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ON THE CLOSURE OF POLYNOMIALS IN $L^t(\mu)$

Abstract. Let μ be a positive Borel measure with compact support. For each $t \in [1,\infty)$, the space $P^t(\mu)$ consists of the functions in $L^t(\mu)$ that belong to the norm closure of the analytic polynomials. Multiplication by z on $P^t(\mu)$ is a bounded operator which we denote by S_μ. We assume that S_μ is irreducible. In 1990, J.E. Thomson has shown that in this case the set of bounded point evaluations for $P^t(\mu)$ is a nonempty simply connected region G. In this article we determine the commutant of M_h, the operator of multiplication by h for certain multiplier functions $h \in P^t(\mu) \cap L^\infty(\mu)$. Let $\varphi : \overline{G} \to \overline{G}$ be a function in $P^t(\mu)$ such that $f \circ \varphi$ is in $P^t(\mu)$ for every $f \in P^t(\mu)$. We investigate the compactness of the composition operator C_φ defined by $C_\varphi(f) = f \circ \varphi$.

1. Introduction

In this section we make a few definitions and set up our notation. Let μ be a positive Borel measure with compact support. For each $t \in [1,\infty)$, we denote by $P^t(\mu)$ the uniform closure of the analytic polynomials in $L^t(\mu)$. A point λ in $\mathbb{C}$ is called a *bounded point evaluation* (bpe) for $P^t(\mu)$ if there is a constant c such that $|p(\lambda)| \le c(\int |p|^t d\mu)^{1/t}$ for every polynomial p.

If λ is a bpe for $P^t(\mu)$, then the linear functional $p \mapsto p(\lambda)$ of evaluation at λ defined on polynomials has a unique extension e_λ to $P^t(\mu)$. We can therefore find a unique element k_λ of $L^s(\mu)$ (with $t^{-1} + s^{-1} = 1$) such that $\int f k_\lambda d\mu = \langle f, e_\lambda \rangle = e_\lambda(f)$ for every f in $P^t(\mu)$. The set of bounded point evaluations for $P^t(\mu)$ is denoted by $B^t(\mu)$. For $f \in P^t(\mu)$ let $\hat{f}(\lambda) = \int f k_\lambda d\mu$. It is not hard to see that $f = \hat{f}$ a.e. $[\mu]$ on the set $B^t(\mu)$. In fact, let $\{p_n\}$ be a sequence of polynomials such that $p_n \to f$ in $P^t(\mu)$ and a.e. Because $p_n(\lambda) = \langle p_n, e_\lambda \rangle$ for $\lambda \in B^t(\mu)$ and $n \ge 1$, we obtain the result by passing to the limit. A point λ in $B^t(\mu)$ is an *analytic bounded point evaluation* (abpe) for $P^t(\mu)$ if $\lambda \in B^t(\mu)^0$, the interior of $B^t(\mu)$, and for every f in $P^t(\mu)$, the map $z \mapsto \hat{f}(z)$ is analytic in a neighborhood of λ. We denote the set of all analytic bpe's for $P^t(\mu)$ by $B^t_a(\mu)$. For more information on bpe's, see [1]. The operator of multiplication by z on $P^t(\mu)$ is a bounded operator which we denote by S_μ. Shields and Wallen [4] studied the commutant of the operator of multiplication by z on the Hilbert spaces of analytic functions. By a slight change in their methods one can obtain the commutant of S_μ.

In 1990, J.E. Thomson [5] proved a remarkable theorem which we state without proof.

Thomson's Theorem. If μ is any compactly supported measure on $\mathbb{C}$ and S_μ is multiplication by z on $P^t(\mu)$, then there exist a Borel partition $\{\Delta_i\}_{i=0}^\infty$ of the support of μ such that if $\mu_n = \mu|_{\Delta_n}$, then the following statements are true:

(a) $P^t(\mu) = L^t(\mu_0) \oplus (\bigoplus_{i=1}^\infty P^t(\mu_i))$.

