Rend. Sem. Mat. Univ. Pol. Torino
Vol. 67, 1 (2009), 41 – 53

P. Bonacini

HILBERT FUNCTIONS OF DECREASING TYPE IN POSITIVE CHARACTERISTIC

Abstract. Let C be an integral curve in $\mathbb{P}^3_k$, with k an algebraically closed field. In the main result of this paper we prove that the Hilbert function of its general plane section $C\cap H$ is of decreasing type, extending to any characteristic a result proved in the case char $k = 0$ by Maggioni and Ragusa. Moreover, the proof given in this paper does not depend on the uniform position property. We also prove in any characteristic that every 0-dimensional differentiable O-sequence of decreasing type is the Hilbert function of a general plane section of an integral smooth ACM curve in $\mathbb{P}^3$. Starting from these results other properties of the Hilbert functions of $C\cap H$ and C are extended to any characteristic.

1. Introduction

Let C be an integral curve in $\mathbb{P}^3_k$, where k is an algebraically closed field of characteristic 0, and let $C\cap H$ be its general plane section. Maggioni and Ragusa in [12] show that the Hilbert function of a general plane section of C is of decreasing type and that every 0-dimensional differentiable O-sequence H of decreasing type is the Hilbert function of a general plane section of an integral smooth arithmetically Cohen-Macaulay (ACM) curve in $\mathbb{P}^3$.

In this paper we extend Maggioni and Ragusa's results to arbirtrary characteristic. More precisely, in Theorem 3 we prove that the Hilbert function of the general plane section of an integral space curve defined over an algebraically closed field of any characteristic is of decreasing type. As consequences of Theorem 3 we extend to any characteristic some bounds for the graded Betti numbers and for the number of minimal generators of the ideal of an integral ACM space curve (Theorem 5). Using Theorem 3 it is also possible to prove in arbitrary characteristic that an integral curve with the Hilbert function of a complete intersection is a complete intersection (Theorem 6).

In Theorem 4 we prove in any characteristic that every 0-dimensional differentiable O-sequence H of decreasing type is the Hilbert function of a general plane section of an integral smooth ACM curve in $\mathbb{P}^3$. This correspondence between the general plane sections of integral curves and the 0-dimensional differentiable O-sequences of decreasing type gives us the possibility of proving the analogue of Sauer's result [15] (Theorem 7) on smoothable ACM curves in $\mathbb{P}^3$ over algebraically closed fields of any characteristic, in which we characterize the degree matrix of the generic plane section of an integral space curve.

If char $k = 0$, the points of a general plane section $C\cap H$ of an integral space curve C lie in uniform position, which means that all the subsets of $C\cap H$ of the same cardinality have the same Hilbert function [7]. This is the basic fact that is used by Maggioni and Ragusa in [12] in order to prove their results and that is commonly used in characteristic 0 for studying properties of the general plane section of an integral space curve. In positive characteristic the Uniform Position Property may not hold (see for example [14, Example 1.2]). An open problem in this sense is to give a characterization of space curves satisfying the Uniform Postion Property in positive characteristic. Some results in this direction have been proved by Rathmann in [14].

Note that Gruson and Peskine in [6, Lemme 3.2] show that, if $\operatorname{char} k = 0$, the numerical character of the generic plane section of C is “sans lacune” a condition equivalent for the Hilbert function of the generic plane section of C to be of decreasing type (see [3, Proposition 1.2]). It seems that the proof of Gruson and Peskine can be extended also to the case of positive characteristic. However the technique used here seems more suitable for proving other results still unknown in positive characteristic.

2. Preliminaries and notation

Let us denote by X a finite set of points in $\mathbb{P}^2_k$, where k is an algebraically closed field. Let us recall that X has the *Uniform Position Property* (briefly UPP) if, for any $n \leq \deg X$, all the subsets of n points have the same Hilbert function. If this happens, the Hilbert function of a set $Z \subset X$ of n points is the following truncated function:

$$H(Z,i) = \min\{H(X,i),n\} \ \forall i.$$

In fact, in [2] it is shown the existence of a subscheme of X with the truncated Hilbert function and so every subscheme of n points of X has that Hilbert function.

Let us consider the first difference of the Hilbert function of X:

$$\Delta H(X,i) = H(X,i) - H(X,i-1).$$

It is known [2] that there exist $a_1 \leq a_2 \leq t$ such that:

$$\Delta H(X,i) = \begin{cases} i+1 & \text{for } i = 0,\ldots,a_1-1 \\ a_1 & \text{for } i = a_1,\ldots,a_2-1 \\ < a_1 & \text{for } i = a_2 \\ \text{non increasing} & \text{for } i = a_2+1,\ldots,t \\ 0 & \text{for } i > t. \end{cases}$$

DEFINITION 1. *We say that X has* the Hilbert function of decreasing type *if for $a_2 \leq i < j < t$ we have $\Delta H(X,i) > \Delta H(X,j)$.*

