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DIRAC STRUCTURES AND GAUGE SYMMETRIES OF PHASE SPACES

Abstract. We study the geometry of the phase space of a particle in a Yang-Mills-Higgs field in the context of the theory of Dirac structures. Several known constructions are merged into the framework of coupling Dirac structures. Functorial properties of our constructions are discussed and examples are provided. Finally, applications to fibered symplectic groupoids are given.

1. Introduction

In symplectic geometry, the formulation of the so-called minimal coupling principle uses principal bundles over symplectic manifolds and Hamiltonian G-spaces (see [St77] and [We78]). By a *classical Yang-Mills-Higgs setup*, we mean a triple (G,P,F) formed by a Lie group G, a principal G-bundle P over a smooth manifold M, and a Hamiltonian Poisson G-space F (see [We87]). Given any classical Yang-Mills-Higgs setup, there is an induced Hamiltonian G-action on $T^*M \times F$ with momentum map $J : T^*M \times F \to \mathfrak{g}^*$, where $\mathfrak{g}^*$ is the dual of the Lie algebra of G. The quotient $J^{-1}(0)/G$ is a Poisson manifold. This is the *Yang-Mills-Higgs phase space of a particle with configuration space M and internal phase space F*. Every connection θ on P, called a *gauge potential*, induces a diffeomorphism $\psi_\theta : J^{-1}(0)/G \to P' \times_G F$, where P' denotes the pull-back of P to T^*M by the canonical projection map $T^*M \to M$. When F is symplectic, one gets the Sternberg-Weinstein phase space of a particle in a Yang-Mills field (see [We78]). Moreover, when F is a Lie-Poisson manifold, one gets a gauged Poisson structure as defined in [MMR84]. The key data in this model for the phase space of a particle interacting with a gauge field is the pull-back principal G-bundle $P' \to (T^*M, \omega_{\text{can}})$ together with the gauge potential θ. The symplectic manifold $(T^*M, \omega_{\text{can}})$ is considered as the canonical phase space of a particle with configuration space M. However, the evolution space in the sense of Souriau [S70] is a presymplectic manifold. Pre-symplectic structures naturally appear in the study of the Hamiltonian dynamics of particles with gauge degrees of freedom.

In this paper, we describe the global geometric object induced on the associated bundle $P \times_G F$ if, instead of the above pull-back bundle $P' \to (T^*M, \omega_{\text{can}})$, one works with a principal bundle P over a given presymplectic manifold $(B = P/G, \omega_B)$. We are naturally led to the theory of Dirac structures on manifolds, which allows us to extend Sternberg's construction of a coupling form. We obtain Theorem 1, which generalizes a result proven in [Wa05] and [BF07]. Connections between the minimal coupling construction and the theory of Dirac structures were first observed in [DuW04], extending Vorobjev's setting [Vo00] from the Poisson category to the Dirac category. In [BF07], Brahic and Fernandès discuss Poisson fibrations based on gauge theory and

Dirac geometry. While they emphasize the integration problem for Poisson fibrations, we focus on the gauge transformations and the functorial property of our construction of Poisson fiber bundles. We show that submersive (resp. immersive) morphisms of Yang-Mills-Higgs setups having presymplectic base together with Ehresmann connections induce forward (resp. backward) Dirac maps of coupling structures. In Section 2, we present basic definitions and results needed here. Our main results with examples are established in Sections 3 and 4. Applications to fibered symplectic groupoids are given in Section 5.

2. Coupling Dirac structures

2.1. Dirac structures

Let N be a smooth finite-dimensional manifold. Given $(X_1,\alpha_1),\ (X_2,\alpha_2)$ smooth sections of $TN \oplus T^*N$ we consider the *symmetric pairing*

$$\langle (X_1,\alpha_1),(X_2,\alpha_2)\rangle = \frac{1}{2}\Big(\alpha_1(X_2)+\alpha_2(X_1)\Big)$$

and the *Courant bracket*

$$[(X_1,\alpha_1),\ (X_2,\alpha_2)] = ([X_1,X_2],\ \mathcal{L}_{X_1}\alpha_2 - i_{X_2}d\alpha_1).$$

A *Dirac structure* on N (see [C90]) is a sub-bundle $L \subset TN \oplus T^*N$ which is maximally isotropic with respect to the symmetric pairing $\langle\cdot,\cdot\rangle$ and whose space of sections is closed under the Courant bracket.