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(b) If $n \geq 1$, then $P^t(\mu_n)$ is irreducible; that is, $P^t(\mu_n)$ contains no nontrivial characteristic functions.

(c) If $n \geq 1$ and $\Omega_n = B_a^t(\mu_n)$, then Ω_n is simply connected with $\mathrm{supp}(\mu_n) \subset \overline{\Omega}_n$.

d) If S_μ is pure (that is, the set Δ_0 is empty) and $f \in P^t(\mu)$ such that f vanishes a.e. $[\mu]$ on the set of abpe's for μ, then $f = 0$. Equivalently, the e_λ for λ an abpe of μ have dense span in $P^s(\mu)$ (with $t^{-1} + s^{-1} = 1$).

(e) If S_μ is pure and Ω is the set of abpe's for μ, then the map $f \mapsto \hat{f}$ is a dual algebra isomorphism of $P^t(\mu) \cap L^\infty(\mu)$ onto $H^\infty(\Omega)$.

If S_μ is pure, by part (e) of Thomson's Theorem the map $f \mapsto \hat{f}$ is an algebraic and isometric isomorphism and a weak star homeomorphism $P^t(\mu) \cap L^\infty(\mu) \to \mathcal{H}^\infty(G)$ and we denote its inverse $\mathcal{H}^\infty(G) \to P^t(\mu) \cap L^\infty(\mu)$ by $f \mapsto \tilde{f}$. Hence $\tilde{\hat{f}} = f$.

Let $t \in [1, \infty)$. Throughout this article unless the contrary is explicitly stated, we assume that S_μ is irreducible (i.e., $P^t(\mu)$ cannot be decomposed into the direct sum of two nonzero closed subspaces which are invariant subspaces of S_μ). Equivalently, $P^t(\mu)$ contains no nontrivial characteristic functions. By Thomson's Theorem, the set of $B_a^t(\mu) = B^t(\mu)$ is a nonempty simply connected region. We let $B_a^t(\mu) = G$. Also we assume that $A(G)$ denotes the subalgebra of $C(\overline{G})$ consisting of those functions that are continuous on $\overline{G}$ and analytic on G. A complex valued function $\varphi \in P^t(\mu)$ is called a multiplier of $P^t(\mu)$ if $\varphi P^t(\mu) \subset P^t(\mu)$ i.e. φf is in $P^t(\mu)$ for every f in $P^t(\mu)$. The set of all multipliers of $P^t(\mu)$ is denoted by $\mathcal{M}(P^t(\mu))$. It is well known that $\mathcal{M}(P^t(\mu)) = P^t(\mu) \cap L^\infty(\mu)$. Given a multiplier φ, let M_φ, which is defined by $M_\varphi(f) = \varphi f$, denote the operator of multiplication by φ. By the closed graph theorem, M_φ is bounded.

A Caratheodory region G is an open simply connected subset of $\mathbb{C}$ whose boundary equals its outer boundary.

In section two of this article we characterize the commutant of the operator M_h under certain conditions on $h \in A(G)$, also we extend the results obtained in [2] concerning the splitting of $\{M_z\}'$ on $P^t(\mu)$. In section three, we give some sufficient conditions under which a composition operator C_ϕ $(\phi \in A(G))$ is compact.

2. On the commutant of certain multiplication operators on $P^t(\mu)$

In this section we first study the commutant of the multiplication operator M_h for certain function h.

LEMMA 1. *For $t \in [1, \infty)$, we have $A(G) \subset P^t(\mu) \cap C(\mathrm{supp}\,\mu)$. Hence every $f \in A(G)$ is a multiplier of $P^t(\mu)$.*

Proof. See [3, proof of Theorem 4.2]. □

LEMMA 2. *If $f \in P^t(\mu)$ and $g \in P^t(\mu) \cap L^\infty(\mu)$, then $\widehat{fg} = \hat{f}\hat{g}$.*

