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THE COHOMOLOGY OF RANK 2 BUNDLES ON $\mathbb{P}^2$

Abstract. We classify all sequences of integers that can be, up to a shift, the cohomology sequence $\{h^1(E(n))\}$ of a rank 2 bundle E on $\mathbb{P}^2$. We show how some of the main invariants of the bundle can be read from the sequence.

1. Introduction

The purpose of this paper is to determine the possible dimensions of the cohomology groups of rank 2 vector bundles on the projective plane, and to describe what information follows from the knowledge of these numbers, concerning the bundles themselves and the geometry of a finite set of points.

The correspondence between rank 2 bundles on the plane and finite sets of points $Z \subset \mathbb{P}^2$ is controlled by Serre's construction and Cayley–Bacharach numbers (CB) of Z. We refer to Definition 1 for a technical setting of what a CB is, in this note. Suffice it to say now that it corresponds to an integer e such that the obvious restriction map $H^0(\mathscr{O}_{\mathbb{P}^2}(e)) \to H^0(\mathscr{O}_Z)$ is uniformly non–surjective.

Once one determines that $e \in \mathbb{Z}$ is a CB for a finite set Z, then there is an extension leading to a rank 2 bundle E on $\mathbb{P}^2$, with first Chern class $c_1 = e+3$, such that Z is the zero locus of a section s of E. The bundle and the set of points are algebraically linked by the exact sequence:

$$0 \to \mathscr{O}_{\mathbb{P}^2} \xrightarrow{s} E \to \mathscr{I}_Z(e+3) \to 0 \tag{1}$$

where $\mathscr{I}_Z$ is the ideal sheaf of Z.

When Z is a complete intersection of curves of degree a,b then $a+b-3$ is CB for Z, and Serre's construction returns the bundle $\mathscr{O}_{\mathbb{P}^2}(a) \oplus \mathscr{O}_{\mathbb{P}^2}(b)$, i.e. a splitting bundle. Conversely if E splits as a sum of line bundles $E = \mathscr{O}_{\mathbb{P}^2}(a) \oplus \mathscr{O}_{\mathbb{P}^2}(b)$, then $a+b-3$ is a CB for Z and Z is a complete intersection of curves of degree a and b.

We have the celebrated Horrocks' criterion for the splitting of E:

THEOREM 1 (Horrocks). *A vector bundle E of rank 2 on $\mathbb{P}^2$ splits if and only if $H^1(E(n)) = 0$, $\forall n$.*

So the groups $H^1(E(n))$, $n \in \mathbb{Z}$, measure, in a certain sense, how far E is from being split, and also how far Z is from being a complete intersection.

The main tool for the study of the geometry of Z is the Hilbert function h_Z, defined as usual by $h_Z(n) = h^0(\mathscr{O}_{\mathbb{P}^2}(n)) - h^0(\mathscr{I}_Z(n))$ or equivalently, $h_Z(n) = d - h^1(\mathscr{I}_Z(n))$, where d is the number of points of Z. Thus the Hilbert function gives essentially the same information of the function $h^1(\mathscr{I}_Z(t))$. The sequence (1) yields that the Hilbert function of Z has an immediate link with the function $h^1(E(t-e-3))$.

So we have some reasons why the study of the dimensions of homogeneous pieces in the module $\oplus H^1(E(n))$ could be of interest. The purpose here is to give an answer to the question:

Q) Which sequences $\{a_n\}$ correspond to the sequence $\{h^1(E(n)) = \dim(H^1(E(n))\}$ of integers for some rank 2 vector bundle E on $\mathbb{P}^2$?

Of course not every sequence of integers is allowed; for instance, we know that a_n must vanish for $n \gg 0$. Furthermore, by duality, $h^1(E(n)) = h^1(E^\vee(-n-3)) = h^1(E(-c_1-n-3))$, thus $\{a_n\}$ must be symmetric around $-\frac{c_1+3}{2}$.

A related question considers how we can detect whether the zero locus of a section of a rank 2 vector bundle is a complete intersection. Beware that this is not equivalent to the splitting of E:

EXAMPLE 1. Fix Z to be one point P and $e = -2$. Then e is a CB for Z. Let E be an associated rank 2 bundle. As $h^2(\mathscr{O}_{\mathbb{P}^2}(-2)) = 0$, then from the sequence

$$0 \to \mathscr{I}_Z \to \mathscr{O}_{\mathbb{P}^2} \to \mathscr{O}_Z \to 0$$

we get $h^1(E(-2)) \neq 0$, i.e. E does not split, while on the other hand Z is complete intersection.

The point in this example is based on the fact that e is not the *maximal* CB for Z.

So we have the following, natural:

QQ) For which sequences $\{a_n\}$ does there exist a non–split rank 2 bundle E, having a global section $s \in E(n_0)$ whose zero locus is a complete intersection, and such that $h^1(E(n)) = a_n \ \forall n$?

In this note we provide an answer to both questions (see the theorems of Section 3 and Theorem 7 below).

The description of the first cohomology also allows us to rephrase in $\mathbb{P}^2$ the fact (valid for bundles in $\mathbb{P}^3$, by [3]) that the vanishing of a single, well-determined cohomology group for E implies that the bundle splits (see Proposition 4).

Also, we are going to show how interesting invariants of the bundle (such as the Atiyah invariant Δ) can be read, in some cases, from the cohomology sequence.

Finally, we will determine some effects on the cohomology and the invariants of sets of points, caused when passing from sections of E to sections of some twist $E(n)$.

2. Preliminaries

Let Z be a set of points in the projective space $\mathbb{P}^2$ over the complex field. Let d denote $\text{length}(Z)$. We will always assume that Z is formed by d distinct points, even if a weaker hypothesis should work.