If (N_1,L_1) and (N_2,L_2) are two Dirac manifolds then a map $\psi : N_1 \to N_2$ is called a *forward Dirac map* if

$$L_2 = \{(T\psi(X),\beta) \mid X \in TN_1,\ \beta \in T^*N_2,\ \text{and}\ (X,(T\psi)^*\beta) \in L_1\}$$

and a *backward Dirac map* if

$$L_1 = \{(X,(T\psi)^*\beta) \mid X \in TN_1,\ \beta \in T^*N_2,\ \text{and}\ (T\psi(X),\beta) \in L_2\}.$$

2.2. Geometric data

Let $\pi : E \to B$ be a smooth fiber bundle. We denote $\mathrm{Vert} = \mathrm{Ker}(T\pi) \subset TE$. An *Ehresmann connection* on E is a surjective bundle map $\Gamma : TE \to \mathrm{Vert}$ such that, at each point $e \in E$, $\Gamma_e^2 = \Gamma_e$ and given any smooth path

$$\begin{array}{rll} c: & [0,1] & \to B \\ & t & \mapsto c(t) \end{array}$$

joining x_0 to x_1 and for any $y_0 \in \pi^{-1}(x_0)$, there exists a unique horizontal lift $t \mapsto \gamma(t)$ in E so that $\gamma(0) = y_0$ and $\pi(\gamma(t)) = c(t)$ for all t. Considering the horizontal sub-bundle $\mathrm{Hor}_\Gamma = \ker\Gamma$, one gets the splitting:

$$TE = \mathrm{Hor}_\Gamma \oplus \mathrm{Vert}.$$

Given any vector field $X \in \mathfrak{X}(B)$, we denote by $hor_\Gamma(X) \in \mathfrak{X}(E)$ its horizontal lift. Define the operator

$$\partial_\Gamma : \Omega^k(B) \otimes C^\infty(E) \to \Omega^{k+1}(B) \otimes C^\infty(E)$$

by setting

$$\begin{aligned} \partial_\Gamma \alpha(X_0, \ldots, X_k) &= \sum_{i=0}^{k} (-1)^i \mathcal{L}_{hor_\Gamma(X_i)}(\alpha(X_0, \ldots, \widehat{X_i}, \ldots, X_k)) \\ &\quad + \sum_{i<j} (-1)^{i+j} \alpha([X_i, X_j], X_0, \ldots, \widehat{X_i}, \ldots, \widehat{X_j}, \ldots, X_k), \end{aligned}$$

where $hor_\Gamma(X)$ is the Γ-horizontal lift of $X \in \mathfrak{X}(B)$ and $\mathcal{L}_{hor_\Gamma(X_i)}$ the Lie derivative along the vector field $hor_\Gamma(X_i)$. Define the curvature of Γ as follows:

$$\mathrm{Curv}_\Gamma(X,Y) = hor_\Gamma([X,Y]) - [hor_\Gamma(X), hor_\Gamma(Y)],$$

for all $X, Y \in \mathfrak{X}(B)$.

DEFINITION 1. Geometric data *on the fiber bundle* $\pi : E \to B$ *consist of a triple* $(\mathcal{V}, \Gamma, \mathbb{F})$ *formed by an Ehresmann connection* Γ*, a vertical bivector field* $\mathcal{V}$*, and a* $C^\infty(E)$*-valued 2-form* $\mathbb{F}$ *on B. It is* integrable *if the following properties are satisfied:*

- $\mathcal{V}$ *is a Poisson tensor, i.e.* $[\mathcal{V}, \mathcal{V}] = 0$*;*
- $\mathcal{V}$ *is preserved by parallel transport, i.e.* $\mathcal{L}_{hor_\Gamma(X)}\ \mathcal{V} = 0,\ \forall X \in \mathfrak{X}(B)$;
- $\partial_\Gamma \mathbb{F} = 0$;
- $\mathrm{Curv}_\Gamma(X,Y) = \mathcal{V}^\sharp(d(\mathbb{F}(X,Y))),\ \forall X, Y \in \mathfrak{X}(B)$.

REMARK 1. **a)** In contrast with Vorobjev's definition [Vo00], we do not assume that $\mathbb{F}$ is non-degenerate in the above definition.
b) The third property of Definition 1 means that the 2-form $\overline{\mathbb{F}}$ defined by

$$\overline{\mathbb{F}}(hor_\Gamma(X), hor_\Gamma(Y))_e = \mathbb{F}(X,Y)_{\pi(e)} \quad \forall X, Y \in \mathfrak{X}(B),\ \forall e \in E \tag{1}$$

is horizontally-closed.

2.3. Coupling Dirac structures

DEFINITION 2. *Let* $\pi : E \to B$ *be a smooth fiber bundle. A Dirac structure L on E is called a* coupling Dirac structure *if there exists geometric data* $(\mathbb{F}, \Gamma, \mathcal{V})$ *such that*

$$L = \left\{ (\overline{X}, i_{\overline{X}} \overline{\mathbb{F}}) + (\mathcal{V}^\sharp \alpha, \alpha) \mid \overline{X} \in \mathrm{Hor}_\Gamma,\ \alpha \in \mathrm{Ann}(\mathrm{Hor}_\Gamma) \right\}. \tag{2}$$

We refer the reader to Vaisman's paper [Va05] for more details about the properties of coupling Dirac structures.

REMARK 2. Given a coupling Dirac structure L on E, the distribution $\mathcal{D}$ generated by the horizontal vector fields $\overline{X}$ satisfying $i_{\overline{X}}\overline{\mathbb{F}} = 0$ defines a foliation $\mathcal{F}$ called the *characteristic foliation* or the *null foliation* of L. When L is reducible [LWX98], the quotient $E/\mathcal{F}$ is a Poisson manifold called the *space of motion*.