Maggioni and Ragusa (see [12, Corollary 2]) show this fact:

COROLLARY 1. *If H is the Hilbert function of a set of points of $\mathbb{P}^2_k$ with the UPP, then H is of decreasing type.*

An important result about the UPP is the following:

THEOREM 1 ([7]). *If $C \subset \mathbb{P}^3_k$ is an integral curve and k is an algebraically closed field of characteristic 0, then the generic plane section of C has the UPP.*

If char $k = p > 0$, then in general $C \cap H$ may not have the UPP, even in the case of complete intersections, as the following example shows:

EXAMPLE 1 (see [14, Example 1.2]). Let C be the curve defined as the complete intersection of ${x_0}^q - x_1{x_3}^{q-1} = {x_1}^q - x_2{x_3}^{q-1} = 0$, where $q = p^f$, for some $f > 0$. On the open affine set defined by $x_3 \neq 0$ C is described by:

$$x_0 = t,\ x_1 = t^q,\ x_2 = t^{q^2}$$

so that C is rational and integral. However the general plane section of C looks like a 2-dimensional vector space over a field with q elements. Indeed, if $P_0, P_1, P_2 \in C$ are linearly independent and H is the plane containing the three points, then the points of $C \cap H$ are given by:

$$P = P_0 + \lambda_1(P_1 - P_0) + \lambda_2(P_2 - P_0)$$

where $\lambda_i^q = \lambda_i$ for $i = 1,2$. Note also that given two points $P_0, P_1 \in C$ the line P_0P_1 meets C in the points given by:

$$P = P_0 + \mu(P_1 - P_0)$$

where $\mu^q = \mu$. So in every generic plane section of C there are, for $q \geq 4$, at least 3 collinear points and 3 linearly independent. It means that the generic plane section of C does not have the UPP. However the Hilbert function of the generic plane section of C is of decreasing type, because it is a complete intersection and the first difference of $H(C \cap H, i)$ looks like:

$$\Delta H(C \cap H, i) = \begin{cases} i+1 & \text{for } i = 0, \ldots, q-1 \\ 2q - i - 1 & \text{for } i = q, \ldots, 2q-2 \\ 0 & \text{for } i > 2q-2. \end{cases}$$

In this example we have shown the existence of an integral curve, in a projective space over a field of positive characteristic, with a generic plane section that does not have the UPP, but whose Hilbert function is of decreasing type. In the next paragraph we will show that the general plane section of an integral curve has always the Hilbert function of decreasing type.

Let us fix some notation.

DEFINITION 2. *If $X \subset \mathbb{P}^2$ is a set of n distinct point, we denote by $\mathscr{I}_X$ the ideal sheaf of X. If $S_l(X)$ is the linear system associated to $H^0(\mathscr{I}_X(l))$, $l \in \mathbb{Z}$, we denote by $Z_l(X)$ the base locus of $S_l(X)$, that is:*

$$Z_l(X) = \{P \in \mathbb{P}^2 \mid f(P) = 0\ \forall f \in S_l(X)\} = V(S_l).$$

Now we put:

$$c_l(X) = \Delta H(X, l)\ \forall l \in \mathbb{N}.$$

We will write c_l instead of $c_l(X)$ if there is no confusion on the zero-dimensional scheme we are considering.

The following theorem, which is true in any characteristic, will be useful in the proof of the main result of the paper:

THEOREM 2 ([10, Theorem 2.9],[1, Corollary 4.2]). *Let X be a set of n points in $\mathbb{P}^2$. Suppose that the first differences of the Hilbert function of X are such that $c_{s-1} = c_s > 0$ for some integer s. Then we have $\dim Z_s(X) > 0$ and, if D is its component of positive dimension, $\deg D = c_s$. Furthermore:*

$$\deg(X \cap D) = c_s(s+1) - \binom{c_s}{2} + \sum_{i=s+1}^{t} c_i.$$

Another result that will be useful in the next section is the following:

LEMMA 1 ([5, Corollaire 3.3.5]). *Let X and Y be two locally noetherian schemes and $f : X \to Y$ be a flat morphism. If Y is reduced in the points of $f(X)$ and if $f^{-1}(y)$ is a reduced $k(y)$-scheme for every $y \in f(X)$, then X is reduced.*

3. The main result

In this section we consider an integral curve $C \subset \mathbb{P}^3_k$, where k is an algebraically closed field of any characteristic. Let $\{\underline{x}_i\}$ be a system of coordinates over $\mathbb{P}^3$ and $\{\underline{t}_j\}$ a system of coordinates over the dual space $\mathbb{P}^{3\vee}$. Denote by $\mathscr{I}_C$ the ideal sheaf of C in $\mathbb{P}^3$ and let us consider the incidence variety $M \subset \mathbb{P}^{3\vee} \times \mathbb{P}^3$ determined by:

$$h := \sum_{i=0}^{3} t_i x_i = 0. \tag{1}$$

We have the two projections:

$$p\colon M \subset \mathbb{P}^{3\vee} \times \mathbb{P}^3 \to \mathbb{P}^3$$

and

$$g\colon M \subset \mathbb{P}^{3\vee} \times \mathbb{P}^3 \to \mathbb{P}^{3\vee}.$$