Proof. See [1, page 62, Proposition 7.3]. □

LEMMA 3. *If $h \in \mathcal{H}^\infty(G)$ and $T \in \{M_{\tilde{h}}\}'$, then $T^*(e_\lambda) \in \ker(M_{\tilde{h}} - h(\lambda))^*$ for every $\lambda \in G$.*

Proof. Let $f \in P^t(\mu)$. We have

$$\begin{aligned}\langle f, M_{\tilde{h}}{}^* T^*(e_\lambda)\rangle &= \langle f, T^* M_{\tilde{h}}{}^*(e_\lambda)\rangle = \langle M_{\tilde{h}} T(f), e_\lambda\rangle \\ &= h(\lambda)\widehat{Tf}(\lambda) = h(\lambda)\langle T(f), e_\lambda\rangle \\ &= h(\lambda)\langle f, T^*(e_\lambda)\rangle = \langle f, h(\lambda) T^*(e_\lambda)\rangle .\end{aligned}$$

Hence, $M_{\tilde{h}}{}^* T^*(e_\lambda) = h(\lambda) T^*(e_\lambda)$, which implies that $T^*(e_\lambda) \in \ker(M - h(\lambda))^*$. □

THEOREM 1. *Let $h \in A(G)$ and let S be a subset of G which has a limit point in G. If $h^{-1}\{h(\lambda)\} = \{\lambda\}$ for each $\lambda \in S$, then $\{M_h\}' = \{M_\psi : \psi \in \mathcal{M}(P^t(\mu))\}$.*

Proof. Let $\lambda \in S$. We shall show that $\operatorname{ran}(M_h - h(\lambda)) = \ker e_\lambda$. Let $g \in P^t(\mu)$. By Lemma 2, we have

$$\langle (h - h(\lambda))g, e_\lambda\rangle = ((h - h(\lambda))\hat{g})(\lambda) = 0,$$

hence $\operatorname{ran}(M_h - h(\lambda)) \subset \ker e_\lambda$.

To show the converse, since $\operatorname{ran}(M_z - \lambda) = \ker e_\lambda$, we have $(h - h(\lambda))(z) = (z - \lambda)\hat{g}(z)$ for some $g \in P^t(\mu)$. As $h \in A(G)$, $\hat{g}$ has a continuous extension on $\overline{G}$. Hence by Thomson's Theorem, without loss of generality we can assume $g \in A(G)$. Since h is not constant we can assume that $h'(\lambda) \neq 0$ for each $\lambda \in S$ so by assumption $g(z) \neq 0$ on $\overline{G}$. Therefore $1/g$ is in $A(G)$ and we have $z - \lambda = (h(z) - h(\lambda))/g(z)$. Now if $f \in \ker e_\lambda$, then $f = (z - \lambda)\varphi$ for some function $\varphi \in P^t(\mu)$. Hence

$$f = \frac{h - h(\lambda)}{g}\varphi = (h - h(\lambda))\frac{\varphi}{g}.$$

Since $\varphi \in P^t(\mu)$ and $1/g \in A(G)$ we conclude that $\ker e_\lambda \subset \operatorname{ran}(M_h - h(\lambda))$ by Lemma 1.

Now by Lemma 3,

$$(M_h - h(\lambda))^*(e_\lambda) = (M_h - h(\lambda))^* T^*(e_\lambda) = 0.$$

Since $\dim \ker(M_h - h(\lambda))^* = 1$, we conclude that $T^*(e_\lambda) = \hat{\psi}(\lambda) e_\lambda$ for some constant $\hat{\psi}(\lambda)$. Hence for every $\lambda \in S$ we have

$$\widehat{T(f)}(\lambda) = \langle T(f), e_\lambda\rangle = \langle f, T^*(e_\lambda)\rangle = \hat{\psi}(\lambda)\langle f, e_\lambda\rangle = \hat{\psi}(\lambda)\hat{f}(\lambda),$$

in particular $\hat{\psi}(\lambda) = \widehat{T(1)}(\lambda)$. Let ψ denote $T(1)$; since S has a limit point in G, we conclude that $\widehat{T(f)}(\lambda) = \hat{\psi}(\lambda)\hat{f}(\lambda)$ for every λ in G. It is then not hard to see that $\|\hat{\psi}\|_\infty \le \|T\|$. Hence, $\hat{\psi} \in \mathcal{H}^\infty(G)$, and it follows by part (e) of Thomson's Theorem that $\psi \in P^t(\mu) \cap L^\infty(\mu)$. The proof is now complete. □