Let h_Z the Hilbert function of Z. We establish the following notation:

$a =$ minimum degree of a (possibly reducible) curve containing Z;

$C_0 =$ one curve of degree a containing Z;

$b =$ minimum degree of a curve containing Z and not containing C_0 (possibly $a = b$).

Then we have the following, almost classical description for the "difference Hilbert function" $\Delta h_Z(n) = h_Z(n) - h_Z(n-1)$.

THEOREM 2. *For any set of d distinct points Z, put $x_n = \Delta h_Z(n)$. Then we have:*

A) $x_n \geq 0\ \forall n$ *and* $\sum_{-\infty}^{\infty} x_n = d$*;*

B) *for* $n \leq -1$ *and* $n \gg 0$, $x_n = 0$*;*

C) *for* $0 \leq n < a$, $x_n = x_{n-1} + 1\ (= n+1)$*;*

D) *for* $a \leq n < b$, $x_n = x_{n-1}\ (= a)$*;*

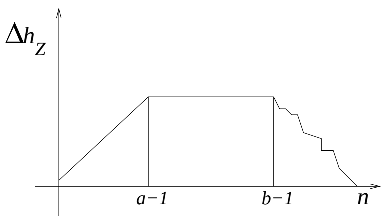
E) *for* $n \geq b$, $x_n \leq x_{n-1}$ *and* $x_b < x_{b-1}$.

If moreover the curve of minimal degree C_0 as above can be chosen irreducible, then we have $\Delta h_Z(n) < \Delta h_Z(n-1)\ \forall n \geq b$, unless $\Delta h_Z(n) = 0$.

Conversely given any sequence of integers $\{x_n\}$ satisfying conditions A)-E), there exists a set of d points Z such that $\Delta h_Z(n) = x_n\ \forall n$.

Proof. See [5]. □

Putting everything into a diagram, we get:



and every *shape* like this can be realized as the difference Hilbert function of a set of points.

The relation between rank 2 bundles and cohomology of points is based on the following:

DEFINITION 1. *An integer e is a Cayley-Bacharach number (CB) for Z if for any subscheme $Z' \subset Z$ of length $d-1$, we have $h^0(\mathscr{I}_{Z'}(e)) = h^0(\mathscr{I}_Z(e))$.*

Equivalently, e is CB if and only if no elements of the canonical basis for $H^0(\mathscr{O}_Z) = \mathbb{C}^d$ lie in the image of the restriction map $H^0(\mathscr{O}_{\mathbb{P}^2}(e)) \to H^0(\mathscr{O}_Z)$. Of course, if e is CB for Z, then $e-1, e-2, \ldots$ are also.

Notice that our definition of Cayley-Bacharach numbers is slightly different from what we find in other related papers. Indeed we do not take a CB to be the *maximum* such that the property holds, but just any number satisfying the property. This justifies our remark that Z has infinitely many CB.

We have (see [1]):

THEOREM 3 (Brun). *If e is CB for Z, then there exists a vector bundle E on $\mathbb{P}^2$ and a section $s \in H^0(E)$ such that Z is, scheme-theoretically, the zero locus of s and the Chern classes of E are $c_1 = e+3$, $c_2 = d$. E and Z are related by the sequence*

$$0 \to \mathcal{O}_{\mathbb{P}^2} \xrightarrow{s} E \to \mathcal{I}_Z(e+3) \to 0.$$

Of course, the bundle E is not unique, neither if we fix the CB number e. Indeed, in order to determine E, we also need to fix an element of $Ext^1(\mathcal{I}_Z(e+3), \mathcal{O}_{\mathbb{P}^2})$, which in general has dimension larger than 1.

The correspondence $\{$sets of points$\} \leftrightarrow \{$rank 2 bundles$\}$ can be more deeply understood as follows: sequence (1) comes about by the choice of an element $\eta \in Ext^1(\mathcal{I}_Z(e+3), \mathcal{O}_{\mathbb{P}^2})$. By the sequence

$$0 \to \mathcal{I}_Z \to \mathcal{O}_{\mathbb{P}^2} \to \mathcal{O}_Z \to 0 \tag{2}$$

we get

$$\begin{array}{ccccc} 0 \to Ext^1(\mathcal{I}_Z(e+3), \mathcal{O}_{\mathbb{P}^2}) \to & Ext^2(\mathcal{O}_Z, \mathcal{O}_{\mathbb{P}^2}) & \xrightarrow{\alpha} & Ext^2(\mathcal{O}_{\mathbb{P}^2}(e+3), \mathcal{O}_{\mathbb{P}^2}) & \to \\ & \| & & \| & \\ & H^0(\mathcal{O}_Z)^\vee & & H^0(\mathcal{O}_{\mathbb{P}^2}(e))^\vee & \end{array} \tag{3}$$

where α is the dual of the obvious restriction map. Further, since E is a vector bundle, $\mathcal{E}xt^1(E, \mathcal{O}_{\mathbb{P}^2})$ must be 0, so η is a local generator for $\mathcal{E}xt^1(\mathcal{I}_Z(e+3), \mathcal{O}_{\mathbb{P}^2}) = \mathcal{E}xt^2(\mathcal{O}_Z, \mathcal{O}_{\mathbb{P}^2}) = \mathcal{O}_Z$. Hence η corresponds to an element η_0 of $H^0(\mathcal{O}_Z)$ which is not in the image of $H^0(\mathcal{O}_{\mathbb{P}^2}(e)) \to H^0(\mathcal{O}_Z)$ and with no vanishing components. The existence of η_0 is precisely the CB property for Z at level e.

We need to point out some relations between CB and the Hilbert function h_Z:

LEMMA 1. *With the previous notation:*

a) $b-2$ is a CB number for Z.

b) If $e \geq 0$ then $\Delta h_Z(e+1) \neq 0$.

