{t\in[0,T]} \|\partial_t^j u(t)\|^2_{H^{m-j}_*(\Omega)}\,. \end{aligned} \tag{18}$$

For the initial datum f we set

$$|||f|||^2_{m,*} := \sum_{j=0}^{m} \|f^{(j)}\|^2_{H^{m-j}_*(\Omega)}\,.$$

3. The scheme of the proof of Theorem 1

The proof of Theorem 1 is made of several steps.

In order to simplify the forthcoming analysis, hereafter we only consider the case when the operator L has smooth coefficients. For the general case of coefficients with the finite regularity prescribed in Theorem 1, we refer the reader to [9]; this case is treated by a reduction to the smooth coefficients case, based upon the stability assumption (F). Thus, from now on, we assume that S_0, A_i, B are given functions in $C^\infty(\overline{Q}_\infty)$.

Just for simplicity, we even assume that the coefficients A_i of L are symmetric matrices (in this case the matrix S_0 reduces to I_N, the identity matrix of size N); the case of a symmetrizable operator can be easily reduced to this one, just by the application of the symmetrizer S_0 to system (1) (see [9] for details).

Below, we introduce the new unknown $u_\gamma(x,t) := e^{-\gamma t}u(x,t)$ and the new data $F_\gamma(x,t) := e^{-\gamma t}F(x,t)$, $G_\gamma(x,t) = e^{-\gamma t}G(x,t)$. Then problem (1)–(3) becomes equivalent to

$$
\begin{aligned}
(\gamma+L)u_\gamma &= F_\gamma && \text{in } Q_T\,,\\
Mu_\gamma &= G_\gamma && \text{on } \Sigma_T\,,\\
u_{\gamma|\,t=0} &= f && \text{in } \Omega\,.
\end{aligned}
\tag{19}
$$

Let us now summarize the main steps of the proof of Theorem 1.

1. We firstly consider the *homogeneous* IBVP

$$
\begin{aligned}
(\gamma+L)u_\gamma &= F_\gamma && \text{in } Q_T\,,\\
Mu_\gamma &= G_\gamma && \text{on } \Sigma_T\,,\\
u_{\gamma|t=0} &= 0 && \text{in } \Omega\,.
\end{aligned}
\tag{20}
$$

   We study (20), by reducing it to a *stationary* boundary value problem (see (26)), for which we deduce the *tangential* regularity. From the tangential regularity of this stationary problem, we deduce the tangential regularity of the homogeneous problem (20) (see the next Theorem 2).

2. We study the general problem (19). The anisotropic regularity, stated in Theorem 1, is obtained in two steps.

   2.i Firstly, from the tangential regularity of problem (20) above, we deduce the *anisotropic* regularity of (19) at order $m = 1$.

   2.ii Eventually, we obtain the anisotropic regularity of (19), at any order $m > 1$, by an induction argument.

3.1. The homogeneous IBVP. Tangential regularity

In this section, we concentrate on the study of the tangential regularity of solutions to the IBVP (19), where the initial datum f is identically zero, and the compatibility conditions are fulfilled in a more restrictive form than the one given in (9). More precisely, we consider the *homogeneous* IBVP (20) where, for a given integer $m \geq 1$, we assume that the data F_γ, G_γ satisfy the following conditions:

$$
\partial_t^h F_{\gamma|\,t=0} = 0\,, \quad \partial_t^h G_{\gamma|\,t=0} = 0\,, \quad h = 0,\dots,m-1\,.
\tag{21}
$$

One can prove that conditions (21) imply the compatibility conditions (9) of order $m-1$, in the case $f = 0$.

THEOREM 2. *Assume that* A_i, B, *for* $i = 1,\dots,n$, *are in* $C^\infty(\overline{Q}_\infty)$, *and that problem* (20) *satisfies assumptions* (A)–(E)*; then for all* $T > 0$ *and* $m \in \mathbb{N}$ *there exist constants* $C_m > 0$ *and* γ_m, *with* $\gamma_m \geq \gamma_{m-1}$, *such that for all* $\gamma \geq \gamma_m$, *for all* $F_\gamma \in H^m_{tan}(Q_T)$ *and all* $G_\gamma \in H^m(\Sigma_T)$ *satisfying* (21) *the unique solution* u_γ *to* (20) *belongs to* $H^m_{tan}(Q_T)$, *the trace of* Pu_γ *on* Σ_T *belongs to* $H^m(\Sigma_T)$ *and the a priori estimate*

$$\gamma \|u_\gamma\|^2_{H^m_{tan}(Q_T)} + \|Pu_{\gamma|\Sigma_T}\|^2_{H^m(\Sigma_T)} \leq C_m \left(\frac{1}{\gamma} \|F_\gamma\|^2_{H^m_{tan}(Q_T)} + \|G_\gamma\|^2_{H^m(\Sigma_T)} \right) \tag{22}$$

is fulfilled.

The first step to prove Theorem 2 is reducing the original mixed *evolution* problem (20) to a *stationary* boundary value problem, where the time is allowed to span the whole real line and it is treated then as an additional tangential variable. To make this reduction, we extend the data F_γ, G_γ and the unknown u_γ of (20) to all positive and negative times, by following methods similar to those of [1, Ch.9]. In the sequel, for the sake of simplicity, we remove the subscript γ from the unknown u_γ and the data F_γ, G_γ.

Because of (21), we extend F and G through $]-\infty, 0]$, by setting them equal to zero for all negative times; then for times $t > T$, we extend them by "reflection", following Lions–Magenes [7, Theorem 2.2]. Let us denote by $\breve{F}$ and $\breve{G}$ the resulting extensions of F and G respectively; by construction, $\breve{F} \in H^m_{tan}(Q)$ and $\breve{G} \in H^m(\Sigma)$.

As we did for the data, the solution u to (20) is extended to all negative times, by setting it equal to zero. To extend u also for times $t > T$, we exploit the assumption (E). More precisely, for every $T' > T$ we consider the mixed problem

$$\begin{array}{lll} (\gamma + L)u & = \breve{F}_{|]0,T'[} & \text{in } Q_{T'}\,, \\ Mu & = \breve{G}_{|]0,T'[}\,, & \text{on } \Sigma_{T'}\,, \\ u_{|\,t=0} & = 0\,, & \text{in } \Omega \end{array} \tag{23}$$

Assumption (E) yields that (23) admits a unique solution $u_{T'} \in C([0,T']; L^2(\Omega))$, such that $Pu_{T'} \in L^2(\Sigma_{T'})$ and the energy estimate

$$\begin{aligned} & \|u_{T'}(T')\|^2_{L^2(\Omega)} + \gamma \|u_{T'}\|^2_{L^2(Q_{T'})} + \|Pu_{T'\,|\Sigma_{T'}}\|^2_{L^2(\Sigma_{T'})} \\ & \leq C' \left(\frac{1}{\gamma} \|\breve{F}_{|]0,T'[}\|^2_{L^2(Q_{T'})} + \|\breve{G}_{|]0,T'[}\|^2_{L^2(\Sigma_{T'})} \right) \end{aligned} \tag{24}$$

is satisfied for all $\gamma \geq \gamma'$ and some constants $\gamma' \geq 1$ and $C' > 0$ depending only on T' (and the norms $\|A_i\|_{\mathrm{Lip}(Q_{T'})}$, $\|B\|_{L^\infty(Q_{T'})}$).

From the uniqueness of the L^2 solution, we infer that for arbitrary $T'' > T' \geq T$ we have $u_{T''} = u_{T'}$ ($u_T := u$) over $]0, T'[$. Therefore, we may extend u beyond T, by setting it equal to the unique solution of (23) over $]0,T'[$ for all $T' > T$. Thus we define

$$\breve{u}(t) := \begin{cases} u_{T'}(t)\,, & \forall t \in]0,T'[\,,\ \forall T' > T\,, \\ 0\,, & \forall t < 0\,. \end{cases} \tag{25}$$

Since $\breve{u}$, $\breve{F}$, $\breve{G}$ are all identically zero for negative times, we can take arbitrary smooth extensions of the coefficients of the differential operator L and the boundary operator M (originally defined on Q_∞ and Σ_∞) on Q and Σ respectively, with the only care to preserve rank $A_\nu = r$ and rank $M = d$ and $\ker A_\nu \subseteq \ker M$ for all $t < 0$. This extensions, that we fix once and for all, are denoted again by A_i, B, M. Moreover, we denote by L the corresponding extension on Q of the differential operator (4).

By construction, we have that $\breve{u}$ solves the boundary value problem (BVP)

$$\begin{aligned} (\gamma+L)u &= \breve{F} \qquad \text{in } Q\,, \\ Mu &= \breve{G}\,, \qquad \text{on } \Sigma\,. \end{aligned} \tag{26}$$

Using the estimate (24), for all $T' > T$, and noticing that the extended data $\breve{F}$, $\breve{G}$, as well as the solution $\breve{u}$, vanish identically for large $t > 0$, we derive that $\breve{u}$ enjoys the following estimate

$$\gamma\|\breve{u}\|^2_{L^2(Q)} + \|P\breve{u}_{|\Sigma}\|^2_{L^2(\Sigma)} \le \breve{C}\left(\frac{1}{\gamma}\|\breve{F}\|^2_{L^2(Q)} + \|\breve{G}\|^2_{L^2(\Sigma)}\right), \tag{27}$$

for all $\gamma \ge \breve{\gamma}$, and suitable constants $\breve{\gamma} \ge 1$, $\breve{C} > 0$.

For the sake of simplicity, in the sequel we remove the superscript from the unknown $\breve{u}$ and the data $\breve{F}$, $\breve{G}$ of (26).

The next step is to move from BVP (26) to a similar BVP posed in the $(n+1)$-dimensional positive half-space $\mathbb{R}^{n+1}_+ := \{(x_1, x', t) : x_1 > 0, (x', t) \in \mathbb{R}^n\}$. To make this reduction into a problem in $\mathbb{R}^{n+1}_+$, we follow a standard localization procedure of the problem (26) near the boundary of the spatial domain Ω; this is done by taking a covering $\{U_j\}_{j=0}^l$ of $\overline{\Omega}$ and a partition of unity $\{\psi_j\}_{j=0}^l$ subordinate to this covering, as in Section 2. Assuming that each patch U_j, $j = 1, \ldots, l$, is sufficiently small, we can write the resulting localized problem in the form

$$\begin{aligned} (\gamma+L)u &= F \qquad \text{in } \mathbb{R}^{n+1}_+\,, \\ Mu &= G\,, \qquad \text{on } \mathbb{R}^n\,. \end{aligned} \tag{28}$$

As a consequence of the localization, the data F and G of the problem (28) are functions in $H^m_{tan}(\mathbb{R}^{n+1}_+)$ and $H^m(\mathbb{R}^n)$ respectively; without loss of generality, we may also assume that the forcing term F and the solution u are supported in the set $\overline{\mathbb{B}}_+ \times [0, +\infty[$, and the boundary datum G is supported in $\partial\mathbb{B}_+ \times [0, +\infty[$. In $(28)_1$, L is now a differential operator in $\mathbb{R}^{n+1}$ of the form

$$L = \partial_t + \sum_{i=1}^n A_i(x,t)\partial_i + B(x,t)\,, \tag{29}$$

where the coefficients A_i, B are matrix-valued functions of (x,t) belonging to the space $C^\infty_{(0)}(\mathbb{R}^{n+1}_+)$ of the restrictions onto $\mathbb{R}^{n+1}_+$ of (matrix-valued) functions in $C^\infty_0(\mathbb{R}^{n+1})$. Let us remark that the boundary matrix of (28) is now $-A_{1\,|\,\{x_1=0\}}$. It is a crucial step that the previously described localization process can be performed in such a way that A_1

has the following block structure

$$(30)\qquad A_1(x,t)=\begin{pmatrix} A_1^{I,I} & A_1^{I,II} \\ A_1^{II,I} & A_1^{II,II}\end{pmatrix}, \quad (x,t)\in\mathbb{R}^{n+1}_+,$$

where $A_1^{I,I}, A_1^{I,II}, A_1^{II,I}, A_1^{II,II}$ are respectively $r\times r$, $r\times(N-r)$, $(N-r)\times r$, $(N-r)\times(N-r)$ sub-matrices. Moreover, $A_1^{I,I}(x,t)$ is invertible over the support of $u(x,t)$ and we have

$$(31)\qquad A_1^{I,II}=0,\quad A_1^{II,I}=0,\quad A_1^{II,II}=0,\quad \text{in}\quad \{x_1=0\}\times\mathbb{R}^n_{x',t}.$$

In view of assumption (C), we may even assume that the matrix M in the boundary condition $(28)_2$ is just $M=(I_d,0)$, where I_d is the identity matrix of size d. According to (30), let us decompose the unknown u as $u=(u^I,u^{II})$; then we have $Pu=(u^I,0)$.

Following the arguments of [3], one can prove that a local counterpart of the global estimate (27), associated to the stationary problem (26), can be attached to the local problem (28). More precisely, there exist constants $C_0>0$ and $\gamma_0\geq 1$ such that for all $\varphi\in L^2(\mathbb{R}^{n+1}_+)$, supported in $\overline{\mathbb{B}}_+\times[0,+\infty[$, such that $L\varphi\in L^2(\mathbb{R}^{n+1}_+)$ and $\gamma\geq\gamma_0$, we have

$$(32)\qquad \begin{aligned}&\gamma\|\varphi\|^2_{L^2(\mathbb{R}^{n+1}_+)}+\|\varphi^I_{|\{x_1=0\}}\|^2_{L^2(\mathbb{R}^n)}\\ &\leq C_0\left(\frac{1}{\gamma}\|(\gamma+L)\varphi\|^2_{L^2(\mathbb{R}^{n+1}_+)}+\|M\varphi_{|\{x_1=0\}}\|^2_{L^2(\mathbb{R}^n)}\right).\end{aligned}$$

Regularity of the stationary problem (28)

The analysis performed in the previous section shows that the tangential regularity of the homogeneous IBVP (20) can be deduced from the study of the regularity of the stationary BVP (28).

For this stationary problem, we are able to show that if the data F and G belong to $H^m_{tan}(\mathbb{R}^{n+1}_+)$ and $H^m(\mathbb{R}^n)$ respectively, and the L^2 a priori estimate (32) is fulfilled, then the L^2 solution of the problem (28) belongs to $H^m_{tan}(\mathbb{R}^{n+1}_+)$, the trace of its non-characteristic part u^I belongs to $H^m(\mathbb{R}^n)$ and the estimate of order m

$$(33)\qquad \gamma\,\|u\|^2_{H^m_{tan}(\mathbb{R}^{n+1}_+)}+\|u^I_{|\{x_1=0\}}\|^2_{H^m(\mathbb{R}^n)}\leq C_m\left(\frac{1}{\gamma}\|F\|^2_{H^m_{tan}(\mathbb{R}^{n+1}_+)}+\|G\|^2_{H^m(\mathbb{R}^n)}\right)$$

is satisfied with some constants $C_m>0$, $\gamma_m\geq 1$ and for all $\gamma\geq\gamma_m$.

Then we recover the tangential regularity of the solution u to problem (26), posed on $Q=\Omega\times\mathbb{R}$, and we find an associated estimate of order m analogous to (33). Recalling that the solution u to (26) is the extension of the solution u_γ of the homogeneous IBVP (20), from the tangential regularity of u we can now derive the tangential regularity of u_γ, namely that $u_\gamma\in H^m_{\tan}(Q_T)$ and $Pu_{\gamma\,|\,\Sigma_T}\in H^m(\Sigma_T)$. To get the energy estimate (22), we observe that the extended data $\breve{F}$ and $\breve{G}$ are defined in such a way that

$$\|\breve{F}\|_{H^m_{\tan}(Q)}\leq C\|F_\gamma\|_{H^m_{\tan}(Q_T)},\quad \|\breve{G}\|_{H^m(\Sigma)}\leq C\|G_\gamma\|_{H^m(\Sigma_T)},$$

with positive constant C independent of F_γ, G_γ and γ.

In order to prove the announced tangential regularity of the BVP (28), we adapt the classical technique of Friedrichs' mollifiers to our setting. More precisely, following Nishitani and Takayama [10], we introduce a "tangential" mollifier J_ε well suited to the tangential Sobolev spaces. Let χ be a function in $C_0^\infty(\mathbb{R}^{n+1})$. For all $0<\varepsilon<1$, we set $\chi_\varepsilon(y) := \varepsilon^{-(n+1)}\chi(y/\varepsilon)$. We define $J_\varepsilon : L^2(\mathbb{R}^{n+1}_+) \to L^2(\mathbb{R}^{n+1}_+)$ by

$$J_\varepsilon w(x) := \int_{\mathbb{R}^{n+1}} w(x_1 e^{-y_1}, x' - y')e^{-y_1/2}\chi_\varepsilon(y)dy\,, \tag{34}$$

which differs from the one introduced in Rauch [15] by the factor $e^{-y_1/2}$. Using J_ε we follow the same lines in Tartakoff [22], Nishitani and Takayama [10] to get regularity of the weak solution u.

Starting from a classical characterization of the ordinary Sobolev spaces given in [6, Theorem 2.4.1], the following characterization of tangential Sobolev spaces $H^m_{tan}(\mathbb{R}^{n+1}_+)$ by means of J_ε can be proved.

PROPOSITION 1. *Assume that $\chi \in C_0^\infty(\mathbb{R}^{n+1})$ satisfies the following conditions:*

$$\widehat{\chi}(\xi) = O(|\xi|^p) \qquad \text{as } \xi \to 0, \qquad \text{for some } p \in \mathbb{N}; \tag{35}$$

$$\widehat{\chi}(t\xi) = 0\,, \qquad \text{for all } t \in \mathbb{R}\,, \qquad \text{implies } \xi = 0. \tag{36}$$

Then for all $m \in \mathbb{N}$ with $m < p$, we have that $u \in H^m_{tan}(\mathbb{R}^{n+1}_+)$ if and only if

a. $u \in H^{m-1}_{tan}(\mathbb{R}^{n+1}_+)$;

b. $\displaystyle\int_0^1 \|J_\varepsilon u\|^2_{L^2(\mathbb{R}^{n+1}_+)} \varepsilon^{-2m}\left(1+\frac{\delta^2}{\varepsilon^2}\right)^{-1}\frac{d\varepsilon}{\varepsilon}$ *is uniformly bounded for* $0<\delta\le 1$.

In view of Proposition 1, showing that the solution $u \in H^{m-1}_{tan}(\mathbb{R}^{n+1}_+)$ of (28) actually belongs to $H^m_{tan}(\mathbb{R}^{n+1}_+)$ amounts to provide a uniform bound, with respect to δ, for the integral quantity appearing in b., computed for the *mollified* solution $J_\varepsilon u$. To get this bound, the scheme is the following:

1. We notice that $J_\varepsilon u$ solves the following BVP

$$\begin{aligned} &(\gamma + L)J_\varepsilon u = J_\varepsilon F + [L, J_\varepsilon]u\,, \ \text{ in } \mathbb{R}^{n+1}_+\,,\\ &MJ_\varepsilon u = G_\varepsilon\,, \quad \text{on } \mathbb{R}^n\,, \end{aligned} \tag{37}$$

   where $[L, J_\varepsilon]$ is the commutator between the operators L and J_ε, and G_ε is a suitable boundary datum that can be computed from the original datum G and the function χ_ε involved in (34) (see [9]).

2. Since the BVP (37) is the same as (28), with data $J_\varepsilon F + [L, J_\varepsilon]u$ and G_ε, the L^2 estimate (32) applied to (37) gives that the L^2 norm of $J_\varepsilon u$ can be estimated by

$$\begin{aligned} &\gamma\,\|J_\varepsilon u\|^2_{L^2(\mathbb{R}^{n+1}_+)} + \|J_\varepsilon u^I_{|\{x_1=0\}}\|^2_{L^2(\mathbb{R}^n)}\\ &\le C_0\left(\frac{1}{\gamma}\|J_\varepsilon F + [L, J_\varepsilon]u\|^2_{L^2(\mathbb{R}^{n+1}_+)} + \|G_\varepsilon\|^2_{L^2(\mathbb{R}^n)}\right). \end{aligned} \tag{38}$$

3. From the preceding estimate, we immediately derive, for the integral quantity in b. and the analogous integral quantity associated to the trace of non characteristic part of the solution, the following bound

$$
\begin{aligned}
&\gamma\int_0^1 \|J_\varepsilon u\|^2_{L^2(\mathbb{R}^{n+1}_+)}\varepsilon^{-2m}\left(1+\tfrac{\delta^2}{\varepsilon^2}\right)^{-1}\frac{d\varepsilon}{\varepsilon}\\
&\qquad+\int_0^1 \|J_\varepsilon u^I_{|\{x_1=0\}}\|^2_{L^2(\mathbb{R}^n)}\varepsilon^{-2m}\left(1+\tfrac{\delta^2}{\varepsilon^2}\right)^{-1}\frac{d\varepsilon}{\varepsilon}\\
&\le C_0\left(\frac{1}{\gamma}\int_0^1 \|J_\varepsilon F\|^2_{L^2(\mathbb{R}^{n+1}_+)}\varepsilon^{-2m}\left(1+\tfrac{\delta^2}{\varepsilon^2}\right)^{-1}\frac{d\varepsilon}{\varepsilon}\right.\\
&\qquad+\frac{1}{\gamma}\int_0^1 \|[L,J_\varepsilon]u\|^2_{L^2(\mathbb{R}^{n+1}_+)}\varepsilon^{-2m}\left(1+\tfrac{\delta^2}{\varepsilon^2}\right)^{-1}\frac{d\varepsilon}{\varepsilon}\\
&\qquad\left.+\int_0^1 \|G_\varepsilon\|^2_{L^2(\mathbb{R}^n)}\varepsilon^{-2m}\left(1+\tfrac{\delta^2}{\varepsilon^2}\right)^{-1}\frac{d\varepsilon}{\varepsilon}\right).
\end{aligned}
\tag{39}
$$

Since $F\in H^m_{tan}(\mathbb{R}^{n+1}_+)$ and $G\in H^m(\mathbb{R}^n)$, the first and the last integrals in the right-hand side of (39) can be estimated by $\|F\|^2_{H^m_{tan}(\mathbb{R}^{n+1}_+)}$ and $\|G\|^2_{H^m(\mathbb{R}^n)}$ respectively.

It remains to provide a uniform estimate for the middle integral involving the commutator $[L,J_\varepsilon]u$. For this term we get the following estimate

$$
\begin{aligned}
&\int_0^1 \|[L,J_\varepsilon]u\|^2_{L^2(\mathbb{R}^{n+1}_+)}\varepsilon^{-2m}\left(1+\tfrac{\delta^2}{\varepsilon^2}\right)^{-1}\frac{d\varepsilon}{\varepsilon}\\
&\le C\int_0^1 \|J_\varepsilon u\|^2_{L^2(\mathbb{R}^{n+1}_+)}\varepsilon^{-2m}\left(1+\tfrac{\delta^2}{\varepsilon^2}\right)^{-1}\frac{d\varepsilon}{\varepsilon}\\
&\qquad+C\gamma^2\|u\|^2_{H^{m-1}_{tan}(\mathbb{R}^{n+1}_+)}+C\|F\|^2_{H^m_{tan}(\mathbb{R}^{n+1}_+)}.
\end{aligned}
\tag{40}
$$

The estimate (40) is obtained by treating separately the different contributions to the commutator $[L,J_\varepsilon]$ associated to the different terms in the expression (29) of L (see [9] for details). The terms of the the commutator involving the tangential derivatives $[A_i\partial_i,J_\varepsilon]$, for $i=2,\dots,n$ (note that $[\partial_t,J_\varepsilon]=0$) and the zero-th order term $[B,J_\varepsilon]$ are estimated by using [10, Lemma 9.2]. The term $[A_1\partial_1,J_\varepsilon]$, involving the normal derivative ∂_1, needs a more careful analysis; to estimate it, it is essential to make use of the structure (30), (31) of the boundary matrix in (28). Actually, by inverting $A_1^{I,I}$ in $(28)_1$, we can write $\partial_1 u^I$ as the sum of *space-time* tangential derivatives by

$$\partial_1 u^I=\Lambda Zu+R,$$

where

$$
\begin{aligned}
&\Lambda Zu=-(A_1^{I,I})^{-1}\left[\left(\partial_t u^I+\sum_{j=2}^n A_jZ_ju\right)^I+A_1^{I,II}\partial_1 u^{II}\right],\\
&R=(A_1^{I,I})^{-1}(F-\gamma u-Bu)^I.
\end{aligned}
$$

Here, we use the fact that, if a matrix A vanishes on $\{x_1=0\}$, we can write $A\partial_1 u=HZ_1u$, where H is a suitable matrix; this trick transforms some normal derivatives into tangential derivatives.

