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**POLENI CURVES ON SURFACES OF CONSTANT CURVATURE**

**Abstract.** In the euclidean plane, a regular curve can be defined through its intrinsic equation which relates its curvature  $k$  to the arc length  $s$ . Elastic plane curves were determined this way. If  $k(s) = \frac{2\alpha}{\cosh(\alpha s)}$ , the curve is known by the name “la courbe des forçats”, introduced in 1729 by Giovanni Poleni in relation with the tractrix [9]. The above equation is yet meaningful on a surface if one interprets  $k$  as the geodesic curvature of the curve. In this paper we solve the above equation on a surface of constant curvature.

**1. Elastic Poleni curves**

In [7] the authors show that on surfaces of Gaussian curvature  $G$ , elastic curves  $\gamma$  are solutions of the intrinsic equation

$$k_g'' + \frac{1}{2}k_g^3 + k_g(G - \lambda) = 0,$$

where  $k_g$  denotes the geodesic curvature of  $\gamma$ ,  $\lambda$  is a constant and the primes indicate derivation with respect to the arc-length parameter  $s$  on  $\gamma$ . As the function  $f(s) = \frac{2}{\cosh(s)}$  satisfies the equation

$$f'' + \frac{1}{2}f^3 - f = 0,$$

there exist on a surface  $S$  of constant curvature elastic curves with intrinsic equation

$$(1) \quad k_g(s) = \frac{2}{\cosh(s)}.$$

If  $S$  is the euclidean plane, the solution of (1) is Giovanni Poleni’s curve introduced in 1729 in relation with the tractrix [9]. Its parametric equations are

$$(2) \quad x(s) = s - 2 \tanh(s), \quad y(s) = \frac{2}{\cosh(s)}.$$

It is plotted in Figure 1. Actually, for every  $\alpha \in \mathbb{R}$ , the function  $f(s) = \frac{2\alpha}{\cosh(\alpha s)}$  satisfies the differential equation

$$f'' + \frac{1}{2}f^3 - \alpha^2 f = 0.$$

Therefore there exist on surface  $S$  elastic curves with intrinsic equation

$$(3) \quad k_g(s) = \frac{2\alpha}{\cosh(\alpha s)}.$$

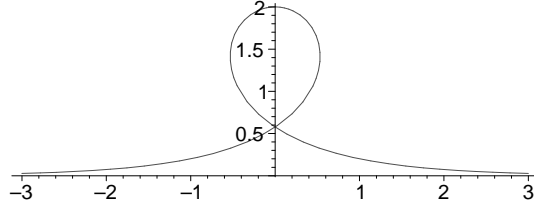


Figure 1: Poleni's curve

If  $S$  is the euclidean plane, equation (3) writes

$$x'' = \frac{-2\alpha y'}{\cosh(\alpha s)}, \quad y'' = \frac{2\alpha x'}{\cosh(\alpha s)}.$$

Integrating these equations, with initial condition

$$x(0) = 0, \quad x'(0) = 1, \quad y(0) = \frac{2}{\alpha}, \quad y'(0) = 0,$$

one finds

$$x(s) = s - \frac{2}{\alpha} \tanh(\alpha s), \quad y(s) = \frac{2}{\alpha \cosh(\alpha s)}.$$

Notice that the curve  $(\alpha x(s/\alpha), \alpha y(s/\alpha))$  is simply the Poleni curve (2), so that the study of equation (3) in the euclidean plane reduces to the study of equation (1). For the sphere and the hyperbolic space, there are no dilatations and equation (3) deserves a special study. In the following, we call "Poleni curves" the solutions of the intrinsic equation (3).

Geodesics on cylinders having as directrix the curve (2) are extremals of Sadowsky's functional

$$S = \int_{\gamma} \kappa^2 \left( 1 + \frac{\tau^2}{\kappa^2} \right)^2 ds$$

where  $\kappa, \tau$  represent the curvature and the torsion of the curve  $\gamma$ , in the three-dimensional euclidean space  $\mathbb{E}^3$ , see [5]. An analogous result is valid for geodesics on cones having as directrix elastic Poleni curves on the sphere  $\mathbb{S}^2$ , see [4]. This example motivates our research.

Notice that the Poleni curve (2) in the euclidean plane has an axe of symmetry and it has an asymptote as  $s \rightarrow \pm\infty$ . In fact, on any surface, a solution of equation (3) is asymptotically geodesic and has an axis of symmetry. If  $S$  is the sphere, the Poleni curve corresponding to  $\alpha = 1$ , is (see Proposition 1):

$$x = 2 \cos(s) \tanh(s/2), \quad y = 2 \sin(s) \tanh(s/2),$$

where  $(x, y)$  are the stereographical coordinates. It is a symmetric double spiral which tends to the circle of radius 2 centered at the origin. Hence (Theorem 1), on the sphere,

the situation is similar to the euclidean case: there exist two geodesics (here circles) which are characterized by the fact that the first one is an axis of symmetry of the curve and the second one is asymptotic to the curve.

If  $S$  is the hyperbolic half plane, the Poleni curve corresponding to  $\alpha = 1$  is (see Proposition 2):

$$x = s, \quad y = \cosh(s).$$

Notice that this is the catenary (or chainette) curve of the euclidean space. Hence (Theorem 2), on the hyperbolic space, the Poleni curve is asymptotic to all geodesics that intersect it transversally at a unique point and among these geodesics there is one which is an axis of symmetry of the curve.

The study of elastic curves goes back to Euler in 1744 [3]. His classification of plane elastic curves can be found in Love's book (see [8], Section 263) or in the historical survey by Truesdell [10]. This study was the birth of the Euler Lagrange calculus of variation. In Euler's classification of plane elastic curves, Poleni's curve, called also "la courbe des forçats", occupies a special place: discarding the circle and the straight line, it is the only elastic plane curve that does not need elliptic functions for its parametrization.