Proof.

a) If not, there is a point $P \in Z$ and a curve C of degree $b-2$ passing through $Z-P$ and missing P. Clearly C cannot contain C_0. But then, for a general line L through P, $C \cup L$ has degree $b-1$, contains Z and not C_0, contrary to the minimality of b.

b) If $\Delta h_Z(e+1)=0$ then $h_Z(e)=h_Z(e+1)=\cdots=h_Z(e+n)\ \forall n$ and this implies $h_Z(e)=d$, that is, the restriction map $H^0(\mathscr{O}_{\mathbb{P}^2}(e))\to H^0(\mathscr{O}_Z)$ is surjective, but this is a contradiction, by definition of CB number.

□

3. The study of the sequence $h^1(E(n))$

The relationship between the cohomology of the bundle E and the Hilbert function of Z is given by:

PROPOSITION 1. *Let Z, e, E be as usual. Then, $\forall n\in\mathbb{Z}$*

$$h^1(E(n))=d-h_Z(-n-3)-h_Z(e+3+n). \tag{4}$$

Proof. By sequence (1) we get on cohomology

$$\begin{array}{cccccccc} 0 & \to & H^1(E(n)) & \to & H^1(\mathscr{I}_Z(e+3+n)) & \stackrel{\beta}{\to} & H^2(\mathscr{O}(n)) \\ & \to & H^2(E(n)) & \to & H^2(\mathscr{I}_Z(e+3+n)) & \to & 0 \end{array} \tag{5}$$

where $h^2(\mathscr{O}(n))=h^0(\mathscr{O}(-n-3))$, $h^2(E(n))=h^0(E(-n-e-6))$ and

$$h^2(\mathscr{I}_Z(e+3+n)=h^2(\mathscr{O}(e+3+n))=h^0(\mathscr{O}(-e-6-n)).$$

Further, $h^0(E(-n-e-6))=h^0(\mathscr{O}(-n-e-6))+h^0(\mathscr{I}_Z(-n-3))$. Putting all this together and recalling the definition of Hilbert function, the formula is settled. □

This relation can be understood in details. Return to sequence (3) of Section 1: the vector bundle E comes from the choice of $\eta\in Ext^1(\mathscr{I}_Z(e+3),\mathscr{O})\subset H^0(\mathscr{O}_Z)^\vee$ and η vanishes under α, hence, in $H^0(\mathscr{O}_Z)$, it is orthogonal to the image of $H^0(\mathscr{O}(e))$. In sequence (5) the dual of β is then, by construction, obtained from the map

$$\gamma: H^0(\mathscr{O}(-n-3))\to Ext^1(\mathscr{I}_Z(e),\mathscr{O}(-n-6))=Ext^1(\mathscr{I}_Z(e+3),\mathscr{O}(-n-3))$$

given by restriction to $H^0(\mathscr{O}_Z)$, followed by tensoring with η. Thus for $n\le -3$,

$$\begin{aligned} h^1(E(n)) &= \dim(Ext^1(\mathscr{I}_Z(e),\mathscr{O}(-n-6)))-\dim(\mathrm{Im}(\gamma)) \\ &= h^1(\mathscr{I}_Z(e+3+n))-h_Z(-n-3)=d-h_Z(e+3+n)-h_Z(-n-3). \end{aligned} \tag{6}$$

On the other hand, for $n>-3$, γ and β vanish, as $h_Z(-n-3)$ does, so:

$$\begin{aligned} h^1(E(n)) &= \dim(Ext^1(\mathscr{I}_Z(e),\mathscr{O}(-n-6)))=h^1(\mathscr{I}_Z(e+3+n)) \\ &= d-h_Z(e+3+n)=d-h_Z(e+3+n)-h_Z(-n-3). \end{aligned} \tag{7}$$

Now one can use the classification of $\Delta h_Z(n)$ to give a description of the "difference cohomology sequence"

$$\Delta h^1E(n)=h^1(E(n))-h^1(E(n-1))$$

which is equal to $\Delta h_Z(-n-2) - \Delta h_Z(e+3+n)$. By Serre duality, the sequence $h^1(E(n))$ is symmetric around $-\frac{e+6}{2}$, thus the sequence $\Delta h^1 E(n)$ is *anti*–symmetric around $-\frac{e+5}{2} = n = -\frac{c_1}{2} - 1$.

LEMMA 2. *Let E be a vector bundle on $\mathbb{P}^2$. Then for $q \gg 0$, $E(q)$ has a section whose zero locus Z is smooth and a general curve of minimal degree passing through Z is irreducible; in this case by Theorem 3, Δh_Z strictly decreases from $b-1$ till it reaches* 0.

Proof. See [7], prop 7.2 and [9]. □

REMARK 1. Since a shifting does not change (but merely shifts) the cohomology modules of a vector bundle, we assume form now on that

(*) $c_1(E) \gg 0$ and E satisfies the thesis of Lemma 2.

We need at this point, to make a distinction between stable and unstable bundles, since they have a quite different behaviour in cohomology.

DEFINITION 2. *A rank* 2 *vector bundle E on $\mathbb{P}^2$ is stable (resp. semistable) if for all invertible subsheaves $\mathcal{L} \subseteq E$, $c_1(\mathcal{L}) < \frac{1}{2}c_1(E)$ (resp. $c_1(\mathcal{L}) \leq \frac{1}{2}c_1(E)$). If E has a section whose zero locus is Z and $c_1(E) \gg 0$, then E is stable (resp semistable) if and only if Z is not contained on curves of degree $\leq \frac{1}{2}c_1(E)$ (resp. $< \frac{1}{2}c_1(E)$) (see* [8], *Section 3).*

If the numbers a, e have the usual meaning, then E is stable (resp. semi-stable) if and only if $a > \frac{e+3}{2}$ (resp. $a \geq \frac{e+3}{2}$).

Now we are ready for classifying the cohomology sequence $\Delta h^1(E(n))$. By the antisymmetry of the sequence, we may equivalently look at $n \leq -\frac{c_1+2}{2}$ or $n \geq -\frac{c_1+2}{2}$; we choose $n \leq -\frac{c_1+2}{2}$ since in this case we get $\Delta h^1(E(n)) \geq 0$.