Now consider the following module over $\mathscr{O}_M$:

$$\mathscr{I}(m,n) = g^\star\left(\mathscr{O}_{\mathbb{P}^{3\vee}}(m)\right) \otimes_{\mathscr{O}_M} p^\star\left(\mathscr{I}_C(n)\right)$$

and let us make the following position:

$$\mathscr{I} = \mathscr{I}(0,0).$$

PROPOSITION 1. *If:*

$$T = p^{-1}(C) \subset M$$

and $\mathscr{I}_T$ is the ideal sheaf of T in $\mathscr{O}_M$, then T is integral and $\mathscr{I} = \mathscr{I}_T$.

Proof. Note that the fibres of the projection $T \to C$ are all integral and equidimensional of dimension 2, being isomorphic to a projective plane. So T is irreducible and $T \to C$ is flat, by [8, Theorem 9.9] (note that $T \subset \mathbb{P}^{3^\vee} \times C$). By Lemma 1 T is reduced.

Now note that:

$$g^\star\left(\mathscr{O}_{\mathbb{P}^{3\vee}}(m)\right) = \mathscr{O}_M(m,0),$$
$$p^\star\left(\mathscr{O}_{\mathbb{P}^3}(n)\right) = \mathscr{O}_M(0,n),$$

so that:

$$p^\star\left(\mathscr{I}_C(n)\right) = p^\star\left(\mathscr{I}_C \otimes_{\mathscr{O}_{\mathbb{P}^3}} \mathscr{O}_{\mathbb{P}^3}(n)\right) = p^\star\left(\mathscr{I}_C\right) \otimes_{\mathscr{O}_M} \mathscr{O}_M(0,n).$$

From this it follows easily that:

$$\mathscr{I}(m,n) = g^\star\left(\mathscr{O}_{\mathbb{P}^{3\vee}}(m)\right) \otimes_{\mathscr{O}_M} p^\star\left(\mathscr{I}_C(n)\right) = \mathscr{O}_M(m,n) \otimes_{\mathscr{O}_M} p^\star\left(\mathscr{I}_C\right)$$

for any $m,n \in \mathbb{Z}$. Now we can observe that p is a smooth morphism, so that it is flat and $\mathscr{O}_M$ is flat over $\mathscr{O}_{\mathbb{P}^3}$. This implies that the following sequence is exact:

$$0 \to p^\star \mathscr{I}_C \to p^\star \mathscr{O}_{\mathbb{P}^3} \to p^\star j_\star \mathscr{O}_C \to 0 \tag{2}$$

where $j : C \hookrightarrow \mathbb{P}^3$. Considering the inclusion $i : T \hookrightarrow \mathbb{P}^3$ and the commutative diagram:

$$\begin{array}{ccc} T & \xrightarrow{p_T} & C \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle j} \\ M & \xrightarrow[p]{} & \mathbb{P}^3 \end{array}$$

by [8, Ch.III, Proposition 9.3] we see that:

$$p^\star j_\star \mathscr{O}_C = i_\star {p_T}^\star \mathscr{O}_C = i_\star \mathscr{O}_T.$$

So from (2), being $p^\star \mathscr{O}_{\mathbb{P}^3} = \mathscr{O}_M$, we get $p^\star \mathscr{I}_C = \mathscr{I}_T$. □

An important tool for the proof of the main theorem of this paper is the following:

PROPOSITION 2. *Let $H \subset \mathbb{P}^3$ be a generic plane for C and let $P_H \in \mathbb{P}^{3^\vee}$ be the corresponding point in the dual projective space. Then for every $n \in \mathbb{N}$ the following evaluation map is surjective:*

$$H^0\left(\mathscr{I}(\alpha,n)\right) \to H^0\left(\mathscr{I}(\alpha,n)|_{g^{-1}(P_H)}\right) \to 0 \;\forall \alpha \gg 0$$

and:

$$H^0\left(\mathscr{I}(\alpha,n)|_{g^{-1}(P_H)}\right) = H^0\left(\mathscr{I}_{C\cap H|H}(n)\right) \forall \alpha.$$

Proof. First of all note that by the projection formula ([8], II, Ex.5.1(d)) $\forall \alpha \in \mathbb{N}$:

$$g_\star \mathscr{I}(\alpha,n) \cong \mathscr{O}_{\mathbb{P}^{3\vee}}(\alpha) \otimes_{\mathscr{O}_{\mathbb{P}^{3\vee}}} g_\star p^\star(\mathscr{I}_C(n)).$$