The study of euclidean elastic curves (together with the extension to the non-euclidean case) was revisited by Langer and Singer [7] and also by Bryant and Griffiths [1], using Griffiths's formalism based on the theory of exterior differential systems, and by Jurdjevic [6], using Pontryagin's Maximum Principle of optimal control theory. The papers [1, 6, 7] contain important results on the classification of elastic curves in the non euclidean case. Even though they mentioned the special case of asymptotically geodesic elastic curves (see especially Figure 8 of [1] and Figure 2.c of [7]), they all don't give explicit formula for the Poleni curve. On the sphere  $\mathbb{S}^2$ , the Poleni curve is explicitly mentioned by Brunnett and Crouch, see Figure 7 in [2], but its double spiral behavior is not apparent in their figure.

This paper is organized as follows. In Section 2 we recall some facts on the riemannian geometry of surfaces. In Section 3, equation (3) of Poleni curves is expressed in the geographical coordinates of the sphere  $\mathbb{S}^2$  and numerical solutions of the obtained equation are given. In Section 4, equation (3) is expressed in the stereographical projection of the sphere and the obtained equation is explicitly solved in the case  $\alpha = 1$ . In Section 5, equation (3) is expressed in the hyperbolic half plane and the obtained equation is explicitly solved in the case  $\alpha = 1$ . In Section 6 we compare our results with the parameterizations given by Jurdjevic in [6]. In Section 7, the Poleni curves of geodesic curvature  $f(s) = \frac{2\alpha}{\cosh(\alpha s)}$  are considered. Discarding the constant curvature curves, these curves are the only elastic curves that can be parametrized by elementary functions.

## 2. Geodesic curvatures and intrinsic equations

On the surface  $S$ , the geodesic curvature  $k_g$  of the curve  $\gamma$  represents the modulus of the normal component of the acceleration vector  $A := \nabla_{\gamma'} \gamma'$ . Clearly,  $s$  being an arc-length

parameters on  $\gamma$ , let  $\gamma' = d\gamma/ds$  be the unit tangent vector field along  $\gamma$ . Denoting by  $g(\cdot, \cdot)$  the scalar product on the tangent bundle of the surface  $S$ , it follows that

$$\|\gamma'\|^2 := g(\gamma', \gamma') = 1.$$

Let  $\nabla$  denotes the operator of covariant (absolute) differentiation on the surface. Then

$$g(\nabla_{\gamma'}\gamma', \gamma') = \frac{1}{2}\nabla_{\gamma'}g(\gamma', \gamma') = 0,$$

so the acceleration vector  $A = \nabla_{\gamma'}\gamma'$  is orthogonal to  $\gamma'$  along the curve. Equation (3) of Poleni curves becomes therefore

$$(4) \quad \|\nabla_{\gamma'}\gamma'\| = \frac{2\alpha}{\cosh(\alpha s)}.$$

If along  $\gamma$ , an orthonormal frame field  $\{T = \gamma', N\}$  is chosen, in order to solve (4) one has to integrate the system

$$(5) \quad \nabla_{\gamma'}\gamma' = \frac{2\alpha}{\cosh(\alpha s)}N.$$

Following [11], we recall some general formulas from the theory of surfaces when the local coordinates  $x_1, x_2$  are used. Let

$$(6) \quad ds^2 = \sum_{i,j=1}^2 g_{ij}dx_i dx_j = g_{11}dx_1^2 + 2g_{12}dx_1 dx_2 + g_{22}dx_2^2$$

be the riemannian metric on the surface  $S$  and let

$$(7) \quad L = \frac{1}{2} \sum_{i,j=1}^2 g_{ij}dx_i dx_j$$

be the Lagrange function associated to (6). The Euler-Lagrange system corresponding to  $L$  is

$$(8) \quad \frac{d}{ds} \frac{\partial L}{\partial x'_i} - \frac{\partial L}{\partial x_i} = 0, \quad i = 1, 2$$

and we use it to deduce the components of the acceleration vector  $A = \nabla_{\gamma'}\gamma'$ . This is done by isolating second order derivatives of the coordinates  $x_i$  in (8) i.e. transforming it in

$$(9) \quad x''_i + \sum_{j,k=1}^2 \Gamma_{jk}^i x'_j x'_k = 0, \quad i = 1, 2.$$

Now the left side part of (9) represents the component  $A_i$  of the acceleration vector  $A$  with respect to  $x_i$ , for  $i = 1, 2$ . In the sequel, we apply this procedure for several curvilinear systems of coordinates on surfaces of constant curvature.

### 3. Geographical coordinates on the sphere $\mathbb{S}^2$

Denote  $\phi$  and  $\theta$  the longitude and the colatitude of a point on the unit sphere  $\mathbb{S}^2$ . The cartesian coordinates of the point are

$$x = \sin \theta \cos \phi, \quad y = \sin \theta \sin \phi, \quad z = \cos \theta, \quad 0 < \theta < \pi$$

and the riemannian metric of  $\mathbb{S}^2$  becomes

$$(10) \quad ds^2 = d\theta^2 + \sin^2 \theta d\phi^2.$$

The Lagrange function (7) associated to  $ds^2$  is

$$(11) \quad L = \frac{1}{2} \left( (\theta')^2 + \sin^2 \theta (\phi')^2 \right)$$

and the Euler-Lagrange equations write

$$\frac{d}{ds} \theta' - \sin \theta \cos \theta (\phi')^2 = 0, \quad \frac{d}{ds} (\sin^2 \theta \phi') = 0$$

so that, transforming in the form (9) gives

$$\theta'' - \sin \theta \cos \theta (\phi')^2 = 0, \quad \phi'' + 2 \frac{\cos \theta}{\sin \theta} \theta' \phi' = 0.$$

