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ON THE FINITENESS OF LOCAL HOMOLOGY MODULES

Abstract. Let $(R,\mathfrak{m})$ be a commutative Noetherian complete local ring, $\mathfrak{a}$ an ideal of R, and A an Artinian R-module with $\text{N-dim}A = d$. We prove that if $d > 0$, then $\text{Cosupp}(H^{\mathfrak{a}}_{d-1}(A))$ is finite and if $d \leq 3$, then the set $\text{Coass}(H^{\mathfrak{a}}_i(A))$ is finite for all i. Moreover, if either $d \leq 2$ or the cohomological dimension $\text{cd}(\mathfrak{a}) = 1$ then $H^{\mathfrak{a}}_i(A)$ is $\mathfrak{a}$-coartinian for all i; that is, $\text{Tor}^R_j(R/\mathfrak{a}, H^{\mathfrak{a}}_i(A))$ is Artinian for all i, j. We also show that if $H^{\mathfrak{a}}_i(A)$ is $\mathfrak{a}$-coartinian for all $i < n$, then $\text{Tor}^R_j(R/\mathfrak{a}, H^{\mathfrak{a}}_n(A))$ is Artinian for $j = 0, 1$. In particular, the set $\text{Coass}(H^{\mathfrak{a}}_n(A))$ is finite.

1. Introduction

Throughout this paper, we assume that $(R,\mathfrak{m})$ is a commutative Noetherian local ring, $\mathfrak{a}$ an ideal of R, and A an Artinian R-module. In [3] Cuong and Nam defined the local homology modules $H^{\mathfrak{a}}_i(A)$ of A with respect to $\mathfrak{a}$ by

$$H^{\mathfrak{a}}_i(A) = \varprojlim_n \text{Tor}^R_i(R/\mathfrak{a}^n, A).$$

This definition is dual to Grothendieck's definition of local cohomology modules (cf. [1]). Also, this definition of local homology modules coincides with the definition of Greenlees and May [7].

The theory of coassociated prime ideals, which is dual to the theory of associated prime ideals, was studied by Macdonald [10], Chambless [2], Zöschinger [18] and Yassemi [17]. In [17] it is shown that, for Noetherian rings, these definitions are equivalent. The concept of cofinite modules introduced by Hartshorne [8] has been proved to be an important tool in the study of modules over a commutative Noetherian ring. Recently, Nam [13] defined the notion of coartinian modules which is in some sense dual to the concept of cofinite modules. There are not many results concerning the finiteness of coassociated primes and coartinianness of local homology modules (cf. [12, 5]). The main aim of this paper is to show some properties about coassociated primes and $\mathfrak{a}$-coartinianness of local homology modules.

2. Preliminaries

We begin by recalling the concept of Noetherian dimension of an R-module M denoted by $\text{N-dim}M$. Note that the Noetherian dimension is introduced by Roberts [14] with the terminology *Krull dimension*. Later, Kirby [9] changed the terminology of Roberts and referred to Noetherian dimension to avoid confusion with the well-known Krull dimension for finitely generated modules. Let M be an R-module. When $M = 0$ we put

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N-dim$M = -1$. Then by induction, for any ordinal α, we put N-dim$M = \alpha$ when (i) N-dim$M < \alpha$ is false, and (ii) for any ascending chain $M_0 \subseteq M_1 \subseteq \ldots$ of submodules of M, there exists a positive integer m_0 such that N-dim$(M_{m+1}/M_m) < \alpha$ for all $m > m_0$. Thus, M is non-zero and Noetherian if and only if N-dim$M = 0$. In case M is an Artinian module, N-dim$M < \infty$ (see [14]). Also, if M is an R-module with N-dim$M = d > 0$ and $x \in R$ such that $xM = M$, then N-dim$(0:_M x) = d-1$ (see [4] and [6]). We now briefly recall some basic properties of local homology modules.

LEMMA 1. ([3]) *Let* $0 \longrightarrow \dot{M} \longrightarrow M \longrightarrow \ddot{M} \longrightarrow 0$ *be a short exact sequence of Artinian modules. Then we have a long exact sequence of local homology modules*
$\ldots \longrightarrow H_i^{\mathfrak{a}}(\dot{M}) \longrightarrow H_i^{\mathfrak{a}}(M) \longrightarrow H_i^{\mathfrak{a}}(\ddot{M}) \longrightarrow \ldots$
$\longrightarrow H_0^{\mathfrak{a}}(\dot{M}) \longrightarrow H_0^{\mathfrak{a}}(M) \longrightarrow H_0^{\mathfrak{a}}(\ddot{M}) \longrightarrow 0.$