For each $\alpha \in \mathbb{N}$ $g_\star \mathscr{I}(\alpha,n)$ is a coherent sheaf over $\mathbb{P}^{3\vee}$ (see [8, Ch. II, Proposition 5.8 and Corollary 5.20]) and so for some $\alpha \gg 0$ $g_\star \mathscr{I}(\alpha,n)$ is generated by its global sections. So, considering the stalk in the point P_H, we find the following exact sequence:

$$H^0 g_\star \mathscr{I}(\alpha,n) \to (g_\star \mathscr{I}(\alpha,n))_{P_H} \to 0.$$

Since the following map is surjective:

$$(g_\star \mathscr{I}(\alpha,n))_{P_H} \to g_\star \mathscr{I}(\alpha,n)|_{P_H} \to 0,$$

combining the two maps we get the exact sequence:

$$H^0 g_\star \mathscr{I}(\alpha,n) \to g_\star \mathscr{I}(\alpha,n)|_{P_H} \to 0. \tag{3}$$

Now we compute $g_\star \mathscr{I}(\alpha,n)|_{P_H}$.

Take an affine open subset $U \subset \mathbb{P}^{3\vee}$ such that $P_H \in U$. Then the inclusion $P_H \hookrightarrow \mathbb{P}^{3\vee}$ can be factored as:

$$P_H \overset{j}{\hookrightarrow} U \overset{i}{\hookrightarrow} \mathbb{P}^{3\vee}.$$

So we want to compute:

$$(i \circ j)^\star [g_\star(\mathscr{I}(\alpha,n))] = j^\star [i^\star g_\star(\mathscr{I}(\alpha,n))]$$

where:

$$j_\star j^\star [i^\star g_\star \mathscr{I}(\alpha,n)] = i^\star g_\star \mathscr{I}(\alpha,n) \otimes_{\mathscr{O}_U} j_\star k(P_H).$$

From the commutative diagram:

$$\begin{array}{ccc} g^{-1}(U) & \xrightarrow{i'} & M \\ \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle g} \\ U & \xrightarrow[i]{} & \mathbb{P}^{3\vee} \end{array}$$

and using [8, Ch. III, Proposition 9.3] we get:

$$i^\star g_\star \mathscr{I}(\alpha,n) \cong g'_\star i'^\star \mathscr{I}(\alpha,n).$$

So:

$$j_\star j^\star [i^\star g_\star \mathscr{I}(\alpha,n)] \cong g'_\star i'^\star \mathscr{I}(\alpha,n) \otimes_{\mathscr{O}_U} j_\star k(P_H)$$

from which it follows that:

$$\begin{aligned} g_\star\mathscr{I}(\alpha,n)|_{P_H} &= H^0\left(j^\star\left(i^\star g_\star\mathscr{I}(\alpha,n)\right)\right) \\ &= H^0\left(j_\star j^\star\left(i^\star g_\star\mathscr{I}(\alpha,n)\right)\right) = H^0\left({g'}_\star {i'}^\star\mathscr{I}(\alpha,n)\otimes_{\mathscr{O}_U} j_\star k(P_H)\right) \end{aligned}$$

where the second equality comes from the definition of the direct image sheaf. Now we can apply [8, Ch. III, Corollary 9.4], observing that ${g'}^{-1}(P_H) = g^{-1}(P_H)$, and we get the equality:

$$H^0\left({g'}_\star {i'}^\star\mathscr{I}(\alpha,s)\otimes_{\mathscr{O}_U} j_\star k(P_H)\right) = H^0\left({j'}^\star {i'}^\star\mathscr{I}(\alpha,n)\right)$$

where $j' : g^{-1}(P_H) \hookrightarrow g^{-1}(U)$. But $i'\circ j' = j_H : g^{-1}(P_H) \hookrightarrow M$ and so:

$$g_\star\mathscr{I}(\alpha,n)|_{P_H} = H^0\left({j_H}^\star\mathscr{I}(\alpha,n)\right).$$

Combining this equality with (3) we get:

$$H^0\left(g_\star\mathscr{I}(\alpha,n)\right) \to H^0\left({j_H}^\star\mathscr{I}(\alpha,n)\right) \to 0.$$

Since:

$$H^0 g_\star\mathscr{I}(\alpha,n) = H^0\mathscr{I}(\alpha,n)$$

we get the surjective map:

(4) $$H^0\mathscr{I}(\alpha,n) \xrightarrow{\phi} H^0 {j_H}^\star\mathscr{I}(\alpha,n) \to 0$$

where ϕ is the map determined by the morphism of sheaves:

$$\mathscr{I}(\alpha,n) \xrightarrow{\phi} {j_H}^\star\mathscr{I}(\alpha,n) = \mathscr{I}(\alpha,n)|_{g^{-1}(P_H)}.$$