The left side parts of these equations represent the components  $A_\theta, A_\phi$  of the acceleration vector  $A$  of a curve  $\gamma$  along which  $\theta$  and  $\phi$  are functions of arc-length parameter  $s$ . The vector fields along the curve,  $T(\theta', \phi'), N(-\phi' \sin \theta, \theta' / \sin \theta)$  are orthonormal as

$$(12) \quad g(T, T) = \|\gamma'\|^2 = (\theta')^2 + \sin^2 \theta (\phi')^2 = 1$$

and

$$\begin{aligned} g(N, N) &= (-\phi' \sin \theta)^2 + \sin^2 \theta (\theta' / \sin \theta)^2 = \sin^2 \theta (\phi')^2 + (\theta')^2 = 1 \\ g(T, N) &= \theta' (-\phi' \sin \theta) + \sin^2 \theta \phi' (\theta' / \sin \theta) = 0. \end{aligned}$$

It follows that the system of differential equations

$$(13) \quad \theta'' - \sin(\theta) \cos(\theta) (\phi')^2 = -\frac{2\alpha}{\cosh(\alpha s)} \phi' \sin(\theta)$$

$$(14) \quad \phi'' + 2 \frac{\cos(\theta)}{\sin(\theta)} \theta' \phi' = \frac{2\alpha}{\cosh(\alpha s)} \frac{\theta'}{\sin(\theta)}$$

translates the vector equation (5).

We have solved numerically (13)–(14), in the case  $\alpha = 1$ , together with the conditions  $\theta(0) = \pi/4, \theta'(0) = 0, \phi(0) = \pi/4$  and  $\phi'(0) = -\sqrt{2}$  by a Runge-Kutta algorithm of fourth order using the symbolic computing system MAPLE. We have obtained  $\theta_i, \theta'_i, \phi_i, \phi'_i$  approximation of  $\theta(i\Delta s), \theta'(i\Delta s), \phi(i\Delta s), \phi'(i\Delta s)$  respectively.

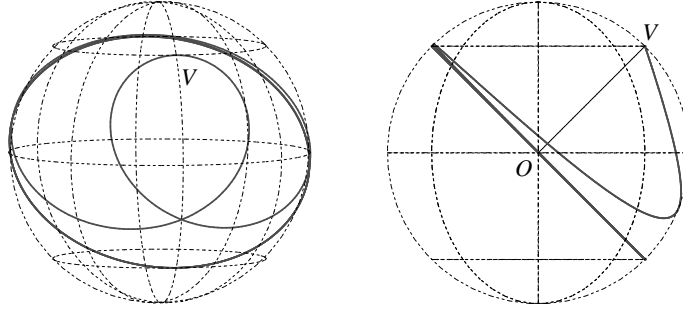


Figure 2: The curve of geographic coordinates  $(\phi_i, \theta_i)$  (left) and its projection on the symmetry plane (right). The asymptotic plane is orthogonal to  $OV$

Let  $V$  be the peak of the curve i.e. the point with maximal curvature 2 (for  $s = 0$ ) and denote by  $t_V$  the tangent at the solution curve in  $V$ . We observe on Figure 2 that the plane orthogonal to  $t_V$  in  $V$  is a plane of symmetry for the curve. It contains a first self-intersecting point. Despite the curve seems to be closed, in fact it is composed by two symmetrical spirals which wind around a diameter of  $\mathbb{S}^2$  and these spirals tend asymptotically to the equatorial circle corresponding to  $V$ , see Theorem 1.

At this stage, the computed  $\phi_i$ ,  $\theta_i$ ,  $\phi'_i$  and  $\theta'_i$  verify the equations (13) and (14). The equation (12) has not been used yet. The computed quantities satisfy

$$-2 \cdot 10^{-6} \leq (\theta'_i)^2 + \sin^2 \theta_i (\phi'_i)^2 - 1 \leq 6 \cdot 10^{-6},$$

in other words, the computed  $\phi_i$ ,  $\theta_i$ ,  $\phi'_i$  and  $\theta'_i$  verify the three equations (12), (13) and (14).

#### 4. Stereographical projection

If one projects the unit sphere  $\mathbb{S}^2$  from its north pole on the plane tangent to  $\mathbb{S}^2$  at its south pole and one transports its metric on that plane, one gets on the plane the Riemann metric

$$ds^2 = \frac{dx^2 + dy^2}{\left[1 + \frac{1}{4}(x^2 + y^2)\right]^2}.$$

The geodesics of this metric are the images of the great circles of  $\mathbb{S}^2$ , these are the circles or straight lines of the plane which contain two diametrically opposed points of the circle of radius 2 centered on the origin. Starting as before with the Lagrange function

$$L = \frac{1}{2} \frac{x^2 + y^2}{\left[1 + \frac{1}{4}(x^2 + y^2)\right]^2},$$

the Euler-Lagrange equations are

$$\frac{d}{ds} \frac{x'}{\left[1 + \frac{1}{4}(x^2 + y^2)\right]^2} + \frac{1}{2} \frac{x(x^2 + y^2)}{\left[1 + \frac{1}{4}(x^2 + y^2)\right]^3} = 0,$$

$$\frac{d}{ds} \frac{y'}{\left[1 + \frac{1}{4}(x^2 + y^2)\right]^2} + \frac{1}{2} \frac{y(x^2 + y^2)}{\left[1 + \frac{1}{4}(x^2 + y^2)\right]^3} = 0.$$