EXAMPLE 1. Let G be a Caratheodory region and let μ be area measure on G. It is well known that $P^t(\mu) = L^t_a(\mu)$, the Bergman space of analytic functions for $t \in [0,\infty)$. Let $h(z) = z^2$, $G = \{z \in \mathbb{C} : 0 \le |z| < 1,\ \alpha \neq \arg z\}$ for some constant $0 < \alpha < \pi$ and $S = \{z \in G : \frac{\alpha}{2} + \pi = \arg z\}$. It is obvious that h is not univalent on G, but by Theorem 1, we have $\{M_{z^2}\}' = \{M_\phi :\ \phi \in \mathcal{H}^\infty(G)\}$. One can easily construct similar examples for $h = z^n$.

By Theorem 1 if $P^t(\mu)$ is pure, and $h \in A(G)$ is a univalent function, then $\{M_h\}' = \{M_\psi :\ \psi \in \mathcal{M}(P^t(\mu))\}$. In the next theorem we show that if $h \in \mathcal{H}^\infty(G)$ is a univalent map, then $\{M_{\tilde{h}}\}' = \{M_\psi :\ \psi \in \mathcal{M}(P^t(\mu))\}$.

THEOREM 2. *Let $P^t(\mu)$ be pure. If $h \in \mathcal{H}^\infty(G)$ is a univalent function, then* $\{M_{\tilde{h}}\}' = \{M_\psi :\ \psi \in \mathcal{M}(P^t(\mu))\}$.

Proof. The proof is similar to the proof of Theorem 1. We replace M_h by $M_{\tilde{h}}$. Since h is univalent $\hat{g}$ is bounded below on G. Hence $1/\hat{g}$ is in $\mathcal{H}^\infty(G)$, now we substitute $1/g$ by $\widetilde{(1/\hat{g})}$ and $h - h(\lambda)$ by $\tilde{h} - h(\lambda)$ in the remainder of the proof. □

THEOREM 3. *Let $h \in \mathcal{H}^\infty(G)$. The operator $M_{\tilde{h}}$ is invertible if and only if there is a constant $c > 0$ such that $|h(z)| > c$ for all $z \in G$. Therefore,* $\operatorname{supp}(M_{\tilde{h}}) = \overline{h(G)}$.

Proof. Let $c > 0$ be a constant such that $|h(z)| > c$ for all $z \in G$. Hence $1/h \in \mathcal{H}^\infty(G)$ and $M_{\widetilde{1/h}}$ is the inverse of $M_{\tilde{h}}$.

To show the converse let $\{\lambda_n\}$ be a sequence in G such that $h(\lambda_n) \to 0$, since $\operatorname{ran}(M_{\tilde{h}} - h(\lambda_n)) \subset \ker e_{\lambda_n}$ it follows that $M_{\tilde{h}-h(\lambda_n)}$ is not invertible, furthermore we have

$$\|M_{\tilde{h}-h(\lambda_n)} - M_{\tilde{h}}\| \le |h(\lambda_n)|,$$

hence $M_{\tilde{h}-h(\lambda_n)}$ tends to $M_{\tilde{h}}$. Since the set of noninvertible operators in $B(P^t(\mu))$ is closed, we conclude that $M_{\tilde{h}}$ is not invertible.