PROPOSITION 2. *Let E be a non–stable rank 2 bundle on $\mathbb{P}^2$ which does not split. Then there are numbers $n_1 \geq n_2$, $n_1 \leq -\frac{c_1+2}{2}$, such that*

A_1) *For $-\frac{c_1+2}{2} \geq n \geq n_1$, $\Delta h^1 E(n) = 0$;*

B_1) *For $n_1 > n \geq n_2$, $\Delta h^1 E(n) = \Delta h^1 E(n+1) + 1$;*

C_1) *For $n < n_2$, $\Delta h^1 E(n) \leq \Delta h^1 E(n+1)$;*

D_1) *For $n \ll 0$, $\Delta h^1 E(n) = 0$.*

(Condition B_1 above may be empty.)

If furthermore $n_1 = -\frac{c_1+2}{2}$, then E is semi-stable.

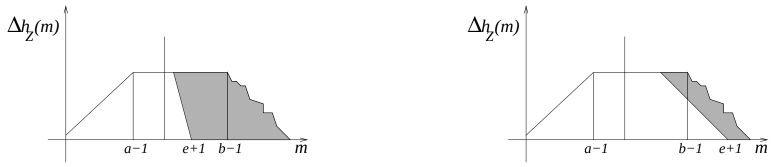
Proof. Twist E until condition (*) is satisfied. Let Z be the zero locus of a general section of E and let a, b, e have their usual meaning. Put $n_1 = a - e - 4$, and $n_2 =$

$\max\{-b-1,-c_1-1\}$. We have $n_2 \leq n_1$. Indeed otherwise, as $a-e-4 = a-c_1-1$, n_2 must be $-b-1$ and $e \geq a+b-3$, but by our assumptions on Z, Δh_Z decreases strictly from $b-1$, going to 0, so that necessarily $e = a+b-3$ and the Hilbert function of Z coincides with the Hilbert function of a complete intersection of curves of degree a and b. But in this case, by [4], Z is complete intersection and E splits.

Now formulas $A_1) - D_1)$ follows by the corresponding formulas A) - E) established in Theorem 2 for Δh_Z, via the equality (4) of Proposition 1.

When $n_1 = -\frac{c_1+2}{2}$, then $a = \frac{e+3}{2}$ so that E is semistable. □

REMARK 2. It is probably unavoidable to draw a picture of the situation: if the following diagrams represent the function Δh_Z; then, by Proposition 1, the non–zero values of $\Delta h^1(E(-n))$, in the range $(\frac{c_1+3}{2}, +\infty)$, are given by the shadowed shape (backward!).



For stable bundles, we have a slightly different description:

PROPOSITION 3. *Let E be a stable rank 2 bundle on $\mathbb{P}^2$ which does not split. Then there are numbers $n_1 \geq n_2$, $n_1 < -\frac{c_1+2}{2}$, such that*

$A_2)$ *if c_1 is odd, then $\Delta h^1 E(-\frac{c_1+3}{2}) = 1$; if c_1 is even, then $\Delta h^1 E(-\frac{c_1+2}{2}) = 0$;*

$B_2)$ *For $-\frac{c_1+4}{2} \geq n \geq n_1$, $\Delta h^1 E(n) = \Delta h^1 E(n+1) + 2$;*

$C_2)$ *For $n_1 > n \geq n_2$, $\Delta h^1 E(n) = \Delta h^1 E(n+1) + 1$;*

$D_2)$ *For $n < n_2$, $\Delta h^1 E(n) \leq \Delta h^1 E(n+1)$;*

$E_2)$ *For $n \ll 0$, $\Delta h^1 E(n) = 0$.*

(Condition C_2 above may be empty.)

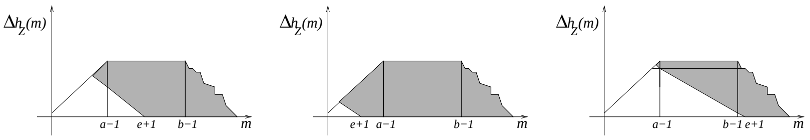
Proof. Twist E until condition (*) is satisfied. Then let Z be the zero locus of a general section of E and let a, b, e have their usual meaning. Put

$$n_1 = \max\{-a-1,-c_1-1\} \quad n_2 = \begin{cases} -a-1 & \text{if } n_1 = -c_1-1 \\ \max\{-b-1,-c_1-1\} & \text{otherwise} \end{cases}$$

The Proposition follows now by Proposition 1, together with the fact that $a-1 > \frac{e+1}{2} = \frac{c_1-2}{2}$ by stability and Δh_Z strictly decreases from $b-1$, going to 0. □

REMARK 3. Here the pictures go as follows:

(8)



$n_1 = -a-1$ $n_2 = -c_1-1$ | $n_1 = -c_1-1$ $n_2 = -a-1$ | $n_1 = -a-1$ $n_2 = -b-1$

The main results of the section say that the two previous propositions are invertible, that is, they describe all the numerical sequences which correspond to $\Delta h^1(E(n))$ for some vector bundle E.

THEOREM 4. *Let $\{a_i\}$ $i \geq 0$ be a sequence of non-negative integers with the following properties: there are numbers $0 < m_1 \leq m_2$ with*

$a_2)$ $a_0 \leq 1$;

$b_2)$ $a_i = a_{i-1} + 2$ *for* $0 < i \leq m_1$;

$c_2)$ $a_i = a_{i-1} + 1$ *for* $m_1 < i \leq m_2$ *(this condition may be empty);*

$d_2)$ $a_i \leq a_{i-1}$ *for* $i > m_2$;

$e_2)$ $a_i = 0$ *for* $i \gg 0$.