Combining the inequalities (39) and (40), and arguing by finite induction on m to estimate $\|u\|_{H^{m-1}_{tan}(\mathbb{R}^{n+1}_+)}$ in the right-hand side of (40), we get the desired uniform bounds of the integrals

$$\int_0^1 \|J_\varepsilon u\|^2_{L^2(\mathbb{R}^{n+1}_+)} \varepsilon^{-2m} \left(1+\frac{\delta^2}{\varepsilon^2}\right)^{-1} \frac{d\varepsilon}{\varepsilon}\,,$$
$$\int_0^1 \|J_\varepsilon u^I_{|\{x_1=0\}}\|^2_{L^2(\mathbb{R}^n)} \varepsilon^{-2m} \left(1+\frac{\delta^2}{\varepsilon^2}\right)^{-1} \frac{d\varepsilon}{\varepsilon}\,,$$

appearing in the left-hand side of (39). From this, in view of Proposition 1 and [6, Theorem 2.4.1], we conclude that $u \in H^m_{tan}(\mathbb{R}^{n+1}_+)$ and $u^I \in H^m(\mathbb{R}^n)$. The a priori estimate (33) is deduced from (39), by following the same arguments.

3.2. The nonhomogeneous IBVP. Case $m = 1$

For *nonhomogeneous* IBVP, we mean the problem (1)–(3) where the initial datum f is different from zero.

As announced before, we firstly prove the statement of Theorem 1 for $m = 1$, namely we prove that, under the assumptions (A)–(F), for all $F \in H^1_*(Q_T)$, $G \in H^1(\Sigma_T)$ and $f \in H^1_*(\Omega)$, with $f^{(1)} \in L^2(\Omega)$, satisfying the compatibility condition $M_{|t=0} f_{|\partial\Omega} = G_{|t=0}$, the unique solution u to (1)–(3), with data (F,G,f), belongs to $\mathcal{C}_T(H^1_*)$ and $Pu_{|\Sigma_T} \in H^1(\Sigma_T)$; moreover, there exists a constant $C_1 > 0$ such that u satisfies the a priori estimate

$$\|u\|_{\mathcal{C}_T(H^1_*)} + \|Pu_{|\Sigma_T}\|_{H^1(\Sigma_T)} \le C_1 \Big(|||f|||_{1,*} + \|F\|_{H^1_*(Q_T)} + \|G\|_{H^1(\Sigma_T)} \Big). \tag{41}$$

To this end, we approximate the data with regularized functions satisfying one more compatibility condition. In this regard we get the following result, for the proof of which we refer to [9] and the references therein.

LEMMA 1. *Assume that problem* (1)–(3) *obeys the assumptions* (A)–(E). *Let* $F \in H^1_*(Q_T)$, $G \in H^1(\Sigma_T)$, $f \in H^1_*(\Omega)$, *with* $f^{(1)} \in L^2(\Omega)$, *such that* $M_{|t=0} f_{|\partial\Omega} = G_{|t=0}$. *Then there exist* $F_k \in H^3(Q_T)$, $G_k \in H^3(\Sigma_T)$, $f_k \in H^3(\Omega)$, *such that* $M_{|t=0} f_k = G_{k|t=0}$, $\partial_t M_{|t=0} f_k + M_{|t=0} f^{(1)}_k = \partial_t G_{k|t=0}$ *on* $\partial\Omega$, *and such that* $F_k \to F$ *in* $H^1_*(Q_T)$, $G_k \to G$ *in* $H^1(\Sigma_T)$, $f_k \to f$ *in* $H^1_*(\Omega)$, $f^{(1)}_k \to f^{(1)}$ *in* $L^2(\Omega)$, *as* $k \to +\infty$.

Given the functions F_k, G_k, f_k as in Lemma 1, we first calculate through equation $Lu = F_k, u_{|t=0} = f_k$, the initial time derivatives $f^{(1)}_k \in H^2(\Omega)$, $f^{(2)}_k \in H^1(\Omega)$. Then we take a function $w_k \in H^3(Q_T)$ such that

$$w_{k|t=0} = f_k\,, \quad \partial_t w_{k|t=0} = f^{(1)}_k\,, \quad \partial^2_{tt} w_{k|t=0} = f^{(2)}_k\,.$$

Notice that this yields

$$(Lw_k)_{|t=0} = F_{k|t=0}\,, \quad \partial_t (Lw_k)_{|t=0} = \partial_t F_{k|t=0}\,. \tag{42}$$

Now we look for a solution u_k of problem (1)–(3), with data F_k, G_k, f_k, of the form $u_k = v_k + w_k$, where v_k is solution to

$$
\begin{aligned}
& Lv_k = F_k - Lw_k, && \text{in } Q_T \\
& Mv_k = G_k - Mw_k, && \text{on } \Sigma_T \\
& v_{k|t=0} = 0, && \text{in } \Omega.
\end{aligned}
\tag{43}
$$

Let us denote again $u_{k\gamma} = e^{-\gamma t}u_k$, $v_{k\gamma} = e^{-\gamma t}v_k$ and so on. Then (43) is equivalent to

$$
\begin{aligned}
& (\gamma + L)v_{k\gamma} = F_{k\gamma} - (\gamma + L)w_{k\gamma}, && \text{in } Q_T \\
& Mv_{k\gamma} = G_{k\gamma} - Mw_{k\gamma}, && \text{on } \Sigma_T \\
& v_{k\gamma|t=0} = 0, && \text{in } \Omega.
\end{aligned}
\tag{44}
$$

We easily verify that (42) yields

$$
\big(F_{k\gamma} - (\gamma+L)w_{k\gamma}\big)_{|t=0} = 0\,, \quad \partial_t\left(F_{k\gamma} - (\gamma+L)w_{k\gamma}\right)_{|t=0} = 0
$$

and $M_{|t=0} f_{k|\partial\Omega} = G_{k|t=0}$, $\partial_t M_{|t=0} f_{k|\partial\Omega} + M_{|t=0} f^{(1)}_{k|\partial\Omega} = \partial_t G_{k|t=0}$ yield

$$
(G_{k\gamma} - Mw_{k\gamma})_{|t=0} = 0, \quad \partial_t (G_{k\gamma} - Mw_{k\gamma})_{|t=0} = 0.
$$

Thus the data of problem (44) obey conditions (21) for $h = 0, 1$; then we may apply to (44) Theorem 2 for γ large enough and find $v_k \in H^2_{tan}(Q_T)$, with $Pv_{k|\Sigma_T} \in H^2(\Sigma_T)$. Accordingly, we infer that $u_k \in H^2_{tan}(Q_T) \hookrightarrow \mathcal{C}_T(H^1_*)$ and $Pu_{k|\Sigma_T} \in H^2(\Sigma_T)$. Moreover $u_k \in L^2(Q_T)$ solves

$$
\begin{aligned}
& Lu_k = F_k, && \text{in } Q_T \\
& Mu_k = G_k, && \text{on } \Sigma_T \\
& u_{k|t=0} = f_k, && \text{in } \Omega.
\end{aligned}
\tag{45}
$$

Arguing as in the previous section, we take a covering $\{U_j\}_{j=0}^l$ of $\overline{\Omega}$ and a related partition of unity $\{\psi_j\}_{j=0}^l$, and we reduce problem (45) into a corresponding problem posed in the positive half-space $\mathbb{R}^n_+$, with new data $F_k \in H^3(\mathbb{R}^n_+ \times]0,T[)$, $G_k \in H^3(\mathbb{R}^{n-1} \times]0,T[)$, $f_k \in H^3(\mathbb{R}^n_+)$, and boundary matrix $M = (I_d, 0)$. We also write the similar problem solved by the first order derivatives $Zu_k = (Z_1u_k, \ldots, Z_{n+1}u_k) \in H^1_{tan}(Q_T) = H^1_*(Q_T)$ (where $Z_{n+1} = \partial_t$). Since assumption (E) is satisfied, applying the a priori estimate (7) to a difference of solutions $u_h - u_k$ of those problems readily gives

$$
\begin{aligned}
& \|u_k - u_h\|_{\mathcal{C}_T(H^1_*)} + \|P(u_k - u_h)_{|\Sigma_T}\|_{H^1(\Sigma_T)} \\
& \le C\Big(|||f_k - f_h|||_{1,*} + \|F_k - F_h\|_{H^1_*(Q_T)} + \|G_k - G_h\|_{H^1(\Sigma_T)}\Big).
\end{aligned}
$$

From Lemma 1, we infer that $\{u_k\}$ is a Cauchy sequence in $\mathcal{C}_T(H^1_*)$ and $\{Pu_{k|\Sigma_T}\}$ is a Cauchy sequence in $H^1(\Sigma_T)$. Therefore there exists a function in $\mathcal{C}_T(H^1_*)$ which is the limit of $\{u_k\}$. Passing to the limit in (45) as $k \to \infty$, we see that this function is a solution to (1)–(3). The uniqueness of the L^2 solution yields $u \in \mathcal{C}_T(H^1_*)$ and

$Pu_{|\Sigma_T} \in H^1(\Sigma_T)$. Applying the a priori estimate (7) to the solution u_k of (45) and its first order derivatives, and passing to the limit finally gives (41). This completes the proof of Theorem 1 for $m = 1$ in the case of C^∞ coefficients. As we already said, here we do not deal with the case of less regular coefficients, for which the reader is referred to [9, Sect. 5].

3.3. The nonhomogeneous IBVP. Proof for $m \geq 2$

Without entering in too many details (we still refer to [9, Sect. 6] for a more extensive discussion), we briefly describe the different steps of the proof, for the reader's convenience.

We proceed by finite induction on m. Assume that Theorem 1 is valid up to $m-1$. Let $f \in H^m_*(\Omega)$, $F \in H^m_*(Q_T)$, $G \in H^m(\Sigma_T)$, with $f^{(k)} \in H^{m-k}_*(\Omega)$, with $k = 1,\ldots,m$. Assume also that the compatibility conditions (9) hold at the order $m-1$. By the inductive hypothesis there exists a unique solution u of problem (1)–(3) such that $u \in \mathcal{C}_T(H^{m-1}_*)$.

In order to show that $u \in \mathcal{C}_T(H^m_*)$, we have to increase the regularity of u by order one, that is by one more tangential derivative and, if m is even, also by one more normal derivative. This can be done as in [16, 17], with the small change of the elimination of the auxiliary system (introduced in [16, 17]) as in [2, 19]. At every step, we can estimate some derivatives of u through equations, where in the right-hand side we can put other derivatives of u that have already been estimated at previous steps. The reason why the main idea in [16] works, even though here we do not have maximally nonnegative boundary conditions, is that for the increase of regularity we consider the problem of the type of (1)–(3), solved by the purely tangential derivatives, where we can use the inductive assumption, and other systems of equations solved by the mixed tangential and normal derivatives where the boundary matrix vanishes identically, so that no boundary condition is needed and we can apply an energy method, under the assumption of the symmetrizable system.

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A TRANSFORMATION OF ALMOST PERIODIC PSEUDODIFFERENTIAL OPERATORS TO FOURIER MULTIPLIER OPERATORS WITH OPERATOR-VALUED SYMBOLS

Dedicated to Professor Luigi Rodino on the occasion of his 60th birthday

Abstract. We present results for pseudodifferential operators on $\mathbb{R}^d$ whose symbol $a(\cdot,\xi)$ is almost periodic (a.p.) for each $\xi \in \mathbb{R}^d$ and belongs to a Hörmander class $S^m_{\rho,\delta}$. We study a linear transformation $a \mapsto U(a)$ from a symbol $a(x,\xi)$ to a frequency-dependent matrix $U(a)(\xi)_{\lambda,\lambda'}$, indexed by $(\lambda,\lambda') \in \Lambda \times \Lambda$ where Λ is a countable set in $\mathbb{R}^d$. The map $a \mapsto U(a)$ transforms symbols of a.p. pseudodifferential operators to symbols of Fourier multiplier operators acting on vector-valued function spaces. We show that the map preserves operator positivity and identity, respects operator composition and respects adjoints.

1. Introduction

The paper concerns pseudodifferential operators (abbreviated to ΨDO) on $\mathbb{R}^d$ in the Kohn–Nirenberg quantization, where the symbol $a(\cdot,\xi)$ is almost periodic (a.p.) for each $\xi \in \mathbb{R}^d$, and belongs to a Hörmander class $S^m_{\rho,\delta}$. This symbol class is denoted $APS^m_{\rho,\delta}$ and the corresponding operators are called a.p. pseudodifferential operators. We study the symbol transformation $a \mapsto U(a)$ given by

$$U(a)(\xi)_{\lambda,\lambda'} = M_x(a(x,\xi-\lambda')e^{-2\pi i x\cdot(\lambda'-\lambda)})$$

where M_x denotes the mean value functional of a.p. functions. This transformation was introduced, for operator kernels rather than symbols, by E. Gladyshev [4, 5], for the purposes of stochastic processes. The connection between stochastic processes and operator theory originates from the fact that the so-called covariance function of a stochastic process is the kernel of a positive operator. Gladyshev studied a particular class of stochastic processes called *almost periodically correlated*, which means that the symbol of the covariance operator is almost periodic in the first variable.

The element $U(a)(\xi)$ can be considered a matrix indexed by $(\lambda,\lambda') \in \Lambda \times \Lambda$ where $\Lambda \subset \mathbb{R}^d$ is the countable set of frequencies that occur in $\{a(\cdot,\xi)\}_{\xi\in\mathbb{R}^d}$. Thus $U(a)(\xi)$ is an operator that acts between sequence spaces and the function $\xi \mapsto U(a)(\xi)$ may be considered the operator-valued symbol of a Fourier multiplier operator denoted $U(a)(D)$.

Let $a \in APS^m_{\rho,\delta}$ and let l^2_s be the space of sequences $(x_\lambda)_{\lambda\in\Lambda}$ such that the weighted norm

$$\|x\|_{l^2_s} = \left(\sum_{\lambda \in \Lambda} (1+|\lambda|^2)^s |x_\lambda|^2 \right)^{1/2}$$

is finite. Using results by M. A. Shubin, we first observe that the norm of the operator $a(x,D) : H^s(\mathbb{R}^d_B) \mapsto H^{s-m}(\mathbb{R}^d_B)$ is equal to the norm of $a(x,D) : H^s(\mathbb{R}^d) \mapsto H^{s-m}(\mathbb{R}^d)$ for any $s \in \mathbb{R}$. Here $H^s(\mathbb{R}^d)$ denotes the classical Sobolev Hilbert space, and $H^s(\mathbb{R}^d_B)$ denotes the Sobolev–Besicovitch space of a.p. functions, completed from the trigonometric polynomials in the norm

$$\|f\|_{H^s(\mathbb{R}^d_B)} = \left(\sum_{\lambda \in \mathbb{R}^d} (1+|\lambda|^2)^s |f_\lambda|^2 \right)^{1/2},$$

where $f_\lambda = M_x(f(x)e^{-2\pi i x\cdot\lambda})$ is the Bohr–Fourier coefficient of an a.p. function f. Then we prove that the norm of the matrix $U(a)(0) : l^2_s \mapsto l^2_{s-m}$ is bounded by the norm of the operator $a(x,D) : H^s(\mathbb{R}^d_B) \mapsto H^{s-m}(\mathbb{R}^d_B)$. We also show that $a(x,D)$ is positive on $\mathscr{S}(\mathbb{R}^d)$ if and only if it is positive on the trigonometric polynomials on $\mathbb{R}^d$ and $a(x,D) \geqslant 0$ on $TP(\Lambda)$ if and only if $U(a)(0)$ is a positive definite matrix. Thus much information about the operator $a(x,D)$ can be read off from the evaluation of the matrix symbol $U(a)$ at the origin.

We prove that $U(a)(\xi)$ is a continuous transformation $l^2_s \mapsto l^2_{s-m}$ for any $\xi \in \mathbb{R}^d$, and the map $\mathbb{R}^d \ni \xi \mapsto U(a)(\xi) \in \mathcal{L}(l^2_s, l^2_{s-m})$ is continuous. Moreover, $U(a)(D) \geqslant 0$ if $a(x,D) \geqslant 0$. The latter result on preservation of positivity was proved by Gladyshev [5] for uniformly continuous operator kernels. Here $U(a)(D)$ acts on vector-valued function spaces like $\mathscr{S}(\mathbb{R}^d, l^2_s)$. Then we show our main result that the transformation $a \mapsto U(a)$ respects operator composition. More precisely, denote the *symbol product*, corresponding to operator composition, by $a(x,D) \circ b(x,D) = (a\#_0 b)(x,D)$. If $a \in APS^{m_1}_{\rho,\delta}$ and $b \in APS^{m_2}_{\rho,\delta}$, $m_1, m_2 \in \mathbb{R}$, then we have

$$U(a\#_0 b)(\xi) = U(a)(\xi) \cdot U(b)(\xi).$$

Finally, we prove that the requirement that the symbol is almost periodic in the first variable is invariant under a common family of quantizations that is defined using a parameter $t \in \mathbb{R}$. The family includes the Kohn–Nirenberg ($t = 0$) and the Weyl ($t = 1/2$) correspondences.

In conclusion, the transformation $a \mapsto U(a)$ is a linear, injective map that preserves operator identity, positivity, adjoint and composition. In the proofs of our results we use mainly results by Shubin [9, 10, 11, 12].

In scalar-valued function spaces, translation-invariant (or convolution or Fourier multiplier) operators commute, but for vector-valued function spaces, the product in $\mathbb{C}$ is replaced by the matrix product, so translation-invariant operators are not commutative. The transformation $a(x,D) \mapsto a \mapsto U(a)(D)$ transfers the non-commutativity of almost periodic pseudodifferential operators with symbols in $S^m_{\rho,\delta}$ into the non-commutativity of the matrix product.

A brief comment on some parts of the literature on a.p. pseudodifferential operators follows. Coburn, Moyer and Singer [1] developed an index theory for pseudodifferential operators on $\mathbb{R}^d$ with almost periodic principal symbol. Shubin has made many important contributions to the theory of partial differential operators with almost periodic coefficients and a.p. pseudodifferential operators. For example, he introduced the Sobolev–Besicovitch spaces [9] and proved the equality of the spectra for a.p. pseudodifferential operators acting on $L^2(\mathbb{R}^d)$ and the Besicovitch space $B^2(\mathbb{R}^d)$, provided the operator is bounded or elliptic [11, 12].

Lately Turunen, Ruzhansky and Vainikko have worked on pseudodifferential operators with symbols that are *periodic* in the first variable [14, 15, 8]. The operators may be considered to act on functions defined on the torus $\mathbb{T}^d$, and the theory of pseudodifferential operators on manifolds may be used. However, the use of Fourier series representations gives a more elementary and global treatment.

2. Notation and preliminaries

We use $\langle x\rangle = (1+|x|^2)^{1/2}$, $x \in \mathbb{R}^d$, and the Fourier transform is defined by

$$\mathscr{F} f(\xi) = \widehat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i x\cdot\xi}\, dx, \quad f \in \mathscr{S}(\mathbb{R}^d).$$

For a multiindex $\alpha = (\alpha_1, \dots, \alpha_d)$, we define the partial differential operator

$$\partial^\alpha f(x) = \partial_x^\alpha f(x) = \frac{\partial^{|\alpha|} f(x)}{\partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d}}, \quad x \in \mathbb{R}^d.$$

We use C for a generic positive constant that may vary over equalities and inequalities, we denote by $C^m(\mathbb{R}^d)$ the space of functions such that $\partial^\alpha f$ is continuous for $|\alpha| \leqslant m$ and $C^\infty = \bigcap_m C^m$ is the space of smooth functions. The symbol $C_b(\mathbb{R}^d)$ stands for the space of continuous and supremum bounded functions, and $C_b^\infty(\mathbb{R}^d)$ is the space of functions whose derivatives of all orders are continuous and bounded in supremum norm. The space of compactly supported smooth (test) functions is denoted $C_c^\infty(\mathbb{R}^d)$. The Schwartz space of smooth rapidly decreasing functions is denoted $\mathscr{S}(\mathbb{R}^d)$ and its dual $\mathscr{S}'(\mathbb{R}^d)$ is the space of tempered distributions. A space of trigonometric polynomials is denoted $TP(S)$ and consists of functions of the form

$$f(x) = \sum_{n=1}^{N} a_n e^{2\pi i \xi_n \cdot x}, \quad a_n \in \mathbb{C}, \quad \xi_n \in S \subseteq \mathbb{R}^d.$$

We will consider functions defined on $\mathbb{R}^d$ and taking values in a Hilbert or Banach space X, and then $C(\mathbb{R}^d, X)$ denotes the space of continuous X-valued functions, and likewise for other function spaces. The space of bounded linear transformations between two Hilbert spaces H and H' is denoted $\mathscr{L}(H,H')$, and $\mathscr{L}(H,H) = \mathscr{L}(H)$. The operator norm is denoted $\|\cdot\|_{\mathscr{L}(H,H')}$ or $\|\cdot\|_{\mathscr{L}(H)}$.

A subset Y of a complete metric space X is *precompact* if it is totally bounded, which means that Y can be covered by a finite union of balls of radius ε, for any $\varepsilon > 0$. This definition is equivalent to the property that the closure of Y is compact.

We define a standard family of symbol classes, the so called Hörmander classes. More precisely, the following symbol classes are global versions of Hörmander spaces [3, 6, 13].

DEFINITION 1. *For $m \in \mathbb{R}$ and $0 \leqslant \rho, \delta \leqslant 1$ the space $S^m_{\rho,\delta}$ is defined as the space of all $a \in C^\infty(\mathbb{R}^{2d})$ such that*

$$\sup_{x,\xi \in \mathbb{R}^d} \langle \xi \rangle^{-m+\rho|\alpha|-\delta|\beta|} |\partial_\xi^\alpha \partial_x^\beta a(x,\xi)| < \infty, \quad \alpha, \beta \in \mathbb{N}^d. \tag{1}$$

We impose the conditions

$$0 < \rho \leqslant 1, \quad 0 \leqslant \delta < 1, \quad \delta \leqslant \rho.$$

Following convention, we set $S^{-\infty}_{\rho,\delta} = \bigcap_{m \in \mathbb{R}} S^m_{\rho,\delta}$ and $S^{\infty}_{\rho,\delta} = \bigcup_{m \in \mathbb{R}} S^m_{\rho,\delta}$.

The space $S^m_{\rho,\delta}$ is a Fréchet space with seminorms defined by (1).

We consider the Kohn–Nirenberg quantization of pseudodifferential operators. A symbol function a defined on the phase space $\mathbb{R}^{2d}$ gives rise to an operator $a(x,D)$ according to the formula

$$a(x,D)f(x) = \int_{\mathbb{R}^{2d}} e^{2\pi i \xi \cdot (x-y)} a(x,\xi) f(y)\, dy\, d\xi, \quad f \in \mathscr{S}(\mathbb{R}^d). \tag{2}$$

When $a \in S^m_{\rho,\delta}$, the corresponding operator class is denoted $L^m_{\rho,\delta}$. For the symbol classes $S^m_{\rho,\delta}$, the oscillatory integral (2) is generally not absolutely convergent and should be read as the iterated integral

$$a(x,D)f(x) = \int_{\mathbb{R}^d} e^{2\pi i \xi \cdot x} a(x,\xi) \widehat{f}(\xi)\, d\xi. \tag{3}$$

In order to extend the operator to act on other function spaces than $\mathscr{S}(\mathbb{R}^d)$ one modifies the definition (2) into

$$a(x,D)f(x) = \lim_{\varepsilon \to +0} \int_{\mathbb{R}^{2d}} \psi(\varepsilon y)\psi(\varepsilon \xi) e^{2\pi i \xi \cdot (x-y)} a(x,\xi) f(y)\, dy\, d\xi \tag{4}$$

where $\psi \in C_c^\infty(\mathbb{R}^d)$ equals one in a neighborhood of the origin. Integrating by parts we may rewrite (4) as

$$\begin{aligned} a(x,D)f(x) = \int_{\mathbb{R}^{2d}} & e^{2\pi i \xi \cdot (x-y)} (1+|\xi|^2)^{-N} (1-\Delta_\xi)^M a(x,\xi) \\ & \times (1-\Delta_y)^N ((1+|x-y|^2)^{-M} f(y))\, dy\, d\xi, \end{aligned}$$

where Δ denotes the normalized Laplacian $\Delta = (2\pi)^{-2}\sum_1^d \partial_j^2$, which is an absolutely convergent integral for $f \in C_b^\infty(\mathbb{R}^d)$ provided that $2M > d$ and $2N > d + m$. By differentiation under the integral it follows that $a(x,D) : C_b^\infty(\mathbb{R}^d) \mapsto C_b^\infty(\mathbb{R}^d)$ continuously. This procedure is standard and fundamental in pseudo-differential calculus [3, 6, 13].