Now let $\lambda \in G$. Hence $M_{\tilde{h}-h(\lambda)}$ is not invertible, and therefore $\overline{h(G)} \subset \operatorname{supp}(M_{\tilde{h}})$. Conversely let $z \in \operatorname{supp}(M_{\tilde{h}})$ and $z \notin \overline{h(G)}$. Hence, by the first part of theorem, $M_{\tilde{h}-z}$ is invertible which is a contradiction. □

THEOREM 4. *Let μ_1 and μ_2 be two measures. Let $B^t_a(\mu_1) = G_1$ and $B^t_a(\mu_2) = G_2$. Also, we assume that $h \in \mathcal{H}^\infty(G_1)$ and $g \in \mathcal{H}^\infty(G_2)$, and there is a bounded operator $T : P^t(\mu_2) \to P^t(\mu_1)$ such that $TM_{\tilde{g}} = M_{\tilde{h}}T$. Now if G_1 contains a subset S which has a limit point in G and $h(S) \cap \overline{g(G_2)} = \emptyset$, then $T = 0$.*

Proof. For $\lambda \in S$, let e^1_λ denote the point evaluation functional at λ. Since $TM_{\tilde{g}} = M_{\tilde{h}}T$, we have $M^*_{\tilde{g}}T^* = T^*M^*_{\tilde{h}}$. Applying the latter on every e^1_λ with $\lambda \in S$, since $(M_{\tilde{h}} - h(\lambda))^*(e^1_\lambda) = 0$, we get

$$M^*_{\tilde{g}}T^*(e^1_\lambda) = T^*M^*_{\tilde{h}}(e^1_\lambda) = h(\lambda)T^*(e^1_\lambda).$$

If $T^*(e^1_\lambda) \neq 0$, then it is an eigenvalue for $M^*_{\tilde{g}}$ corresponding to the eigenvalue $h(\lambda)$. Because $h(S)\cap\overline{g(G_2)} = \emptyset$, we get a contradiction. Hence $T^*(e^1_\lambda) = 0$ for every $\lambda \in S$ which implies that

$$\widehat{T(f)}(\lambda) = \left\langle T(f), e^1_\lambda \right\rangle = \left\langle f, T^*(e^1_\lambda) \right\rangle = 0$$

for every $f \in P^t(\mu_2)$ and every $\lambda \in S$. Therefore $\widehat{T(f)}(\lambda) = 0$ for every $\lambda \in G$. Now since $\widehat{T(f)} = T(f)$ a.e. $[\mu]$ on G, by Thomson's Theorem, $T(f) = 0$. Hence $T = 0$. □

Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be Banach spaces. Every operator X acting on $\mathcal{B} = \mathcal{B}_1 \oplus \mathcal{B}_2$ can be written in the form

$$X = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} \tag{1}$$

where $X_{ij} : \mathcal{B}_j \to \mathcal{B}_i$, $i, j = 1,2$ is defined by $X_{ij} = P_i X|_{\mathcal{B}_j}$ and $P_i : \mathcal{B} \to \mathcal{B}_i$ is the projection onto $\mathcal{B}_i$ for $i = 1,2$.

THEOREM 5. *Suppose, μ_1 and μ_2 are measures and $\mathcal{B} = P^t(\mu_1) \oplus P^t(\mu_2)$. Let $B^t_a(\mu_1) = G$ and $B^t_a(\mu_2) = W$. Also assume that $g \in \mathcal{H}^\infty(G)$ and $h \in \mathcal{H}^\infty(W)$ are univalent functions, and $A = M_{\tilde{h}} \oplus M_{\tilde{g}}$. If $\overline{h(W)} \cap g(S_1) = \emptyset$, where S_1 is a subset of G which has a limit point in G and $\overline{g(G)} \cap h(S_2) = \emptyset$, where S_2 is a subset of W which has a limit point in W, then $X \in \{A\}'$ if and only if*

$$X = \begin{bmatrix} M_\phi & 0 \\ 0 & M_\psi \end{bmatrix}$$

in which $\phi \in \mathcal{M}(P^t(\mu_1))$ and $\psi \in \mathcal{M}(P^t(\mu_2))$. That is, $\{M_{\tilde{h}} \oplus M_{\tilde{g}}\}' = \{M_{\tilde{h}}\}' \oplus \{M_{\tilde{g}}\}'$.