LEMMA 2. ([3]) *Let M be an Artinian R-module. Then the following holds: If $i > 0$, then $H_i^{\mathfrak{a}}(\cap_{n>0}\mathfrak{a}^n M) \cong H_i^{\mathfrak{a}}(M)$ and $H_i^{\mathfrak{a}}(\cap_{n>0}\mathfrak{a}^n M) = 0$ whenever $i = 0$.*

LEMMA 3. ([7]) *Let M be an R-module and that the local cohomology module $H_{\mathfrak{a}}^i(R) = 0$ for all $i > n$. Then the local homology module $H_i^{\mathfrak{a}}(M) = 0$ for all $i > n$.*

LEMMA 4. ([3]) *Let M be an Artinian R-module with* N-dim$M = d$. *Then $H_i^{\mathfrak{a}}(M) = 0$ for all $i > d$.*

LEMMA 5. ([4]) *Let M be an Artinian R-module. Then $H_0^{\mathfrak{a}}(M) = 0$ if and only if $xM = M$ for some $x \in \mathfrak{a}$.*

DEFINITION 1. ([7]) *An R-module M is called cocyclic if it is a submodule of $E(R/\mathfrak{m})$. A prime ideal $\mathfrak{p}$ is called coassociated to a non-zero R-module N if there is a cocyclic homomorphic image T of N with $\mathfrak{p} = \mathrm{Ann}(T)$. The set of coassociated primes of N is denoted by* $\mathrm{Coass}_R(N)$ *(or simply* $\mathrm{Coass}(N)$ *if there is no ambiguity about the under ring). Note that* $\mathrm{Coass}(N) = \emptyset$ *if and only if $N = 0$.*

Recall that the Krull dimension of M, denoted by $\dim M$, is the Krull dimension of the Noetherian ring $R/\mathrm{Ann}(M)$. For convenient, we stipulate that $\dim M = -1$ if $M = 0$. Note that if M is Artinian then the set of minimal prime ideals of $\mathrm{Ann}(M)$ is just the set of minimal elements of $\mathrm{Coass}(M)$ (cf. [17]). Therefore $\dim M$ is the supremum of $\dim R/\mathfrak{p}$, where $\mathfrak{p}$ runs over $\mathrm{Coass}(M)$.

DEFINITION 2. ([17]) *Let N be an R-module. The cosupport of N, written* $\mathrm{Cosupp}(N)$*, is the set of prime ideals $\mathfrak{p}$ such that there exists a cocyclic homomorphic image T of N with $\mathfrak{p} \supseteq \mathrm{Ann}(T)$.*

Let S be a multiplicative set in R and N an R-module. In [11] Melkersson and Schenzel called the module ${}_SN = \mathrm{Hom}_R(R_S, N)$ the co-localization of N with respect to S and $\mathrm{Cos}(N) = \{\mathfrak{p} \in \mathrm{Spec}(R) : {}_{\mathfrak{p}}N \neq 0\}$ the cosupport of N. Yassemi [17] proved that $\mathrm{Cos}(N) = \mathrm{Cosupp}(N)$ in case N is an Artinian R-module. However, the equality is in general not true, but we always have $\mathrm{Cos}(N) \subseteq \mathrm{Cosupp}(N)$.

LEMMA 6. ([6]) *If M is an Artinian R-module, then it has a natural structure as an $\hat{R}$-module and* $\text{N-dim}_R M = \text{N-dim}_{\hat{R}} M = \dim_{\hat{R}} M$.

LEMMA 7. ([17]) *Let* $0 \longrightarrow \dot{M} \longrightarrow M \longrightarrow \ddot{M} \longrightarrow 0$ *be an exact sequence of R-modules. Then the following statements hold:*
(i) $\text{Coass}(\ddot{M}) \subseteq \text{Coass}(M) \subseteq \text{Coass}(\dot{M}) \cup \text{Coass}(\ddot{M})$*;*
(ii) $\text{Cosupp}(M) = \text{Cosupp}(\dot{M}) \cup \text{Cosupp}(\ddot{M})$.

LEMMA 8. ([5]) *Let M be an R-module. Then, for all i,* $\text{Cos}(H_i^{\mathfrak{a}}(M)) \subseteq V(\mathfrak{a})$.

COROLLARY 1. *Let M be an Artinian R-module. Then, for all i,* $\text{Cosupp}(H_i^{\mathfrak{a}}(M)) = \text{Cos}(H_i^{\mathfrak{a}}(M))$*. In particular,* $\text{Cosupp}(H_i^{\mathfrak{a}}(M)) \subseteq V(\mathfrak{a})$.