For an admissible pair of symbols a, b we define the *symbol product* $\#_0$ by

$$c = a\#_0 b \quad \Longleftrightarrow \quad c(x,D) = a(x,D)b(x,D).$$

We have the following well-known result in the theory of pseudodifferential operators [3, 6]. The symbol product is a continuous bilinear map from $S^{m_1}_{\rho,\delta} \times S^{m_2}_{\rho,\delta}$ to $S^{m_1+m_2}_{\rho,\delta}$,

$$S^{m_1}_{\rho,\delta}\#_0 S^{m_2}_{\rho,\delta} \subseteq S^{m_1+m_2}_{\rho,\delta}, \quad m_1, m_2 \in \mathbb{R}. \tag{5}$$

3. Almost periodic functions and pseudodifferential operators

We will work with spaces of almost periodic functions [2, 7, 12]. The basic space of uniform almost periodic functions is denoted $CAP(\mathbb{R}^d)$ and defined as follows. A set $U \subset \mathbb{R}^d$ is called *relatively dense* if there exists a compact set $K \subset \mathbb{R}^d$ such that $(x+K) \cap U \neq \emptyset$ for any $x \in \mathbb{R}^d$. An element $\tau \in \mathbb{R}^d$ is called an ε-almost period of a function $f \in C_b(\mathbb{R}^d)$ if $\sup_x |f(x+\tau) - f(x)| < \varepsilon$. Then $CAP(\mathbb{R}^d)$ is defined as the space of all $f \in C_b(\mathbb{R}^d)$ such that, for any $\varepsilon > 0$, the set of ε-almost periods of f is relatively dense. With the assumption that the uniform almost periodic functions is a subspace of $C_b(\mathbb{R}^d)$, this original definition by H. Bohr is equivalent to the following three [2, 7, 12]:

(i) the set of translations $\{f(\cdot - x)\}_{x\in\mathbb{R}^d}$ is precompact in $C_b(\mathbb{R}^d)$;

(ii) $f = g \circ i_B$ where i_B is the canonical homomorphism from $\mathbb{R}^d$ into the Bohr compactification $\mathbb{R}^d_B$ of $\mathbb{R}^d$ and $g \in C(\mathbb{R}^d_B)$. Hence f can be extended to a continuous function on $\mathbb{R}^d_B$;

(iii) f is the uniform limit of trigonometric polynomials.

The space $CAP(\mathbb{R}^d)$ is a conjugate-invariant complex algebra of uniformly continuous functions. For $f \in CAP(\mathbb{R}^d)$ the mean value functional

$$M(f) = \lim_{T\to+\infty} T^{-d} \int_{s+K_T} f(x)\,dx, \tag{6}$$

where $K_T = \{x \in \mathbb{R}^d :\ 0 \leqslant x_j \leqslant T,\ j = 1,\dots,d\}$, exists uniformly over $s \in \mathbb{R}^d$. By M_x we understand the mean value in the variable x of a function of several variables. The Bohr (–Fourier) transformation [7] is defined by

$$f_\lambda = M_x(f(x)e^{-2\pi i\lambda\cdot x}), \quad \lambda \in \mathbb{R}^d,$$

and $f_\lambda \neq 0$ for at most countably many $\lambda \in \mathbb{R}^d$. The set $\{\lambda \in \mathbb{R}^d : f_\lambda \neq 0\}$ is called the set of frequencies for f.

A function $f \in CAP(\mathbb{R}^d)$ may be reconstructed from its Bohr–Fourier coefficients $(f_\lambda)_{\lambda \in \Lambda}$ using Bochner–Fejér polynomials [7, 12]. We give a brief overview of the results we need. Let $\beta_n \in \mathbb{R}^d$, $n = 1, 2, \ldots$, be a *rational basis* for the set of frequencies Λ for f. This means that $(\beta_n)_{n=1}^\infty$ is linearly independent over $\mathbb{Q}$ and each $\lambda \in \Lambda$ can be written

$$\lambda = \sum_{n=1}^N q_n \beta_n, \quad q_n \in \mathbb{Q},$$

with unique coefficients $(q_n)_{n=1}^N$. Every countable set $\Lambda \subset \mathbb{R}^d$ has a rational basis contained in Λ [7]. The composite Bochner–Fejér kernel is defined as

$$\begin{aligned} K_{n;\beta_1,\ldots,\beta_n}(x) = &\sum_{|\nu_1| \leqslant (n!)^2, \ldots, |\nu_n| \leqslant (n!)^2} \left(1 - \frac{|\nu_1|}{(n!)^2}\right) \cdots \left(1 - \frac{|\nu_n|}{(n!)^2}\right) \\ &\times \exp\left(2\pi i \left(\frac{\nu_1}{n!}\beta_1 + \cdots + \frac{\nu_n}{n!}\beta_n\right) \cdot x\right). \end{aligned}$$

We denote its coefficients

$$K_{n;\nu_1,\ldots,\nu_n} = \left(1 - \frac{|\nu_1|}{(n!)^2}\right) \cdots \left(1 - \frac{|\nu_n|}{(n!)^2}\right), \quad |\nu_j| \leqslant (n!)^2, \quad 1 \leqslant j \leqslant n. \tag{7}$$

Since $(\beta_n)_{n=1}^\infty$ is linearly independent over $\mathbb{Q}$, and since $M_x(e^{2\pi i \lambda \cdot x}) = 0$ when $\lambda \neq 0$, we have $M(K_{n;\beta_1,\ldots,\beta_n}) = 1$.

For a given $f \in CAP(\mathbb{R}^d)$ the Bochner–Fejér polynomial of order n is defined by

$$\begin{aligned} P_n(f)(x) &= M_y\left(f(y) K_{n;\beta_1,\ldots,\beta_n}(x - y)\right) \\ &= \sum_{|\nu_1| \leqslant (n!)^2, \ldots, |\nu_n| \leqslant (n!)^2} K_{n;\nu_1,\ldots,\nu_n} f_{\frac{\nu_1}{n!}\beta_1 + \cdots + \frac{\nu_n}{n!}\beta_n} \\ &\quad \times \exp\left(2\pi i \left(\frac{\nu_1}{n!}\beta_1 + \cdots + \frac{\nu_n}{n!}\beta_n\right) \cdot x\right). \end{aligned} \tag{8}$$

It follows from $M(K_{n;\beta_1,\ldots,\beta_n}) = 1$ and $K_{n;\beta_1,\ldots,\beta_n}(x) \geqslant 0$ [7] that

$$\|P_n(f)\|_{L^\infty} \leqslant \|f\|_{L^\infty}. \tag{9}$$

If we define the function on Λ

$$K_n(\lambda) = \begin{cases} K_{n;\nu_1,\ldots,\nu_n} & \text{if } \lambda = \frac{\nu_1}{n!}\beta_1 + \cdots + \frac{\nu_n}{n!}\beta_n, \quad |\nu_j| \leqslant (n!)^2, \quad 1 \leqslant j \leqslant n, \\ 0 & \text{otherwise}, \end{cases}$$

then we may write (8) in shorter form as

$$P_n(f)(x) = \sum_{\lambda \in \Lambda} K_n(\lambda) f_\lambda e^{2\pi i \lambda \cdot x}. \tag{10}$$

We observe that $K_n(\lambda)$ has finite support and $0 \leqslant K_n(\lambda) \leqslant 1$. For an arbitrary $\lambda \in \Lambda$ we may write for some $n > 0$ and $|\nu_j| \leqslant (n!)^2$, $1 \leqslant j \leqslant n$,

$$\begin{aligned}\lambda &= \frac{\nu_1}{n!}\beta_1 + \cdots + \frac{\nu_n}{n!}\beta_n \\ &= \frac{\nu_1(n+m)!/n!}{(n+m)!}\beta_1 + \cdots + \frac{\nu_n(n+m)!/n!}{(n+m)!}\beta_n + 0\cdot\beta_{n+1} + \cdots + 0\cdot\beta_{n+m},\end{aligned}$$

where $m \geqslant 0$ is arbitrary. It follows that

$$K_{n+m}(\lambda) = K_{n+m;\nu_1(n+m)!/n!,\ldots,\nu_n(n+m)!/n!,0,\ldots,0}.$$

For n and $\nu_1,\ldots,\nu_n$ fixed, it follows from (7) that the right hand side approaches 1 as $m \to \infty$, because

$$1 - \frac{|\nu_j|(n+m)!/n!}{((n+m)!)^2} = 1 - \frac{|\nu_j|}{n!(n+m)!} \to 1, \quad m \to \infty, \quad 1 \leqslant j \leqslant n.$$

We may conclude that $K_n(\lambda) \to 1$ as $n \to +\infty$, for any $\lambda \in \Lambda$.

We state the fundamental approximation result for the Bochner–Fejér polynomials [7, 12]. If $f \in CAP(\mathbb{R}^d)$ then we have the uniform limit

$$\sup_{x\in\mathbb{R}^d} |P_n(f)(x) - f(x)| \to 0, \quad n \to \infty. \tag{11}$$

The limit in (11) holds for any $f \in CAP(\mathbb{R}^d)$ whose set of frequencies is contained in Λ.

The next lemma resembles [12, Corollary 2.1]. We give a proof for completeness.

LEMMA 1. *For a precompact set $\mathcal{F} \subset CAP(\mathbb{R}^d)$, the limit*

$$\sup_{x\in\mathbb{R}^d} |P_n(f)(x) - f(x)| \to 0, \quad n \to \infty$$

is uniform over $f \in \mathcal{F}$.

Proof. Denote $\|\cdot\| = \|\cdot\|_{L^\infty}$. Due to the assumption that $\mathcal{F}$ is precompact, there exists for each integer $k > 0$ a finite set $\{f_{k,j}\}_{j=1}^{N_k} \subset \mathcal{F}$ such that $\|f - f_{k,j}\| < 1/k$ holds for each $f \in \mathcal{F}$ for some j, $1 \leqslant j \leqslant N_k$. Let Λ_k be the union of the frequencies that occur in $\{f_{k,j}\}_{j=1}^{N_k}$ and let Λ be the linear hull over $\mathbb{Q}$ of $\bigcup_{k\geqslant 1}\Lambda_k$. Define the Bochner–Fejér kernels $\{K_{n;\beta_1,\ldots,\beta_n}(x)\}_{n\geqslant 1}$ as above from the countable set Λ.

Let $\varepsilon > 0$ and pick an integer $k > \varepsilon^{-1}$. According to limit (11) we have $\|f_{k,j} - P_n(f_{k,j})\| < \varepsilon$ for all $1 \leqslant j \leqslant N_k$ if $n \geqslant N_\varepsilon$ for a sufficiently large integer N_ε. Let $f \in \mathcal{F}$ and pick an $f_{k,j}$ such that $\|f - f_{k,j}\| < 1/k < \varepsilon$. We have, using (9),

$$\begin{aligned}\|f - P_n(f)\| &\leqslant \|f - f_{k,j}\| + \|f_{k,j} - P_n(f_{k,j})\| + \|P_n(f_{k,j} - f)\| \\ &\leqslant \|f - f_{k,j}\| + \|f_{k,j} - P_n(f_{k,j})\| + \|f_{k,j} - f\| < 3\varepsilon, \quad n \geqslant N_\varepsilon.\end{aligned}$$

□

For $m \in \mathbb{N}$, the space $CAP^m(\mathbb{R}^d)$ is defined as all $f \in C^m(\mathbb{R}^d)$ such that $\partial^\alpha f \in CAP(\mathbb{R}^d)$ for $|\alpha| \leqslant m$, and $CAP^\infty(\mathbb{R}^d) = \bigcap_{m\in\mathbb{N}} CAP^m(\mathbb{R}^d)$. Then $CAP^\infty = CAP \cap C_b^\infty$ [12].

The mean value defines an inner product

$$(f,g)_B = M(f\overline{g}), \quad f,g \in CAP(\mathbb{R}^d). \tag{12}$$

The completion of $CAP(\mathbb{R}^d)$ in the norm $\|\cdot\|_B$ is the Hilbert space of Besicovitch a.p. functions $B^2(\mathbb{R}^d)$ [12].

Inspired by the usual Sobolev space norm

$$\|f\|_{H^s(\mathbb{R}^d)} = \left(\int_{\mathbb{R}^d} (1+|\xi|^2)^s |\widehat{f}(\xi)|^2 \, d\xi\right)^{1/2},$$

Shubin [9] has defined Sobolev–Besicovitch spaces of a.p. functions $H^s(\mathbb{R}^d_B)$ for $s \in \mathbb{R}$, as the completion of $TP(\mathbb{R}^d)$ in the norm corresponding to the inner product

$$(f,g)_{H^s(\mathbb{R}^d_B)} = \sum_{\xi\in\mathbb{R}^d} (1+|\xi|^2)^s f_\xi \overline{g}_\xi, \quad f,g \in TP(\mathbb{R}^d).$$

The spaces $H^s(\mathbb{R}^d_B)$ are Hilbert spaces containing $TP(\mathbb{R}^d)$ as a dense subspace, $H^0(\mathbb{R}^d_B) = B^2(\mathbb{R}^d)$, and one defines

$$H^\infty(\mathbb{R}^d_B) = \bigcap_{s\in\mathbb{R}} H^s(\mathbb{R}^d_B), \quad H^{-\infty}(\mathbb{R}^d_B) = \bigcup_{s\in\mathbb{R}} H^s(\mathbb{R}^d_B).$$

We have the inclusion $CAP^\infty(\mathbb{R}^d) \subset H^\infty(\mathbb{R}^d_B)$, but there is no result corresponding to the Sobolev embedding theorem for the Sobolev–Besicovitch spaces. In fact, $H^\infty(\mathbb{R}^d_B)$ is not embedded in $CAP(\mathbb{R}^d)$ [12]. The reason is that the frequencies may be contained in a bounded set, for example as in

$$f(x) = \sum_{k=1}^\infty \frac{1}{k} e^{2\pi i \xi_k \cdot x}, \quad |\xi_k| = 1.$$

This function is clearly a member of $H^\infty(\mathbb{R}^d_B)$, and if the frequencies $\{\xi_k\}_{k=1}^\infty$ are linearly independent over $\mathbb{Z}$, then $\|f\|_{L^\infty} = \sum_{k=1}^\infty 1/k = \infty$ [12].

Next we define the symbol spaces for almost periodic pseudodifferential operators.

DEFINITION 2. *For $m \in \mathbb{R}$, the space $APS^m_{\rho,\delta}$ is defined as the space of all $a \in S^m_{\rho,\delta}(\mathbb{R}^{2d})$ such that $a(\cdot,\xi) \in CAP(\mathbb{R}^d)$ for all $\xi \in \mathbb{R}^d$. The corresponding operator class in the Kohn–Nirenberg quantization is denoted $APL^m_{\rho,\delta}$, and its members are called almost periodic pseudodifferential operators.*

For fixed $\xi \in \mathbb{R}^d$, we denote the Bohr–Fourier coefficients of $a(\cdot,\xi)$ by

$$a_\lambda(\xi) = (a(\cdot,\xi))_\lambda = M_x(a(x,\xi)e^{-2\pi i\lambda\cdot x}), \quad \xi \in \mathbb{R}^d, \quad \lambda \in \mathbb{R}^d. \tag{13}$$

LEMMA 2. *For $a \in APS^m_{\rho,\delta}$ the set of frequencies*

$$\Lambda = \Lambda(a) = \{\lambda \in \mathbb{R}^d : \exists \xi \in \mathbb{R}^d : a_\lambda(\xi) \neq 0\}$$

is countable.

Proof. As already mentioned $\Lambda_\xi = \{\lambda \in \mathbb{R}^d : a_\lambda(\xi) \neq 0\}$ is countable for each $\xi \in \mathbb{R}^d$. Using $\Lambda = \bigcup_{\xi \in \mathbb{R}^d} \Lambda_\xi$, it suffices to show that $\bigcup_{\xi \in \mathbb{R}^d} \Lambda_\xi \subset \bigcup_{\xi \in \mathbb{Q}^d} \Lambda_\xi$. If $\lambda \in \bigcup_{\xi \in \mathbb{R}^d} \Lambda_\xi$ there exists $\xi \in \mathbb{R}^d$ such that $a_\lambda(\xi) \neq 0$. By the mean value theorem we have

$$a(x,\xi+\eta) - a(x,\xi) = (\nabla_2 \operatorname{Re} a(x,\xi+\theta_1\eta) + i\nabla_2 \operatorname{Im} a(x,\xi+\theta_2\eta)) \cdot \eta \tag{14}$$

where ∇_2 denotes the gradient in the second $\mathbb{R}^d$ variable and $0 \leqslant \theta_1, \theta_2 \leqslant 1$. It follows that $|a_\lambda(\xi+\eta) - a_\lambda(\xi)| \leqslant M_x(|a(x,\xi+\eta) - a(x,\xi)|) \leqslant C|\eta|$. Hence there exists $\xi' \in \mathbb{Q}^d$ such that $a_\lambda(\xi') \neq 0$. □

Without loss of generality we may assume that Λ is a linear space over $\mathbb{Q}$. Furthermore it follows from (14) that $\partial_\xi^\alpha a(\cdot,\xi) \in CAP(\mathbb{R}^d)$ for all $\alpha \in \mathbb{N}^d$ and $\xi \in \mathbb{R}^d$, since a ξ-derivative is a uniform limit of $CAP(\mathbb{R}^d)$ functions. Thus $\partial_\xi^\alpha \partial_x^\beta a(\cdot,\xi) \in CAP(\mathbb{R}^d)$ for all $\alpha, \beta \in \mathbb{N}^d$ and $\xi \in \mathbb{R}^d$.

LEMMA 3. *Suppose $a \in APS^m_{\rho,\delta}$ and $\lambda \in \Lambda$. Then $a_\lambda \in C^\infty(\mathbb{R}^d)$ and*

$$\partial^\alpha(a_\lambda)(\xi) = (\partial_\xi^\alpha a)_\lambda(\xi), \quad \alpha \in \mathbb{N}^d, \tag{15}$$

$$(\partial_x^\beta a)_\lambda(\xi) = (2\pi i\lambda)^\beta a_\lambda(\xi), \quad \beta \in \mathbb{N}^d. \tag{16}$$

Proof. By differentiation under the mean value we obtain (15). To prove (16), we integrate by parts which gives

$$\begin{aligned}(\partial_x^\beta a)_\lambda(\xi) &= M_x((\partial_x^\beta a)(x,\xi)e^{-2\pi i\lambda\cdot x}) \\ &= M_x(a(x,\xi)(-\partial_x)^\beta(e^{-2\pi i\lambda\cdot x})) \\ &= (2\pi i\lambda)^\beta a_\lambda(\xi).\end{aligned}$$

□

Lemma 3 gives

$$\partial^\alpha(a_\lambda)(\xi) = (\partial_\xi^\alpha a)_\lambda(\xi) = (2\pi i\lambda)^{-\beta}(\partial_\xi^\alpha \partial_x^\beta a)_\lambda(\xi), \quad \lambda \neq 0.$$

From (13) and Definition 1 we thus obtain the estimate

$$|\partial^\alpha(a_\lambda)(\xi)| \leqslant C_{k,\alpha}\langle\lambda\rangle^{-k}\langle\xi\rangle^{m-\rho|\alpha|+\delta k}, \quad k \in \mathbb{N}, \quad \alpha \in \mathbb{N}^d. \tag{17}$$

LEMMA 4. *If $a \in APS^m_{\rho,\delta}$ and $f \in TP(\mathbb{R}^d)$ then*

$$a(x,D)f(x) = \sum_{\lambda \in \mathbb{R}^d} e^{2\pi i x\cdot\lambda} a(x,\lambda) f_\lambda. \tag{18}$$

Proof. Since $f(x) = \sum_\lambda f_\lambda e^{2\pi i x\cdot\lambda}$ is a finite sum we have by the definition (4)

$$\begin{aligned} a(x,D)f(x) &= \sum_\lambda f_\lambda \lim_{\varepsilon\to+0} \int_{\mathbb{R}^{2d}} \psi(\varepsilon y)\psi(\varepsilon\xi) e^{2\pi i(\xi\cdot x - y\cdot(\xi-\lambda))} a(x,\xi)\, dy\, d\xi \\ &= \sum_\lambda f_\lambda \lim_{\varepsilon\to+0} \int_{\mathbb{R}^d} a(x,\xi) e^{2\pi i \xi\cdot x} \psi(\varepsilon\xi) \left(\int_{\mathbb{R}^d} \psi(\varepsilon y) e^{-2\pi i y\cdot(\xi-\lambda)} dy \right) d\xi \\ &= \sum_\lambda f_\lambda \lim_{\varepsilon\to+0} \int_{\mathbb{R}^d} a(x,\xi+\lambda) e^{2\pi i x\cdot(\xi+\lambda)} \psi(\varepsilon(\xi+\lambda))\, \varepsilon^{-d} \widehat{\psi}(\xi/\varepsilon)\, d\xi. \end{aligned} \tag{19}$$

Let us define $g(\xi) = a(x,\xi+\lambda)e^{2\pi i x\cdot(\xi+\lambda)} \in C^\infty(\mathbb{R}^d)$. Using the fact that $\int \varepsilon^{-d}\widehat{\psi}(\xi/\varepsilon) d\xi = \psi(0) = 1$ we obtain

$$\begin{aligned} &\left| g(0) - \int_{\mathbb{R}^d} g(\xi)\psi(\varepsilon(\xi+\lambda))\varepsilon^{-d}\widehat{\psi}(\xi/\varepsilon)\, d\xi \right| \\ &\leqslant \int_{\mathbb{R}^d} |g(0)-g(\xi)| \varepsilon^{-d} |\widehat{\psi}(\xi/\varepsilon)|\, d\xi + \int_{\mathbb{R}^d} |1-\psi(\varepsilon(\xi+\lambda))||g(\xi)|\varepsilon^{-d}|\widehat{\psi}(\xi/\varepsilon)|\, d\xi \\ &= \int_{\mathbb{R}^d} |g(0)-g(\varepsilon\xi)||\widehat{\psi}(\xi)|\, d\xi + \int_{\mathbb{R}^d} |1-\psi(\varepsilon(\varepsilon\xi+\lambda))||g(\varepsilon\xi)||\widehat{\psi}(\xi)|\, d\xi. \end{aligned}$$

The integrand of the first term tends to zero as $\varepsilon \to 0$ for each $\xi \in \mathbb{R}^d$. For $0 < \varepsilon < 1$ it is dominated by $C(1+\langle\xi\rangle^{|m|}\langle\lambda\rangle^{|m|})|\widehat{\psi}(\xi)|$ which is integrable, so by Lebesgue's dominated convergence theorem the first integral approaches zero as $\varepsilon \to 0$. Likewise, the second integral approaches zero as $\varepsilon \to 0$, since the integrand approaches zero as $\varepsilon \to 0$ for each $\xi \in \mathbb{R}^d$, and is dominated by $C|\widehat{\psi}(\xi)|\langle\xi\rangle^{|m|}\langle\lambda\rangle^{|m|}$ which is integrable. We conclude that

$$\lim_{\varepsilon\to+0} \int_{\mathbb{R}^d} a(x,\xi+\lambda) e^{2\pi i x\cdot(\xi+\lambda)} \psi(\varepsilon(\xi+\lambda))\varepsilon^{-d}\widehat{\psi}(\xi/\varepsilon) d\xi = a(x,\lambda)e^{2\pi i x\cdot\lambda}$$

which inserted into (19) proves (18). □

As Shubin has shown [9, 12], most of the basic results of pseudodifferential calculus with symbols in $S^m_{\rho,\delta}$, such as asymptotic expansions, the formula for composition of two operators and the formal adjoint of an operator, are true for $APS^m_{\rho,\delta}$, with the conclusion that all involved symbols satisfy $a(\cdot,\xi) \in CAP(\mathbb{R}^d)$ for all $\xi \in \mathbb{R}^d$. In particular we have [12, Theorem 3.1]: If $a \in APS^{m_1}_{\rho,\delta}$ and $b \in APS^{m_2}_{\rho,\delta}$ then $a\#_0 b \in APS^{m_1+m_2}_{\rho,\delta}$.