Proof. Let $X \in \{A\}'$, and represent X as in (1). Then we have the following relations:

$$X_{11}M_{\tilde{g}} = M_{\tilde{g}}X_{11}, \quad X_{22}M_{\tilde{h}} = M_{\tilde{h}}X_{22},$$
$$X_{12}M_{\tilde{h}} = M_{\tilde{g}}X_{12}, \quad X_{21}M_{\tilde{g}} = M_{\tilde{h}}X_{21}.$$

Now apply Theorem 2 and Theorem 4. □

LEMMA 4. *Let μ be a measure such that $P^t(\mu)$ is pure. If $\{W_i\}_{i=1}^\infty$ are components of $B^t_a(\mu) = G$, then there is a Borel partition $\{\Delta_i\}_{i=1}^\infty$ of* supp μ *such that*

$$P^t(\mu) = P^t(\mu_1) \oplus P^t(\mu_2) \oplus \cdots \tag{2}$$

In which $\mu_i = \mu|_{\Delta_i}$, each $P^t(\mu_i)$ is irreducible and $\Delta_i \subset \overline{W_i}$.

Proof. See the proof of Thomson's Theorem in the pure case. □

THEOREM 6. *Let μ be a measure such that $P^t(\mu)$ is pure and let G, $\{W_i\}_{i=1}^\infty$ and $\{\Delta_i\}_{i=1}^\infty$ be as in Theorem 6. If $h \in \mathcal{H}^\infty(G)$ is a univalent map and $X \in \{M_{\tilde{h}}\}'$,* *then*

$$X=\begin{bmatrix} M_{\phi_1} & 0 & 0 & . & . & . \\ 0 & M_{\phi_2} & 0 & . & . & . \\ 0 & 0 & M_{\phi_3} & . & . & . \\ 0 & 0 & 0 & . & . & . \\ 0 & 0 & 0 & . & . & . \end{bmatrix}$$

where each ϕ_i*,* $i=1,2,...,$ *belongs to* $P^t(\mu_i)\cap L^\infty(\mu_i)$*. Moreover if* $\{h_i\}_{i=1}^\infty$ *is the decomposition of h with respect to (2), then*

$$\{M_{\tilde{h}}\}'=\bigoplus_{i=1}^{\infty}\{M_{\tilde{h}_i}\}'.$$

Proof. Set $B_a^t(\mu_i)=G_i$. Since $h_i=h|_{\overline{G_i}}\in\mathcal{H}^\infty(G_i)$, we have that $\tilde{h}_i$ belongs to the intersection $P^t(\mu_i)\cap L^\infty(\mu_i)$. Now let $X\in\{M_{\tilde{h}}\}'$. Because $XM_{\tilde{h}}=M_{\tilde{h}}X$, we have $X_{ij}M_{\tilde{h}_j}=M_{\tilde{h}_i}X_{ij}$. Also since h is univalent, $\overline{h_i(G_i)}\cap h_j(G_j)=\emptyset$ for $i\neq j$. Hence by Theorem 4, $X_{ij}=0$ for $i\neq j$. Since $X_{ii}\in\{M_{\tilde{h}_i}\}'$ by Theorem 2, it follows that $X_{ii}=M_{\phi_i}$ for some $\phi_i\in P^t(\mu_i)\cap L^\infty(\mu_i)$. □

Let μ be a compactly supported positive Borel measure such that S_μ is irreducible and $B_a^t(\mu)=G$. Assume μ_0 is a measure with $\operatorname{supp}\mu_0\cap G=\emptyset$ and $AS_\mu=N_{\mu_0}A$, where $N_{\mu_0}:L^t(\mu_0)\to L^t(\mu_0)$ with $f\mapsto zf$. Then $A^*S_\mu{}^*=N_{\mu_0}{}^*A^*$. Letting this act on $e_\lambda,(\lambda\in G)$ and using the fact that $\sigma(N_{\mu_0})\subset\operatorname{supp}\mu_0$ and $\operatorname{supp}\mu_0\cap G=\emptyset$, we conclude that $A=0$.