Isolating the second order derivatives  $x''$ ,  $y''$  one gets the components of the acceleration

$$A_x = x'' - \frac{x(x^2 - y^2) + 2yx'y'}{2 + \frac{1}{2}(x^2 + y^2)}, \quad A_y = y'' - \frac{y(y^2 - x^2) + 2xx'y'}{2 + \frac{1}{2}(x^2 + y^2)}.$$

The vector fields  $T(x', y')$ ,  $N(-y', x')$  along the curve  $\gamma$  are unitary and orthogonal as

$$(15) \quad \|\gamma'(s)\|^2 = \frac{(x'(s))^2 + (y'(s))^2}{\left[1 + \frac{1}{4}((x(s))^2 + (y(s))^2)\right]^2} = 1$$

and equation (5) is expressed by the system

$$(16) \quad x'' - \frac{x(x^2 - y^2) + 2yx'y'}{2 + \frac{1}{2}(x^2 + y^2)} = \frac{-2\alpha y'}{\cosh(\alpha s)},$$

$$(17) \quad y'' - \frac{y(y^2 - x^2) + 2xx'y'}{2 + \frac{1}{2}(x^2 + y^2)} = \frac{2\alpha x'}{\cosh(\alpha s)}.$$

PROPOSITION 1. *Up to isometry, the parametric equations of the Poleni curve with  $\alpha = 1$  on the sphere are*

$$(18) \quad x(s) = 2 \cos(s) \tanh(s/2), \quad y(s) = 2 \sin(s) \tanh(s/2).$$

*Proof.* Up to isometry, one can restrict to the initial condition

$$x(0) = 0, \quad x'(0) = 1, \quad y(0) = 0, \quad y'(0) = 0.$$

We use the symbolic computing system MAPLE to write Taylor series of the solution of (16,17), with  $\alpha = 1$ , satisfying the previous initial conditions. At order 7 one obtains

$$x(s) = s - \frac{7}{12}s^3 + \frac{11}{120}s^5 + 0(s^7), \quad y(s) = -s^2 + \frac{1}{4}s^4 - \frac{11}{360}s^6 + 0(s^7).$$

Then we compute

$$\frac{x(s)}{\sqrt{x^2(s) + y^2(s)}} = 1 - \frac{s^2}{2} + \frac{s^4}{24} + 0(s^6) = \cos(s),$$

$$\frac{y(s)}{\sqrt{x^2(s)+y^2(s)}} = s - \frac{s^3}{6} + \frac{s^5}{120} + 0(s^6) = \sin(s),$$

$$\frac{\sqrt{x^2(s)+y^2(s)}}{2} = \frac{s}{2} - \frac{s^3}{24} + \frac{s^5}{240} + 0(s^6) = \tanh(s/2).$$

Hence formulas (18) are conjectured. By straightforward calculations we verify that (18) is actually the solution of (16,17), with  $\alpha = 1$ , satisfying the specified initial conditions.  $\square$

**THEOREM 1.** *Let  $V$  be the peak of the Poleni curve, with  $\alpha = 1$ , on the sphere. Then the plane orthogonal to the Poleni curve at  $V$  is a symmetry plane for the curve. The curve has an infinite number of self-intersecting points of arc-length  $\pm s_n$  given by*

$$s_n = \frac{\pi}{2} + n\pi, \quad n \in \mathbb{N}$$

*contained in the symmetry plane. The Poleni curve is composed of two symmetrical spirals which wind around the diameter determined by  $V$  and tend asymptotically to a circle of radius 1.*

*Proof.* Let us project the unit sphere  $S^2$  from point  $-V$  on the plane tangent to  $S^2$  at  $V$ . We chose cartesian coordinates  $x$  and  $y$  on this plane such that  $t_V$  projects on the vector  $(1,0)$ . Then the Poleni curve is given by equations (18). Thus  $x^2(s)+y^2(s)$  tends to 4. The circle of radius 2 correspond to the circle of radius 1 on the sphere, orthogonal to  $V$ . The arc-lengths  $\pm s_n$  of the self-intersecting points are given by equation  $\cos(s_n) = 0$ . Thus  $s_n = \frac{\pi}{2} + n\pi$ .  $\square$

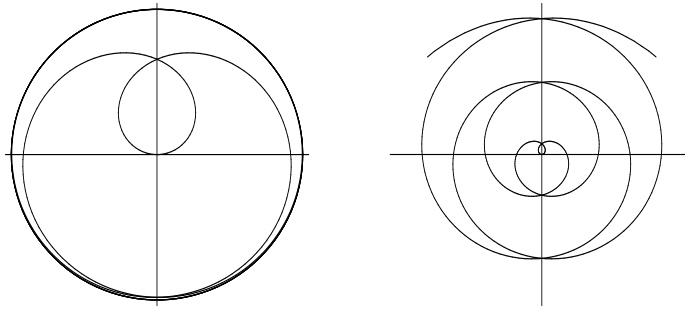


Figure 3: The Poleni curve (18) and its image via the transformation (19), showing the self-intersection points on the  $y$  axis

Figure 3 shows the Poleni curve (18) and its image via the transformation

$$(19) \quad (x, y) \mapsto \left( xf \left( \sqrt{x^2 + y^2} \right), yf \left( \sqrt{x^2 + y^2} \right) \right),$$



where  $f(r) = -\frac{\ln(1-r/2)}{r} = \frac{1}{2} + O(r)$ . This transformation is a magnifying glass of the neighborhood of the circle of radius 2. It shows many self-intersection points of the Poleni curve. In the geographic coordinates  $(\phi, \theta)$  the curve (18) is given by

$$\phi(s) = s, \quad \theta(s) = \arccos\left(\frac{1}{\cosh(s)}\right).$$

In the cartesian coordinates  $(x, y, z)$  such that  $x^2 + y^2 + z^2 = 1$ , the Poleni curve has the following parametric equations

$$(20) \quad x(s) = \cos(s) \tanh(s), \quad y(s) = \sin(s) \tanh(s), \quad z(s) = \frac{1}{\cosh(s)}.$$