Then there is a stable rank 2 bundle E on $\mathbb{P}^2$ and a constant $k \in \mathbb{Z}$ such that $\forall i \geq 0$ $\Delta h^1 E(-i+k) = a_i$.

Proof. First note that, by Proposition 3, if E is a stable bundle, the sequence $a_i = \Delta h^1(E(-i+k))$, where k is the first integer such that $k \leq -\frac{c_1+2}{2}$, satisfies $a_2) - e_2)$.

Now, set $e = 2m_2 + a_0 - 3$ and let q be the first integer such that $q \geq \frac{e+1}{2}$ $(q = m_2 - 1)$. Define the sequence $\{x_m\}$ as follows:

$$x_m = \begin{cases} m+1 & \text{for } 0 \leq m < q \\ e+2-m+a_{m-q} & \text{for } q \leq m \leq e+1 \\ a_{m-q} & \text{for } m > e+1. \end{cases}$$

Using conditions $a_2) - e_2)$ one can see immediately that $\{x_m\}$ satisfies the hypothesis of the last part of Theorem 2, hence there exists a set of points Z which is smooth and $\forall m \geq 0$ $\Delta h_Z(m) = x_m$.

By construction, $x_m \geq x_{m-1}$ for $m \leq e+2$ (for instance, $x_{e+2} = a_{e+2-q} \leq a_{m_2-1} + 1 = x_{e+1}$), hence the usual number b for Z is greater than or equal to $e+2$; it follows by Lemma 1, part $a)$ that e is a CB number for Z giving rise to a vector bundle E such that, by Propositions 2 and 3 and by construction $c_1(E) = e+3$, $\Delta h^1(E(-i+k)) = a_i$ if we let k be the first integer such that $k \leq -\frac{c_1+2}{2}$.

Finally E is stable by construction and by Proposition 5. □

There is the obvious analogue for semi-stable and unstable bundles.

THEOREM 5. *Let* $\{a_i\}$ $i \geq 0$ *be a sequence of non-negative integers with the following properties: there are numbers* $0 < m_1 \leq m_2$ *with:*

$a_1)$ $a_i = 0$ *for* $0 \leq i \leq m_1$*;*

$b_1)$ $a_i = a_{i-1} + 1$ *for* $m_1 < i \leq m_2$ *(this condition may be empty);*

$c_1)$ $a_i \leq a_{i-1}$ *for* $i > m_2$*;*

$d_1)$ $a_i = 0$ *for* $i \gg 0$.

Then there is a non-stable rank 2 bundle E on $\mathbb{P}^2$ *and a constant* $k \in \mathbb{Z}$ *such that* $\forall i \geq 0$ $\Delta h^1 E(-i+k) = a_i$*. If further* $m_1 = 0$*, then E can be chosen semi-stable.*

Proof. Put $e = 2m_2 - 3$ or $e = 2m_2 - 2$ and let q be the minimum integer such that $q \geq \frac{e+1}{2}$ Define

$$x_m = \begin{cases} m+1 & \text{for } 0 \leq m \leq \frac{e+1}{2} - m_1 \\ x_{m-1} & \text{for } \frac{e+1}{2} - m_1 < m \leq \frac{e+2}{2} + m_1 \\ e+2-m-a_{m-q} & \text{for } \frac{e+2}{2} + m_1 < m \leq e+1 \\ a_{m-q} & \text{for } m > e+1. \end{cases}$$

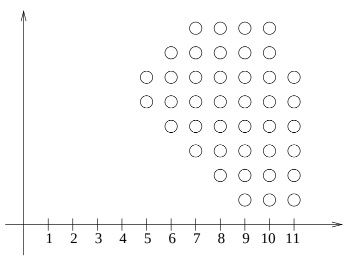
Now the proof goes just as in Theorem 4 and k is the minimum integer such that $k \leq -\frac{c_1+2}{2}$. The bundle E we get here is not stable by Proposition 5, but when $m_1 = 0$, by choosing $e = 2m_2 - 3$ we get a semi-stable bundle. □

EXAMPLE 2. It could be interesting to show directly how the construction works. Let us begin with the stable bundle case.

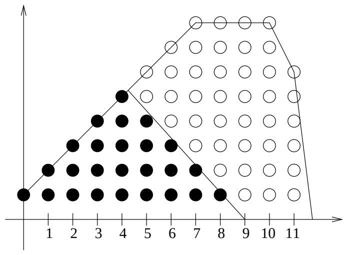
Pick a sequence satisfying $a_2) - e_2)$, for instance

i	0	1	2	3	4	5	6	7	≥ 8
a_i	0	2	4	6	7	8	8	6	0

which satisfies the inequalities, with $m_1 = 3$, $m_2 = 5$. Here $e = 7$, $q = 4$. One can, by hand, build up the sequence $\{x_m\}$ as follows: for $i \leq m_2$, starting from $q+1$, write a vertical row of a_i points, pushing for every i the row one step downward, in such a way that the row of a_{m_2} begins from 0, then align on the right the rows representing the a_i's for $i > m_2$.



Then fill the triangular shape at the bottom: this represents the sequence $\Delta h_Z(m)$.



The set of points Z here is given by 66 points lying on a curve of degree 8 (and not less) and having $b = 11$.

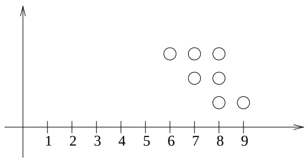
The corresponding bundle E has $c_1 = e+3 = 10$, $c_2 = 66$.

For the unstable case, take

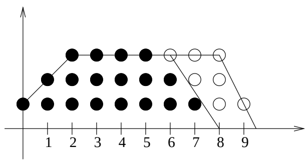
i	0	1	2	3	4	5	≥ 6
a_i	0	0	1	2	3	1	0

which satisfies $a_1) - d_1)$ of Theorem 5. Choose $e = 2m_2 - 2 = 6$ (here $m_1 = 1$, $m_2 = 4$).