We will use the following result that can be found in [Wa05].

PROPOSITION 1 ([Wa05]). *Let $\pi : E \to B$ be a smooth fiber bundle. The integrability of geometric data $(\mathcal{V}, \Gamma, \mathbb{F})$ is equivalent to the fact that the space of smooth sections of the corresponding sub-bundle $L \subset TE \oplus T^*E$ (defined as in Equation (2)) is closed under the Courant bracket.*

3. Construction of phase spaces

First, we will prove the following result:

THEOREM 1. *Let (G, P, F) be a classical Yang-Mills-Higgs setup. Assume that the base manifold $B = P/G$ is equipped with a pre-symplectic form ω_B. Then every connection θ on P induces a coupling Dirac structure on the associated bundle $E = P \times_G F$ which restricts to the Poisson structure along the fibers of E inherited from the Poisson manifold $(F, \mathcal{V}_F)$.*

Under the notations and assumptions of Theorem 1, the connection θ on P induces a connection Γ on $E = P \times_G F$. We have the splitting

$$TE = \mathrm{Hor}_\Gamma \oplus \mathrm{Vert}.$$

Moreover, the Γ-horizontal lift of $X \in \mathfrak{X}(B)$ is given by

$$hor_\Gamma(X)_{[p,f]} = T_{(p,f)}\pi_{P\times F}(\overline{X}_p, \mathbf{0}_f), \tag{3}$$

where $[p, f] \in E = P \times_G F$ is the equivalent class of $(p, f) \in P \times F$, $\pi_{P\times F} : P \times F \to P \times_G F$ is the canonical projection, $\overline{X}$ is the θ-horizontal lift of $X \in \mathfrak{X}(B)$ and $\mathbf{0}_f$ is the zero tangent vector at f. Define the vertical bivector field $\mathcal{V}$ as follows

$$\mathcal{V} = (\pi_{P\times F})_* \mathcal{V}_F \tag{4}$$

We have the following lemma:

LEMMA 1. *Under the above notations, we have*

- *$\mathcal{V}$ is a vertical Poisson tensor;*
- *$\mathcal{L}_{hor_\Gamma(X)}\, \mathcal{V} = 0$, for any $X \in \mathfrak{X}(B)$.*

Proof. The fact that the Schouten bracket $[\mathcal{V}, \mathcal{V}]$ vanishes follows immediately from $[\mathcal{V}_F, \mathcal{V}_F] = 0$. Furthermore, it follows from Equations (3)-(4) that one has $\mathcal{L}_{hor_\Gamma(X)}\, \mathcal{V} = 0$, for any $X \in \mathfrak{X}(B)$. □

LEMMA 2. *Under the assumptions of Theorem 1, there exist induced integrable geometric data on E.*

Proof. Taking into account Lemma 1, we only have to find a suitable 2-form $\mathbb{F} \in \Omega^2(B) \times C^\infty(E)$. To construct such a 2-form we will use the momentum map associated with the Hamiltonian G-action on F, denoted by $J : F \to \mathfrak{g}^*$, where $\mathfrak{g}^*$ is the dual of the Lie algebra of G. The infinitesimal action $\rho_F : \mathfrak{g} \to \mathfrak{X}(F)$ transforms an element $\xi \in \mathfrak{g}$ into a Hamiltonian vector field $\mathcal{V}_F^\sharp(dJ_\xi)$, where

$$J_\xi(f) = \langle J(f),\ \xi\rangle,$$

for all $f \in F$. Consider the connection 1-form $\theta \in \Omega^1_{\text{Vert}}(P) \otimes \mathfrak{g}$. The curvature of θ is the horizontal $\mathfrak{g}$-valued 2-form given by

$$\text{Curv}_\theta\Big(\overline{X}_p, \overline{Y}_p\Big) = \theta_p\Big([\overline{X}, \overline{Y}]_p - \overline{[X,Y]}_p\Big), \tag{5}$$

where $\overline{X}, \overline{Y} \in \mathfrak{X}(P)$ are the θ-horizontal lifts of $X, Y \in \mathfrak{X}(B)$, respectively. Next, we define

$$\Big(\mathbb{G}(X,Y)\Big)([p,f]) = \Big\langle J(f),\ \text{Curv}_\theta\Big(\overline{X}_p,\ \overline{Y}_p\Big)\Big\rangle, \tag{6}$$

for all $X, Y \in \mathfrak{X}(B)$. Let ω_B be the pre-symplectic form on B. Define

$$\mathbb{F} = \omega_B \otimes 1 + \mathbb{G}. \tag{7}$$

We will show that $\text{Curv}_\Gamma(X,Y) = \mathcal{V}^\sharp\Big(d(\mathbb{F}(X,Y))\Big)$, for all $X, Y \in \mathfrak{X}(B)$. Notice that the curvature of θ and that of Γ are related as follows

$$\Big(\text{Curv}_\Gamma(X,Y)\Big)([p,f]) = T_{(p,f)}\pi_{P\times F}\Big(\mathbf{0}_p,\ (\rho_F \circ \text{Curv}_\theta(\overline{X}_p, \overline{Y}_p))(f)\Big), \tag{8}$$

where $\mathbf{0}_p$ is the zero tangent vector at p, ρ_F is the infinitesimal action associated to the G-action on F. Since it is enough to work with local coordinates, we pick a local system of coordinates $(x_1, \ldots, x_{2s})$ on B. Set

$$\mathbb{G}_{ij} = \mathbb{G}\Big(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\Big) \quad \text{and} \quad \mathbb{F}_{ij} = \mathbb{F}\Big(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\Big).$$