LEMMA 5. *Let* μ *be a compactly supported positive Borel measure and let* μ_0*,* $\mu_1,\mu_2,\ldots$ *be as in Thomson's Theorem. If* $A_i:P^t(\mu_i)\to L^t(\mu_0)$ *is an operator such that* $A_iS_{\mu_i}=N_{\mu_0}A_i$*, then* $A_i=0$ *for* $i=1,2,\ldots.$

Proof. For every polynomial p it is easy to see that $A_i(pf)=pA_if$. Hence $A_ip=p\psi_i$, where $\psi_i=A_i1\in L^t(\mu_0)$. Now let $f\in P^t(\mu_i)$ and choose a sequence $\{p_n\}$ of polynomials such that $p_n\to f$ in $P^t(\mu)$ and so in $P^t(\mu_i)$. By the continuity of A_i it follows that $A_ip_n\to A_if$ in $L^t(\mu_0)$. By passing to subsequences we may assume that $p_n\to f$ a.e. $[\mu]$ and $p_n\psi_i\to A_if$ a.e. $[\mu_0]$. Because $p_n\to 0$ a.e. $[\mu_0]$ we have $A_if=0$ for all $f\in P^t(\mu_i)$. Hence $A_i=0$. □

THEOREM 7. *Let* μ *be a compactly supported positive Borel measure. Relative to the decomposition in part (a) of Thomson's Theorem* $\{S_\mu\}'$ *consists of all elements* $X=(X_{ij})_{i,j=0}^\infty$ *such that* $X_{ii}=M_{\phi_i}$ *for* $i=0,1,2,\ldots.$ *Here,* $\phi_0\in L^\infty(\mu_0)$ *and* $\phi_i\in\mathcal{M}(P^t(\mu_i))$ *and* $X_{ij}=0$ *for* $i\neq j$*,* $i=0,1,2,\ldots,$ $j=0,1,2,\ldots.$ *In fact*

$$X=\begin{bmatrix} M_{\phi_1} & 0 & 0 & . & . & . \\ 0 & M_{\phi_2} & 0 & . & . & . \\ 0 & 0 & M_{\phi_3} & . & . & . \\ 0 & 0 & 0 & . & . & . \\ 0 & 0 & 0 & . & . & . \end{bmatrix}.$$

Hence

$$P^t(\mu)\cap L^\infty(\mu) \;=\; L^\infty(\mu_0)\oplus(P^t(\mu_1)\cap L^\infty(\mu_1))\oplus\cdots\oplus(P^t(\mu_n)\cap L^\infty(\mu_n))\oplus\cdots.$$

3. On the compactness of composition operators on $P^t(\mu)$

In this section we give some sufficient conditions under which a composition operator C_φ ($\varphi\in A(G)$) is compact. First we give several lemmas that we use later.

Let D^t denote the unit ball of $P^t(\mu)$. We set $\hat{D}^t=\{\hat{f}: f\in D^t\}$.

LEMMA 6. *$\hat{D}^t$ is a normal family on G for every $1\le t\le\infty$. Hence if $f_n\to f$ weakly, then $\hat{f}_n\to\hat{f}$ uniformly on each compact subset of G.*

Proof. The necessary and sufficient condition for normality of $\hat{D}^t$ is that the functions in the family be uniformly bounded on each compact subset of G. Let $K\subseteq G$ be a compact set. Then we have $\sup\{|e_\lambda(f)|:\lambda\in K\}=\sup\{|\hat{f}(\lambda)|:\lambda\in K\}<\infty$ for each $f\in D^t$. Hence by the principle of uniform boundedness there is a constant c_K such that $\|e_\lambda\|\le c_K$ for all $\lambda\in K$ thus $|\hat{f}(\lambda)|\le c_K$ for each f in the unit ball of $P^t(\mu)$ and all $\lambda\in K$. Now let $\{f_n\}$ be a sequence in $P^t(\mu)$ such that $f_n\to f$ weakly. Then by the principle of uniform boundedness $\{f_n\}$ is bounded and therefore by the first part of theorem the proof is complete. □