We will need three more results from Shubin's article [12].

THEOREM 1 (M.A. Shubin). *Let $A \in APL^m_{\rho,\delta}$.*

(i) *If* $u, v \in CAP^\infty(\mathbb{R}^d)$ *then*

$$(Au, v)_B = \lim_{R \to +\infty} |B_R|^{-1} (A(\varphi_R u), \varphi_R v)_{L^2}$$

where $\{\varphi_R\}_{R \geqslant 1} \subset C_c^\infty(\mathbb{R}^d)$ *is a family of functions that satisfy*

$$\varphi_R(x) = \begin{cases} 1 & \text{for } |x| \leqslant R, \\ 0 & \text{for } |x| \geqslant R + R^\kappa, \end{cases}$$
$$|\partial^\alpha \varphi_R(x)| \leqslant C_\alpha R^{-\kappa|\alpha|},$$

where $0 < \kappa < 1$. *Here* $B_R \subset \mathbb{R}^d$ *denotes the ball of radius* R *centered at the origin and* $|B_R|$ *its volume.*

(ii) *If* $u \in \mathscr{S}(\mathbb{R}^d)$ *and* $u_k = u * \psi_k \in CAP^\infty(\mathbb{R}^d)$, *where* $\{\psi_k\}_{k=1}^\infty \subset CAP(\mathbb{R}^d)$ *are chosen in a particular way (see [12, Lemma 4.3]), then*

$$(Au, u)_{L^2} = \lim_{k \to +\infty} (Au_k, u_k)_B.$$

(iii) $\|A\|_{\mathcal{L}(L^2(\mathbb{R}^d))} = \|A\|_{\mathcal{L}(B^2(\mathbb{R}^d))}$.

The result (iii) is an immediate consequence of (i) and (ii).

From Lemma 4 we see that $\langle D \rangle^s$ is a unitary operator from $H^s(\mathbb{R}^d_B)$ to $H^0(\mathbb{R}^d_B) = B^2(\mathbb{R}^d)$, just as in the case of $H^s(\mathbb{R}^d)$. The well-known result that $a \in S^0_{\rho,\delta}$ implies $a(x,D) \in \mathcal{L}(L^2(\mathbb{R}^d))$ [6] has the following consequence.

COROLLARY 1. *If* $a \in APS^m_{\rho,\delta}$ *then for any* $s \in \mathbb{R}$

$$\|a(x,D)\|_{\mathcal{L}(H^s(\mathbb{R}^d), H^{s-m}(\mathbb{R}^d))} = \|a(x,D)\|_{\mathcal{L}(H^s(\mathbb{R}^d_B), H^{s-m}(\mathbb{R}^d_B))} < \infty.$$

Proof. We have

$$\begin{aligned}
\|a(x,D)\|_{\mathcal{L}(H^s(\mathbb{R}^d), H^{s-m}(\mathbb{R}^d))} &= \sup_{\|f\|_{H^s(\mathbb{R}^d)} \leqslant 1} \|a(x,D) f\|_{H^{s-m}(\mathbb{R}^d)} \\
&= \sup_{\|\langle D \rangle^s f\|_{L^2(\mathbb{R}^d)} \leqslant 1} \|\langle D \rangle^{s-m} a(x,D) \langle D \rangle^{-s} \langle D \rangle^s f\|_{L^2(\mathbb{R}^d)} \\
&= \sup_{\|f\|_{L^2(\mathbb{R}^d)} \leqslant 1} \|\langle D \rangle^{s-m} a(x,D) \langle D \rangle^{-s} f\|_{L^2(\mathbb{R}^d)} \\
&= \sup_{\|f\|_{B^2(\mathbb{R}^d)} \leqslant 1} \|\langle D \rangle^{s-m} a(x,D) \langle D \rangle^{-s} f\|_{B^2(\mathbb{R}^d)} \\
&= \sup_{\|f\|_{H^s(\mathbb{R}^d_B)} \leqslant 1} \|a(x,D) f\|_{H^{s-m}(\mathbb{R}^d_B)} \\
&= \|a(x,D)\|_{\mathcal{L}(H^s(\mathbb{R}^d_B), H^{s-m}(\mathbb{R}^d_B))}.
\end{aligned}$$

In fact, the fourth equality is Theorem 1 (iii). The finiteness of the operator norm follows from the observation that the symbol

$$\langle\xi\rangle^{s-m} \#_0\, a \,\#_0 \langle\xi\rangle^{-s} \in S^0_{\rho,\delta},$$

due to (5), and the above mentioned $L^2(\mathbb{R}^d)$-continuity for operators with symbol in $S^0_{\rho,\delta}$. □

4. A transformation of symbols for a.p. pseudodifferential operators

DEFINITION 3. *Let $a \in APS^m_{\rho,\delta}$ and let $\Lambda = \Lambda(a)$ denote the frequencies whose Bohr–Fourier coefficients a_λ are not identically zero. We set*

$$U(a)(\xi)_{\lambda,\lambda'} = a_{\lambda'-\lambda}(\xi - \lambda'), \quad \lambda,\lambda' \in \Lambda, \quad \xi \in \mathbb{R}^d, \tag{20}$$

where $a_\lambda(\xi)$ is the Bohr–Fourier coefficient defined in (13)*.*

We note the property

$$U(a)(\xi)_{\lambda,\lambda'} = U(a)(\xi+\mu)_{\lambda+\mu,\lambda'+\mu}, \quad \mu \in \Lambda.$$

By Lemma 1 the inverse transformation of $a \mapsto U(a)_{\lambda,\lambda'}$ is

$$a(x,\xi) = \lim_{n\to\infty} \sum_{\lambda\in\Lambda} K_n(\lambda) U(a)(\xi)_{-\lambda,0}(\xi) e^{2\pi i\lambda\cdot x}$$

which converges uniformly in x for each ξ. For $a \in S^m_{\rho,\delta}$ the map $a \mapsto U(a)_{\lambda,\lambda'}$ is thus injective.

For fixed $\xi \in \mathbb{R}^d$ we may look upon $U(a)(\xi)$ as a matrix,

$$U(a)(\xi) = [U(a)(\xi)_{\lambda,\lambda'}]_{\lambda,\lambda'\in\Lambda},$$

indexed by $(\lambda,\lambda') \in \Lambda \times \Lambda$. This matrix defines an operator on complex-valued sequences defined on Λ, which are denoted $z = (z_\lambda)_{\lambda\in\Lambda}$, according to

$$(U(a)(\xi)\cdot z)_\lambda = \sum_{\lambda'\in\Lambda} U(a)(\xi)_{\lambda,\lambda'} z_{\lambda'}.$$

It follows from (15) that

$$\partial^\alpha_\xi (U(a))(\xi) = U(\partial^\alpha_\xi a)(\xi). \tag{21}$$

Moreover, denoting translation by $(T_{0,-\eta}a)(x,\xi) = a(x,\xi+\eta)$ we have

$$U(T_{0,-\eta}a)(\xi)_{\lambda,\lambda'} = (T_{0,-\eta}a)_{\lambda'-\lambda}(\xi-\lambda') = U(a)(\xi+\eta)_{\lambda,\lambda'}. \tag{22}$$

Since the operator-valued function $U(a)$ depends on the frequency variable only, it may be used to define a Fourier multiplier operator for vector-valued functions according to

$$U(a)(D)F(x) = \int_{\mathbb{R}^d} e^{2\pi i\xi\cdot x} U(a)(\xi)\cdot \widehat{F}(\xi)\, d\xi, \tag{23}$$

where $F(x) = (F_\lambda(x))_{\lambda\in\Lambda}$ is the vector-valued function

$$\mathbb{R}^d \ni x \mapsto (F_\lambda(x))_{\lambda\in\Lambda}.$$

The inner product for vector-valued functions is

$$\begin{aligned}(F,G)_{L^2(\mathbb{R}^d,l^2)} &= (F,G)_{L^2(\mathbb{R}^d,l^2(\Lambda))} = \int_{\mathbb{R}^d} (F(x),G(x))_{l^2}\,dx \\ &= \int_{\mathbb{R}^d} \sum_{\lambda\in\Lambda} F_\lambda(x)\overline{G_\lambda(x)}\,dx, \quad F,G \in L^2(\mathbb{R}^d,l^2).\end{aligned}$$

If the symbol a does not depend on x, i.e. $a(x,D)$ is a Fourier multiplier (convolution) operator, then $a_\lambda(\xi) = 0$ when $\lambda \neq 0$ follows from (13). Thus $U(a)(\xi)$ is the pointwise multiplier operator

$$(U(a)(\xi)\cdot z)_\lambda = \sum_{\lambda'\in\Lambda} a_{\lambda'-\lambda}(\xi-\lambda')z_{\lambda'} = a_0(\xi-\lambda)z_\lambda = a(\xi-\lambda)z_\lambda,$$

and

$$(U(a)(D)F(x))_\lambda = \int_{\mathbb{R}^d} e^{2\pi i\xi\cdot x} a(\xi-\lambda)\widehat{F_\lambda}(\xi)\,d\xi = (T_\lambda a)(D)F_\lambda(x).$$

Thus $U(a)(D)$ acts pointwise in the λ variable by a convolution in x. If a does not depend on ξ, then $U(a)$ does not depend on ξ either, and $U(a)_{\lambda,\lambda'} = a_{\lambda'-\lambda}$. Thus, in this case we have

$$(U(a)(D)F(x))_\lambda = (U(a)\cdot F(x))_\lambda = \sum_{\lambda'\in\Lambda} a_{\lambda'-\lambda}F_{\lambda'}(x),$$

which is an operator that acts pointwise in x, by a convolution over the index set Λ. In particular we have $U(1)(\xi)_{\lambda,\lambda'} = \delta_{\lambda'-\lambda}$ which denotes the Kronecker delta. This means that $U(1)(D) = I$.

The above discussion is not precise since we have not yet proved in what sense $U(a)(\xi)$ is a continuous operator for fixed $\xi \in \mathbb{R}^d$, and whether the operator-valued function $\xi \mapsto U(a)(\xi)$ is continuous and bounded. Let us therefore address these questions.

We shall first evaluate the operator-valued function $U(a)(\xi)$ in the origin. It will turn out that $U(a)(0)$ contains much information about continuity, positivity and invertibility of $a(x,D)$. We need the sequence spaces

$$l_s^p = l_s^p(\Lambda) = \left\{ (x_\lambda)_{\lambda\in\Lambda} : \|x\|_{l_s^p} = \left(\sum_{\lambda\in\Lambda} \langle\lambda\rangle^{ps}|x_\lambda|^p \right)^{1/p} < \infty \right\}, \tag{24}$$

parametrized by $s \in \mathbb{R}$ and normed by $\|\cdot\|_{l_s^p}$ where $1 \leqslant p \leqslant \infty$. In some places we will use the symbol l_c^2 which denotes the space of square-summable sequences with compact support.

PROPOSITION 1. *For $a \in APS^m_{\rho,\delta}$ we have for any $s \in \mathbb{R}$*

$$\|U(a)(0)\|_{\mathcal{L}(l^2_s, l^2_{s-m})} \leqslant \|a(x,D)\|_{\mathcal{L}(H^s(\mathbb{R}^d_B), H^{s-m}(\mathbb{R}^d_B))} < \infty. \tag{25}$$

Proof. Let $f, g \in TP(\Lambda)$. Lemma 4 gives

$$\begin{aligned}(a(x,D)f,g)_B &= \sum_{\lambda,\lambda'} M_x(a(x,\lambda)e^{2\pi i x\cdot(\lambda-\lambda')}) f_\lambda \overline{g}_{\lambda'} \\ &= \sum_{\lambda,\lambda'} a_{\lambda'-\lambda}(\lambda) f_\lambda \overline{g}_{\lambda'} \\ &= (U(a)(0)\cdot \check{f}, \check{g})_{l^2}\end{aligned} \tag{26}$$

where $\check{f}_\lambda = f_{-\lambda}$. We abbreviate $H^s = H^s(\mathbb{R}^d_B)$. Using the duality $(H^s)' = H^{-s}$ under the form $(\cdot,\cdot)_B$, we obtain

$$\begin{aligned}\|a(x,D)\|_{\mathcal{L}(H^s,H^{s-m})} &= \sup_{\|f\|_{H^s}\leqslant 1} \|a(x,D)f\|_{H^{s-m}} \\ &= \sup_{\|f\|_{H^s}\leqslant 1, \|g\|_{H^{m-s}}\leqslant 1} |(a(x,D)f,g)_B| \\ &\geqslant \sup_{\|f_\cdot\|_{l^2_s}\leqslant 1, \|g_\cdot\|_{l^2_{m-s}}\leqslant 1} |(U(a)(0)\cdot \check{f}, \check{g})_{l^2}| \\ &= \|U(a)(0)\|_{\mathcal{L}(l^2_s, l^2_{s-m})},\end{aligned}$$

where we denote $\|f_\cdot\|^2_{l^2_s} = \sum_\lambda \langle\lambda\rangle^{2s}|f_\lambda|^2$. □

As a consequence of (26) and Theorem 1 (i) and (ii) we have the following result on positivity. As customary we say that A is a positive operator on a topological vector space X if $(Af,f)_H \geqslant 0$ for all $f \in X$, where $X \subset H$ and H is a Hilbert space, naturally associated with X. (We avoid the requirement $(Af,f)_H \geqslant 0$ for all $f \in H$ since the expression $(Af,f)_H$ may not be well-defined if A is not a bounded operator on H.) This is denoted $A \geqslant 0$ (where the spaces X and H are understood from the context). We will use the following pairs (X,H): $(\mathscr{S}(\mathbb{R}^d), L^2(\mathbb{R}^d))$, $(TP(\mathbb{R}^d), B^2(\mathbb{R}^d))$, (l^2_c, l^2) and $(\mathscr{S}(\mathbb{R}^d, l^2_c), L^2(\mathbb{R}^d, l^2))$.

COROLLARY 2. *If $a \in APS^m_{\rho,\delta}$ then $a(x,D) \geqslant 0$ on $\mathscr{S}(\mathbb{R}^d)$ if and only if $a(x,D)$ $\geqslant 0$ on $TP(\mathbb{R}^d)$. Moreover, $a(x,D) \geqslant 0$ on $TP(\Lambda)$ if and only if $U(a)(0) \geqslant 0$ on l^2_c.*

The next result gives a continuity statement of the operator-valued map $\xi \mapsto U(a)(\xi)$.

PROPOSITION 2. *If $a \in APS^m_{\rho,\delta}$ then we have*

$$\|U(a)(\xi)\|_{\mathcal{L}(l^1_{|m|}, l^\infty)} \leqslant C\langle\xi\rangle^m, \tag{27}$$

$$U(a) \in C(\mathbb{R}^d, \mathcal{L}(l^1_{|m|}, l^\infty)). \tag{28}$$

Proof. Using the inequality $\langle x+y\rangle^u \leqslant C\langle x\rangle^u\langle y\rangle^{|u|}$, Definition 3 and (17) we obtain

$$|U(a)(\xi)_{\lambda,\lambda'}| \leqslant C\langle \xi-\lambda'\rangle^m \leqslant C\langle\xi\rangle^m\langle\lambda'\rangle^{|m|}.$$

Hence

$$\|U(a)(\xi)\cdot x\|_{l^\infty} \leqslant C\langle\xi\rangle^m\|x\|_{l^1_{|m|}}$$

which proves (27). To prove (28), we note that

$$(U(a)(\xi)-U(a)(\xi+\eta))_{\lambda,\lambda'} = U(a-T_{0,-\eta}a)(\xi)_{\lambda,\lambda'} \tag{29}$$

follows from (22). Thus, by the mean value theorem (14), and again Definition 3 and (17),

$$\begin{aligned}
&\left|(U(a)(\xi)-U(a)(\xi+\eta))_{\lambda,\lambda'}\right| \\
&\leqslant |\eta|\left|(\nabla_2\operatorname{Re}a)_{\lambda'-\lambda}(\xi-\lambda'+\theta_1\eta)+i(\nabla_2\operatorname{Im}a)_{\lambda'-\lambda}(\xi-\lambda'+\theta_2\eta)\right| \\
&\leqslant C|\eta|\left(\langle\xi-\lambda'+\theta_1\eta\rangle^{m-\rho}+\langle\xi-\lambda'+\theta_2\eta\rangle^{m-\rho}\right) \\
&\leqslant C|\eta|\langle\lambda'\rangle^{m-\rho}\left(\langle\xi+\theta_1\eta\rangle^{|m-\rho|}+\langle\xi+\theta_2\eta\rangle^{|m-\rho|}\right) \\
&\leqslant C|\eta|\langle\lambda'\rangle^{|m|}\langle\eta\rangle^{|m-\rho|}\langle\xi\rangle^{|m-\rho|},
\end{aligned}$$

and therefore

$$\begin{aligned}
&\|U(a)(\xi)-U(a)(\xi+\eta)\|_{\mathcal{L}(l^1_{|m|},l^\infty)} \\
&= \sup_{\|x\|_{l^1_{|m|}}\leqslant 1}\ \sup_{\lambda\in\Lambda}|((U(a)(\xi)-U(a)(\xi+\eta))\cdot x)_\lambda| \\
&\leqslant C|\eta|\langle\eta\rangle^{|m-\rho|}\langle\xi\rangle^{|m-\rho|} \\
&\to 0, \quad |\eta|\to 0.
\end{aligned}$$

This proves (28). □

The next result gives a sharpening of condition (28), since we have $l^1_{|m|}\subset l^2_{|m|}$ and $l^2_{|m|-m}\subset l^\infty$.

PROPOSITION 3. *If $a\in APS^m_{\rho,\delta}$ then we have for any $s\in\mathbb{R}$*

$$U(a)(\xi)\in\mathcal{L}(l^2_s,l^2_{s-m}),\quad \xi\in\mathbb{R}^d, \tag{30}$$

$$U(a)\in C(\mathbb{R}^d,\mathcal{L}(l^2_s,l^2_{s-m})). \tag{31}$$

Proof. From (22) we see that $U(a)(\xi)=U(T_{0,-\xi}a)(0)$. Since $T_{0,-\xi}a\in APS^m_{\rho,\delta}$ for any $\xi\in\mathbb{R}^d$, (30) follows from Proposition 1.

In order to prove (31), it suffices to prove continuity in the origin, since

$$U(a)(\xi+\eta)-U(a)(\eta)=U(T_{0,-\eta}a)(\xi)-U(T_{0,-\eta}a)(0).$$

We use (29) and again Proposition 1 and Corollary 1, which give

$$\begin{aligned}\|U(a)(\xi)-U(a)(0)\|_{\mathscr{L}(l_s^2,l_{s-m}^2)} &= \|U(T_{0,-\xi}a-a)(0)\|_{\mathscr{L}(l_s^2,l_{s-m}^2)}\\ &\leqslant \|(T_{0,-\xi}a-a)(x,D)\|_{\mathscr{L}(H^s(\mathbb{R}^d),H^{s-m}(\mathbb{R}^d))}.\end{aligned}$$

In the next step we use

$$\|b(x,D)\|_{\mathscr{L}(H^s(\mathbb{R}^d),H^{s-m}(\mathbb{R}^d))} = \|\langle D\rangle^{s-m}b(x,D)\langle D\rangle^{-s}\|_{\mathscr{L}(L^2)}$$

for $b\in S^m_{\rho,\delta}$, and the fact that the $\mathscr{L}(L^2)$-norm of an operator with symbol in $S^0_{\rho,\delta}$ may be estimated by a finite sum of seminorms of the symbol in $S^0_{\rho,\delta}$ (see [6, Theorem 18.1.11] and [3, Theorem 2.80]). By (5) it thus suffices to prove that

$$T_{0,-\xi}a-a\to 0 \quad \text{in} \quad S^m_{\rho,\delta} \quad \text{as} \quad \xi\to 0. \tag{32}$$

The mean value theorem (14) gives

$$a(x,\eta+\xi)-a(x,\eta) = (\nabla_2 \operatorname{Re} a(x,\eta+\theta_1\xi)+i\nabla_2 \operatorname{Im} a(x,\eta+\theta_2\xi))\cdot\xi$$

with $0\leqslant\theta_1,\theta_2\leqslant 1$, so we have

$$\begin{aligned}&\left|\partial_\eta^\alpha\partial_x^\beta(T_{0,-\xi}a-a)(x,\eta)\right|\\ &\leqslant |\xi|\left|\partial_\eta^\alpha\partial_x^\beta\nabla_2\operatorname{Re} a(x,\eta+\theta_1\xi)+i\partial_\eta^\alpha\partial_x^\beta\nabla_2\operatorname{Im} a(x,\eta+\theta_2\xi)\right|\\ &\leqslant C|\xi|\left(\langle\eta+\theta_1\xi\rangle^{m-\rho(|\alpha|+1)+\delta|\beta|}+\langle\eta+\theta_2\xi\rangle^{m-\rho(|\alpha|+1)+\delta|\beta|}\right)\\ &\leqslant C|\xi|\langle\xi\rangle^{|m-\rho(|\alpha|+1)+\delta|\beta||}\langle\eta\rangle^{m-\rho(|\alpha|+1)+\delta|\beta|}.\end{aligned}$$

This proves (32), and therefore (31). ☐

The following result concerns positivity.

PROPOSITION 4. *If $a\in APS^m_{\rho,\delta}$ then we have: $a(x,D)\geqslant 0$ on $\mathscr{S}(\mathbb{R}^d)$ implies $U(a)\,(D)\geqslant 0$ on $\mathscr{S}(\mathbb{R}^d,l^2_c)$. Moreover, $U(a)(D)\geqslant 0$ on $\mathscr{S}(\mathbb{R}^d,l^2_c)$ implies $a(x,D)\geqslant 0$ on $TP(\Lambda)$.*

Proof. Suppose $a(x,D)\geqslant 0$ on $\mathscr{S}(\mathbb{R}^d)$. For $f\in\mathscr{S}(\mathbb{R}^d)$ and $M_\eta f(x)=e^{2\pi i\eta\cdot x}f(x)$ we have, for any $\eta\in\mathbb{R}^d$,

$$\begin{aligned}0&\leqslant (a(x,D)M_\eta f,M_\eta f)_{L^2(\mathbb{R}^d)}\\ &=\iint_{\mathbb{R}^{2d}} e^{2\pi i x\cdot(\xi-\eta)}a(x,\xi)\widehat{f}(\xi-\eta)\overline{f(x)}\,dx\,d\xi\\ &=\iint_{\mathbb{R}^{2d}} e^{2\pi i x\cdot\xi}a(x,\xi+\eta)\widehat{f}(\xi)\overline{f(x)}\,dx\,d\xi\\ &=((T_{0,-\eta}a)(x,D)f,f)_{L^2(\mathbb{R}^d)}.\end{aligned}$$

Thus $(T_{0,-\eta}a)(x,D) \geqslant 0$ on $\mathscr{S}(\mathbb{R}^d)$ for all $\eta \in \mathbb{R}^d$. By Corollary 2 and (22) it follows that $U(a)(\xi) \geqslant 0$ on l_c^2 for all $\xi \in \mathbb{R}^d$. If $F \in \mathscr{S}(\mathbb{R}^d, l_c^2)$ we obtain

$$(U(a)(D)F,F)_{L^2(\mathbb{R}^d,l^2)} = \int_{\mathbb{R}^d} (U(a)(\xi)\cdot \widehat{F}(\xi), \widehat{F}(\xi))_{l^2}\, d\xi \geqslant 0,$$

since the integrand is nonnegative everywhere. Thus $U(a)(D) \geqslant 0$ on $\mathscr{S}(\mathbb{R}^d, l_c^2)$.