Note now that the morphism $g\circ j_H : g^{-1}(P_H) \to \mathbb{P}^{3\vee}$ can be factored in this way:

$$g^{-1}(P_H) \to P_H \to \mathbb{P}^{3\vee}$$

and, since $\mathscr{O}_{\mathbb{P}^{3\vee}}(\alpha)|_{P_H} = \mathscr{O}_{P_H}$, we get:

$$(g\circ j_H)^\star\left(\mathscr{O}_{\mathbb{P}^{3\vee}}(\alpha)\right) = \mathscr{O}_{g^{-1}(P_H)}.$$

So:

$$\begin{aligned} H^0\left({j_H}^\star\mathscr{I}(\alpha,n)\right) &= H^0\left((g\circ j_H)^\star\left(\mathscr{O}_{\mathbb{P}^{3\vee}}\right)\otimes_{\mathscr{O}_{g^{-1}(P_H)}}(p\circ j_H)^\star\left(\mathscr{I}_C(n)\right)\right) \\ &= H^0\left(\mathscr{O}_{g^{-1}(P_H)}\otimes_{\mathscr{O}_{g^{-1}(P_H)}}(p\circ j_H)^\star\left(\mathscr{I}_C(n)\right)\right) \\ &= H^0\left((p\circ j_H)^\star\left(\mathscr{I}_C(n)\right)\right) = H^0\mathscr{I}_{C\cap H|H}(n). \end{aligned}$$

From (4) we get the following surjective map:

$$H^0\mathscr{I}(\alpha,n) \to H^0\mathscr{I}_{C\cap H|H}(n) \to 0.$$

□

THEOREM 3. *Let C be a reduced and irreducible curve in a projective space $\mathbb{P}^3$ defined over an algebraically closed field of any characteristic. Then the Hilbert function of the generic plane section of C is of decreasing type.*

Proof. Take a generic plane H and consider the Hilbert function of $C \cap H$. Obviously if $c_{s-1} > c_s \; \forall s = a_1 + 1, \ldots, t$, there is nothing to prove. So, we can suppose that there exists $s \in \{a_1 + 1, \ldots, t\}$ such that $c_{s-1} = c_s > 0$. In this case we apply Theorem 2 and we can say that the curves in H of degree s containing $C \cap H$ have a common factor of degree c_s.

We will prove that $c_s = a_1$ and this will give us the statement.

First note that by the Künneth formula (see [13, Ch.VI, Corollary 8.13]) we get:

$$H^1\left(\mathscr{O}_{\mathbb{P}^{3\vee} \times \mathbb{P}^3}(m,n)\right) = 0 \; \forall m,n.$$

This implies that we have the following surjective map:

$$H^0\left(\mathscr{O}_{\mathbb{P}^{3\vee} \times \mathbb{P}^3}(m,n)\right) \to H^0\left(\mathscr{O}_M(m,n)\right) \to 0 \; \forall m,n \geq 0.$$

This means that any element in $H^0(\mathscr{O}_M(m,n))$ can be lifted to a homogeneous form in $H^0(\mathscr{O}_{\mathbb{P}^{3\vee} \times \mathbb{P}^3}(m,n))$ for any $m,n \geq 0$.

By Proposition 2 we get the surjective map:

$$H^0\left(M, \mathscr{I}(\alpha, s)\right) \to H^0\left(H, \mathscr{I}_{C \cap H|H}(s)\right) \to 0.$$

In this way, taken any nonzero element in $H^0(\mathscr{I}_{C \cap H|H}(s))$, we find a hypersurface $V \subset M$ with a fibre with respect to the projection over $\mathbb{P}^{3\vee}$ that is a curve of degree s containing a generic plane section of C. Note that the projection of V over $\mathbb{P}^{3\vee}$ is dominant, because otherwise each fibre would have dimension at least 2, while we know that there exists one of dimension 1. So, taken a generic $P_L \in \mathbb{P}^{3\vee}$, the image of V in $H^0(\mathscr{I}_{C \cap L|L}(s))$ is nonzero and is exactly the fibre over the point P_L. And so:

$$p\left(g^{-1}(P_L) \cap V\right) \supset C \cap L$$

for L generic plane.

In particular, we consider a basis of the vector space $H^0(\mathscr{I}_{C \cap H|H}(s))$, whose associate plane curves are $D_1, \ldots, D_r$, and in correspondence we find $S_1, \ldots, S_r$ hypersurfaces of the incidence variety M. By Proposition 1 we get that $T = p^{-1}(C) \subset S_i$ for each i. Note that, since $s > a_1$, $r > 1$.