Proof. This is immediate by [4] (Proposition 3.3) and [13] (Theorem 3.8). □

LEMMA 9. ([17]) *Let M be an R-module. Then the following statements hold:*
(i) $\text{Coass}(M) \subseteq \text{Cosupp}(M)$*;*
(ii) Every minimal element of the set $\text{Cosupp}(M)$ *belongs to* $\text{Coass}(M)$*;*
(iii) $\text{Coass}(M \otimes_R N) = \text{Coass}(M) \cap \text{Supp}(N)$ *if N is a finitely generated R-module;*
(iv) If M is finitely generated, then $\text{Cosupp}(M) \subseteq \{\mathfrak{m}\}$.

3. The results

In this section, we first recall the definition of coartinian modules which is in some sense dual to the concept of cofinite modules.

DEFINITION 3. *An R-module M is said to be $\mathfrak{a}$-coartinian if* $\text{Cosupp}(M) \subseteq V(\mathfrak{a})$ *and* $\text{Tor}_i^R(R/\mathfrak{a}, M)$ *is an Artinian R-module for each i.*

It is clear that if M is of finite length, then M is $\mathfrak{a}$-coartinian for all ideals $\mathfrak{a}$ of R.

PROPOSITION 1. *Let t be a non-negative integer such that $H_i^{\mathfrak{a}}(A)$ is $\mathfrak{a}$-coartinian for all $i \neq t$. Then $H_i^{\mathfrak{a}}(A)$ is $\mathfrak{a}$-coartinian for all i.*

Proof. By using [15] (Theorem 11.39), [3] (Lemma 4.3) and [16] (Corollary 2.4) there is a Grothendieck spectral sequence

$$E_{p,q}^2 := \text{Tor}_p^R(R/\mathfrak{a}, H_q^{\mathfrak{a}}(A)) \underset{p}{\Longrightarrow} \text{Tor}_{p+q}^R(R/\mathfrak{a}, A).$$

For each $r \geq 2$, we consider the exact sequence

$$0 \longrightarrow \ker d_{p,t}^r \longrightarrow E_{p,t}^r \longrightarrow E_{p-r,t+r-1}^r. \tag{1}$$

It follows from the hypotheses that the R-module $E_{p-r,t+r-1}^r$ is Artinian. Note that $E_{p,q}^r$ is a subquotient of $E_{p,q}^2$ for all $p, q \in \mathbb{N}_0$. There is an integer s such that $E_{p,q}^\infty \cong E_{p,q}^r$ for all $r \geq s$. Also, for each $n \in \mathbb{N}_0$ there is a bounded filtration

$$0 = \phi^{-1}H^n \subseteq \phi^0 H^n \subseteq \ldots \subseteq \phi^{n-1}H^n \subseteq \phi^n H^n = H^n$$

for the module $H^n = \mathrm{Tor}_n^R(R/\mathfrak{a}, A)$ such that $E^\infty_{p,n-q} \cong \phi^p H^n/\phi^{p-1}H^n$ for $p = 0,1,\ldots,n$. Thus $E^\infty_{p,q}$ is Artinian for all p,q. Since $E^s_{p,t} = \ker d^{s-1}_{p,t}/\mathrm{im}\, d^{s-1}_{p+s-1,t-s+2}$, it follows that $\ker d^{s-1}_{p,t}$ is Artinian. Hence by using the exact sequence (1) for $r = s-1$, we deduce that $E^{s-1}_{p,t}$ is Artinian. By continuing this argument repeatedly for integers $s-1, s-2, \ldots, 3$ instead of s, we obtain that $E^2_{p,t}$ is Artinian for all $p \geq 0$. This completes the proof. □

COROLLARY 2. *Let* $\mathfrak{a}$ *be an ideal of* R *with* $\mathrm{cd}(\mathfrak{a}) = 1$. *Then* $H^{\mathfrak{a}}_i(A)$ *is* $\mathfrak{a}$-*coartinian for all* i.

Proof. This is immediate by Proposition 1 and Lemma 3. □

COROLLARY 3. *Let* A *be an Artinian* R-*module with* N-dim$A = n$. *If* $n \leq 2$, *then* $H^{\mathfrak{a}}_i(A)$ *is* $\mathfrak{a}$-*coartinian for all* i.

Proof. By [13] (Corollary 4.15) $H^{\mathfrak{a}}_2(A)$ is $\mathfrak{a}$-coartinian. On the other hand $H^{\mathfrak{a}}_0(A)$ is Artinian and hence by Proposition 1 and Lemma 4 the result follows. □

LEMMA 10. *Let* $x \in \mathfrak{a}$. *Then the following statements hold:*
(i) $\mathrm{Coass}(H^{\mathfrak{a}}_i(A)) = \mathrm{Coass}(H^{\mathfrak{a}}_i(A)/xH^{\mathfrak{a}}_i(A))$;
(ii) $\mathrm{Cosupp}(H^{\mathfrak{a}}_i(A)) = \mathrm{Cosupp}(H^{\mathfrak{a}}_i(A)/xH^{\mathfrak{a}}_i(A))$.