Suppose on the other hand that $U(a)(D) \geqslant 0$ on $\mathscr{S}(\mathbb{R}^d, l_c^2)$. Let $z \in l_c^2$ and pick $\varphi \in C_c^\infty(\mathbb{R}^d)$ with support in the unit ball such that $\varphi \geqslant 0$ and $\|\varphi\|_{L^2} = 1$. With $\varphi_\varepsilon(x) = \varepsilon^{-d/2}\varphi(x/\varepsilon)$ and $F_\varepsilon(x)_\lambda = \mathscr{F}^{-1}\varphi_\varepsilon(x) z_\lambda$ we then have

$$\begin{aligned} 0 \leqslant (U(a)(D)F_\varepsilon, F_\varepsilon)_{L^2(\mathbb{R}^d,l^2)} &= \int_{\mathbb{R}^d} (U(a)(\xi)\cdot z, z)_{l^2} \varphi_\varepsilon(\xi)^2\, d\xi \\ &\to (U(a)(0)\cdot z, z)_{l^2}, \quad \varepsilon \to 0, \end{aligned}$$

where we have used (31) and the shrinking support of φ_ε. Therefore $U(a)(0) \geqslant 0$ on l_c^2 which implies that $a(x,D) \geqslant 0$ on $TP(\Lambda)$ according to Corollary 2. □

The previous result is similar to Gladyshev's results [4, 5], which were formulated in the framework of almost periodically correlated (or cyclostationary) stochastic processes and vector-valued weakly stationary stochastic processes. The so-called covariance operator of a second-order stochastic process is a positive operator, and an almost periodically correlated stochastic process has a covariance operator whose symbol is almost periodic in the first variable. Weakly stationary stochastic processes have translation invariant covariance operators, that is, they are convolution (or Fourier multiplier) operators. Gladyshev showed that the transformation (20), $a \mapsto U(a)$, which he formulated in terms of operator kernels, transforms a uniformly continuous kernel corresponding to a positive a.p. pseudodifferential operator to the kernel of a positive translation-invariant operator acting on vector-valued function spaces. The kernel of the operator (2) is

$$k_a(x,y) = \int_{\mathbb{R}^d} e^{2\pi i \xi\cdot(x-y)} a(x,\xi)\, d\xi = (\mathscr{F}_2^{-1}a)(x, x-y),$$

understood as an oscillatory integral. Here $\mathscr{F}_2$ denotes partial Fourier transform in the second $\mathbb{R}^d$ variable. The study of almost periodically correlated stochastic processes is in many respects rather similar to the theory of positive a.p. pseudodifferential operators. The symbol classes $S^m_{\rho,\delta}$ are however rarely used for stochastic processes. One usually restricts to operators whose kernels are continuous functions.

The next result concerns composition.

THEOREM 2. *If $a \in APS^{m_1}_{\rho,\delta}$ and $b \in APS^{m_2}_{\rho,\delta}$, $m_1, m_2 \in \mathbb{R}$, then*

$$U(a\#_0 b)(\xi) = U(a)(\xi)\cdot U(b)(\xi), \quad \xi \in \mathbb{R}^d. \tag{33}$$

Proof. Let Λ denote the linear hull over $\mathbb{Q}$ of $\Lambda(a) \cup \Lambda(b)$. According to (30) in Proposition 3, $U(a)(\xi) \in \mathscr{L}(l_s^2, l_{s-m_1}^2)$ and $U(b)(\xi) \in \mathscr{L}(l_s^2, l_{s-m_2}^2)$ for any $s \in \mathbb{R}$. Therefore the sum

$$(34)\qquad \begin{aligned}(U(a)(\xi)\cdot U(b)(\xi))_{\lambda,\lambda'} &= \sum_{\mu\in\Lambda} U(a)(\xi)_{\lambda,\mu} U(b)(\xi)_{\mu,\lambda'} \\ &= \sum_{\mu\in\Lambda} a_{\mu-\lambda}(\xi-\mu) b_{\lambda'-\mu}(\xi-\lambda')\end{aligned}$$

is absolutely convergent for all $(\lambda,\lambda')\in\Lambda\times\Lambda$, and the matrix $U(a)(\xi)\cdot U(b)(\xi)$ maps l_s^2 to $l_{s-m_1-m_2}^2$ continuously for any $s\in\mathbb{R}$ and any $\xi\in\mathbb{R}^d$.

We study the left hand side of (33) by regularizing the symbol b in two steps. First we pick a test function $\varphi\in C_c^\infty(\mathbb{R}^d)$ which equals one in a neighborhood of the origin, set $\varphi_\varepsilon(\xi)=\varphi(\varepsilon\xi)$ and define

$$b_\varepsilon(x,\xi)=b(x,\xi)\varphi_\varepsilon(\xi)\in S_{\rho,\delta}^{-\infty},\quad 0\leqslant\varepsilon\leqslant 1.$$

By [6, Proposition 18.1.2] $\varphi_\varepsilon\to 1$ in $S_{1,0}^\theta$ as $\varepsilon\to 0$ for any $\theta>0$. Since convergence in $S_{1,0}^\theta$ implies convergence in $S_{\rho,\delta}^\theta$ and $b_\varepsilon=b\#_0\varphi_\varepsilon$, it follows from (5) that $b_\varepsilon\to b$ in $S_{\rho,\delta}^{m_2+\theta}$ as $\varepsilon\to 0$, and

$$a\#_0 b=\lim_{\varepsilon\to 0} a\#_0 b_\varepsilon \quad\text{in}\quad S_{\rho,\delta}^{m_1+m_2+\theta},\quad \theta>0.$$

Convergence in $S_{\rho,\delta}^m$ for any $m\in\mathbb{R}$ implies the uniform convergence

$$\sup_{x\in\mathbb{R}^d}|a\#_0 b(x,\xi)-a\#_0 b_\varepsilon(x,\xi)|\to 0,\quad \varepsilon\to 0,$$

for any $\xi\in\mathbb{R}^d$, and therefore we have for the Bohr–Fourier coefficients

$$(35)\qquad (a\#_0 b)_\mu(\xi)=\lim_{\varepsilon\to 0}(a\#_0 b_\varepsilon)_\mu(\xi),\quad \mu\in\mathbb{R}^d,\quad \xi\in\mathbb{R}^d.$$

In the second step we regularize the symbol b_ε. Fix $\alpha,\beta\in\mathbb{N}^d$ and define the family of functions $\mathcal{F}=\{\partial_\xi^\alpha\partial_x^\beta b_\varepsilon(\cdot,\xi)\}_{\xi\in\mathbb{R}^d}\subset CAP(\mathbb{R}^d)$. The family $\mathcal{F}$ depends continuously in the $CAP(\mathbb{R}^d)$ norm on ξ by (14), and has compact support with respect to ξ. Thus $\mathcal{F}$ is precompact, and by Lemma 1 the Fourier series reconstruction with the Bochner–Fejér polynomials

$$(36)\qquad \begin{aligned}\partial_\xi^\alpha\partial_x^\beta b_\varepsilon(x,\xi) &= \lim_{n\to\infty} P_n(\partial_\xi^\alpha\partial_x^\beta b_\varepsilon(\cdot,\xi))(x) \\ &= \lim_{n\to\infty}\sum_{\lambda\in\Lambda} K_n(\lambda)(\partial_\xi^\alpha\partial_x^\beta b_\varepsilon)_\lambda(\xi)e^{2\pi i\lambda\cdot x}\end{aligned}$$

is uniformly convergent in both variables, i.e. in $\mathbb{R}^{2d}$. By Lemma 3 we have

$$(\partial_\xi^\alpha\partial_x^\beta b_\varepsilon)_\lambda(\xi)=\partial_\xi^\alpha(\partial_x^\beta b_\varepsilon)_\lambda(\xi)=(2\pi i\lambda)^\beta\partial_\xi^\alpha(b_\varepsilon)_\lambda(\xi),$$

which means that we can rewrite (36) as the uniform limit over $\mathbb{R}^{2d}$

$$(37)\qquad \partial_\xi^\alpha\partial_x^\beta b_\varepsilon(x,\xi)=\lim_{n\to\infty}\partial_\xi^\alpha\partial_x^\beta\left(\sum_{\lambda\in\Lambda}K_n(\lambda)(b_\varepsilon)_\lambda(\xi)e^{2\pi i\lambda\cdot x}\right).$$

Let us denote, observing that $(b_\varepsilon)_\lambda(\xi) = b_\lambda(\xi)\varphi(\varepsilon\xi)$,

$$b_{\varepsilon,n}(x,\xi) = \varphi(\varepsilon\xi)\sum_{\lambda\in\Lambda} K_n(\lambda) b_\lambda(\xi) e^{2\pi i\lambda\cdot x}.$$

The fact that $b_\varepsilon(x,\cdot)$ and $b_{\varepsilon,n}(x,\cdot)$ have support in a compact set, common for all $x\in\mathbb{R}^d$, in combination with the uniform limit (37), implies that

$$\sup_{x,\xi\in\mathbb{R}^d} \langle\xi\rangle^{-m+\rho|\alpha|-\delta|\beta|} \left|\partial_\xi^\alpha\partial_x^\beta\left(b_{\varepsilon,n}(x,\xi) - b_\varepsilon(x,\xi)\right)\right| \to 0, \quad n\to\infty,$$

for any $m\in\mathbb{R}$. This holds for any $\alpha,\beta\in\mathbb{N}^d$, and hence $b_{\varepsilon,n}\to b_\varepsilon$ in $S^m_{\rho,\delta}$ as $n\to\infty$ for any $m\in\mathbb{R}$. This means by (5) that $a\#_0 b_{\varepsilon,n}\to a\#_0 b_\varepsilon$ in $S^m_{\rho,\delta}$ as $n\to\infty$ for any $m\in\mathbb{R}$. As above we thus obtain

$$(a\#_0 b)_\mu(\xi) = \lim_{\varepsilon\to 0}\lim_{n\to\infty}(a\#_0 b_{\varepsilon,n})_\mu(\xi), \quad \mu\in\mathbb{R}^d, \quad \xi\in\mathbb{R}^d, \tag{38}$$

using (35).

Since the symbol $c_\lambda(x,\xi) = e^{2\pi i\lambda\cdot x} b_\lambda(\xi)\varphi(\varepsilon\xi)$ gives the pseudodifferential operator

$$c_\lambda(x,D)g(x) = \int_{\mathbb{R}^d} e^{2\pi i\xi\cdot x} b_\lambda(\xi-\lambda)\varphi(\varepsilon(\xi-\lambda))\widehat{g}(\xi-\lambda)\,d\xi, \quad g\in\mathscr{S}(\mathbb{R}^d), \tag{39}$$

it follows that

$$\begin{aligned} a(x,D)(c_\lambda(x,D)g)(x) &= \int_{\mathbb{R}^d} e^{2\pi i x\cdot\xi} a(x,\xi)\mathscr{F}(c_\lambda(x,D)g)(\xi)\,d\xi \\ &= \int_{\mathbb{R}^d} e^{2\pi i x\cdot\xi} a(x,\xi) b_\lambda(\xi-\lambda)\varphi(\varepsilon(\xi-\lambda))\widehat{g}(\xi-\lambda)\,d\xi \\ &= \int_{\mathbb{R}^d} e^{2\pi i x\cdot(\xi+\lambda)} a(x,\xi+\lambda) b_\lambda(\xi)\varphi(\varepsilon\xi)\widehat{g}(\xi)\,d\xi, \end{aligned}$$

and thus

$$a\#_0 c_\lambda(x,\xi) = a(x,\xi+\lambda) b_\lambda(\xi)\varphi(\varepsilon\xi) e^{2\pi i\lambda\cdot x}.$$

This gives

$$(a\#_0 b_{\varepsilon,n})(x,\xi) = \sum_{\lambda\in\Lambda} K_n(\lambda) a(x,\xi+\lambda) b_\lambda(\xi)\varphi(\varepsilon\xi) e^{2\pi i\lambda\cdot x}.$$

Hence

$$\begin{aligned} \lim_{n\to\infty}(a\#_0 b_{\varepsilon,n})_\mu(\xi) &= \varphi(\varepsilon\xi)\lim_{n\to\infty}\sum_{\lambda\in\Lambda} K_n(\lambda) a_{\mu-\lambda}(\xi+\lambda) b_\lambda(\xi) \\ &= \varphi(\varepsilon\xi)\sum_{\lambda\in\Lambda} a_{\mu-\lambda}(\xi+\lambda) b_\lambda(\xi), \end{aligned} \tag{40}$$

due to $0\leqslant K_n\leqslant 1$, $K_n(\lambda)\to 1$ as $n\to\infty$ for all $\lambda\in\Lambda$, the absolutely convergent sum (34), and the dominated convergence theorem. Now (38) and (40) yield

$$(a\#_0 b)_\mu(\xi) = \sum_{\lambda\in\Lambda} a_{\mu-\lambda}(\xi+\lambda) b_\lambda(\xi), \quad \mu\in\Lambda, \quad \xi\in\mathbb{R}^d.$$

Finally we have

$$\begin{aligned} U(a\#_0 b)(\xi)_{\lambda,\lambda'} &= (a\#_0 b)_{\lambda'-\lambda}(\xi-\lambda') \\ &= \sum_{\mu\in\Lambda} a_{\lambda'-\lambda-\mu}(\xi-\lambda'+\mu) b_\mu(\xi-\lambda') \\ &= \sum_{\mu\in\Lambda} a_{\mu-\lambda}(\xi-\mu) b_{\lambda'-\mu}(\xi-\lambda'). \end{aligned}$$

A comparison with (34) completes the proof. □

To summarize our findings hitherto, the transformation $a \mapsto U(a)$ maps a symbol $a \in S^m_{\rho,\delta}$ defined on the phase space $\mathbb{R}^d \times \mathbb{R}^d$ to an operator-valued symbol $U(a)$ that depends on the frequency variable $\xi \in \mathbb{R}^d$ only. The operator corresponding to the symbol $U(a)$ acts on sequence-space valued function spaces, e.g. $\mathscr{S}(\mathbb{R}^d, l^2_c)$. The operator corresponding to the symbol $U(a)$ is thus a convolution (Fourier multiplier) operator. The map $a(x,D) \mapsto U(a)(D)$ is linear, injective, preserves identity and positivity, and respects operator composition,

$$a(x,D)b(x,D) \mapsto U(a\#_0 b)(D) = U(a)(D)\cdot U(b)(D).$$

Convolution operators do not commute when function spaces are vector-valued as they do for scalar-valued function spaces. The transformation $a \mapsto U(a)$ encodes the non-commutativity of $a(x,D)$ and $b(x,D)$ in the matrix product of the symbols $U(a)$ and $U(b)$. That is, with the notation for the commutator $[A,B] = AB - BA$, we have

$$[a(x,D), b(x,D)] \mapsto U(a)(D)\cdot U(b)(D) - U(b)(D)\cdot U(a)(D),$$

where the right hand side operator acts by

$$\begin{aligned} &[U(a)(D), U(b)(D)]F(x) \\ &= \int_{\mathbb{R}^d} e^{2\pi i x\cdot\xi} \left(U(a)(\xi)\cdot U(b)(\xi) - U(b)(\xi)\cdot U(a)(\xi)\right)\cdot \widehat{F}(\xi)\, d\xi. \end{aligned}$$

In our final result we show that the basic assumption of this paper, i.e. that symbols are almost periodic in the first variable, is invariant under the quantization. More precisely, let us introduce the family of quantizations

$$a_t(x,D) f(x) = \int_{\mathbb{R}^{2d}} e^{2\pi i \xi\cdot(x-y)} a((1-t)x + ty, \xi) f(y)\, dy\, d\xi \tag{41}$$

parametrized by $t \in \mathbb{R}$. The Kohn–Nirenberg quantization is obtained for $t = 0$ and the Weyl quantization has $t = 1/2$. The following result says that if an operator is expressed in two different quantizations, then if its symbol is almost periodic in the first variable in one quantization, it will have the same property in any other quantization. In other words, the fact that we have worked in the Kohn–Nirenberg quantization is not essential.

PROPOSITION 5. *If* $a \in APS^m_{\rho,\delta}$*,* $s,t \in \mathbb{R}$*,* $s \neq t$*, and* $a_t(x,D) = b_s(x,D)$*, then* $b \in APS^m_{\rho,\delta}$*.*

Proof. We use a technique that is similar to the proof of Theorem 2. If $a,b \in \mathscr{S}(\mathbb{R}^{2d})$ and $f \in \mathscr{S}(\mathbb{R}^d)$ then the integral over ξ in (41) is a partial Fourier transform, so we get

$$\begin{aligned} a_t(x,D)f(x) &= \int_{\mathbb{R}^d} \mathscr{F}_2 a((1-t)x+ty, y-x) f(y)\,dy \\ &= \int_{\mathbb{R}^d} \mathscr{F}_2 a(x+ty, y) f(y+x)\,dy \\ &= \iint_{\mathbb{R}^{2d}} \widehat{a}(z,y) e^{2\pi i t z\cdot y} e^{2\pi i z\cdot x} f(y+x)\,dy\,dz. \end{aligned}$$

Thus if $a_t(x,D) = b_s(x,D)$ we have

$$\widehat{b}(x,\xi) = e^{-2\pi i(s-t)x\cdot\xi}\widehat{a}(x,\xi),$$

which extends by continuity to $a,b \in \mathscr{S}'(\mathbb{R}^{2d})$ [3]. This transformation is often denoted [6]

$$b(x,\xi) = e^{-2\pi i(s-t)D_x\cdot D_\xi} a(x,\xi) := (Ta)(x,\xi). \tag{42}$$

According to [3, Theorem 2.37], we have

$$e^{-2\pi i(s-t)D_x\cdot D_\xi} : S^m_{\rho,\delta} \mapsto S^m_{\rho,\delta} \quad \text{continuously}, \quad m \in \mathbb{R}. \tag{43}$$

Therefore it suffices to prove that $(Ta)(\cdot,\xi) \in CAP(\mathbb{R}^d)$ for all $\xi \in \mathbb{R}^d$.

We proceed with a regularization of the symbol a as in the proof of Theorem 2. Thus let $\varphi \in C_c^\infty(\mathbb{R}^d)$ equal one in a neighborhood of the origin, set $\varphi_\varepsilon(\xi) = \varphi(\varepsilon\xi)$ and define $a_\varepsilon(x,\xi) = a(x,\xi)\varphi_\varepsilon(\xi)$. Then $a_\varepsilon \to a$ in $S^{m+\theta}_{\rho,\delta}$ as $\varepsilon \to 0$ for any $\theta > 0$. By the continuity (43) we have $Ta_\varepsilon \to Ta$ in $S^{m+\theta}_{\rho,\delta}$ as $\varepsilon \to 0$. Moreover, if we define

$$a_{\varepsilon,n}(x,\xi) = \varphi(\varepsilon\xi) \sum_{\lambda\in\Lambda} K_n(\lambda) a_\lambda(\xi) e^{2\pi i\lambda\cdot x}$$

then we obtain $a_{\varepsilon,n} \to a_\varepsilon$ in $S^{m'}_{\rho,\delta}$ as $n \to \infty$ for any $m' \in \mathbb{R}$, as in the proof of Theorem 2. Again by the continuity (43) it follows that $Ta_{\varepsilon,n} \to Ta_\varepsilon$ in $S^{m'}_{\rho,\delta}$ as $n \to \infty$. It follows that for each fixed $\xi \in \mathbb{R}^d$ we have the uniform limits

$$(Ta)(\cdot,\xi) = \lim_{\varepsilon\to 0} \lim_{n\to\infty} (Ta_{\varepsilon,n})(\cdot,\xi).$$

Since $CAP(\mathbb{R}^d)$ is closed under uniform convergence [7], the proof is complete if we show that $(Ta_{\varepsilon,n})(\cdot,\xi) \in CAP(\mathbb{R}^d)$ for any $\xi \in \mathbb{R}^d$, $\varepsilon > 0$ and $n \in \mathbb{N}$.

We have, since $(a_\varepsilon)_\lambda(\xi) = a_\lambda(\xi)\varphi(\varepsilon\xi)$,

$$\mathscr{F}(a_{\varepsilon,n})(\eta,z) = \sum_{\lambda\in\Lambda} K_n(\lambda)\delta_\lambda(\eta)\mathscr{F}(a_\varepsilon)_\lambda(z),$$

where $\delta_\lambda = \delta_0(\cdot - \lambda)$ denotes a translated Dirac distribution. Hence we have

$$e^{-2\pi i(s-t)\eta\cdot z}\mathscr{F}(a_{\varepsilon,n})(\eta,z) = \sum_{\lambda\in\Lambda} K_n(\lambda)e^{-2\pi i(s-t)\lambda\cdot z}\delta_\lambda(\eta)\mathscr{F}(a_\varepsilon)_\lambda(z) = \sum_{\lambda\in\Lambda} K_n(\lambda)\delta_\lambda(\eta)\mathscr{F}(T_{(s-t)\lambda}(a_\varepsilon)_\lambda)(z)$$

and, since $Ta = \mathscr{F}^{-1}M\mathscr{F}$ where $(Mf)(\eta,z) = e^{-2\pi i(s-t)\eta\cdot z}f(\eta,z)$,

$$(Ta_{\varepsilon,n})(x,\xi) = \sum_{\lambda\in\Lambda} K_n(\lambda)(T_{(s-t)\lambda}(a_\varepsilon)_\lambda)(\xi)e^{2\pi i\lambda\cdot x} = \sum_{\lambda\in\Lambda} K_n(\lambda)(a_\varepsilon)_\lambda(\xi-(s-t)\lambda)e^{2\pi i\lambda\cdot x}.$$

Hence $(Ta_{\varepsilon,n})(\cdot,\xi)$ is a trigonometric polynomial, because the sum is finite, so we may conclude that $(Ta_{\varepsilon,n})(\cdot,\xi)\in CAP(\mathbb{R}^d)$ for any $\xi\in\mathbb{R}^d$, $\varepsilon>0$ and $n\in\mathbb{N}$. □

REMARK 1. We have worked in the Kohn–Nirenberg quantization and the transformation $a\mapsto U(a)$. For the Weyl quantization, the corresponding transformation is $a\mapsto V(a)$ where

$$V(a)(\xi)_{\lambda,\lambda'} = a_{\lambda'-\lambda}\left(\xi - \frac{\lambda+\lambda'}{2}\right).$$

With the Weyl product defined by $a_{1/2}(x,D)b_{1/2}(x,D) = (a\#b)_{1/2}(x,D)$, we then have $V(a\#b)(\xi) = V(a)(\xi)\cdot V(b)(\xi)$, corresponding to Theorem 2. Moreover, $V(\overline{a})(\xi)_{\lambda,\lambda'} = \overline{V(a)(\xi)}_{\lambda',\lambda}$, i.e. $V(\overline{a})(\xi) = V(a)(\xi)^*$ where A^* denotes the Hermitian (conjugate transpose) matrix, which gives $V(\overline{a})(D) = V(a)(D)^*$. Since $\overline{a}_{1/2}(x,D) = a_{1/2}(x,D)^*$, we obtain as a consequence that the transformation $a_{1/2}(x,D)\mapsto V(a)(D)$, as well as $a(x,D)\mapsto U(a)(D)$, respects adjoints.

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THE KLEIN-GORDON EQUATION IN ANTI-DE SITTER SPACETIME

Dedicated to Professor Luigi Rodino on the occasion of his 60th birthday

Abstract. In this article, we apply the fundamental solution constructed in a previous paper, and obtain the representation of the solution to the Cauchy problem for the Klein-Gordon equation $\Box_g \phi - m^2\phi = f$ in anti-de Sitter spacetime.

1. Introduction

In this article, we study the Cauchy problem for the Klein–Gordon equation, namely $\Box_g \phi - m^2\phi = f$, in anti-de Sitter spacetime.

In the model of the universe proposed by de Sitter, the line element has the form

$$\begin{aligned} ds^2 &= -\left(1-\frac{2M_{bh}}{r}-\frac{\Lambda r^2}{3}\right)c^2dt^2+\left(1-\frac{2M_{bh}}{r}-\frac{\Lambda r^2}{3}\right)^{-1}dr^2 \\ &\quad +r^2(d\theta^2+\sin^2 d\phi^2). \end{aligned}$$

The constant M_{bh} may have an interpretation as the "mass of the black hole", while Λ is the *cosmological constant*. The corresponding metric with this line element is called the Schwarzschild–de Sitter metric. Hubble's discovery in 1929 of an expanding universe (see, e.g., [9]), which can be understood as due to a cosmological constant, has initiated a lot of work with the aim to study how Λ affects, e.g., quantum mechanics, quantum field theory, and celestial mechanics. In principle, the cosmological constant should take part in phenomena on every physical scale [10].

The Cauchy problem for the linear and semilinear Klein–Gordon equation in Minkowski spacetime ($M_{bh} = \Lambda = 0$) is well investigated. (See, e.g., [8], [11] and references therein.) In particular, for the semilinear equation $u_{tt} - \Delta u + u = F(u)$, with initial conditions $u(0,x) = \varepsilon\varphi_0(x)$, $u_t(0,x) = \varepsilon\varphi_1(x)$, Keel and Tao [8] proved that if $n = 1,2,3$ and $1 < p < 1+2/n$, then there exists a (non-Hamiltonian) nonlinearity F satisfying $|D^\alpha F(u)| \le C|u|^{p-|\alpha|}$ for $0 \le \alpha \le [p]$ and such that there is no finite energy global solution supported in the forward light cone, for any nontrivial smooth compactly supported φ_0 and φ_1 and for any $\varepsilon > 0$.