From Equations (4) and (6), one gets

$$\begin{aligned}\text{Curv}_\Gamma\Big(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\Big)([p,f]) &= T_{(p,f)}\pi_{P\times F}\Big(\mathbf{0}_p,\ \mathcal{V}_F^\sharp(d\mathbb{G}_{ij})(f)\Big)\\ &= T_{(p,f)}\pi_{P\times F}\Big(\mathbf{0}_p,\ \mathcal{V}_F^\sharp(d\mathbb{F}_{ij})(f)\Big),\end{aligned}$$

since the components $(\omega_B)_{ij}$ of ω_B satisfy $\mathcal{V}_F^\sharp(d(\omega_B)_{ij}) = 0$. Therefore,

$$\text{Curv}_\Gamma(X,Y) = \mathcal{V}^\sharp(d(\mathbb{F}(X,Y))),\ \forall X, Y \in \mathfrak{X}(B).$$

Moreover,

$$\partial_\Gamma \mathbb{F} = d\omega_B \otimes 1 + \partial_\Gamma \mathbb{G}.$$

Since ω_B is closed, we get $\partial_\Gamma \mathbb{F} = \partial_\Gamma \mathbb{G}$. To show that $\partial_\Gamma \mathbb{G} = 0$, we use classical arguments. Precisely, we define the vertical 1-form Φ on $P \times_G F$ as follows:

$$\Phi_{[p,f]}([(X_p, Z_f)]) = \left\langle J(f),\ \theta_p(X_p) \right\rangle,$$

where $[(X_p, Z_f)] = T_{(p,f)}\pi_{P\times F}(X_p, Z_f)$. One can easily check that

$$\mathbb{G}(X,Y) = d\Phi(hor_\Gamma(X), hor_\Gamma(Y)).$$

Consequently $\partial_\Gamma \mathbb{G} = 0$ since $d^2\Phi = 0$. Hence the triple $(\mathcal{V}, \Gamma, \mathbb{F})$ defines integrable geometric data on $E = P \times_G F$. □

Theorem 1 follows immediately from Proposition 1 and Lemma 2.

EXAMPLE 1. Consider the trivial principal bundle $SU(2) \times \mathbb{R}^3 \to \mathbb{R}^3$ together with the Hamiltonian $SU(2)$-space $\mathfrak{su}(2)^*$, that is, the dual of the Lie algebra $\mathfrak{su}(2)$ endowed with its canonical linear Poisson structure. Here $\mathbb{R}^3$ is equipped with the exact form $\omega = d\eta$ defined in the standard linear coordinates (x,y,z) by $\eta = x \wedge dy$. Given any connection θ on $SU(2) \times \mathbb{R}^3$, we can apply Theorem 1 which provides a Dirac structure on $\left(SU(2) \times \mathbb{R}^3\right) \times_{SU(2)} \mathfrak{su}(2)^*$.

EXAMPLE 2. Consider the principal $U(1)$-bundle $S^5 \to \mathbb{C}P^2$ and $F = S^2$. We equip $\mathbb{C}P^2$ with its canonical homogeneous Kähler metric and the 2-sphere with its canonical symplectic structure. By Theorem 1, any connection on $S^5 \to \mathbb{C}P^2$ determines a Dirac structure on the associated bundle $S^5 \times_{U(1)} S^2$.

REMARK 3. The proof of Theorem 1 is quite similar to that of Theorem 3.2 in [Wa05]. However, the fact that the base manifold B is presymplectic leads to a major improvement. Indeed, Theorem 1 includes both the construction of the phase space for minimal coupling for gauge fields and Theorem 3.2 in [Wa05]. These two known results correspond to the extremal cases where the rank of the presymplectic form ω_B is equal to zero or dim B.

Another consequence of Theorem 1 is:

COROLLARY 1 (Weak coupling Poisson structures). *Let (G,P,F) be a classical Yang-Mills-Higgs setup and θ a connection on $P \to B$. Assume that both F and P are compact and the base B is equipped with a symplectic form ω_B. Then there is 1-parameter family of Poisson structures Π_ε on the associated bundle $E = P \times_G F$ such that each Poisson structure Π_ε restricts to the Poisson structure along the fibers of E which is inherited from $(F, \mathcal{V}_F)$.*