### 5. The Poincaré model of hyperbolic geometry

In the half-plane  $\mathbb{P} = \{(x, y) \in \mathbb{R}^2 : y > 0\}$  the riemannian metric and the Lagrange function are

$$ds^2 = \frac{dx^2 + dy^2}{y^2}, \quad L = \frac{1}{2} \frac{x'^2 + y'^2}{y^2}.$$

The Euler-Lagrange equations are

$$\frac{d}{ds} \frac{x'}{y^2} = 0, \quad \frac{d}{ds} \frac{y'}{y^2} + \frac{x'^2 + y'^2}{y^3} = 0.$$

The  $x, y$  components of the acceleration are

$$A_x = x'' - 2\frac{x'y'}{y}, \quad A_y = y'' + \frac{x'^2 - y'^2}{y}$$

and equation (5) is expressed by the system

$$(21) \quad x'' - 2\frac{x'y'}{y} = \frac{-2\alpha y'}{\cosh(\alpha s)}, \quad y'' + \frac{x'^2 - y'^2}{y} = \frac{2\alpha x'}{\cosh(\alpha s)}.$$

The Poleni curves  $(x(s), y(s))$  are the solutions of (21) satisfying

$$(22) \quad \frac{(x'(s))^2 + (y'(s))^2}{(y(s))^2} = 1.$$

**PROPOSITION 2.** *Up to isometry, the parametric equations of the Poleni curve, with  $\alpha = 1$ , on the hyperbolic half-plane are*

$$(23) \quad x(s) = s, \quad y(s) = \cosh(s).$$

*Proof.* Up to isometry, one can restrict to the initial condition

$$x(0) = 0, \quad x'(0) = 1, \quad y(0) = 1, \quad y'(0) = 0.$$

As in the proof of Proposition 1, we use the symbolic computing system MAPLE to write Taylor series of the solution and we conjecture formulas (23). By straightforward calculations we verify that (23) is actually the solution of (21), with  $\alpha = 1$ , satisfying the previous initial conditions.  $\square$

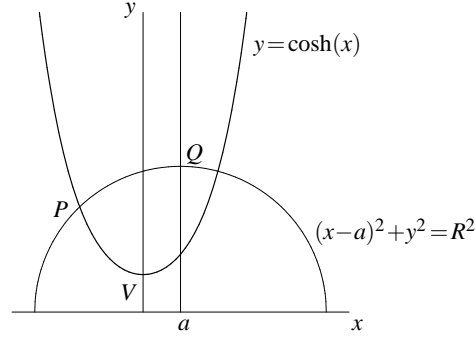


Figure 4: Vertical lines are asymptotic to the Poleni curve  $y = \cosh(x)$  in  $\mathbb{P}$

**THEOREM 2.** *Let  $V$  be the peak of the Poleni curve. The geodesic which is orthogonal to the Poleni curve at point  $V$  is an axis of geodesical symmetry of the curve and it is asymptotic to the curve as  $s \rightarrow \pm\infty$ . A geodesic is asymptotic to the Poleni curve as  $s \rightarrow +\infty$  or  $s \rightarrow -\infty$  if and only if it intersects transversally the Poleni curve at an unique point.*

*Proof.* Up to isometry, the Poleni curve  $\mathcal{P}$  is the graph  $y = \cosh(x)$ , with  $V = (0, 1)$ . The geodesic which is orthogonal to  $\mathcal{P}$  at  $V$  is the  $y$  axis. It is an axis of symmetry of  $\mathcal{P}$ . A vertical line is the only geodesic that can be asymptotic to  $\mathcal{P}$ . Let us prove that any vertical line  $x = a$  is asymptotic to  $\mathcal{P}$ . Let  $P = (s, \cosh(s))$  be a point of  $\mathcal{P}$ . The geodesic passing by  $P$  and orthogonal to the vertical line  $x = a$  intersects this line at point  $Q$  (see Figure 4). Its equation is

$$(x-a)^2 + y^2 = R^2, \quad \text{where} \quad R^2 = (s-a)^2 + \cosh^2(s).$$

Hence the distance from point  $P$  to the geodesic  $x = a$  is

$$d_{\mathbb{P}}(P, Q) = \left| \int_s^a \frac{\sqrt{1 + (f'(x))^2}}{f(x)} dx \right|, \quad f(x) = \sqrt{R^2 - (x-a)^2}.$$

Thus

$$d_{\mathbb{P}}(P, Q) = \operatorname{argtanh} \frac{|s-a|}{\sqrt{(s-a)^2 + \cosh^2(s)}}$$

tends to 0 as  $s \rightarrow \pm\infty$ . □

Figure 5 shows various images of the Poleni curve (23) under the hyperbolic rotations

$$z = x + iy \mapsto \frac{az + b}{-bz + a}, \quad a^2 + b^2 = 1,$$

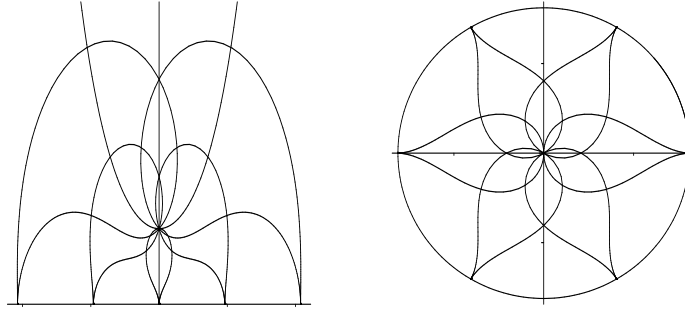


Figure 5: Isometric Poleni curves in  $\mathbb{P}$  and  $\mathbb{D}$

together with their corresponding curves in the unit disk  $\mathbb{D} = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$  via the transformation

$$z = x + iy \in \mathbb{P} \mapsto \frac{z - i}{z + i} = \frac{x^2 + y^2 - 1 - 2ix}{x^2 + (y + 1)^2} \in \mathbb{D}.$$