The Cauchy problem for the linear wave equation ($m = 0$) without source term on the maximally extended Schwarzschild–de Sitter spacetime in the case of non-extremal black-hole corresponding to parameter values $0 < M_{bh} < 1/3\sqrt{\Lambda}$, is considered by Dafermos and Rodnianski [5]. They proved that in the region bounded by a set of black/white hole horizons and cosmological horizons, solutions converge pointwise to a constant faster than any given polynomial rate, where the decay is measured with respect to natural future-directed advanced and retarded time coordinates.

Catania and Georgiev [4] studied the Cauchy problem for the semilinear wave equation $\square_g \phi = |\phi|^p$ in the Schwarzschild metric $(3+1)$-dimensional spacetime, that is the case of $\Lambda = 0$ in $0 < M_{bh} < 1/3\sqrt{\Lambda}$. They established that the problem in the Regge–Wheeler coordinates is locally well-posed in H^σ for any $\sigma \in [1, p+1)$. Then for the special choice of the initial data they proved the blow-up of the solution in two cases: (a) $p \in (1, 1+\sqrt{2})$ and small initial data supported far away from the black hole; (b) $p \in (2, 1+\sqrt{2})$ and large data supported near the black hole. In both cases, they also gave an estimate from above for the lifespan of the solution.

In the present paper we focus on another limit case as $M_{bh} \to 0$ in the interval $0 < M_{bh} < 1/3\sqrt{\Lambda}$, namely, we set $M_{bh} = 0$ to ignore completely influence of the black hole. Thus, the line element in the de Sitter spacetime has the form

$$ds^2 = -\left(1 - \frac{r^2}{R^2}\right) c^2\, dt^2 + \left(1 - \frac{r^2}{R^2}\right)^{-1} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta\, d\phi^2\right) .$$

The Lamaître–Robertson transformation [9]

$$r' = \frac{r}{\sqrt{1 - r^2/R^2}} e^{-ct/R}, \quad t' = t + \frac{R}{2c} \ln\left(1 - \frac{r^2}{R^2}\right), \quad \theta' = \theta, \quad \phi' = \phi,$$

leads to the following form for the line element:

$$ds^2 = -c^2\, dt'^2 + e^{2ct'/R}\left(dr'^2 + r'^2\, d\theta'^2 + r'^2 \sin^2\theta'\, d\phi'^2\right).$$

By defining coordinates x', y', z' connected with r', θ', ϕ' by the usual equations connecting Cartesian coordinates and polar coordinates in a Euclidean space, the line element may be written [9, Sec.134]

$$ds^2 = -c^2\, dt'^2 + e^{2ct'/R}\left(dx'^2 + dy'^2 + dz'^2\right) .$$

The new coordinates r', θ', ϕ', t' can take all values from $-\infty$ to ∞. Here R is the "radius" of the universe. In the Robertson–Walker spacetime [3, 7] one can choose coordinates so that the metric has the form

$$ds^2 = -dt^2 + S^2(t) d\sigma^2 .$$

In particular, the metric in the de Sitter and anti-de Sitter spacetime in the Lamaître–Robertson coordinates [9] has this form with $S(t) = e^t$ and $S(t) = e^{-t}$, respectively.

In the paper [16], we study the Cauchy problem for the Klein–Gordon equation in Robertson–Walker spacetime by applying the Lamaître-Robertson transformation and by employing the fundamental solutions constructed there for the Klein–Gordon operator in Robertson–Walker spacetime, that is for $\mathcal{S} := \partial_t^2 - e^{-2t} \triangle + M^2$. The fundamental solution $\mathcal{E} = \mathcal{E}(x, t; x_0, t_0)$, that is solution of $\mathcal{S}\mathcal{E} = \delta(x - x_0, t - t_0)$, with a support in the forward light cone and the fundamental solution with a support in the backward light cone are constructed in [16]. The fundamental solution with the support in the forward light cone has been used in [16] to represent solutions of the Cauchy problem and to prove $L^p - L^q$ estimates for the solutions of the equation with and without a source term.

The matter waves in the de Sitter spacetime are described by the function ϕ, which satisfies equations of motion. In the de Sitter and anti-de Sitter spacetime the equation for the scalar field with mass m is the covariant Klein–Gordon equation

$$\square_g \phi - m^2\phi = f \quad \text{or} \quad \frac{1}{\sqrt{|g|}}\frac{\partial}{\partial x^i}\left(\sqrt{|g|}g^{ik}\frac{\partial \phi}{\partial x^k}\right) - m^2\phi = f\,,$$

with the usual summation convention, where $x = (x^0, x^1, \ldots, x^n)$ and g^{ik} is a metric tensor. Written explicitly in coordinates in the de Sitter spacetime it has the form

$$\phi_{tt} + n\phi_t - e^{-2t}\Delta\phi + m^2\phi = f\,. \tag{1}$$

Here t is x^0, while $\triangle$ is the Laplace operator on the flat metric in $\mathbb{R}^n$. If we introduce the new unknown function $u = e^{\frac{n}{2}t}\phi$, then the equation (1) takes the form of the linear Klein–Gordon equation for u on de Sitter spacetime

$$u_{tt} - e^{-2t}\triangle u + M^2 u = f, \tag{2}$$

where the "curved mass" M is defined by the equation $M^2 := m^2 - n^2/4$. In the case of $0 \le m \le n/2$, equation (2) can be regarded as Klein–Gordon equation with imaginary mass. Equations with imaginary mass appear in several physical models such as the ϕ^4 field model, tachion (super-light) fields, Landau–Ginzburg–Higgs equation and others.

The equation (2) is strictly hyperbolic. That implies the well-posedness of the Cauchy problem in the different functional spaces. Consequently, the solution operator is well-defined in those functional spaces. Then, the speed of propagation is variable, namely, it is equal to e^{-t}. The second-order strictly hyperbolic equation (2) possesses two fundamental solutions resolving the Cauchy problem without source term f. They can be written in terms of the Fourier integral operators, which give complete description of the wave front sets of the solutions. Moreover, the integrability of the characteristic roots, $\int_0^\infty |\lambda_i(t,\xi)|dt < \infty$, $i = 1,2$, leads to the existence of the so-called "horizon" for that equation. More precisely, any signal emitted from the spatial point $x_0 \in \mathbb{R}^n$ at time $t_0 \in \mathbb{R}$ remains inside the ball $B^n_{t_0}(x_0) := \{x \in \mathbb{R}^n \,|\, |x - x_0| < e^{-t_0}\}$ for all time $t \in (t_0, \infty)$. In particular, it can cause a nonexistence of the $L^p - L^q$ decay for the solutions. In [14] this phenomenon is illustrated by a model equation with a permanently bounded domain of influence, power decay of characteristic roots, and without $L^p - L^q$ decay. The above mentioned $L^p - L^q$ decay estimates are one of the important tools for studying nonlinear problems (see, e.g. [11]). Equation (2) is neither Lorentz invariant nor invariant with respect to usual scaling and that creates additional difficulties.

In the present paper we consider Klein–Gordon operator in anti-de Sitter spacetime, that is

$$\mathcal{S} := \partial_t^2 - e^{2t}\triangle + M^2\,,$$

where M is the curved mass, $M \in \mathbb{C}$, and $x \in \mathbb{R}^n$, $t \in \mathbb{R}$. Results of [16] (by means of the time inversion transformation $t \to -t$) provide us with the fundamental solution $\mathcal{E} = \mathcal{E}(x,t;x_0,t_0)$,

$$\mathcal{E}_{tt} - e^{2t}\Delta\mathcal{E} + M^2\mathcal{E} = \delta(x - x_0, t - t_0),$$

with support in the "forward light cone" $D_+(x_0,t_0)$, $x_0 \in \mathbb{R}^n$, $t_0 \in \mathbb{R}$, and for the fundamental solution with support in the "backward light cone" $D_-(x_0,t_0)$, $x_0 \in \mathbb{R}^n$, $t_0 \in \mathbb{R}$, defined as follows

$$D_\pm(x_0,t_0) \;:=\; \left\{(x,t) \in \mathbb{R}^{n+1}\,;\, |x - x_0| \le \pm(e^{t_0} - e^t)\right\}. \tag{3}$$

In fact, any intersection of $D_-(x_0,t_0)$ with the hyperplane $t = const < t_0$ determines the so-called dependence domain for the point (x_0,t_0), while the intersection of $D_+(x_0,t_0)$ with the hyperplane $t = const > t_0$ is the so-called domain of influence of the point (x_0,t_0). The equation (2) is non-invariant with respect to time inversion. Moreover, the dependence domain is wider than any given ball if time $const > t_0$ is sufficiently large, while the domain of influence is permanently, for all time $const < t_0$, in the ball of the radius e^{t_0}. In fact, the representation formulas obtained in [16] for the solution of the Cauchy problem in the de Sitter spacetime cannot be applied to the solutions of the Cauchy problem for the equation in the anti-de Sitter spacetime. The present paper is aimed to fill up that gap.

Define for $t_0 \in \mathbb{R}$ in the domain $D_+(x_0,t_0) \cup D_-(x_0,t_0)$ the function

$$\begin{aligned} E(x,t;x_0,t_0) \;=\; & (4e^{t_0+t})^{iM}\Big((e^t + e^{t_0})^2 - (x - x_0)^2\Big)^{-\frac{1}{2}-iM} \\ & \times F\left(\frac{1}{2} + iM, \frac{1}{2} + iM; 1; \frac{(e^{t_0} - e^t)^2 - (x - x_0)^2}{(e^{t_0} + e^t)^2 - (x - x_0)^2}\right), \end{aligned} \tag{4}$$

where $F\big(a,b;c;\zeta\big)$ is the hypergeometric function (See, e.g., [2].). In (4) we use the notation $x^2 = |x|^2$ for $x \in \mathbb{R}^n$. Let $E(x,t;0,t_0)$ be function (4), and set

$$\mathcal{E}_\pm(x,t;0,t_0) := \begin{cases} E(x,t;0,t_0) & \text{in } D_\pm(0,t_0), \\ 0 & \text{elsewhere.} \end{cases}$$

Since the function $E = E(x,t;0,t_0)$ is smooth in $D_\pm(0,t_0)$ and is locally integrable, it follows that $\mathcal{E}_+(x,t;0,\, t_0)$ and $\mathcal{E}_-(x,t;0,\, t_0)$ are distributions whose supports are in $D_+(0,t_0)$ and $D_-(0,t_0)$, respectively. In order to make the present paper self-contained we make the transformation $t \to -t$ in Theorem 0.1 [16] and introduce the next result.

THEOREM 1 ([16]). *Suppose that $M \in \mathbb{C}$. The distributions $\mathcal{E}_+(x,t;0,t_0)$ and $\mathcal{E}_-(x,t;0,t_0)$ are the fundamental solutions for the operator $\mathcal{S} = \partial_t^2 - e^{2t}\partial_x^2 + M^2$ relative to the point $(0,t_0)$, that is $\mathcal{S}\,\mathcal{E}_\pm(x,t;0,t_0) = \delta(x,t - t_0)$, or*

$$\frac{\partial^2}{\partial t^2}\mathcal{E}_\pm(x,t;0,t_0) - e^{2t}\frac{\partial^2}{\partial x^2}\mathcal{E}_\pm(x,t;0,t_0) + M^2\mathcal{E}_\pm(x,t;0,t_0) = \delta(x,t - t_0).$$

To motivate our construction for the higher-dimensional case $n \geq 2$ we follow the approach suggested in [13] and represent the fundamental solution $\mathcal{E}_+(x,t;0,t_0)$ as follows

$$\begin{aligned} \mathcal{E}_+(x,t;0,t_0) \;=\; & \int_{e^{t_0}-e^t}^{e^t-e^{t_0}} (4e^{t_0+t})^{iM}\Big((e^{t_0}+e^t)^2-r^2\Big)^{-\frac{1}{2}-iM} \\ & \times F\Big(\tfrac{1}{2}+iM,\tfrac{1}{2}+iM;1;\frac{(e^{t_0}-e^t)^2-r^2}{(e^{t_0}+e^t)^2-r^2}\Big)\mathcal{E}^{string}(x,r)\,dr, \quad t>t_0, \end{aligned}$$

where the distribution $\mathcal{E}^{string}(x,t)$ is the fundamental solution of the Cauchy problem for the string equation:

$$\frac{\partial^2}{\partial t^2}\mathcal{E}^{string}-\frac{\partial^2}{\partial x^2}\mathcal{E}^{string}=0, \qquad \mathcal{E}^{string}(x,0)=\delta(x),\ \mathcal{E}_t^{string}(x,0)=0\,.$$

Hence, $\mathcal{E}^{string}(x,t)=\frac{1}{2}\{\delta(x+t)+\delta(x-t)\}$. The integral makes sense in the topology of the space of distributions. The fundamental solution $\mathcal{E}_-(x,t;0,t_0)$ for $t<t_0$ admits a similar representation.

We appeal to the wave equation in Minkowski spacetime to obtain in the next theorem very similar representations of the fundamental solutions of the higher-dimensional equation in the anti-de Sitter spacetime. In fact, the transformation $t\to -t$ in Theorem 0.2 [16] implies the next theorem.

THEOREM 2 ([16]). *If $x\in\mathbb{R}^n$, $n\geq 2$, and $M\in\mathbb{C}$, then for the operator $\mathcal{S}=\partial_t^2-e^{2t}\triangle+M^2$ the fundamental solution $\mathcal{E}_{+,n}(x,t;x_0,t_0)$ with support in the forward cone $D_+(x_0,t_0)$, $x_0\in\mathbb{R}^n$, $t_0\in\mathbb{R}$, $supp\,\mathcal{E}_{+,n}\subseteq D_+(x_0,t_0)$, is given by the following integral $(t>t_0)$*

$$\begin{aligned} \text{(5)}\quad \mathcal{E}_{+,n}(x,t;x_0,t_0) \;=\; & 2\int_0^{e^t-e^{t_0}} dr\,(4e^{t_0+t})^{iM}\Big((e^{t_0}+e^t)^2-r^2\Big)^{-\frac{1}{2}-iM} \\ & \times F\Big(\tfrac{1}{2}+iM,\tfrac{1}{2}+iM;1;\frac{(e^{t_0}-e^t)^2-r^2}{(e^{t_0}+e^t)^2-r^2}\Big)\mathcal{E}^{w}(x-x_0,r). \end{aligned}$$

Here the distribution $\mathcal{E}^w(x,t)$ is a fundamental solution to the Cauchy problem for the wave equation

$$\mathcal{E}^w_{tt}-\triangle\mathcal{E}^w=0\,,\quad \mathcal{E}^w(x,0)=\delta(x)\,,\quad \mathcal{E}^w_t(x,0)=0\,.$$

The fundamental solution $\mathcal{E}_{-,n}(x,t;x_0,t_0)$ with support in the backward cone, that is, $supp\,\mathcal{E}_{-,n}\subseteq D_-(x_0,t_0)$, $x_0\in\mathbb{R}^n$, $t_0\in\mathbb{R}$, is given by the following integral $(t<t_0)$

$$\begin{aligned} \text{(6)}\quad \mathcal{E}_{-,n}(x,t;x_0,t_0) \;=\; & -2\int_{e^t-e^{t_0}}^0 dr\,(4e^{t_0+t})^{iM}\Big((e^{t_0}+e^t)^2-r^2\Big)^{-\frac{1}{2}-iM} \\ & \times F\Big(\tfrac{1}{2}+iM,\tfrac{1}{2}+iM;1;\frac{(e^{t_0}-e^t)^2-r^2}{(e^{t_0}+e^t)^2-r^2}\Big)\mathcal{E}^{w}(x-x_0,r). \end{aligned}$$

In particular, formula (5) shows that Huygens's Principle is not valid for waves propagating in the anti-de Sitter spacetime (cf. [12]).

Next we use Theorem 1 to solve the Cauchy problem for the one-dimensional equation

$$u_{tt} - e^{2t}u_{xx} + M^2u = f(x,t)\,, \qquad t>0\,, \quad x\in\mathbb{R}\,, \tag{7}$$

with vanishing initial data:

$$u(x,0) = u_t(x,0) = 0\,. \tag{8}$$

THEOREM 3. *Assume that $f \in C^\infty$ and that for every fixed t it has compact support, $\operatorname{supp} f(\cdot,t) \subset \mathbb{R}$. Then the function $u = u(x,t)$ defined by*

$$\begin{aligned} u(x,t) \;=\; & \int_0^t db \int_{x-(e^t-e^b)}^{x+e^t-e^b} dy\, f(y,b)(4e^{b+t})^{iM}\Big((e^t+e^b)^2-(x-y)^2\Big)^{-\frac{1}{2}-iM} \\ & \times F\left(\tfrac{1}{2}+iM, \tfrac{1}{2}+iM; 1; \frac{(e^b-e^t)^2-(x-y)^2}{(e^b+e^t)^2-(x-y)^2}\right) \end{aligned}$$

is a C^∞ solution to the Cauchy problem for equation (7) with vanishing initial data (8).

The representation of the solution of the Cauchy problem for the one-dimensional case of equation without source term is given by the next theorem.

THEOREM 4. *The solution $u = u(x,t)$ of the Cauchy problem*

$$u_{tt} - e^{2t}u_{xx} + M^2u = 0\,, \qquad u(x,0) = \varphi_0(x)\,, \qquad u_t(x,0) = \varphi_1(x)\,, \tag{9}$$

with $\varphi_0, \varphi_1 \in C_0^\infty(\mathbb{R})$ can be represented as follows

$$\begin{aligned} u(x,t) \;=\; & \tfrac{1}{2}e^{-\frac{t}{2}}\Big[\varphi_0(x+e^t-1)+\varphi_0(x-e^t+1)\Big] \\ & + \int_0^{e^t-1} \big[\varphi_0(x-z)+\varphi_0(x+z)\big]K_0(z,t)\,dz \\ & + \int_0^{e^t-1} \Big[\varphi_1(x-z)+\varphi_1(x+z)\Big]K_1(z,t)dz\,, \end{aligned}$$

where the kernels $K_0(z,t)$ and $K_1(z,t)$ are defined respectively by

$$\begin{aligned} K_0(z,t) \;:=\; & -\Big[\frac{\partial}{\partial b}E(z,t;0,b)\Big]_{b=0} \\ =\; & -(4e^t)^{iM}\big((e^t+1)^2-z^2\big)^{-iM}\frac{1}{[(e^t-1)^2-z^2]\sqrt{(e^t+1)^2-z^2}} \\ & \times \Bigg[\big(e^t-1-iM(e^{2t}-1-z^2)\big)F\Big(\tfrac{1}{2}+iM, \tfrac{1}{2}+iM; 1; \frac{(e^t-1)^2-z^2}{(e^t+1)^2-z^2}\Big) \\ & + \big(1-e^{2t}+z^2\big)\Big(\tfrac{1}{2}-iM\Big)F\Big(-\tfrac{1}{2}+iM, \tfrac{1}{2}+iM; 1; \frac{(e^t-1)^2-z^2}{(e^t+1)^2-z^2}\Big)\Bigg], \\ & \qquad\qquad 0 \le z < e^t-1, \end{aligned}$$

$$K_1(z,t) := E(z,t;0,0) = (4e^t)^{iM}\left((e^t+1)^2-z^2\right)^{-\frac{1}{2}-iM}$$
$$\times F\left(\tfrac{1}{2}+iM, \tfrac{1}{2}+iM; 1; \frac{(e^t-1)^2-z^2}{(e^t+1)^2-z^2}\right), \quad 0 \le z \le e^t-1.$$

The kernels $K_0(z,t)$ and $K_1(z,t)$ play leading roles in the derivation of $L^p - L^q$ estimates. Their main properties follow from the ones of the function $E(x,t;x_0,t_0)$, which are listed in Proposition 1 of Section 2.
Next we turn to the higher-dimensional equation with $n \ge 2$.

THEOREM 5. *If n is odd, $n = 2m+1$, $m \in \mathbb{N}$, then the solution $u = u(x,t)$ to the Cauchy problem*

$$u_{tt} - e^{2t}\Delta u + M^2 u = f(x,t), \quad u(x,0) = 0, \quad u_t(x,0) = 0, \tag{10}$$

with $f \in C^\infty(\mathbb{R}^{n+1})$ and with vanishing initial data is given by

$$\begin{aligned} u(x,t) = {} & 2\int_0^t db \int_0^{e^t-e^b} dr_1 \left(\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{\frac{n-3}{2}} \frac{r^{n-2}}{\omega_{n-1}c_0^{(n)}} \int_{S^{n-1}} f(x+ry,b)\, dS_y \right)_{r=r_1} \\ & \times (4e^{b+t})^{iM}\left((e^t+e^b)^2-r_1^2\right)^{-\frac{1}{2}-iM} \\ & \times F\left(\tfrac{1}{2}+iM, \tfrac{1}{2}+iM; 1; \frac{(e^b-e^t)^2-r_1^2}{(e^b+e^t)^2-r_1^2}\right), \end{aligned} \tag{11}$$

where $c_0^{(n)} = 1 \cdot 3 \cdot \ldots \cdot (n-2)$, and ω_{n-1} is the area of the unit sphere $S^{n-1} \subset \mathbb{R}^n$.

If n is even, $n = 2m$, $m \in \mathbb{N}$, then the solution $u = u(x,t)$ is given by

$$\begin{aligned} u(x,t) = {} & 2\int_0^t db \int_0^{e^t-e^b} dr_1 \left(\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{\frac{n-2}{2}} \frac{2r^{n-1}}{\omega_{n-1}c_0^{(n)}} \int_{B_1^n(0)} \frac{f(x+ry,b)}{\sqrt{1-|y|^2}}\, dV_y \right)_{r=r_1} \\ & \times (4e^{b+t})^{iM}\left((e^t+e^b)^2-r_1^2\right)^{-\frac{1}{2}-iM} \\ & \times F\left(\tfrac{1}{2}+iM, \tfrac{1}{2}+iM; 1; \frac{(e^b-e^t)^2-r_1^2}{(e^b+e^t)^2-r_1^2}\right). \end{aligned} \tag{12}$$

Here $B_1^n(0) := \{|y| \le 1\}$ is the unit ball in $\mathbb{R}^n$, while $c_0^{(n)} = 1 \cdot 3 \cdot \ldots \cdot (n-1)$.

Thus, in both cases, of even and odd n, one can write

$$\begin{aligned} u(x,t) &= 2\int_0^t db \int_0^{e^t-e^b} dr \quad v(x,r;b)(4e^{b+t})^{iM}\left((e^t+e^b)^2-r^2\right)^{-\frac{1}{2}-iM} \\ & \qquad \times F\left(\tfrac{1}{2}+iM, \tfrac{1}{2}+iM; 1; \frac{(e^b-e^t)^2-r^2}{(e^b+e^t)^2-r^2}\right), \end{aligned} \tag{13}$$

where the function $v(x,t;b)$ is a solution to the Cauchy problem for the wave equation

$$v_{tt} - \triangle v = 0\,, \quad v(x,0;b) = f(x,b)\,, \quad v_t(x,0;b) = 0\,.$$

The next theorem represents the solutions of the equation with the initial data prescribed at $t = 0$.