Proof. The additional condition saying that P and F are compact ensures that we can choose a real number $\varepsilon > 0$ sufficiently small so that the 2-form

$$\mathbb{F}_\varepsilon = \omega_B \otimes 1 + \varepsilon \mathbb{G}$$

is non-degenerate. Replacing $\mathbb{F}$ by $\mathbb{F}_\varepsilon$ in the proof of Theorem 1, we conclude that the triple $(\mathcal{V},\Gamma,\mathbb{F}_\varepsilon)$ is integrable. Furthermore, its corresponding coupling Dirac structure L_ε satisfies

$$L_\varepsilon \cap (TE \oplus \{0\}) = \{0\}.$$

This is equivalent to the fact that L_ε is the graph of a Poisson structure Π_ε. There follows Corollary 1. □

REMARK 4. The associated bundle $P\times_G F$ obtained in Theorem 1 is a Poisson fiber bundle (also called Poisson fibration) [BF07, Wa05]. Recall that a Poisson fibration is a fiber bundle whose fiber is a Poisson manifold and which has an atlas of local trivializations whose transition maps induce Poisson isomorphisms of the fibers.

4. Morphisms

4.1. Gauge transformations

Let $\lambda = (G,P,F)$ be a classical Yang-Mills-Higgs setup together with a pre-symplectic form on $B = P/G$ and a connection θ on P. Consider a diffeomorphism $h : P \to P$ which descends to the identity map on B. The map $h\times id : P\times F \to P\times F$, canonically, induces a fiberwise Poisson diffeomorphism $\psi : P\times_G F \to P\times_G F$. Moreover, we have two connections on P, namely θ and $\theta' = h^*\theta$. They induce connections on $P\times_G F$ denoted by Γ_θ and $\Gamma_{\theta'}$. Applying Theorem 1, one gets two Dirac structures on L_1 and L_2 corresponding to geometric data $(\mathcal{V},\Gamma_\theta,\overline{\mathbb{F}})$ and $(\mathcal{V},\Gamma_{\theta'},\overline{\mathbb{F}}')$, respectively. Precisely,

$$L_1 = \Big\{(\overline{X}, i_{\overline{X}}\overline{\mathbb{F}}) + (\mathcal{V}^\sharp\alpha,\alpha) \mid \overline{X}\in \mathrm{Hor}_\Gamma,\ \alpha\in \mathrm{Ann}(\mathrm{Hor}_\Gamma)\Big\}$$

and

$$L_2 = \Big\{(\overline{X}', i_{\overline{X}'}\overline{\mathbb{F}}') + (\mathcal{V}^\sharp\alpha,\alpha) \mid \overline{X}'\in \mathrm{Hor}_{\Gamma'},\ \alpha\in \mathrm{Ann}(\mathrm{Hor}_{\Gamma'})\Big\}.$$

We know that there are 2-forms $\mathbb{F}$ and $\mathbb{F}'\in\Omega^2(B)\times C^\infty(P\times_G F)$ such that

$$\overline{\mathbb{F}}\Big(hor_\Gamma(X), hor_\Gamma(Y)\Big) = \mathbb{F}(X,Y) \quad \text{and} \quad \overline{\mathbb{F}}'\Big(hor_{\Gamma'}(X), hor_{\Gamma'}(Y)\Big) = \mathbb{F}'(X,Y).$$

Setting

$$\mathcal{B} = \mathbb{F} - \mathbb{F}',$$

we can rewrite L_1 as

$$L_1 = \mathrm{Span}\Big\{(hor_\Gamma(X),\ \overline{i_X\mathbb{F}'} + \overline{i_X\mathcal{B}}) + (\mathcal{V}^\sharp\alpha,\alpha) \mid X\in\mathfrak{X}(B),\ \alpha\in\mathrm{Ann}(\mathrm{Hor}_\Gamma)\Big\},$$

where $\overline{i_X\mathbb{F}'}$ and $\overline{i_X\mathcal{B}}$ are the horizontal Γ-lift of $i_X\mathbb{F}'$ and $i_X\mathcal{B}$, respectively, i.e.

$$\overline{i_X\mathbb{F}'}(hor_\Gamma(Y)) = F'(X,Y) \quad \text{and} \quad \overline{i_X\mathcal{B}}(hor_\Gamma(Y)) = \mathcal{B}(X,Y).$$

We say that L_1 and L_2 are *gauge equivalent* Dirac structures.

4.2. Functorial Property

We begin by recalling classical facts on principal fiber bundles with connection 1-forms. Given a principal fiber bundle with connection 1-form (G,P,θ) the curvature Curv_θ defined in Equation (5) satisfies the *structure equation*

$$2\mathrm{Curv}_\theta(X,Y) = d\theta(X,Y) + [\theta(X),\theta(Y)] \quad \forall X,Y \in \mathfrak{X}(P). \tag{9}$$

DEFINITION 3. *A* morphism from a principal fiber bundle with connection 1-form (G,P,θ) to (G',P',θ') *is a pair* (ϕ,h)*, where* $\phi : G \to G'$ *is a homomorphism of Lie groups, and* $h : P \to P'$ *is a* ϕ*-equivariant map such that the connection 1-forms satisfy the pullback condition*

$$h^*\theta' = \phi_* \circ \theta. \tag{10}$$

From equation (10) it follows that the induced map $\underline{h} : P/G \to P'/G'$ satisfies

$$\overline{\underline{h}_* X} = h_* \overline{X}, \quad \forall X \in \mathfrak{X}(B). \tag{11}$$