In particular, the isometric image of (23) under the isometry  $z \mapsto -\frac{1}{z}$  of  $\mathbb{P}$  is the curve.

$$x(s) = \frac{-s}{s^2 + \cosh^2(s)}, \quad y(s) = \frac{\cosh(s)}{s^2 + \cosh^2(s)}.$$

The geodesics which are asymptotic to this curve are the  $y$  axis and all the circles with center on the  $x$  axis and tangent to the  $y$ -axis. All these geodesics intersect the Poleni curve transversally at a unique point.

### 6. Jurdjevic's parametrization of non euclidean elastic curves

Jurdjevic gave the following parametrization (see [6], Formula (10) page 117) of an elastic curve on the sphere of geodesical curvature  $k(s)$ :

$$(24) \quad x = \pm \cos(\phi) \sqrt{1 - \frac{k^2}{M}}, \quad y = \sin(\phi) \sqrt{1 - \frac{k^2}{M}}, \quad z = \frac{k}{\sqrt{M}},$$

where  $\phi(s)$  is defined by

$$\phi' = \frac{\sqrt{MH_1}}{M - k^2}, \quad H_1(s) = H - \frac{1}{2}k^2(s).$$

The constants  $H$  and  $M$  in these formulas are given by (see [6], Proposition 4)

$$(u')^2 + u^3 - 4u^2(H - 1) + 4u(H^2 - M) = 0, \quad u(s) = k^2(s).$$

In the case  $k(s) = \frac{2}{\cosh(s)}$ , the function  $u(s) = k^2(s)$  is a solution of

$$(25) \quad (u')^2 + u^3 - 4u^2 = 0.$$

Hence  $H = 2$ ,  $M = H^2 = 4$  and

$$\phi'(s) = 2 \frac{2 - \frac{2}{\cosh^2(s)}}{4 - \frac{4}{\cosh^2(s)}} = 1.$$

Thus  $\phi(s) = s$ . On the other hand

$$\sqrt{1 - \frac{k^2}{M}} = \sqrt{1 - \frac{1}{\cosh^2(s)}} = |\tanh(s)|.$$

Consequently formula (24) becomes

$$(26) \quad x(s) = \pm \cos(s)|\tanh(s)|, \quad y(s) = \sin(s)|\tanh(s)|, \quad z(s) = \frac{1}{\cosh(s)}.$$

Actually, these formulas define two symmetrical Poleni curves on the sphere, compare with (20). Notice that there is no choice of the sign  $\pm$  in (26) which gives a complete Poleni curve, see Figure 6.

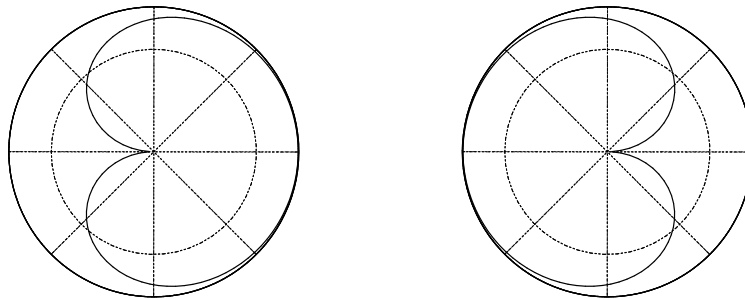


Figure 6: The curve (26) with sign  $-$  (left) and sign  $+$  (right), seen from the north pole

Jurdjevic gave also the parametrization of elastic curve on the hyperbolic space

$$\mathbb{H} = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 - z^2 = -1, z > 0\}.$$

He distinguishes three cases. The first case corresponds to  $M < 0$ , see [6], Formula (10), page 117. Notice that this formula contains a misprint:  $M$  should be replaced by

$|M|$ . The parametrization of the elastic curve is

$$(27) \quad x = \pm \cos(\phi) \sqrt{\frac{k^2}{-M} - 1}, \quad y = \sin(\phi) \sqrt{\frac{k^2}{-M} - 1}, \quad z = \frac{k}{\sqrt{-M}},$$

where  $\phi(s)$  is defined by

$$\phi' = \frac{\sqrt{-MH_1}}{M - k^2}.$$

The second case corresponds to  $M > 0$ , see [6], Formula (12) page 120. The elastic curve is parametrized by

$$(28) \quad x = -\frac{k}{\sqrt{M}}, \quad y = \cosh(\phi) \sqrt{1 + \frac{k^2}{M}}, \quad z = \sinh(\phi) \sqrt{1 + \frac{k^2}{M}},$$

where  $\phi(s)$  is defined by

$$\phi' = -\frac{\sqrt{MH_1}}{M + k^2}.$$

In the third case, corresponding to  $M = 0$ , the parametrization (which appears without number in page 122 of [6]) is

$$(29) \quad x = \frac{k^2\phi - k^2 + 1}{2k}, \quad y = k\phi, \quad z = \frac{k^2\phi + k^2 + 1}{2k},$$

where  $\phi(s)$  is defined by

$$\phi' = \frac{H_1}{k^2}.$$

In the hyperbolic case  $H_1(s) = H - k^2(s)/2$ , as in the spherical case, but constants  $H$  and  $M$  are given now by (see [6], Proposition 4)

$$(u')^2 + u^3 - 4u^2(H + 1) + 4u(H^2 - M) = 0.$$

The Poleni curve satisfies (25). Thus, it corresponds to  $H = 0$ ,  $M = H^2 = 0$ . In this case  $\phi' = -1/2$ . Thus  $\phi(s) = -s/2$ . Consequently formula (29) writes:

$$(30) \quad x(s) = \frac{s^2 + \cosh^2(s) - 4}{4 \cosh(s)}, \quad y(s) = \frac{-s}{\cosh(s)}, \quad z(s) = \frac{s^2 + \cosh^2(s) + 4}{4 \cosh(s)}.$$