Consider now the scheme-theoretic intersection $S = S_1 \cap \cdots \cap S_r$ and the projection:

$$g_S : S \to \mathbb{P}^{3\vee}$$

restriction of g to S. The fibre ${g_S}^{-1}(P_H) \supset D$, where D is the plane curve of degree c_s associated to the common factor of $H^0(\mathscr{I}_{C \cap H|H}(s))$. Indeed:

$${g_S}^{-1}(P_H) = \bigcap_{i=1}^{r} {g_i}^{-1}(P_H) = D_1 \cap \cdots \cap D_r \supset D.$$

So ${g_S}^{-1}(P_H)$ is the plane curve D plus some possible additional points on the plane. Besides for a generic plane L and for the corresponding point $P_L \in \mathbb{P}^{3\vee}$,

$$g_S^{-1}(P_L) = g_{S_1}^{-1}(P_L) \cap \cdots \cap g_{S_r}^{-1}(P_L) \supset D_L$$

where D_L is the plane curve on L of degree c_s associated to the common factor of $H^0(\mathscr{I}_{C\cap L|L}(s))$. So g_S is dominant and for any $P_L \in \mathbb{P}^{3\vee}$ generic the fibre $g_S^{-1}(P_L)$ is 1-dimensional, since it contains a plane curve and is contained in plane curves, from which it follows that:

$$\dim S = 4$$

even if S might not be equidimensional.

Since S is the intersection of some 4-dimensional hypersurfaces in M, its 4-dimensional part is a common component S' of $S_1, \dots, S_r$. So for each i:

(5) $$S_i = S' \cup V_i$$

where V_i is equidimensional of dimension 4. Considering now the projections $g_{S'} : S' \to \mathbb{P}^{3\vee}$ and $g_{V_i} : V_i \to \mathbb{P}^{3\vee}$ for each i, since S' and V_i are equidimensional of dimension 4, we have that all the components of $g_{S'}^{-1}(P_H)$ and $g_{V_i}^{-1}(P_H)$ have dimension at least 1. Since for each i $g_{S_i}^{-1}(P_H)$ is a curve on H, that is equidimensional of dimension 1, from (5) we get that the fibres $g_{S'}^{-1}(P_H)$ and $g_{V_i}^{-1}(P_H)$ must be curves on H, that is they are equidimensional of dimension 1. Since $S' \subset S$, $g_{S'}^{-1}(P_H) \subset g_S^{-1}(P_H)$ and so $g_{S'}^{-1}(P_H)$ is a part of D, that is the plane curve of degree c_s associated to the common factor of $H^0(\mathscr{I}_{C\cap H|H}(s))$.

Since:

$$T \subset S = S_1 \cap \cdots \cap S_r \text{ and } S_i = S' \cup V_i \ \forall i$$

and T is integral, then:

$$\text{either } T \subset S' \text{ or } T \subset \bigcap_{i=1}^{r} V_i.$$

If $T \subset \bigcap_{i=1}^{r} V_i$, then $T \subset V_i$, $\forall i = 1, \dots, r$. From this is follows that $C\cap H \subset g_{V_i}^{-1}(P_H)\ \forall i = 1, \dots, r$. Note some of the $g_{V_i}^{-1}(P_H)$ may have $g_{S'}^{-1}(P_H)$ as a component, but not all of them. In fact, otherwise we find that any element in $H^0(\mathscr{I}_{C\cap H|H}(s))$ is not reduced and it leads us to a contradiction, because $C \cap H$ is reduced. So there exists an $\bar{i}$ such that $C \cap H \subset g_{V_{\bar{i}}}^{-1}(P_H)$, where $g_{V_{\bar{i}}}^{-1}(P_H)$ is a curve of degree less than s not containing $g_{S'}^{-1}(P_H)$. Since $g_{S'}^{-1}(P_H)$ is a component of D, then $g_{V_{\bar{i}}}^{-1}(P_H)$ contains $C \cap H$ and does not contain D. It follows that there exist curves of degree s containing $C \cap H$ and not containing D. This contradicts the choice of s.

So it must be $T \subset S'$. So $C \cap H \subset g_{S'}^{-1}(P_H)$ and, since $g_{S'}^{-1}(P_H)$ is equidimensional of dimension 1 and is a part of D, the only possibility is that $g_{S'}^{-1}(P_H) = D$, because any curve of degree $\leq s$ must contain D and so:

$$C \cap H \subset D.$$

So obviously D is the minimal curve containing the generic plane section of C. In this case $a_1 = c_s$ and so:

$$\Delta(C \cap H, c_s) = c_s$$

while by definition

$$\Delta(C \cap H, s) = c_s$$

where $s \geq c_s$. Since at the beginning of the proof we chose any s such that $c_s = c_{s-1}$, then the Hilbert function is of decreasing type. □

4. Further results

In this section we show some results that are well known in characteristic 0 and that we prove in any characteristic, most of them as consequences of Theorem 3.

Let us first recall that a 0-dimensional O-sequence $H = \{h_i\}$ is a sequence of positive integers such that there exists a set of points of $\mathbb{P}^2$ whose Hilbert function is H (see [16] and [2] for more details). Then it is possible to extend to arbitrary characteristic the following result:

THEOREM 4. *If H is a 0-dimensional O-sequence of decreasing type, then there exists an irreducible smooth ACM curve of $\mathbb{P}^3$ whose generic plane section has H as its Hilbert function.*

Proof. We can proceed as in [12, Theorem 4], just observing that as a Bertini type theorem we have to use [4, Corollary 2.11(ii)]. □

Now we must fix some notation.