The structure equation (9) and (11) imply that the curvatures are related as follows:

$$\mathrm{Curv}_{\theta'}\left(\overline{\underline{h}_* X_1}, \overline{\underline{h}_* X_2}\right) = \phi_*\left(\mathrm{Curv}_\theta\left(\overline{X_1}, \overline{X_2}\right)\right) \quad \forall X_1, X_2 \in \mathfrak{X}(B). \tag{12}$$

Let $\lambda = (G,P,F)$ and $\lambda' = (G',P',F')$ be two classical Yang-Mills-Higgs setups. Suppose that the principal bundles are equipped with connections θ (resp. θ') on P (resp. P'), and assume that $B = P/G$ and $B' = P'/G'$ are equipped with pre-symplectic forms ω_B and $\omega_{B'}$, respectively.

DEFINITION 4. *A* morphism from $(\lambda,\omega_B,\theta)$ to $(\lambda',\omega_{B'},\theta')$ *is a triple* (ϕ,f,h)*, where*

- (ϕ,h) *is a morphism of principal bundles with connection 1-forms such that the induced map* $\underline{h} : B \to B'$ *is a morphism of pre-symplectic manifolds, that is,* $\underline{h}^*\omega_{B'} = \omega_B$*;*
- $f : F \to F'$ *is a* ϕ*-equivariant Poisson map such the following diagram commutes*

$$\begin{array}{ccc} F & \xrightarrow{f} & F' \\ {\scriptstyle J}\downarrow & & \downarrow{\scriptstyle J'} \\ \mathfrak{g}^* & \xrightarrow[\phi^*]{} & \mathfrak{g}'^* \end{array} \tag{13}$$

where $\phi^* : \mathfrak{g}'^* \to \mathfrak{g}$ *is the canonical map induced by* ϕ.

Given two such classical Yang-Mills-Higgs setups, we denote by $L = L(\lambda, \omega_B, \theta)$ and $L' = L(\lambda', \omega_{B'}, \theta')$ the corresponding Dirac structures on $E = P \times_G F$ and $E' = P' \times_{G'} F'$, respectively. We denote by $\psi = [h, f] : (E, L) \to (E', L')$ the canonical map induced on the associated bundles endowed with their coupling Dirac structure. Then the diagram below commutes.

$$\begin{array}{ccc} E & \xrightarrow{\psi} & E' \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi'} \\ B & \xrightarrow[\underline{h}]{} & B' \end{array} \tag{14}$$

LEMMA 3. *Under the above notations, one has $\mathbb{F} = (\psi, \underline{h})^*\mathbb{F}'$ where $(\psi, \underline{h}) : E \times B \to E' \times B'$ is the canonical product map.*

Proof. Let $X_1, X_2 \in T_bB$, let $e = [p, f]$ be a point in the fiber of $E = P \times_G F$ over b, and let $e' = [p', f'] = \psi(e)$. We compute that

$$\begin{array}{rcll} (\psi,\underline{h})^*\mathbb{F}'(X_1,X_2)(e) & = & \mathbb{F}'(\underline{h}_*X_1, \underline{h}_*X_2)(\psi(e)) & \\ & = & \omega'(\underline{h}_*X_1, \underline{h}_*X_2) + \mathbb{G}'(\underline{h}_*X_1, \underline{h}_*X_2)([p', f']) & \text{by (7)} \\ & = & (\underline{h}^*\omega')(X_1,X_2) + \langle J'(f'), \mathrm{Curv}_{\theta'}(\overline{\underline{h}_*X_1}_{p'}, \overline{\underline{h}_*X_2}_{p'})\rangle & \text{by (6)} \\ & = & \omega(X_1,X_2) + \langle (J' \circ f)(f), \phi_*\mathrm{Curv}_\theta(\overline{X_1}_p, \overline{X_2}_p)\rangle & \text{by (12)} \\ & = & \omega(X_1,X_2) + \langle (\phi^* \circ J' \circ f)(f), \mathrm{Curv}_\theta(\overline{X_1}_p, \overline{X_2}_p)\rangle & \\ & = & \omega(X_1,X_2) + \langle J(f), \mathrm{Curv}_\theta(\overline{X_1}_p, \overline{X_2}_p)\rangle & \text{by (13)} \\ & = & \omega(X_1,X_2) + \mathbb{G}(X_1,X_2)([p,f]) & \text{by (6)} \\ & = & \mathbb{F}(X_1,X_2)(e) & \text{by (7)} \end{array}$$

which was to be shown. □

PROPOSITION 2. *Under the above notations, we assume that (ϕ, f, h) is a morphism from $(\lambda, \omega_B, \theta)$ to $(\lambda', \omega_{B'}, \theta')$. Then the induced map on coupling Dirac structures*

$$\psi = [h, f] : (E, L) \to (E', L')$$

is forward (resp. backward) Dirac if ψ is a submersion (resp. immersion).