In a diagram, starting from $\frac{e+1}{2} = \frac{7}{2}$ leave $m_1 + 2 = 2$ free places on the right and perform the same procedure as above.



Then fill the trapezoidal shape on the left; this gives $\{x_m\}$.



Here Z is formed by 25 points on a curve of degree 3 (and not less) and it has $b = 9$. The vector bundle E has Chern classes $c_1 = e+3 = 9$, $c_2 = 25$.

4. Some consequences of the theorems

Through this section, we write down a series of corollaries, which can be used to build several cohomological criteria for rank 2 bundles in $\mathbb{P}^2$.

We point out first the following (more or less known) criterion for the splitting of a bundle E. It is the analogue of the splitting criterion of [3], but it does not follow from it.

PROPOSITION 4. *Let q be the maximum integer such that $q \leq -\frac{c_1(E)+2}{2}$ then $h^1(E(q)) = 0$ if and only if E splits.*

Proof. If E splits, then $h^1(E(n)) = 0\ \forall n$. Conversely assume $h^1(E(q)) = 0$; by Propositions 2 and 3, $\Delta h^1(E(n)) \geq 0$ for $n \leq -\frac{c_1(E)+2}{2}$. Hence $h^1(E(q)) = 0$ implies $h^1(E(n)) = 0$ for $n \leq -\frac{c_1(E)+2}{2}$, thus, by antisymmetry, $h^1(E(n))$ vanishes for all n. It follows that E splits, by Horrocks' criterion. □

Next, we show how the cohomology sequence can be used to decide whether a bundle is stable or not.

PROPOSITION 5.

a) *A vector bundle E of rank 2 on $\mathbb{P}^2$ is stable if and only if:*

$$\begin{array}{ll} \Delta h^1(E(-\frac{c_1+4}{2})) = 2 & \text{when } c_1 \text{ is even} \\ \Delta h^1(E(-\frac{c_1+3}{2})) = 1 & \text{when } c_1 \text{ is odd.} \end{array}$$

b) *A vector bundle as above is semi-stable but not stable if and only if c_1 is even and $\Delta h^1(E(-\frac{c_1+2}{2})) = 1$.*

Proof. This is immediate from condition B_1 of Proposition 2 and condition B_2 of Proposition 3. □

The previous criterion can be extended to determine precisely the *stability index* of a rank 2 bundle.

DEFINITION 3. *Let k be the minimum integer such that $E(k)$ has a non-zero section. We define the* stability index *$s(E)$ as the number $k + \frac{c_1}{2}$ which is invariant under twisting of E.*

The stability index gives a measure of the "stability" of E, indeed one can see that E is stable (resp. semi-stable) if and only if $s(E) > 0$ (resp. $s(E) \geq 0$).

PROPOSITION 6. *Let x be the minimum integer such that*

$$x < -\frac{c_1+2}{2} \text{ and } \Delta h^1(E(x)) = \Delta h^1(E(x-1)) + 1.$$

Then $|s(E)| = -\left(x + 2 + \frac{c_1}{2}\right)$.

Proof. Twisting E the number $x + 2 + \frac{c_1}{2}$ does not change, so assume E satisfies condition (*) and further $h^0(E(-1)) \neq 0$. Take the zero locus Z of a general section of E; looking at

$$0 \to \mathscr{O} \to E \to \mathscr{I}_Z(c_1) \to 0,$$

with the usual notation we get $k = a - c_1$; if E is stable then, by Proposition 3, $x = -a-2$ so that $s(E) = -x-2-\frac{c_1}{2}$ which is positive by our assumption on x.

If E is not stable, $x = a - c_1 - 2$ by Proposition 2, so that $s(E) = x+2+\frac{c_1}{2}$. □

There are several other characteristic numbers of the bundle E which can be read soon in the sequence $\Delta h^1(E)$.

For instance, let us look at the Atiyah invariant $\Delta(E) = c_1^2 - 4c_2$ (see [8]). It is well know that if E is stable, then $\Delta < 0$.

PROPOSITION 7. *Let E be stable. Then*

$$\Delta(E) = \begin{cases} -4\sum_{n\leq -\frac{c_1+2}{2}} \Delta h^1(E(n)) & \text{if } c_1 \text{ is even} \\ -1-4\sum_{n\leq -\frac{c_1+2}{2}} \Delta h^1(E(n)) & \text{if } c_1 \text{ is odd.} \end{cases}$$

Proof. Twist E until it satisfies (*); then $c_2 = \text{degree}(Z)$ is the area enclosed in the shape of diagram (8), while the non-shadowed area in the diagrams represents $\frac{c_1^2}{4}$ or $\frac{c_1^2-1}{4}$ according to the residue of c_1 mod 2. □

EXAMPLE 3. There is not an analogue of the above formula for unstable bundles. Indeed choose bundles E and E', built as in Theorem 5, whose cohomology sequences (in the range $(\frac{c_1+2}{2}, +\infty)$) are respectively:

$$0 \mid 0 \mid 1 \mid 2 \mid 3 \mid 0 \mid \dots$$

and

$$0 \mid 1 \mid 2 \mid 3 \mid 0 \mid \dots$$

We get Chern classes $c_1(E) = 9, c_2(E) = 24, c_1(E') = 7, c_2(E') = 18$ hence $\sum \Delta h^1(E(n)) = \sum \Delta h^1(E'(n))$ but $\Delta(E) = -15 \neq \Delta(E') = -23$.

5. Cayley-Bacharach numbers and the effect of twisting

Let Z be a smooth set of points, e a CB for Z and let E be the associated bundle. If, for some n, $E(n)$ has a section whose zero locus is itself a smooth set of points Z', then one would expect strong relations between the sequence $\Delta h_Z(m)$ and $\Delta h_{Z'}(m)$. In fact, $\Delta h_{Z'}(m)$ can be reconstructed once we know $\Delta h_Z(m)$, as the following proposition shows.