LEMMA 7. *Let $P^t(\mu)$ be pure and let $\{f_n\}$ be a bounded sequence in $P^t(\mu)$ such that $\hat{f}_n\to f$ pointwise on G. Then there exists a function $\tilde{f}$ in $P^t(\mu)$ such that $f_n\to\tilde{f}$ weakly and $\tilde{f}=f$ on G a.e. $[\mu]$.*

Proof. Since $P^t(\mu)$ is a separable reflexive Banach space, the unit ball of $P^t(\mu)$ in the weak topology is compact. Assume there are two subsequences $\{f_{n_k}\}$ and $\{f_{m_k}\}$ of $\{f_n\}$ and functions h and g in $P^t(\mu)$ such that $f_{n_k}\to h$ and $f_{m_k}\to g$ weakly. Since for each $\lambda\in G$, e_λ is bounded we have $f(\lambda)=\hat{g}(\lambda)=\hat{h}(\lambda)$ a.e. $[\mu]$ on G. As $P^t(\mu)$ is pure and $h-g=0$ a.e. $[\mu]$ on G, we conclude that $h=g$ and h has desired properties. □

THEOREM 8. *Let $\varphi:\overline{G}\to\overline{G}$ be in $A(G)$, and let φ induce a composition operator C_φ on $P^t(\mu)$. If $\varphi(\overline{G})\subset G$, then C_φ is compact.*

Proof. We show that C_φ is completely continuous. Since $\varphi(\overline{G})\subset G$, so $\varphi(\overline{G})$ is a compact subset of G. Now let $\{f_n\}$ be a sequence of functions in $P^t(\mu)$ such that $\hat{f}_n\to 0$ in compact open topology of G. Let

$$E=\{\lambda\in G:\; f_n(\lambda)\neq\hat{f}_n(\lambda)\;\text{ for some positive integer } n\}.$$

Since $\mu(E)=0$, χ_E is the zero function in $P^t(\mu)$, hence $\mu(\varphi^{-1}(E))=\|C_\varphi\chi_E\|^t=0$. Using the change of variables formula, we have

$$\int_{\varphi(\overline{G})}|f_n|^t\,d\mu\circ\varphi^{-1}=\int_{\overline{G}}|f_n\circ\varphi|^t\,d\mu=\int_{\overline{G}-\varphi^{-1}(E)}|\hat{f}_n\circ\varphi|\,d\mu.$$

Hence, we conclude that $\|C_\varphi f_n\| \to 0$ and the proof is complete. □

LEMMA 8. *Let U be a simply connected region containing G, and let f be a Riemann map of U onto $\mathbb{D}$. Then $|\tilde{f}(z)| = 1$ a.e. with respect to $\mu|_{\partial G}$.*

Proof. See [3, Lemma 2.2]. □

THEOREM 9. *Let C_φ be a compact composition operator on $P^t(\mu)$. If we set $A = \{z \in \overline{G} : \varphi(z) \in \partial G\}$, then $\mu(A) = 0$.*

Proof. Let ϕ be the Riemann map from G onto $\mathbb{D}$. It is easy to see that $\{\tilde{\phi}^n\}$ tends to zero weakly. Now we have

$$\|\tilde{\phi}^n \circ \varphi\|^t = \int_A |\tilde{\phi}^n \circ \varphi|^t d\mu + \int_{A^c} |\tilde{\phi}^n \circ \varphi|^t d\mu = \mu(A) + \int_{A^c} |\tilde{\phi}^n \circ \varphi|^t d\mu.$$

Hence $\|\tilde{\phi}^n \circ \varphi\|^t \to \mu(A)$ and since C_φ is compact, we conclude that $\mu(A) = 0$. □

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