If $X \subset \mathbb{P}^2$ is a reduced 0-dimensional scheme, then the ideal sheaf $\mathscr{I}_X$ of X has a minimal free resolution of the type (because X is ACM of codimension 2):

$$0 \to \bigoplus_{i=1}^{m-1} \mathscr{O}_{\mathbb{P}^2}(-b_i) \to \bigoplus_{i=1}^{m} \mathscr{O}_{\mathbb{P}^2}(-a_i) \to \mathscr{I}_X \to 0. \tag{6}$$

In this resolution $a_1 \leq a_2 \leq \cdots \leq a_m$ are the degrees of a minimal set of generators of I_X and $b_1 \leq b_2 \leq \cdots \leq b_{m-1}$ the degrees of a minimal set of generators for the syzygy module.

Given the Hilbert function $H(X,i)$ of X we use the following notation:

$$h_i = H(X,i)$$

$$c_i = \Delta H(X,i) = h_i - h_{i-1}$$

$$d_i = \Delta^2 H(X,i) = c_i - c_{i-1}$$

$$e_i = \Delta^3 H(X,i) = d_i - d_{i-1}.$$

We call α_j (resp. β_j) the number of the a_i (resp. b_i) equal to j. From (6) it follows that there exist at least $-e_i$ generators of degree i, if $e_i < 0$, and at least e_i syzygies of degree i, if $e_i > 0$.

Now we can prove the following result in arbitrary characteristic:

THEOREM 5. *Let $C \subset \mathbb{P}^3$ be an integral ACM curve and X a generic plane section of C with Hilbert function $H = \{h_i\}$. Then the numbers α_i of elements of degree i in any minimal system of generators of I_C satisfy:*

$$\alpha_{a_1} = -e_{a_1},\ \alpha_{a_2} = -e_{a_2} \textit{ and } \max\{0, -e_i\} \leq \alpha_i \leq -d_i - 1 \textit{ for } i > a_2.$$

In particular, denoting by $\nu(I_C)$ the number of elements of a minimal set of generators of $\mathscr{I}_C$, we have:

$$\nu(I_C) \leq a_1 + a_2 - t.$$

Proof. For the first part of the statement note that by Theorem 3 $H(X,i)$ is of decreasing type and, since C is ACM, $H(X,i) = \Delta H(C,i)$ for each i. So $X \subseteq CI(a_1,a_2)$ and we can apply [11, Proposition 1.4].

The second part of the statement now follows as in [11, Corollary 1.6]. □

Following the notation of [12], let us denote by $CI(a_1,a_2)$ a complete intersection of two plane curves of degrees a_1 and a_2. We get:

THEOREM 6. *Let $C \subset \mathbb{P}^3$ be an integral curve such that $H(C,i) = H(CI(a,b),i)$ for each i. Then C is a $CI(a,b)$.*

Proof. We follow the proof of [11, Proposition 3.2]. By hypothesis $\deg C = ab$ and, if X is a generic plane section of C, $\deg X = \deg C$. Since $\Delta H(C,i) \geq H(X,i)$ for each i, if $\Delta H(C,i) > H(X,i)$ for some i, then $\deg X = \deg C < ab$, because $H(X,i)$ is of decreasing type by Theorem 3. So C is an ACM curve and we can apply Theorem 5. In fact in our case $t = a_1 + a_2 - 2$, so that it must be $\nu(I_C) = 2$, which means that C is a $CI(a,b)$. □

Another consequence of our main result is the following:

PROPOSITION 3. *Let $X \subset \mathbb{P}^2$ be a reduced 0-dimensional scheme such that X is a generic plane section of an integral curve $C \subset \mathbb{P}^3$. Then X is contained in a complete intersection of type (a_1,a_2).*

Proof. From the definition of a_1 and a_2 we see that a_1 is the least degree of a curve F containing X and that there exists a curve G of degree a_2 containing X, but not F. If X were not contained in a complete intersection of type (a_1,a_2), then the linear system of curves of degree a_2 containing X had in its base locus a component $\overline{F}$ of degree $1 \leq r < a_1$.

If $X = C \cap L$, where L is a generic plane for the curve C, then C is such that the plane curves of degree a_2 containing X have a common factor $\overline{F}$ of degree r. So proceding as in Theorem 3 we see that $X \subset \overline{F}$. However $r < a_1$ and this is a contradiction. $\square$

The last relevant result we prove concerns the degree matrix of a set of points in $\mathbb{P}^2$. Let us recall some definitions. Let X be a set of points in $\mathbb{P}^2$ and let us consider its minimal free resolution:

$$0 \to \bigoplus_{i=1}^{m-1} \mathscr{O}_{\mathbb{P}^2}(-p_i) \xrightarrow{\varphi} \bigoplus_{i=1}^{m} \mathscr{O}_{\mathbb{P}^2}(-d_i) \to \mathscr{I}_X \to 0$$

with $d_1 \geq \cdots \geq d_m$ and $p_1 \geq \cdots \geq p_{m-1}$. Let us recall that $\sum_{i=1}^m d_i = \sum_{j=1}^{m-1} p_j$. If A is matrix associated to φ, then the entries in A have degrees u_{ij}, with $u_{ij} = p_i - d_j$. The matrix of the degrees of A is denoted by $\partial A = (u_{ij})$ and is called the *degree matrix* of I_X.