PROPOSITION 8. *In the above setting, put* $n \geq 0$. *Then*

$$\Delta h_{Z'}(m) - \Delta h_Z(m-n) = \begin{cases} 0 & \text{if } m < 0 \\ m+1 & \text{if } 0 \leq m < n \\ n & \text{if } n \leq m < e+3+n \\ 2n+e+2-m & \text{if } e+3+n \leq m < e+3+2n \\ 0 & \text{if } m \geq 2n+e+3. \end{cases}$$

Proof. By
$$0 \to \mathscr{O} \to E(n) \to \mathscr{I}_{Z'}(e+3+2n) \to 0$$
we get:
$$\begin{aligned}\Delta h_{Z'}(m) &= h^0(\mathscr{O}(m)) - h^0(\mathscr{I}_{Z'}(m)) - h^0(\mathscr{O}(m-1)) + h^0(\mathscr{I}_{Z'}(m-1))\\ &= h^0(\mathscr{O}(m)) - h^0(\mathscr{O}(m-1)) - h^0(\mathscr{O}(m-e-3-2n))\\ &\quad -h^0(\mathscr{O}(m-e-4-2n)) - h^0(E(m-e-3-n))\\ &\quad +h^0(E(m-e-4-n)).\end{aligned}$$

Similarly
$$\begin{aligned}\Delta h_Z(m-n) &= h^0(\mathscr{O}(m-n)) - h^0(\mathscr{I}_Z(m-n)) - h^0(\mathscr{O}(m-n-1))\\ &\quad +h^0(\mathscr{I}_Z(m-n-1))\\ &= h^0(\mathscr{O}(m-n)) - h^0(\mathscr{O}(m-n-1)) - h^0(\mathscr{O}(m-e-3-n))\\ &\quad -h^0(\mathscr{O}(m-e-4-n)) - h^0(E(m-e-3-n))\\ &\quad +h^0(E(m-e-4-n))\end{aligned}$$

from which
$$\begin{aligned}\Delta h_{Z'}(m) - \Delta h_Z(m-n) &= [h^0(\mathscr{O}(m)) - h^0(\mathscr{O}(m-1))]\\ &\quad +[h^0(\mathscr{O}(m-e-3-2n)) - h^0(\mathscr{O}(m-e-4-2n))]\\ &\quad -[h^0(\mathscr{O}(m-n)) - h^0(\mathscr{O}(m-e-4-n))]\\ &\quad -[h^0(\mathscr{O}(m-e-3-n)) - h^0(\mathscr{O}(m-e-4-n))]\end{aligned}$$

and the proposition follows. □

Let Z be a smooth set of points in $\mathbb{P}^2$ and assume we know the whole Hilbert function of Z.

Are we able to establish which number is a CB number for Z ?

The answer is negative. In the following example we produce two smooth sets of points with the same Hilbert function but with a different behaviour with respect to the Cayley-Bacharach property.

EXAMPLE 4. Consider the sequence

m	0	1	2	3	4	$\cdots$
x_m	1	2	3	1	0	$\cdots$

which satisfies the hypothesis of Theorem 6. We know that 2 is a CB-number of a smooth set of points Z such that $x_m = \Delta h_Z(m)$, $\forall m$: it is enough to take Z to be seven points in general position.

In fact, going through the proof of Theorem 6, we get a sequence y_m which is 1 at 0 and vanishes elsewhere. Hence Z' is formed by one point with -2 as CB-number and gives a vector bundle E and a sequence $0 \to \mathscr{O} \to E \to \mathscr{I}_{Z'}(1) \to 0$. Since $\mathscr{O}(2)$ and $\mathscr{I}_{Z'}(3)$ are generated by global sections and $H^1(\mathscr{O}(2)) = 0$, then also

$E(2)$ is generated by global sections. A general section of $E(2)$ has as zero locus $Z = \{7 \text{ general points in } \mathbb{P}^2\}$, as one can see by comparing the above sequence with $0 \to \mathscr{O} \to E(2) \to \mathscr{I}_Z(5) \to 0$.

On the other hand, not every smooth set of points having $\{x_m\}$ as difference Hilbert function admits 2 as a CB. For instance, one gets an example by taking

$$Z = \{6 \text{ points on a conic }\} \cup \{\text{ one general point in } \mathbb{P}^2\}.$$

Observe that, as explained in [2] (but see also [6]), there are in fact sets of points with the same Hilbert function as a complete intersection, which are not complete intersections themselves.

In order to see an example, just take the union of three points on a line and one general point.

A different question arises if, instead of fixing the set Z, we just fix a "difference Hilbert function" (that is, a sequence $\{x_m\}$ satisfying conditions A)-E) of Theorem 2). We know that there exists some smooth set of points Z with $\Delta h_Z(m) = x_m$. Now we ask:

QQQ) For which numbers e does there exists a smooth set of points Z with $\Delta h_Z(m) = x_m$ $\forall m$, such that e is a CB for Z?

We give a partial answer to this question here.

From now on, let $\{x_m\}$ be a sequence, satisfying condition A)-E) of Theorem 2:

A) $x_m \geq 0 \ \forall m$;

B) $x_m = 0$ for $m < 0$ and $m \gg 0$;

C) there is $a > 0$ such that $x_m = x_{m-1} + 1$ for $0 \leq m < a$;

D) there is $b \geq a$ such that $x_m = x_{m-1} = a$ for $a \leq m < b$;

E) $x_b < x_{b-1}$ and $\forall m > b \ x_m \leq x_{m-1}$.