Proof. We show here that ψ is forward Dirac when ψ is a submersion. We must verify that

$$L' = \{(\psi_*X, \beta) | X \in TE, \beta \in T^*E', (X, \psi^*\beta) \in L\}.$$

Recall that the coupling Dirac structure L splits as a vector bundle into horizontal and vertical components as

$$L = \{(\overline{X}, i_{\overline{X}}\overline{\mathbb{F}}) | \overline{X} \in \mathrm{Hor}_\Gamma\} \oplus \{(\mathcal{V}^\#\alpha, \alpha) | \alpha \in \mathrm{Hor}^0_\Gamma\} \text{ by (2)},$$

and L' splits similarly. We will verify in turn that the forward Dirac condition is satisfied for each component of the splitting.

(i) Suppose that $X \in TE$, $\beta \in T^*E'$, $Y \in TB$ and

$$(X, \psi^*\beta) = (\overline{Y}, i_{\overline{Y}}\overline{\mathbb{F}}). \tag{15}$$

We wish to show that $(\psi_* X, \beta) \in L'$, that is,

$$\begin{aligned} \beta &= i_{\psi_* X}\overline{\mathbb{F}'} && \text{by (2)} \\ &= i_{\psi_* \overline{Y}}(\pi'^*\mathbb{F}') && \text{by (15)} \\ &= \pi'^*(i_{\underline{h}_* Y}\mathbb{F}') && \text{by (14).} \end{aligned}$$

To this end, suppose that $Z'_{e'} \in T_{e'}E'$. Since $\psi_* : T_eE \to T_{e'}E'$ is surjective there exists $Z_e \in T_eE$ such that

$$Z'_{e'} = \psi_* Z_e. \tag{16}$$

We compute that

$$\begin{aligned} \beta(Z'_{e'}) &= \beta(\psi_* Z_e) && \text{by (16)} \\ &= (\psi^*\beta)(Z_e) \\ &= (i_{\overline{Y}}\overline{\mathbb{F}})(Z_e) && \text{by (15)} \\ &= \mathbb{F}(\pi_*\overline{Y}, \pi_* Z_e)(e) && \text{by (1)} \\ &= (\psi^*, \underline{h})^*\mathbb{F}'(Y, \pi_* Z_e)(e) && \text{by Lemma 3} \\ &= \mathbb{F}'(\underline{h}_* Y, \underline{h}_*\pi_* Z_e)(\psi(e)) \\ &= \mathbb{F}'(\underline{h}_* Y, \pi'_*\psi_* Z_e)(\psi(e)) && \text{by (14)} \\ &= \pi'^*(i_{\underline{h}_* Y}\mathbb{F}')(\psi_* Z_e) \\ &= \pi'^*(i_{\underline{h}_* Y}\mathbb{F}')(Z'_{e'}) && \text{by (16)} \end{aligned}$$

which was to be shown. Moreover, the entire horizontal component of L' is hit as the map ψ_* surjects Hor_Γ onto $\mathrm{Hor}_{\Gamma'}$.

(ii) Suppose that $X \in TE$, $\alpha' \in T^*E'$, and $(X, \psi^*\alpha') = (\mathcal{V}^\sharp\alpha, \alpha)$ where $\alpha \in \mathrm{Ann}(\mathrm{Hor}_\Gamma)$. Then

$$\mathcal{V}'^\sharp\alpha' = (\psi_*\mathcal{V})^\sharp\alpha' = \psi_*(\mathcal{V}^\sharp\psi^*\alpha') = \psi_*(\mathcal{V}^\sharp\alpha) = \psi_* X.$$

Hence $(\psi_* X, \alpha') = (\mathcal{V}'^\sharp\alpha', \alpha')$. We note that $\alpha' \in \mathrm{Ann}(\mathrm{Hor}_{\Gamma'})$ since $\psi^*\alpha' \in \mathrm{Ann}(\mathrm{Hor}_\Gamma)$ and the map ψ_* surjects Hor_Γ onto $\mathrm{Hor}_{\Gamma'}$. This also shows that the entire vertical component of L' is hit.

The proof that ψ is backward Dirac if ψ is an immersion is a similar computation. In particular, the injectivity of ψ_* ensures that X is horizontal if $(\psi_* X, \beta)$ is in the horizontal component of L'. $\square$

5. Applications to fibered symplectic groupoids

Recall that a Lie groupoid over a smooth manifold M is given by a smooth manifold $\mathcal{G}$ together with two surjective submersions $\alpha, \beta : \mathcal{G} \to M$ called the source map and the target map, a multiplication $m : \mathcal{G}_2 \to \mathcal{G}$, a unit section $\varepsilon : M \to \mathcal{G}$ and an inversion map $i : \mathcal{G} \to \mathcal{G}$, where $\mathcal{G}_2 = \{(g,h) \in \mathcal{G} \times \mathcal{G} \mid \beta(g) = \alpha(h)\}$ is the set of composable pairs and the following properties are satisfied:

1. $\alpha(m(g,h)) = \alpha(h)$ and $\beta(m(g,h)) = \beta(g)$, $\forall (g,h) \in \mathcal{G}_2$,
2. $m(g,m(h,k)) = m(m(g,h),k)$, $\forall g,h,k \in \mathcal{G}$ such that $\alpha(g) = \beta(h)$ and $\alpha(h) = \beta(k)$,
3. $\alpha(\varepsilon(x)) = x$ and $\beta(\varepsilon(x)) = x$, $\forall x \in M$,
4. $m(g,\varepsilon(\alpha(g))) = g$ and $m(\varepsilon(\beta(g)),g) = g$, $\forall g \in \mathcal{G}$,
5. $m(g,\iota(g)) = \varepsilon(\beta(g))$ and $m(\iota(g),g) = \varepsilon(\alpha(g))$, $\forall g \in \mathcal{G}$.