The coordinates  $(x_1, y_1)$  in the Poincaré hyperbolic half space  $\mathbb{P}$  are linked to the coordinates  $(x, y, z)$  by the relations

$$x_1 = \frac{y}{x - z}, \quad y_1 = \frac{1}{z - x}.$$

Thus, the curve in  $\mathbb{P}$  corresponding to the curve (30) is

$$x_1 = \frac{s}{2}, \quad y_1(s) = \frac{\cosh(s)}{2}.$$

This curve is isometric to the Poleni curve (23) since the transformation  $x_1 = x/2$ ,  $y_1 = y/2$  is an isometry of  $\mathbb{P}$ .

### 7. Poleni curves of geodesic curvature $f(s) = \frac{2\alpha}{\cosh(\alpha s)}$

We analyze the spherical case first. We use cartesian coordinates. Here we have

$$H = 1 + \alpha^2, \quad M = H^2 = (1 + \alpha^2)^2.$$

Using Jurdjevic's parametrization (24), we obtain, after some lengthy computations and simplifications, the following formulas (the details will appear in a forthcoming paper)

PROPOSITION 3. *Up to isometry, the parametric equations of the Poleni curve on the sphere  $\mathbb{S}^2$  are*

$$x(s) = \frac{(1 - \alpha^2) \sin(s) + 2\alpha \cos(s) \tanh(\alpha s)}{1 + \alpha^2},$$

$$y(s) = \frac{(\alpha^2 - 1) \cos(s) + 2\alpha \sin(s) \tanh(\alpha s)}{1 + \alpha^2}, \quad z(s) = \frac{2\alpha}{(1 + \alpha^2) \cosh(\alpha s)}.$$

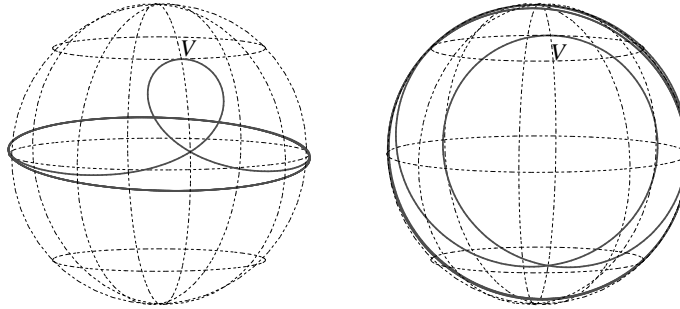


Figure 7: Poleni curves in the sphere with  $\alpha = 2$  (left) and  $\alpha = 0.5$  (right)

Figures 7 and 8 show the curves corresponding to  $\alpha = 2$  and  $\alpha = 1/2$ . From this parametrization we deduce the following result

THEOREM 3. *The plane which is orthogonal to the Poleni curve at its peak  $V$  is a plane of symmetry for the curve. The diameter contained in the plane which is orthogonal to the symmetry plane and makes the angle*

$$\beta(\alpha) = 2 \arctan(\alpha)$$

*with the vector  $OV$ , is asymptotic to the curve. The Poleni curve has an infinite number of self-intersection points of arc-lengths  $\pm s_n$  which are given by the solutions of equation*

$$s_n + \arctan \frac{2\alpha \tanh(\alpha s_n)}{1 - \alpha^2} = n\pi, \quad n \in \mathbb{N}.$$

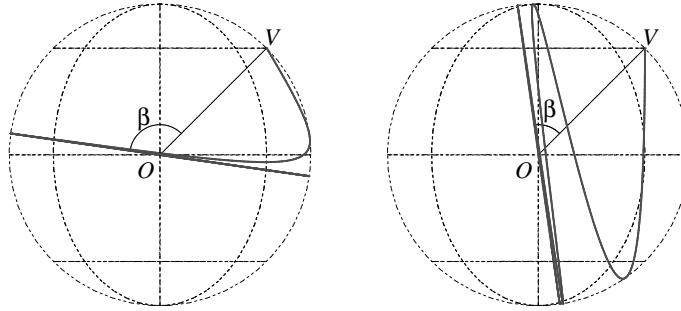


Figure 8: Orthogonal projection on the symmetry plane of the Poleni curve in the sphere with  $\alpha = 2$  (left) and  $\alpha = 0.5$  (right), showing the angle  $\beta = \beta(\alpha)$

We analyze now the hyperbolic case. We use cartesian coordinates on  $\mathbb{H}^2$ . Here we have

$$H = \alpha^2 - 1, \quad M = H^2 = (\alpha^2 - 1)^2.$$

Using Jurdjevic's parametrization (28) in the case  $\alpha^2 \neq 1$ , we obtain the following formulas (the details will appear in a forthcoming paper)

PROPOSITION 4. *Up to isometry, the parametric equations of the Poleni curve, with  $\alpha^2 \neq 1$ , on the hyperbolic space are*

$$x(s) = \frac{2\alpha}{|\alpha^2 - 1| \cosh(\alpha s)}, \quad y(s) = \frac{(1 + \alpha^2) \sinh(s) - 2\alpha \cosh(s) \tanh(\alpha s)}{|\alpha^2 - 1|},$$

$$z(s) = \frac{(1 + \alpha^2) \cosh(s) - 2\alpha \sinh(s) \tanh(\alpha s)}{|\alpha^2 - 1|}.$$

Notice that this family of elastic curves does not contain the Poleni curve (23) which corresponds to  $\alpha = 1$ . From the parametrization we deduce the following result.