THEOREM 6. *Let $\{x_m\}$ be a sequence as above, and define $\omega = \min\{m \geq 0 : x_m = 0\}$. Assume further that $\forall m \geq b$, $m \leq \omega$, we have $x_m < x_{m-1}$. Then there is a smooth set of points Z such that $\forall m \ \Delta h_Z(m) = x_m$ and $w - 2$ is a CB-number for Z.*

First observe that if Z is a set of points with $\Delta h_Z(m) = x_m$, $\forall m$, then $\Delta h_Z(\omega) = 0$ implies that $\omega - 1$ is not a CB-number of Z by Lemma 1,b), and $\omega - 2$ is the maximal candidate to be a CB-number for Z.

Secondly, note that the condition $x_m < x_{m-1}$ for $b \leq m \leq \omega$ is satisfied when $\{x_m\}$ is the difference Hilbert function of a set of points contained in a curve of minimal degree which is irreducible.

Proof. Put $l = \omega - b + 1$; clearly $l > 0$ by construction of ω and b. Consider the function

$$\varphi_{\omega,l}(m) = \begin{cases} 0 & \text{if } m < 0 \text{ or } m \geq \omega \\ m+1 & \text{for } 0 \leq m < l \\ l & \text{for } l \leq m \leq \omega - l \\ \omega - m & \text{for } \omega - l < m < \omega \end{cases}$$

which is the difference Hilbert function of a complete intersection of curves of degree l and $\omega - l$.

By the assumption $x_m < x_{m-1}$ for $b \leq m \leq \omega$ we see immediately that $a \geq l$ and $a = l$ if and only if $\varphi_{\omega,l}(m) = x_m$, $\forall m$, in which case the Theorem is trivial.

If $a > l$, then it is easy to see that the function $y_m = x_{m+l} - \varphi_{\omega,l}(m+l)$ also satisfies conditions A)-E) above, where the new critical numbers now are $a' = a - l$, $b' = b - l$. Let Z' be a smooth set of points such that $\Delta h_{Z'}(m) = y_m$, $\forall m$. We have $\omega - 2l < b' - l$ hence $w - 2l - 2$ is a CB-number for Z' by Lemma 1a). Let E be the associated bundle. If we can prove that $E(l)$ has a section whose zero locus Z is smooth then, by Proposition 8, we must have $\Delta h_Z(m) = x_m$, $\forall m$, and we are done.

To prove the existence of such a Z, we will show that $E(l)$ is generated by global sections, then apply [8, 1.4]. We have $h^2(E(l-2)) = h^0(E(l-\omega-2))$ by duality, and the last one is 0 by $0 \to \mathcal{O} \to E \to \mathcal{I}_{Z'}(\omega - 2l + 1) \to 0$ since $0 < l < a \leq \omega$. Moreover $h^1(E(l-1)) = h^1(E(l-\omega-3))$ but, $\forall n \gg 0$, $\Delta h^1 E(l-\omega-3-n) = \Delta h_{Z'}(\omega+n-l)$ by Proposition 1, and $\Delta h_{Z'}(\omega+n-l) = y_{\omega+n-l} = x_{\omega+n} - \varphi_{\omega,l}(\omega+n) = 0$; this implies $h^1(E(l-1)) = 0$. Now we can apply to $E(l)$ the Castelnuovo-Mumford criterion ([10, §14]), which implies that E is generated by global sections. □

As pointed out in the introduction (Example 1), there are non–split rank 2 bundles with sections vanishing on a complete intersection set of points Z. This situation happens when one starts with the complete intersection Z of type a, b and chooses $e < a + b - 3$ as a CB for Z.

In this situation, however, we have the following restriction for the sequence $a_n = h^1(E(n))$.

THEOREM 7. *Let* $\{a_i\}$ $i \geq 0$ *be a sequence of non-negative integers.*

There exists a stable *rank 2 bundle E on* $\mathbb{P}^2$*, associated with a complete intersection set Z, and a constant* $k \in \mathbb{Z}$ *such that* $\forall i \geq 0$ $\Delta h^1 E(-i+k) = a_i$*, if and only if* $\{a_i\}$ *satisfies the following property: there are numbers* $0 < m_1 \leq m_2 \leq m_3$ *with*

$a_s)$ $a_0 \leq 1$

$b_s)$ $a_i = a_{i-1} + 2$ *for* $0 < i \leq m_1$

$c_s)$ $a_i = a_{i-1} + 1$ *for* $m_1 < i \leq m_2$ *(this condition may be empty)*

$d_s)$ $a_i = a_{i-1}$ *for* $m_2 < i \leq m_3$ *(this condition may be empty)*

$e_s)$ $a_i = a_{i-1} - 1$ *for* $i > m_3$*, unless* $a_{i-1} = 0$.

There exists a non–stable *rank 2 bundle E on* $\mathbb{P}^2$*, associated with a complete intersection set Z, and a constant* $k \in \mathbb{Z}$ *such that* $\forall i \geq 0$ $\Delta h^1 E(-i+k) = a_i$*, if and only if* $\{a_i\}$ *satisfies the following property: there are numbers* $0 < m_1 \leq m_2 \leq m_3$ *with*

$a_u)$ $a_i = 0$ *for* $0 \leq i \leq m_1$

$b_u)$ $a_i = a_{i-1} + 1$ *for* $m_1 < i \leq m_2$

$c_u)$ $a_i = a_{i-1}$ *for* $m_2 < i \leq m_3$

$d_u)$ $a_i = a_{i-1} - 1$ *for* $i > m_3$*, unless* $a_{i-1} = 0$.

Proof. This is a simple computational consequence of the theorems of section 3, and the fact that a sequence of integers is the Hilbert function of a complete intersection set of points if and only if it satisfies the numerical conditions of theorem 2 with E) replaced by:

E′) for $n \geq b$, either $x_n = 0$ or $x_n = x_{n-1} - 1$.

□

Notice that, as there are sets of points which have the same Hilbert function as a complete intersection, one cannot push forward the task of characterizing rank 2 bundles associated to a complete intersection set of points from the dimensions of its cohomology groups.

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