In other words, a Lie groupoid is a small category such that all morphisms are invertible, the spaces are smooth manifolds and all structure maps are smooth. Here, the base manifold M, the α-fibers and the β-fibers are supposed to be Hausdorff but $\mathcal{G}$ is not necessarily Hausdorff. A symplectic groupoid $\mathcal{G}$ over a Poisson manifold M is a Lie groupoid such that source and target maps are Poisson (resp. anti-Poisson) maps and such that the graph of the groupoid multiplication is Lagrangian.

Given a smooth manifold B, we denote by $\mathbf{Fib}(B)$ the category of locally trivial fiber bundles over B whose objects are the locally trivial fiber bundles $F \to E \to B$ and whose morphisms are the fiber preserving maps $\psi : E \to E'$ over the identity map $id : B \to B$. By a *fibered Lie groupoid* $\mathcal{G} \rightrightarrows E$ over B, we mean a small sub-category in $\mathbf{Fib}(B)$.

DEFINITION 5 ([BF07]). *A* fibered symplectic groupoid $\mathcal{G} \rightrightarrows E$ *over* B *is fibered Lie groupoid whose fiber* $\mathcal{F} \rightrightarrows F$ *is a symplectic groupoid.*

It is known that the base $E \to B$ of any fibered symplectic groupoid $\mathcal{G}$ has naturally a Poisson fibration structure (see [BF07]). A Poisson structure on F is *integrable* if its induced Lie algebroid is isomorphic to the Lie algebroid of some Lie groupoid. Up to isomorphism, there is a unique source-simply connected Lie groupoid $\mathcal{G}(A)$ corresponding to an integrable Lie algebroid A. By a source-simply connected Lie groupoid, we mean that the source-fibers are simply connected. Given any classical Yang-Mills-Higgs setup (G,P,F) such that the fiber F admits an integrable Poisson structure $\mathcal{V}_F$, the associated bundle

$$\mathcal{G} = P \times_G \mathcal{F} \rightrightarrows P \times_G F$$

is a source 1-connected fibered symplectic groupoid, where $\mathcal{F}$ is the unique source 1-connected symplectic groupoid integrating F. We will use that fact as well as the following definition:

DEFINITION 6 ([CF05]). *A Poisson manifold* $(F,\mathcal{V}_F)$ *is said to be of* compact type *if it is integrable and its 1-source connected symplectic groupoid* $\mathcal{F}$ *is compact.*

We have the following result:

PROPOSITION 3. *Let* (G,P,F) *be a classical Yang-Mills-Higgs setup and* θ *a connection on P. Assume that P is compact, F is of compact type and the base B is equipped with a symplectic form* ω_B*. Then the associated bundle* $E = P\times_G F$ *integrates into a source 1-connected fibered symplectic groupoid* $\mathcal{G} \rightrightarrows E$*. Furthermore, there is 1-parameter family of symplectic forms on the total space* $\mathcal{G}$ *such that each of them restricts to the symplectic structure along the fibers of* $\mathcal{F}_b$*.*

Proof. The Poisson fiber bundle $E = P\times_G F \to B$ integrates into a source 1-connected fibered symplectic groupoid $\mathcal{G} \rightrightarrows E$ since F is of compact type. The existence of a 1-parameter family of symplectic forms on the total space of $\mathcal{G} \rightrightarrows E$ which is compatible with its fibered symplectic groupoid structure can we shown by using exactly the same method as in the proof of Corollary 1. □

EXAMPLE 3. Consider the principal $U(1)$-bundle $S^{2n+1} \to \mathbb{C}P^n$, where the projective space $\mathbb{C}P^n$ is endowed with its natural Kähler form. Let F be the 2-sphere S^2, with its canonical symplectic structure. It's known that S^2 is of compact type and its associated source 1-connected symplectic groupoid is the pair groupoid $\mathcal{F} = S^2\times\overline{S^2}$, where the notation $\overline{S^2}$ means that we consider the opposite of the symplectic form ω_0 on S^2. Moreover the associated fibered fibered symplectic groupoid is:

$$\mathcal{G} = S^{2n+1}\times_{U(1)}(S^2\times\overline{S^2}) \rightrightarrows S^{2n+1}\times_{U(1)} S^2.$$

It follows from Proposition 3 that, given any connection θ on our principal circle bundle, there is a 1-parameter family of symplectic forms on the total space of $\mathcal{G} \rightrightarrows E$ which is compatible with its induced fibered symplectic groupoid structure.

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