THEOREM 6. *The solution $u = u(x,t)$ to the Cauchy problem*

$$u_{tt} - e^{2t} \triangle u + M^2 u = 0\,, \quad u(x,0) = \varphi_0(x)\,, \quad u_t(x,0) = \varphi_1(x)\,, \tag{14}$$

with φ_0, $\varphi_1 \in C_0^\infty(\mathbb{R}^n)$, *$n \geq 2$, can be represented as follows:*

$$\begin{aligned} u(x,t) &= e^{-\frac{t}{2}} v_{\varphi_0}(x,\phi(t)) + 2\int_0^1 v_{\varphi_0}(x,\phi(t)s) K_0(\phi(t)s,t)\phi(t)\,ds \\ &\quad + 2\int_0^1 v_{\varphi_1}(x,\phi(t)s) K_1(\phi(t)s,t)\phi(t)\,ds, \quad x \in \mathbb{R}^n,\ t > 0\,, \end{aligned} \tag{15}$$

$\phi(t) := e^t - 1$, and where the kernels K_0 and K_1 have been defined in Theorem 4. Here for $\varphi \in C_0^\infty(\mathbb{R}^n)$ and for $x \in \mathbb{R}^n$, $n = 2m+1$, $m \in \mathbb{N}$,

$$v_\varphi(x,\phi(t)s) := \left(\frac{\partial}{\partial r} \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{\frac{n-3}{2}} \frac{r^{n-2}}{\omega_{n-1} c_0^{(n)}} \int_{S^{n-1}} \varphi(x+ry)\, dS_y \right)_{r=\phi(t)s}$$

while for $x \in \mathbb{R}^n$, $n = 2m$, $m \in \mathbb{N}$,

$$v_\varphi(x,\phi(t)s) := \left(\frac{\partial}{\partial r} \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{\frac{n-2}{2}} \frac{2r^{n-1}}{\omega_{n-1} c_0^{(n)}} \int_{B_1^n(0)} \frac{1}{\sqrt{1-|y|^2}} \varphi(x+ry) dV_y \right)_{r=s\phi(t)}.$$

The function $v_\varphi(x,\phi(t)s)$ coincides with the value $v(x,\phi(t)s)$ of the solution $v(x,t)$ of the Cauchy problem

$$v_{tt} - \triangle v = 0, \quad v(x,0) = \varphi(x), \quad v_t(x,0) = 0\,.$$

As a consequence of the above theorems, we obtain in a forthcoming paper the following $L^p - L^q$ decay estimate for the particles with "large" mass m, $m \geq n/2$, that is, with nonnegative curved mass $M \geq 0$.

$$\begin{aligned} & \|(-\triangle)^{-s} u(x,t)\|_{L^q(\mathbb{R}^n)} \\ & \leq\ C e^{t\left(2s - n(\frac{1}{p}-\frac{1}{q})\right)} \int_0^t \|f(x,b)\|_{L^p(\mathbb{R}^n)} (1+t-b)^{1-\operatorname{sgn} M} db \\ & \quad + C(1+t)^{1-\operatorname{sgn} M} (e^t - 1)^{2s - n(\frac{1}{p}-\frac{1}{q})} \left\{ \|\varphi_0(x)\|_{L^p(\mathbb{R}^n)} + (1-e^{-t}) \|\varphi_1\|_{L^p(\mathbb{R}^n)} \right\} \end{aligned} \tag{16}$$

provided that $1 < p \leq 2$, $\frac{1}{p} + \frac{1}{q} = 1$, $\frac{1}{2}(n+1)\left(\frac{1}{p} - \frac{1}{q}\right) \leq 2s \leq n\left(\frac{1}{p} - \frac{1}{q}\right) < 2s+1$.

We emphasize that the estimate (16) implies exponential decay for large time. It is essentially different from the decay estimate obtained in [16] for the wave equation in the de Sitter spacetime. This difference is caused by the striking difference between the global geometries of the forward and backward light cones of the equation (7).

The paper is organized as follows. In Section 2 we apply the fundamental solutions to solve the Cauchy problem with the source term and with vanishing initial data given at $t = 0$. More precisely, we give a representation formula for the solutions. In that section we also give several basic properties of the function $E(x,t;x_0,t_0)$. In Sections 3–4, we use the formulas of Section 2 to derive and to complete the list of representation formulas for the solutions of the Cauchy problem for the case of one-dimensional spatial variable. The higher-dimensional equation with the source term is considered in Section 5, where we derive a representation formula for the solutions of the Cauchy problem with the source term and with vanishing initial data given at $t = 0$. In the same section this formula is used to complete the proof of Theorem 6. Applications of all these results to the nonlinear equations will be done in a forthcoming paper.

2. Application to the Cauchy problem: source term and $\mathbf{n} = 1$

Consider now the Cauchy problem for the equation (7) with vanishing initial data (8). The coefficients of the equation (7) are independent of x, therefore the equation is translation invariant in x that implies $\mathcal{E}_+(x,t;y,b) = \mathcal{E}_+(x-y,t;0,b)$. Using the fundamental solution from Theorem 1 one can write the convolution

$$\text{(17)} \qquad \begin{aligned} u(x,t) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathcal{E}_+(x,t;y,b) f(y,b)\, db\, dy \\ &= \int_0^t db \int_{-\infty}^{\infty} \mathcal{E}_+(x-y,t;0,b) f(y,b)\, dy, \end{aligned}$$

which is well-defined since $\operatorname{supp} f \subset \{t \geq 0\}$. Then according to the definition of the distribution $\mathcal{E}_+$ we obtain the statement of Theorem 3. Thus, Theorem 3 is proven.

The following corollary implies the existence of an operator transforming the solutions of the Cauchy problem for the string equation to the solutions of the Cauchy problem for the inhomogeneous equation with time-dependent speed of propagation.

COROLLARY 1. *The solution $u = u(x,t)$ of the Cauchy problem (7)-(8) can be represented by (13), where the functions $v(x,t;\tau) := \frac{1}{2}(f(x+t,\tau) + f(x-t,\tau))$, $\tau \in [0,\infty)$, form a one-parameter family of solutions to the Cauchy problem for the string equation, that is, $v_{tt} - v_{xx} = 0$, $v(x,0;\tau) = f(x,\tau)$, $v_t(x,0;\tau) = 0$.*

Proof. From the convolution (17) we derive

$$\begin{aligned} u(x,t) &= \int_0^t db \int_{e^b-e^t}^{e^t-e^b} dy\, f(x+y,b)(4e^{b+t})^{iM} \left((e^t+e^b)^2 - y^2\right)^{-\frac{1}{2}-iM} \\ &\quad \times F\left(\tfrac{1}{2}+iM, \tfrac{1}{2}+iM; 1; \frac{(e^b-e^t)^2-y^2}{(e^b+e^t)^2-y^2}\right) \end{aligned}$$

$$= \; 2\int_0^t db \int_0^{e^t-e^b} dy\, \tfrac{1}{2}\{ f(x+y,b) + f(x-y,b)\}$$
$$\times\; (4e^{b+t})^{iM} \left((e^{-t}+e^{-b})^2 - y^2\right)^{-\frac{1}{2}-iM} F\left(\tfrac{1}{2}+iM, \tfrac{1}{2}+iM; 1; \frac{(e^b-e^t)^2-y^2}{(e^b+e^t)^2-y^2}\right).$$

The corollary is proven. □

In the next proposition we collect some elementary auxiliary formulas in order to make the proofs of the main theorems more transparent. The proof of that proposition is straightforward and we omit it.

PROPOSITION 1. *Let $E(x,t;x_0,t_0)$ be function defined by (4). One has*

$$E(x,t;y,b) \;=\; E(y,b;x,t), \tag{18}$$

$$E(x,t;y,b) = E(x-y,t;0,b), \quad E(x,t;0,b) = E(-x,t;0,b), \tag{19}$$

$$E(x,t;0,\ln(e^t-x)) \;=\; \frac{1}{2}\frac{1}{\sqrt{e^t}\sqrt{e^t-x}}, \tag{20}$$

$$\frac{\partial}{\partial b}\Big(e^b E(e^b-e^t,t;0,b)\Big) \;=\; \tfrac{1}{4}e^{-t/2}e^{b/2}, \tag{21}$$

$$\frac{\partial}{\partial b}\Big(be^b E(e^b-e^t,t;0,b)\Big) \;=\; \tfrac{1}{4}e^{-t/2}e^{b/2}(2+b), \tag{22}$$

$$\lim_{y\to x+e^t-e^b}\frac{\partial}{\partial x}E(x-y,t;0,b) \;=\; \tfrac{1}{16}(1+4M^2)e^{-2(b+t)}e^{b/2}e^{t/2}(e^b-e^t), \tag{23}$$

$$\lim_{y\to x-e^t+e^b}\frac{\partial}{\partial x}E(x-y,t;0,b) \;=\; \tfrac{1}{16}(1+4M^2)e^{-2(b+t)}e^{t/2}e^{b/2}(e^t-e^b), \tag{24}$$

$$\left[\frac{\partial}{\partial b}E(x,t;0,b)\right]_{b=\ln(e^t-x)} \;=\; \tfrac{1}{16}e^{-3t/2}\frac{(-4e^t+x(1+4M^2))}{\sqrt{e^t-x}}, \tag{25}$$

$$\begin{aligned}\left[\frac{\partial}{\partial b}E(x,t;0,b)\right]_{b=0} \;=&\; \frac{(4e^t)^{iM}\left((1+e^t)^2-x^2\right)^{-iM}}{2[(e^t-1)^2-x^2]\sqrt{(1+e^t)^2-x^2}} \\ &\times\left\{(2iM-1)\left(e^{2t}-1-x^2\right)F\left(-\tfrac{1}{2}+iM,\tfrac{1}{2}+iM,1,\frac{(-1+e^t)^2-x^2}{(1+e^t)^2-x^2}\right)\right. \\ &\left.-2\left(1-e^t+iM\left(e^{2t}-1-x^2\right)\right)F\left(\tfrac{1}{2}+iM,\tfrac{1}{2}+iM,1,\frac{(-1+e^t)^2-x^2}{(1+e^t)^2-x^2}\right)\right\}.\end{aligned} \tag{26}$$

3. The Cauchy problem: second data and $\mathbf{n=1}$

In this section we prove Theorem 4 in the case of $\varphi_0(x)=0$. More precisely, we have to prove that the solution $u(x,t)$ of the Cauchy problem (9) with $\varphi_0(x)=0$ and $\varphi_1(x)=\varphi(x)$ can be represented as follows

$$(27)\qquad u(x,t) = \int_0^{e^t-1}\Big[\varphi(x+z)+\varphi(x-z)\Big]K_1(z,t)\,dz = \int_0^1\Big[\varphi(x+\phi(t)s)+\varphi(x-\phi(t)s)\Big]K_1(\phi(t)s,t)\phi(t)\,ds,$$

where $\phi(t)=e^t-1$. The proof of the theorem is divided into several steps.

PROPOSITION 2. *The solution $u=u(x,t)$ of the Cauchy problem (9) for which $\varphi_0(x)=0$ and $\varphi_1(x)=\varphi(x)$ can be represented as follows*

$$(28)\qquad \begin{aligned} u(x,t) &= \int_0^t db\Big[\tfrac{1}{4}e^{-t/2}e^{b/2}(2+b)+\tfrac{1}{16}be^{-3t/2}e^{b/2}(e^b-e^t)(1+4M^2)\Big]\\ &\qquad\times\Big[\varphi(x+e^t-e^b)+\varphi(x-e^t+e^b)\Big]\\ &\quad+\int_0^t db\int_{x-(e^t-e^b)}^{x+e^t-e^b}dy\,\varphi(y)b\Big[e^{2b}\left(\frac{\partial}{\partial y}\right)^2E(x-y,t;0,b)-M^2E(x-y,t;0,b)\Big].\end{aligned}$$

Proof. We look for the solution $u=u(x,t)$ of the form $u(x,t)=w(x,t)+t\varphi(x)$. Then (9) implies

$$w_{tt}-e^{2t}w_{xx}+M^2w=te^{2t}\varphi^{(2)}(x)-M^2t\varphi(x),\qquad w(x,0)=0,\ w_t(x,0)=0.$$

We set $f(x,t)=te^{2t}\varphi^{(2)}(x)-M^2t\varphi(x)$ and due to Theorem 3 obtain

$$w(x,t)=\widetilde{w(x,t)}-M^2\int_0^t b\,db\int_{x-(e^t-e^b)}^{x+e^t-e^b}dy\,\varphi(y)E(x-y,t;0,b),$$

where

$$\widetilde{w(x,t)} := \int_0^t be^{2b}\,db\int_{x-(e^t-e^b)}^{x+e^t-e^b}dy\,\varphi^{(2)}(y)E(x-y,t;0,b)\,.$$

Then we integrate by parts:

$$\begin{aligned}\widetilde{w(x,t)} &= \int_0^t be^{2b}\,db\Big[\varphi^{(1)}(x+e^t-e^b)E(e^b-e^t,t;0,b)\\ &\qquad\qquad -\varphi^{(1)}(x-e^t+e^b)E(e^t-e^b,t;0,b)\Big]\\ &\quad-\int_0^t be^{2b}\,db\int_{x-(e^t-e^b)}^{x+e^t-e^b}dy\,\varphi^{(1)}(y)\frac{\partial}{\partial y}E(x-y,t;0,b)\,.\end{aligned}$$

But $\varphi^{(1)}(x+e^t-e^b)=-e^{-b}\frac{\partial}{\partial b}\varphi(x+e^t-e^b)$, $\varphi^{(1)}(x-e^t+e^b)=e^{-b}\frac{\partial}{\partial b}\varphi(x-e^t+e^b)$. Then, $E(e^b-e^t,t;0,b)=E(-e^b+e^t,t;0,b)$ due to (19), and we obtain

$$
\begin{aligned}
\widetilde{w(x,t)} &= -\int_0^t db\,\Big(-\varphi(x+e^t-e^b)\frac{\partial}{\partial b}\Big(be^bE(-e^t+e^b,t;0,b)\Big)\\
&\qquad -\varphi(x-e^t+e^b)\frac{\partial}{\partial b}\Big(be^bE(e^t-e^b,t;0,b)\Big)\Big)\\
&\quad -\int_0^t be^{2b}\,db\int_{x-(e^t-e^b)}^{x+e^t-e^b} dy\,\varphi^{(1)}(y)\frac{\partial}{\partial y}E(x-y,t;0,b)\,.
\end{aligned}
$$

One more integration by parts leads to

$$
\begin{aligned}
\widetilde{w(x,t)} &= -2te^t\varphi(x)E(0,t;0,t)\\
&\quad -\int_0^t db\,\Big(-\varphi(x+e^t-e^b)\frac{\partial}{\partial b}\Big(be^bE(-e^t+e^b,t;0,b)\Big)\\
&\qquad -\varphi(x-e^t+e^b)\frac{\partial}{\partial b}\Big(be^bE(e^t-e^b,t;0,b)\Big)\Big)\\
&\quad -\int_0^t be^{2b}\,db\int_{x-(e^t-e^b)}^{x+e^t-e^b} dy\,\varphi^{(1)}(y)\frac{\partial}{\partial y}E(x-y,t;0,b)\,.
\end{aligned}
$$

Since $E(0,t;0,t)=e^{-t}/2$ we use (22) of Proposition 1 to derive the representation

$$
\begin{aligned}
\widetilde{w(x,t)}+t\varphi(x) &= \int_0^t db\tfrac{1}{4}e^{-t/2}e^{b/2}(2+b)\Big(\varphi(x+e^t-e^b)+\varphi(x-e^t+e^b)\Big)\\
&\quad -\int_0^t be^{2b}\,db\int_{x-(e^t-e^b)}^{x+e^t-e^b} dy\,\varphi^{(1)}(y)\frac{\partial}{\partial y}E(x-y,t;0,b)\,.
\end{aligned}
$$

Integration by parts in the second term leads to

$$
\begin{aligned}
\widetilde{w(x,t)}+t\varphi(x) &= \int_0^t db\tfrac{1}{4}e^{-t/2}e^{b/2}(2+b)\Big(\varphi(x+e^t-e^b)+\varphi(x-e^t+e^b)\Big)\\
&\quad +\int_0^t be^{2b}\,db\,\varphi(x+e^t-e^b)\Big[\frac{\partial}{\partial x}E(x-y,t;0,b)\Big]_{y=x+e^t-e^b}\\
&\quad -\int_0^t be^{2b}\,db\,\varphi(x-e^t+e^b)\Big[\frac{\partial}{\partial x}E(x-y,t;0,b)\Big]_{y=x-e^t+e^b}\\
&\quad +\int_0^t be^{2b}\,db\int_{x-(e^t-e^b)}^{x+e^t-e^b} dy\,\varphi(y)\Big(\frac{\partial}{\partial y}\Big)^2E(x-y,t;0,b)\,.
\end{aligned}
$$

Applying (23) and (24) of Proposition 1 and $\frac{\partial}{\partial y}E(x-y,t;0,b)=-\frac{\partial}{\partial x}E(x-y,t;0,b)$,

$$
\begin{aligned}
\widetilde{w(x,t)}+t\varphi(x) &= \int_0^t \Big[\tfrac{1}{4}e^{-t/2}e^{b/2}(2+b)+\tfrac{1}{16}be^{-3t/2}e^{b/2}(e^b-e^t)(1+4M^2)\Big]\\
&\qquad\times\Big[\varphi(x+e^t-e^b)+\varphi(x-e^t+e^b)\Big]db\\
&\quad +\int_0^t be^{2b}\,db\int_{x-(e^t-e^b)}^{x+e^t-e^b} dy\,\varphi(y)\left(\frac{\partial}{\partial y}\right)^2E(x-y,t;0,b)\,.
\end{aligned}
$$

Finally, we get (28). The proposition is proven. □

COROLLARY 2. *The solution $u = u(x,t)$ of the Cauchy problem (9) with $\varphi_0(x) = 0$ and $\varphi_1(x) = \varphi(x)$ can be represented by*

$$\begin{aligned} u(x,t) \;=\; & \int_0^t \left[\tfrac{1}{4}e^{-t/2}e^{b/2}(2+b)+\tfrac{1}{16}be^{-3t/2}e^{b/2}(e^b-e^t)(1+4M^2)\right] \\ & \qquad \times \left[\varphi(x+e^t-e^b)+\varphi(x-e^t+e^b)\right] db \\ & + \int_0^t db \int_0^{e^t-e^b} dz \left[\varphi(x-z)+\varphi(x+z)\right] \\ & \qquad \times b\left[e^{2b}\left(\frac{\partial}{\partial z}\right)^2 E(z,t;0,b) - M^2E(z,t;0,b)\right], \end{aligned}$$

as well as by (27), where

$$\begin{aligned} K_1(z,t) \;=\; & \left[\tfrac{1}{4}e^{-t/2}(2+\ln(e^t-z)) - \tfrac{1}{16}(1+4M^2)e^{-3t/2}z\ln(e^t-z)\right]\frac{1}{\sqrt{e^t-z}} \\ & + \int_0^{\ln(e^t-z)} b\left[e^{2b}\left(\frac{\partial}{\partial z}\right)^2 E(z,t;0,b) - M^2E(z,t;0,b)\right] db. \end{aligned} \tag{29}$$

Proof of the Corollary. By means of the statement (28) of Proposition 2, the change $y = x - z$, and (19) we obtain

$$\begin{aligned} u(x,t) \;=\; & \int_0^t db\left[\tfrac{1}{4}e^{-t/2}e^{b/2}(2+b)+\tfrac{1}{16}be^{-3t/2}e^{b/2}(e^b-e^t)(1+4M^2)\right] \\ & \qquad \times \left[\varphi(x+e^t-e^b)+\varphi(x-e^t+e^b)\right] \\ & - \int_0^t db \int_0^{-(e^t-e^b)} dz\,\varphi(x-z)\left[be^{2b}\left(\frac{\partial}{\partial z}\right)^2 E(z,t;0,b) - M^2bE(z,t;0,b)\right] \\ & + \int_0^t db \int_{-(e^t-e^b)}^0 dz\,\varphi(x+z)\left[be^{2b}\left(\frac{\partial}{\partial z}\right)^2 E(z,t;0,b) - M^2bE(z,t;0,b)\right]. \end{aligned}$$

To prove (27) with $K_1(z,t)$ defined by (29) we apply (19) and write

$$\begin{aligned} u(x,t) \;=\; & \int_0^t db\left[\tfrac{1}{4}e^{-t/2}e^{b/2}(2+b)+\tfrac{1}{16}be^{-3t/2}e^{b/2}(e^b-e^t)(1+4M^2)\right] \\ & \qquad \times \left[\varphi(x+e^t-e^b)+\varphi(x-e^t+e^b)\right] \\ & + \int_0^t db \int_0^{e^t-e^b} dz \left[\varphi(x-z)+\varphi(x+z)\right] \\ & \qquad \times \left[be^{2b}\left(\frac{\partial}{\partial z}\right)^2 E(z,t;0,b) - M^2bE(z,t;0,b)\right]. \end{aligned}$$

Next we make change $z = e^t - e^b$, $dz = -e^b db$, $db = -(e^t-z)^{-1}dz$, and $b = \ln(e^t - z)$ in the first integral:

$$\begin{aligned}
&\int_0^t db\Big[\tfrac{1}{4}e^{-t/2}e^{b/2}(2+b)+\tfrac{1}{16}(1+4M^2)be^{-3t/2}e^{b/2}(e^b-e^t)\Big]\\
&\qquad\qquad\times\Big[\varphi(x+e^t-e^b)+\varphi(x-e^t+e^b)\Big]\\
&= \int_0^{e^t-1}\Big[\varphi(x+z)+\varphi(x-z)\Big]\Big[\tfrac{1}{4}e^{-t/2}(2+\ln(e^t-z))\\
&\qquad\qquad -\tfrac{1}{16}(1+4M^2)e^{-3t/2}z\ln(e^t-z)\Big]\frac{1}{\sqrt{e^t-z}}dz.
\end{aligned}$$

Then

$$\begin{aligned}
u(x,t) &= \int_0^{e^t-1}\Big[\varphi(x+z)+\varphi(x-z)\Big]\Big[\tfrac{1}{4}e^{-t/2}(2+\ln(e^t-z))\\
&\qquad -\tfrac{1}{16}(1+4M^2)e^{-3t/2}z\ln(e^t-z)\Big]\frac{1}{\sqrt{e^t-z}}dz\\
&\quad +\int_0^t db\int_0^{e^t-e^b} dz\Big[\varphi(x+z)+\varphi(x-z)\Big]\\
&\qquad \times\Big[be^{2b}\Big(\frac{\partial}{\partial z}\Big)^2E(z,t;0,b)-M^2bE(z,t;0,b)\Big].
\end{aligned}$$

In the last integral we change the order of integration,

$$\begin{aligned}
u(x,t) &= \int_0^{e^t-1}\Big[\varphi(x+z)+\varphi(x-z)\Big]\Big[\tfrac{1}{4}e^{-t/2}(2+\ln(e^t-z))\\
&\qquad -\tfrac{1}{16}(1+4M^2)e^{-3t/2}z\ln(e^t-z)\Big]\frac{1}{\sqrt{e^t-z}}dz\\
&\quad +\int_0^{e^t-1} dz\Big[\varphi(x+z)+\varphi(x-z)\Big]\int_0^{\ln(e^t-z)} db\, b\\
&\qquad \times\Big[e^{2b}\Big(\frac{\partial}{\partial z}\Big)^2E(z,t;0,b)-M^2E(z,t;0,b)\Big],
\end{aligned}$$

and obtain (27), where $K_1(z,t)$ is defined by (29). The corollary is proven. □

The next lemma completes the proof of Theorem 4 in the case of $\varphi_0=0$.

LEMMA 1. *The kernel $K_1(z,t)$ defined by (29) coincides with one given in Theorem 4.*

Proof. Due to Lemma 1.2 [16], (19), and by integration by parts, we have

$$\begin{aligned}
&\int_0^{\ln(e^t-z)} b\left[e^{2b}\Big(\frac{\partial}{\partial z}\Big)^2E(z,t;0,b)-M^2E(z,t;0,b)\right]db\\
&= \ln(e^t-z)\Big[\frac{\partial}{\partial b}E(z,t;0,b)\Big]_{b=\ln(e^t-z)}-E(z,t;0,\ln(e^t-z))+E(z,t;0,0).
\end{aligned}$$

On the other hand, (20) and (25) of Proposition 1 imply

$$\begin{aligned}
&\int_0^{\ln(e^t-z)} b\left[e^{2b}\left(\frac{\partial}{\partial z}\right)^2 E(z,t;0,b)-M^2E(z,t;0,b)\right]db\\
= \quad &\ln(e^t-z)\frac{e^{-2t}\sqrt{e^t}\left(-4e^t+z(1+4M^2)\right)}{16\sqrt{e^t-z}}-\tfrac{1}{2}e^{-\frac{t}{2}}(e^t-z)^{-\frac{1}{2}}+E(z,t;0,0).
\end{aligned}$$

Thus, for the kernel $K_1(z,t)$ defined by (29) we have

$$\begin{aligned}
K_1(z,t) &= \left[\tfrac{1}{4}e^{-t/2}(2+\ln(e^t-z))-\tfrac{1}{16}(1+4M^2)e^{-3t/2}z\ln(e^t-z)\right]\frac{1}{\sqrt{e^t-z}}\\
&\quad+\int_0^{\ln(e^t-z)} b\left[e^{2b}\left(\frac{\partial}{\partial z}\right)^2 E(z,t;0,b)-M^2E(z,t;0,b)\right]db\\
&= \left[\tfrac{1}{4}e^{-t/2}(2+\ln(e^t-z))-\tfrac{1}{16}(1+4M^2)e^{-3t/2}z\ln(e^t-z)\right]\frac{1}{\sqrt{e^t-z}}\\
&\quad+\ln(e^t-z)\frac{e^{-2t}\sqrt{e^t}\left(-4e^t+z(1+4M^2)\right)}{16\sqrt{e^t-z}}-\tfrac{1}{2}e^{-\frac{t}{2}}(e^t-z)^{-\frac{1}{2}}+E(z,t;0,0)\\
&= E(z,t;0,0)\,.
\end{aligned}$$

The last line coincides with $K_1(z,t)$ of Theorem 4. The lemma is proven. □

4. The Cauchy problem: first data and n = 1

In this section, we prove Theorem 4 in the case of $\varphi_1(x)=0$. Thus, we have to prove the representation given by Theorem 4 for the solution $u=u(x,t)$ of the Cauchy problem (9) with $\varphi_1(x)=0$, that is equivalent to

$$\begin{aligned}
u(x,t) &= \tfrac{1}{2}e^{-\frac{t}{2}}\left[\varphi_0(x+e^t-1)+\varphi_0(x-e^t+1)\right]\\
&\quad+\int_0^1\left[\varphi_0(x-\phi(t)s)+\varphi_0(x+\phi(t)s)\right]K_0(\phi(t)s,t)\phi(t)\,ds,
\end{aligned}$$

where $\phi(t)=e^t-1$. The proof of this case consists of several steps.