From Theorem 3 we are able to prove in any characteristic the following result, well known in characteristic 0:

THEOREM 7. *Let $d_1 \geq \ldots d_m$ and $p_1 \geq \cdots \geq p_{m-1}$ be two sequences of positive integers such that $\sum_{i=1}^m d_i = \sum_{j=1}^{m-1} p_j$. Let $u_{ij} = p_i - d_j$ and let us consider the matrix $A = (u_{ij})$. Then A is the degree matrix of the generic plane section of an integral curve $C \subset \mathbb{P}^3$ if and only if $u_{i,i-1} > 0$ for all $i = 1, \ldots, m-1$.*

Proof. If $C \subset \mathbb{P}^3$ is an integral curve and X is a general plane section of C, then by [9, Lemma 2.1] and Proposition 3 we see that it must be $u_{i,i-1} > 0$ for $i = 1, \ldots, m-1$. For the converse see [15, Proposition 2] and also [9, Corollary 3.2]. $\square$

Acknowledgements. I would like to thank my advisor Rosario Strano for many stimulating discussions and Riccardo Re for all the helpful conversations and encouragement.

References

[1] DAVIS E. D., *Complete intersections of codimension 2 in $\mathbb{P}^r$: the Bezout-Jacobi-Segre theorem revisited*, Rend. Sem. Mat. Univ. Politec. Torino **43** 2 (1985), 333–353.

[2] GERAMITA A. V., MAROSCIA P. AND ROBERTS L. G. *The Hilbert function of a reduced k-algebra*, J. London Math. Soc. **28** 2 (1983), 443–452.

[3] GERAMITA A. V. AND MIGLIORE J. C., *Hyperplane sections of a smooth curve in $\mathbb{P}^3$*, Comm. Alg. **17** 12 (1989), 3129–3164.

[4] GRECO S. AND VALABREGA P., *On the singular locus of a general complete intersection through a variety in a projective space*, B.U.M.I. Alg. e Geom. VI, v.II n.1 (1983), 113–145.

[5] GROTHENDIECK A., *Éléments de géométrie algébrique: IV. Étude locale de schémas et des morphismes de schémas, Seconde Partie*, Publ. Math. IHES **24**, 1965.

[6] GRUSON L. AND PESKINE P., *Genre de courbes de l'espace projectif*, Algebraic geometry, Lecture Notes in Math. **687**, Springer, Berlin 1978, 31–59.

[7] HARRIS J., *The genus of space curves*, Math. Ann. **249** (1980), 191–204.

[8] HARTSHORNE R., *Algebraic geometry*, Springer Verlag, Graduate Texts in Mathematics **52**, 1977.

[9] HERZOG J., TRUNG N. V. AND VALLA G., *On hyperplane sections of reduced irreducible varieties of low codimension*, J. Math. Kyoto Univ. **34** 1 (1994), 47–72.

[10] MAGGIONI R. AND RAGUSA A., *Connections between Hilbert function and geometric properties for a finite set of points in* $\mathbb{P}^2$, Le Matematiche **39** (1984), 153–170.

[11] MAGGIONI R. AND RAGUSA A., *Construction of smooth curves of* $\mathbb{P}^3$ *with assigned Hilbert function and generators' degrees*, Le Matematiche **42** (1987), 195–210.

[12] MAGGIONI R. AND RAGUSA A., *The Hilbert function of generic plane sections of curves of* $\mathbb{P}^3$, Inv. Math. **91** (1988), 253–258.

[13] MILNE J. S., *Étale cohomology*, Princeton University Press, Princeton, N.J., Princeton Mathematical Series **33** 1980.

[14] RATHMANN J., *The uniform position lemma for curves in characteristic p*, Math. Ann. **276** (1987), 565–579.

[15] SAUER T., *Smoothing projectively Cohen-Macaulay space curves*, Math. Ann., **272** (1985), 83–90.

[16] STANLEY R. P., *The Hilbert functions of graded algebras*, Adv. in Math. **28** (1978), 57–82.

AMS Subject Classification: 13D40, 14H50

Paola BONACINI,
Dipartimento di Matematica e Informatica, Università degli Studi di Catania,
Viale A. Doria 6, 95124 Catania, ITALIA
e-mail: `bonacini@dmi.unict.it`

Lavoro pervenuto in redazione il 11.09.2006 e, in forma definitiva, il 03.12.2007.