THEOREM 4. *For all  $\alpha \neq \pm 1$ , the geodesic which is orthogonal to the Poleni curve at its peak, is an axis of symmetry of the curve and there exist a unique geodesic, which is asymptotic to the curve as  $s \rightarrow \pm\infty$ . When  $|\alpha| < 1$  the Poleni curve has no self-intersection. When  $|\alpha| > 1$  the Poleni curve has a unique self-intersection point. The arc-lengths  $\pm s_0$  of the self-intersecting point are given by the solution of equation*

$$\tanh(s_0) = \frac{2\alpha}{1 + \alpha^2} \tanh(\alpha s_0).$$

The Poleni curves corresponding to  $\alpha$  and  $-\alpha$  respectively, are isometric. Recall that in Theorem 2 we proved that in the case  $\alpha = 1$ , all the geodesics which intersect the Poleni curve transversally at a unique point are asymptotic to the curve as  $s \rightarrow \pm\infty$ .

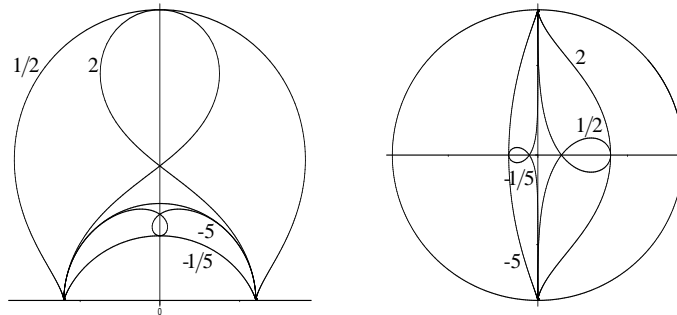


Figure 9: Poleni curves in  $\mathbb{P}$  and  $\mathbb{D}$ , corresponding to the curve given by the parametrization of Proposition 4, with  $\alpha = 2$ ,  $\alpha = 1/2$ ,  $\alpha = -5$  and  $\alpha = -1/5$

Figure 9 shows the curves in  $\mathbb{P}$  and  $\mathbb{D}$  corresponding to the Poleni curve of  $\mathbb{H}^2$  defined by the parametrization given in Proposition 4, with various values of  $\alpha$ . In  $\mathbb{P}$ , the peak  $V$  of the curve and its tangent vector  $t_V$  are

$$V = \left( 0, \left| \frac{\alpha - 1}{\alpha + 1} \right| \right), \quad t_V = \left( \frac{\alpha - 1}{\alpha + 1}, 0 \right),$$

thus  $V$  tends to the “boundary” as  $\alpha \rightarrow \pm 1$ . In  $\mathbb{P}$  the asymptotic geodesic is the circle of radius 1 centered at the origin. In  $\mathbb{D}$  the asymptotic geodesic is the vertical axis.

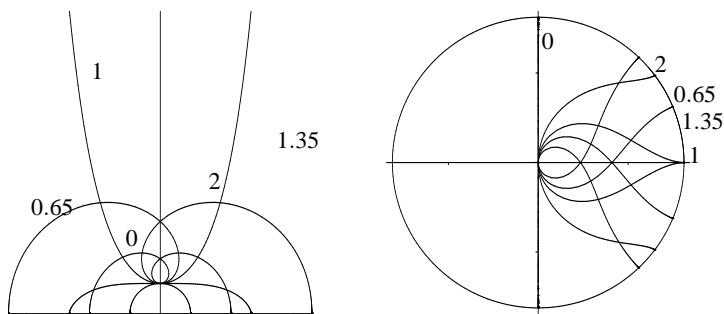


Figure 10: Poleni curves in  $\mathbb{P}$  and  $\mathbb{D}$ , corresponding to  $\alpha = 0$  (a geodesic),  $\alpha = 0.65$ ,  $\alpha = 1$  (the Poleni curve),  $\alpha = 1.35$  and  $\alpha = 2$

Figure 10 shows various Poleni curves in the hyperbolic half-space  $\mathbb{P}$ , and the



corresponding curves in the disk  $\mathbb{D}$ , where the initial condition is fixed at

$$x_1(0) = 0, x_1'(0) = 1, y_1(0) = 0, y_1'(0) = 0,$$

where  $(x_1, y_1)$  are the coordinates in  $\mathbb{P}$ . The parametrization of these curves is

$$x_1(s) = \frac{(\alpha + 1)(\alpha^2 + 1) \frac{\sinh[(\alpha-1)s]}{\alpha-1} - (\alpha^2 - 1) \cosh(s) \sinh(\alpha s)}{2\alpha (\cosh[(\alpha-1)s] + 1) + (\alpha - 1)^2 \cosh(s) \cosh(\alpha s)},$$

$$y_1(s) = \frac{(\alpha + 1)^2 \cosh(\alpha s)}{2\alpha (\cosh[(\alpha-1)s] + 1) + (\alpha - 1)^2 \cosh(s) \cosh(\alpha s)}.$$

This parametrization reduces to the Poleni curve (23) when  $\alpha = 1$ . It shows how the Poleni curve (23) is the limiting case between Poleni curves with self-intersection and those without self-intersection.

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