PROPOSITION 3. *The solution $u=u(x,t)$ of the Cauchy problem (9) can be represented as follows*

$$\begin{aligned}
u(x,t) &= \tfrac{1}{2}e^{-\frac{t}{2}}\left[\varphi_0(x+e^t-1)+\varphi_0(x-e^t+1)\right]\\
&\quad+\int_0^t db\left[\tfrac{1}{4}e^{\frac{b}{2}}e^{-\frac{t}{2}}+\tfrac{1}{16}(1+4M^2)e^{-2t}e^{\frac{b}{2}}e^{\frac{t}{2}}(e^b-e^t)\right]\\
&\qquad\qquad\times\left[\varphi_0(x+e^t-e^b)+\varphi_0(x-e^t+e^b)\right]\\
&\quad+\int_0^t db\int_{x-(e^t-e^b)}^{x+e^t-e^b}dy\,\varphi_0(y)\left[e^{2b}\left(\frac{\partial}{\partial y}\right)^2E(x-y,t;0,b)-M^2E(x-y,t;0,b)\right].
\end{aligned}$$

Proof. We set $u(x,t) = w(x,t) + \varphi_0(x)$, then

$$w_{tt} - e^{2t} w_{xx} + M^2 w = e^{2t} \varphi_0^{(2)}(x) - M^2 \varphi_0(x)\,, \quad w(x,0) = 0\,,\; w_t(x,0) = 0.$$

Next we plug $f(x,t) = e^{2t}\varphi_0^{(2)}(x) - M^2\varphi_0(x)$ into the formula given by Theorem 3 and obtain

$$w(x,t) \quad = \quad \widetilde{w(x,t)} - \int_0^t db \int_{x-(e^t-e^b)}^{x+e^t-e^b} dy\, M^2 \varphi_0(y) E(x-y,t;0,b), \tag{30}$$

where we have denoted

$$\widetilde{w(x,t)} \quad := \quad \int_0^t e^{2b}\, db \int_{x-(e^t-e^b)}^{x+e^t-e^b} dy\, \varphi_0^{(2)}(y) E(x-y,t;0,b)\,.$$

Next we integrate by parts and apply (19):

$$\begin{aligned}
\widetilde{w(x,t)} \quad = \quad & \int_0^t e^{2b}\, db \Big(\varphi_0^{(1)}(x+e^t-e^b) E(-e^t+e^b,t;0,b) \\
& \qquad - \varphi_0^{(1)}(x-e^t+e^b) E(e^t-e^b,t;0,b) \Big) \\
& - \int_0^t e^{2b}\, db \int_{x-(e^t-e^b)}^{x+e^t-e^b} dy\, \varphi_0^{(1)}(y) \frac{\partial}{\partial y} E(x-y,t;0,b)\,.
\end{aligned}$$

On the other hand,

$$\varphi_0^{(1)}(x+e^t-e^b) = -e^{-b}\frac{\partial}{\partial b}\varphi_0(x+e^t-e^b),\; \varphi_0^{(1)}(x-e^t+e^b) = e^{-b}\frac{\partial}{\partial b}\varphi_0(x-e^t+e^b)$$

implies that

$$\begin{aligned}
\widetilde{w(x,t)} \quad = \quad & \int_0^t e^{b}\, db \Big(-\frac{\partial}{\partial b}\varphi_0(x+e^t-e^b) E(-e^t+e^b,t;0,b) \\
& \qquad - \frac{\partial}{\partial b}\varphi_0(x-e^t+e^b) E(e^t-e^b,t;0,b) \Big) \\
& - \int_0^t e^{2b}\, db \int_{x-(e^t-e^b)}^{x+e^t-e^b} dy\, \varphi_0^{(1)}(y) \frac{\partial}{\partial y} E(x-y,t;0,b)\,.
\end{aligned}$$

One more integration by parts leads to

$$\begin{aligned}
\widetilde{w(x,t)} + \varphi_0(x) \quad = \quad & \tfrac{1}{2} e^{-\frac{t}{2}} \Big(\varphi_0(x+e^t-1) + \varphi_0(x-e^t+1) \Big) \\
& + \int_0^t db \Big(\varphi_0(x+e^t-e^b) \frac{\partial}{\partial b} \Big(e^b E(-e^t+e^b,t;0,b) \Big) \\
& \qquad + \varphi_0(x-e^t+e^b) \frac{\partial}{\partial b} \Big(e^b E(e^t-e^b,t;0,b) \Big) \Big) \\
& - \int_0^t e^{2b}\, db \int_{x-(e^t-e^b)}^{x+e^t-e^b} dy\, \varphi_0^{(1)}(y) \frac{\partial}{\partial y} E(x-y,t;0,b)\,,
\end{aligned}$$

where $E(0,t;0,t)=e^{-t}/2$, and $E(e^t-1,t;0,0)=E(1-e^t,t;0,0)=e^{-\frac{t}{2}}/2$ have been used. Next we apply (21) of Proposition 1 and an integration by parts to obtain

$$\begin{aligned}\widetilde{w(x,t)}+\varphi_0(x) &= \tfrac{1}{2}e^{-\frac{t}{2}}\Big(\varphi_0(x+e^t-1)+\varphi_0(x-e^t+1)\Big)\\ &\quad+\int_0^t db\tfrac{1}{4}e^{\frac{b}{2}}e^{-\frac{t}{2}}\Big(\varphi_0(x+e^t-e^b)+\varphi_0(x-e^t+e^b)\Big)\\ &\quad-\int_0^t e^{2b}\,db\Big[\varphi_0(y)\frac{\partial}{\partial y}E(x-y,t;0,b)\Big]_{y=x-(e^t-e^b)}^{y=x+e^t-e^b}\\ &\quad+\int_0^t e^{2b}\,db\int_{x-(e^t-e^b)}^{x+e^t-e^b}dy\,\varphi_0(y)\Big(\frac{\partial}{\partial y}\Big)^2E(x-y,t;0,b).\end{aligned}$$

From (23) and (24) of Proposition 1, we have

$$\begin{aligned}\widetilde{w(x,t)}+\varphi_0(x) &= \tfrac{1}{2}e^{-\frac{t}{2}}\Big[\varphi_0(x+e^t-1)+\varphi_0(x-e^t+1)\Big]\\ &\quad+\int_0^t db\tfrac{1}{4}e^{\frac{b}{2}}e^{-\frac{t}{2}}\Big[\varphi_0(x+e^t-e^b)+\varphi_0(x-e^t+e^b)\Big]\\ &-\int_0^t e^{2b}\,db\tfrac{1}{16}(1+4M^2)e^{-2(b+t)}e^{b/2}e^{t/2}(e^t-e^b)\Big[\varphi_0(x+e^t-e^b)+\varphi_0(x-(e^t-e^b))\Big]\\ &\quad+\int_0^t e^{2b}\,db\int_{x-(e^t-e^b)}^{x+e^t-e^b}dy\,\varphi_0(y)\Big(\frac{\partial}{\partial y}\Big)^2E(x-y,t;0,b),\end{aligned}$$

Then the last equation together with (30) proves the desired representation. The proposition is proven. □

Completion of the proof of Theorem 4. We make the change $z=e^t-e^b$, $dz=-e^b db$, and $b=\ln(e^t-z)$ in the second term of the representation given by the previous proposition:

$$\begin{aligned}&\int_0^t\Big[\tfrac{1}{4}e^{\frac{b}{2}}e^{-\frac{t}{2}}+\tfrac{1}{16}(1+4M^2)e^{-2t}e^{\frac{b}{2}}e^{\frac{t}{2}}(e^b-e^t)\Big]\\ &\qquad\times\Big[\varphi_0(x+e^t-e^b)+\varphi_0(x-e^t+e^b)\Big]db\\ &=\int_0^{e^t-1}\Big[\tfrac{1}{4}e^{-\frac{t}{2}}-\tfrac{1}{16}(1+4M^2)e^{-2t}e^{\frac{t}{2}}z\Big]\frac{1}{\sqrt{e^t-z}}\Big[\varphi_0(x-z)+\varphi_0(x+z)\Big]dz.\end{aligned}$$

Next we apply (19) to the last term of that representation, and then we change the order of integration:

$$\begin{aligned}&\int_0^t db\int_{x-(e^t-e^b)}^{x+e^t-e^b}dy\,\varphi_0(y)\left[e^{2b}\left(\frac{\partial}{\partial y}\right)^2E(x-y,t;0,b)-M^2E(x-y,t;0,b)\right]\\ &=\int_0^{e^t-1}dz\Big[\varphi_0(x+z)+\varphi_0(x-z)\Big]\int_0^{\ln(e^t-z)}db\\ &\qquad\times\left[e^{2b}\left(\frac{\partial}{\partial z}\right)^2E(z,t;0,b)-M^2E(z,t;0,b)\right].\end{aligned}$$

On the other hand, since the function $E(z,t;0,b)$ solves the Klein–Gordon equation, the last integral is equal to

$$\int_0^{e^t-1} dz\Big[\varphi_0(x+z)+\varphi_0(x-z)\Big]\int_0^{\ln(e^t-z)} db\left(\frac{\partial}{\partial b}\right)^2 E(z,t;0,b)\,db$$
$$=\int_0^{e^t-1} dz\Big[\varphi_0(x+z)+\varphi_0(x-z)\Big]\Big[\frac{\partial}{\partial b}E(z,t;0,\ln(e^t-z))-\frac{\partial}{\partial b}E(z,t;0,0)\Big].$$

Application of (25) and (26) gives

$$\Big[\tfrac{1}{4}e^{-\frac{t}{2}}-\tfrac{1}{16}(1+4M^2)e^{-2t}e^{\frac{t}{2}}z\Big]\frac{1}{\sqrt{e^t-z}}+\frac{\partial}{\partial b}E(z,t;0,\ln(e^t-z))-\frac{\partial}{\partial b}E(z,t;0,0)$$
$$=\Big[\tfrac{1}{4}e^{-\frac{t}{2}}-\tfrac{1}{16}(1+4M^2)e^{-2t}e^{\frac{t}{2}}z\Big]\frac{1}{\sqrt{e^t-z}}+\tfrac{1}{16}e^{-3t/2}\frac{(-4e^t+z(1+4M^2))}{\sqrt{e^t-z}}$$
$$-\left(4e^t\right)^{iM}\left((1+e^t)^2-z^2\right)^{-iM}\frac{1}{2[(e^t-1)^2-z^2]\sqrt{(1+e^t)^2-z^2}}$$
$$\times\Big\{(2iM-1)\left(e^{2t}-1-z^2\right)F\Big(-\tfrac{1}{2}+iM,\tfrac{1}{2}+iM,1,\frac{(-1+e^t)^2-z^2}{(1+e^t)^2-z^2}\Big)$$
$$-2\left(1-e^t+iM\left(e^{2t}-1-z^2\right)\right)F\Big(\tfrac{1}{2}+iM,\tfrac{1}{2}+iM,1,\frac{(-1+e^t)^2-z^2}{(1+e^t)^2-z^2}\Big)\Big\}.$$

The terms on the line after the last equality all cancel out, leaving the last three lines that add up to $K_0(z,t)$. This completes the proof of Theorem 4. □

5. The n-dimensional case, n ≥ 2

Proof of Theorem 5. Let us consider the case of $x\in\mathbb{R}^n$, where first $n=2m+1,\ m\in\mathbb{N}$. First, for the given function $u=u(x,t)$, we define the spherical means of u about the point x:

$$I_u(x,r,t) \quad = \quad \frac{1}{\omega_{n-1}}\int_{S^{n-1}} u(x+ry,t)\,dS_y\,,$$

where ω_{n-1} denotes the area of the unit sphere $S^{n-1}\subset\mathbb{R}^n$. Then we define an operator Ω_r by

$$\Omega_r(u)(x,t):=\left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{m-1} r^{2m-1}I_u(x,r,t)\,.$$

One can show that there are constants $c_j^{(n)},\ j=0,\dots,m-1$, where $n=2m+1$, with $c_0^{(n)}=1\cdot3\cdot5\cdots(n-2)$, such that

$$\left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{m-1} r^{2m-1}\varphi(r)=r\sum_{j=0}^{m-1}c_j^{(n)}r^j\frac{\partial^j}{\partial r^j}\varphi(r)\,.$$

One can recover the functions according to

$$u(x,t) = \lim_{r\to 0} I_u(x,r,t) = \lim_{r\to 0} \frac{1}{c_0^{(n)} r}\Omega_r(u)(x,t)\,, \tag{31}$$

$$u(x,0) = \lim_{r\to 0} \frac{1}{c_0^{(n)} r}\Omega_r(u)(x,0), \quad u_t(x,0) = \lim_{r\to 0} \frac{1}{c_0^{(n)} r}\Omega_r(\partial_t u)(x,0). \tag{32}$$

It is well known that $\Delta_x \Omega_r h = \frac{\partial^2}{\partial r^2}\Omega_r h$ for every function $h \in C^2(\mathbb{R}^n)$. Therefore we arrive at the following mixed problem for the function $v(x,r,t) := \Omega_r(u)(x,r,t)$:

$$\begin{cases} v_{tt}(x,r,t) - e^{2t} v_{rr}(x,r,t) + M^2 v(x,r,t) = F(x,r,t), & t \geq 0,\ r \geq 0,\ x \in \mathbb{R}^n, \\ v(x,0,t) = 0, & \text{for all } t \geq 0,\ x \in \mathbb{R}^n, \\ v(x,r,0) = 0, \quad v_t(x,r,0) = 0, & \text{for all } r \geq 0,\ x \in \mathbb{R}^n, \\ F(x,r,t) := \Omega_r(f)(x,t), \quad F(x,0,t) = 0, & \text{for all } x \in \mathbb{R}^n. \end{cases}$$

It must be noted here that the spherical mean I_u defined for $r > 0$ has an extension as an even function for $r < 0$ and hence $\Omega_r(u)$ has a natural extension as an odd function. That allows replacing the mixed problem with the Cauchy problem. Namely, let functions $\widetilde{v}$ and $\widetilde{F}$ be the continuations of the functions v and the forcing term F, respectively, by

$$\widetilde{v}(x,r,t) = \begin{cases} v(x,r,t) & \text{if } r \geq 0 \\ -v(x,-r,t) & \text{if } r \leq 0, \end{cases} \qquad \widetilde{F}(x,r,t) = \begin{cases} F(x,r,t) & \text{if } r \geq 0 \\ -F(x,-r,t) & \text{if } r \leq 0. \end{cases}$$

Then $\widetilde{v}$ solves the Cauchy problem

$$\begin{cases} \widetilde{v}_{tt}(x,r,t) - e^{2t}\widetilde{v}_{rr}(x,r,t) + M^2\widetilde{v}(x,r,t) = \widetilde{F}(x,r,t), & t \geq 0,\ r \in \mathbb{R},\ x \in \mathbb{R}^n, \\ \widetilde{v}(x,r,0) = 0, \quad \widetilde{v}_t(x,r,0) = 0 & \text{for all } r \in \mathbb{R},\ x \in \mathbb{R}^n. \end{cases}$$

Hence, according to Theorem 3, one has the representation

$$\widetilde{v}(x,r,t) = \int_0^t db \int_{r-(e^t-e^b)}^{r+e^t-e^b} \widetilde{F}(x,r_1,b) E(r,t;r_1,b)\, dr_1.$$

Since $u(x,t) = \lim_{r\to 0}\big(\widetilde{v}(x,r,t)/(c_0^{(n)} r)\big)$, we consider the case of $r < t$ in the above representation to obtain:

$$u(x,t) = \frac{1}{c_0^{(n)}} \int_0^t db \int_0^{e^t-e^b} dr_1 E(0,t;r_1,b) \lim_{r\to 0} \frac{\widetilde{F}(x,r+r_1,b) + \widetilde{F}(x,r-r_1,b)}{r}.$$

Then by definition of $\widetilde{F}$, we replace $\lim_{r\to 0}\Big\{\widetilde{F}(x,r-r_1,b) + \widetilde{F}(x,r+r_1,b)\Big\}/r$ with $2\Big(\frac{\partial}{\partial r}F(x,r,b)\Big)_{r=r_1}$ in the last formula. The definitions of $F(x,r,t)$ and of the operator Ω_r yield:

$$u(x,t) = \frac{2}{c_0^{(n)}} \int_0^t db \int_0^{e^t-e^b} \left(\frac{\partial}{\partial r} \Big(\frac{1}{r}\frac{\partial}{\partial r} \Big)^{m-1} r^{2m-1} I_f(x,r,t) \right)_{r=r_1} E(0,t;r_1,b)\, dr_1,$$

where $x \in \mathbb{R}^n$, $n = 2m+1$, $m \in \mathbb{N}$. Thus, the solution to the Cauchy problem is given by (11). We employ the method of descent to complete the proof for the case of even n, $n = 2m$, $m \in \mathbb{N}$. Theorem 5 is proven. □

Proof of Theorem 6. First we consider the case of $\varphi_0(x) = 0$. More precisely, we have to prove that the solution $u(x,t)$ of the Cauchy problem (14) with $\varphi_0(x) = 0$ can be represented by (15) with $\varphi_0(x) = 0$. The next lemma will be used in both cases.

LEMMA 2. *Consider the mixed problem*

$$\begin{cases} v_{tt} - e^{2t} v_{rr} + M^2 v = 0, & \text{for all } t \geq 0,\ r \geq 0, \\ v(r,0) = \tau_0(r), \quad v_t(r,0) = \tau_1(r) & \text{for all } r \geq 0, \\ v(0,t) = 0, & \text{for all } t \geq 0, \end{cases}$$

and denote by $\widetilde{\tau}_0(r)$ and $\widetilde{\tau}_1(r)$ the continuations of the functions $\tau_0(r)$ and $\tau_1(r)$ for negative r as odd functions: $\widetilde{\tau}_0(-r) = -\tau_0(r)$ and $\widetilde{\tau}_1(-r) = -\tau_1(r)$ for all $r \geq 0$, respectively. Then the unique solution $v(r,t)$ to the mixed problem is given by the restriction of (27) to $r \geq 0$:

$$\begin{aligned} v(r,t) &= \tfrac{1}{2} e^{-\frac{t}{2}} \Big[\widetilde{\tau}_0(r + e^t - 1) + \widetilde{\tau}_0(r - e^t + 1)\Big] \\ &\quad + \int_0^1 \big[\widetilde{\tau}_0(r - \phi(t)s) + \widetilde{\tau}_0(r + \phi(t)s)\big] K_0(\phi(t)s,t)\phi(t)\, ds \\ &\quad + \int_0^1 \Big[\widetilde{\tau}_1\Big(r + \phi(t)s\Big) + \widetilde{\tau}_1\Big(r - \phi(t)s\Big)\Big] K_1(\phi(t)s,t)\phi(t)\, ds, \end{aligned}$$

where $K_0(z,t)$ and $K_1(z,t)$ are defined in Theorem 4 and $\phi(t) = e^t - 1$.

Proof. This lemma is a direct consequence of Theorem 4. □

Now let us consider the case of $x \in \mathbb{R}^n$, where $n = 2m+1$. First for the given function $u = u(x,t)$ we define the spherical means of u about point x. One can recover the functions by means of (31), (32), and

$$\varphi_i(x) = \lim_{r\to 0} I_{\varphi_i}(x,r) = \lim_{r\to 0} \frac{1}{c_0^{(n)} r} \Omega_r(\varphi_i)(x), \quad i = 0,1.$$

Then we arrive at the following mixed problem

$$\begin{cases} v_{tt}(x,r,t) - e^{2t} v_{rr}(x,r,t) + M^2 v(x,r,t) = 0, & \text{for all } t \geq 0,\ r \geq 0,\ x \in \mathbb{R}^n, \\ v(x,0,t) = 0 & \text{for all } t \geq 0,\ x \in \mathbb{R}^n, \\ v(x,r,0) = 0, \quad v_t(x,r,0) = \Phi_1(x,r) & \text{for all } r \geq 0,\ x \in \mathbb{R}^n, \end{cases}$$

with the unknown function $v(x,r,t) := \Omega_r(u)(x,r,t)$, where

$$\Phi_i(x,r) := \Omega_r(\varphi_i)(x) = \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{m-1} r^{2m-1} \frac{1}{\omega_{n-1}} \int_{S^{n-1}} \varphi_i(x+ry)\, dS_y, \tag{33}$$

$$\Phi_i(x,0) = 0, \quad i = 0,1, \qquad \text{for all} \quad x \in \mathbb{R}^n. \tag{34}$$

Then, due to Lemma 2 and to $u(x,t) = \lim_{r\to 0}\left(v(x,r,t)/(c_0^{(n)}r)\right)$, we obtain:

$$u(x,t) = \frac{1}{c_0^{(n)}} \lim_{r\to 0} \frac{1}{r} \int_0^1 \left[\widetilde{\Phi}_1\left(x, r+\phi(t)s\right) + \widetilde{\Phi}_1\left(x, r-\phi(t)s\right)\right] K_1(\phi(t)s,t)\phi(t)ds.$$

The last limit is equal to

$$\begin{aligned}
&2\int_0^1 \left(\frac{\partial}{\partial r}\Phi_1(x,r)\right)_{r=\phi(t)s} K_1(\phi(t)s,t)\phi(t)\,ds \\
&= 2\int_0^1 \left(\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{\frac{n-3}{2}} \frac{r^{n-2}}{\omega_{n-1}} \int_{S^{n-1}} \varphi_1(x+ry)\,dS_y\right)_{r=\phi(t)s} K_1(\phi(t)s,t)\phi(t)ds.
\end{aligned}$$

Thus, Theorem 6 in the case of $\varphi_0(x) = 0$ is proven.

Now we turn to the case of $\varphi_1(x) = 0$. Thus, we arrive at the following mixed problem

$$\begin{cases}
v_{tt}(x,r,t) - e^{2t}v_{rr}(x,r,t) + M^2 v(x,r,t) = 0, & \text{for all } t \geq 0,\ r \geq 0,\ x \in \mathbb{R}^n, \\
v(x,r,0) = \Phi_0(x,r), \quad v_t(x,r,0) = 0 & \text{for all } r \geq 0,\ x \in \mathbb{R}^n, \\
v(x,0,t) = 0 & \text{for all } t \geq 0,\ x \in \mathbb{R}^n,
\end{cases}$$

with the unknown function $v(x,r,t) := \Omega_r(u)(x,r,t)$ defined by (33), (34). Then, according to Lemma 2 and to $u(x,t) = \lim_{r\to 0} v(x,r,t)/(c_0^{(n)}r)$, we obtain:

$$\begin{aligned}
u(x,t) &= \frac{1}{c_0^{(n)}} e^{-\frac{t}{2}} \left(\frac{\partial}{\partial r}\Phi_0(x,r)\right)_{r=\phi(t)} \\
&\quad + \frac{2}{c_0^{(n)}} \int_0^1 \left(\frac{\partial}{\partial r}\Phi_0(x,r)\right)_{r=\phi(t)s} K_0(\phi(t)s,t)\phi(t)\,ds \\
&= e^{-\frac{t}{2}} v_{\varphi_0}(x,\phi(t)) + 2\int_0^1 v_{\varphi_0}(x,\phi(t)s) K_0(\phi(t)s,t)\phi(t)\,ds\,.
\end{aligned}$$

Theorem 6 is proven. □

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