Proof. (i) This is immediate by Corollary 1 and Lemma 9.
(ii) This is clear by Lemmas 7 and 9. □

THEOREM 1. *Let* $(R,\mathfrak{m})$ *be a complete local ring and that* A *be an Artinian* R-*module with* N-dim$A = n$. *Then the following statements hold:*
(i) $\mathrm{Cosupp}(H^{\mathfrak{a}}_n(A)) \subseteq \{\mathfrak{m}\}$;
(ii) $\mathrm{Cosupp}(H^{\mathfrak{a}}_{n-1}(A))$ *is finite.*

Proof. The proof of cases (i) and (ii) are similar. Hence we only to prove (ii). We proceed by induction on n. When $n = 1$, $\mathrm{Cosupp}(H^{\mathfrak{a}}_0(A)) \subseteq V(\mathrm{Ann}(A))$ and hence the result in this case is true. So let $n > 1$ and suppose that the claim is true for $n-1$. By Lemma 2.2 we can replace A by $\cap_{n>0}\mathfrak{a}^n A$ and note that N-dim$A \geq$ N-dim$(\cap_{n>0}\mathfrak{a}^n A)$. The last module is just equal to $\mathfrak{a}^n A$ for some enough large n. Therefore we may assume that $\mathfrak{a}A = A$. Since A is Artinian, there is an element $x \in \mathfrak{a}$ such that $xA = A$. Thus the exact sequence

$$0 \longrightarrow (0 :_A x) \longrightarrow A \xrightarrow{x} A \longrightarrow 0$$

gives rise by Lemma 2.1 to an exact sequence

$$0 \longrightarrow H^{\mathfrak{a}}_{n-1}(A)/xH^{\mathfrak{a}}_{n-1}(A) \longrightarrow H^{\mathfrak{a}}_{n-2}(0 :_A x).$$

Since N-dim$(0 :_A x) = n-1$, $\mathrm{Cosupp}(H^{\mathfrak{a}}_{n-2}(0 :_A x))$ is finite by induction hypothesis.

Hence, $\operatorname{Cosupp}(H^{\mathfrak{a}}_{n-1}(A)/xH^{\mathfrak{a}}_{n-1}(A))$ is finite and so by Lemma 3.5 the result follows. □

THEOREM 2. *Let* $(R,\mathfrak{m})$ *be a local ring and that* n *be a non-negative integer such that* $H^{\mathfrak{a}}_i(A)$ *is* $\mathfrak{a}$*-coartinian for all* $i<n$*. Then* $\operatorname{Tor}^R_i(R/\mathfrak{a},H^{\mathfrak{a}}_n(A))$ *is Artinian for* $i=0,1$*. In particular,* $\operatorname{Coass}(H^{\mathfrak{a}}_n(A))$ *is finite.*

Proof. By [15] (Theorem 11.39), [3] (Lemma 4.3) and [16] (Corollary 2.4) there is a Grothendieck spectral sequence

$$E^2_{p,q}:=\operatorname{Tor}^R_p(R/\mathfrak{a},H^{\mathfrak{a}}_q(A))\underset{p}{\Longrightarrow}\operatorname{Tor}^R_{p+q}(R/\mathfrak{a},A).$$

Since $E^i_{p,q}$ is a submodule of $E^2_{p,q}$ for all $i\geq 2$, our hypothesis give us that $E^i_{p,q}$ is Artinian for all $i\geq 2,p\geq 0$, and $q<n$. For each $i\geq 2$ and $p=0,1$, we consider the exact sequence

$$(2)\qquad 0\longrightarrow \operatorname{im} d^i_{p+i,n-i+1}\longrightarrow \ker d^i_{p,n}\longrightarrow E^{i+1}_{p,n}\longrightarrow 0$$

where $\operatorname{im} d^i_{p+i,n-i+1}=\operatorname{im}(E^i_{p+i,n-i+1}\longrightarrow E^i_{p,n})$ and $\ker d^i_{p,n}=\ker(E^i_{p,n}\longrightarrow E^i_{p-i,n+i-1})$. Now $\operatorname{im} d^i_{p+i,n-i+1}$ is Artinian for all $i\geq 2$, $E^{\infty}_{p,n}$ is isomorphic to a subquotient of $\operatorname{Tor}^R_{p+n}(R/\mathfrak{a},A)$ and thus is Artinian for all p. Since $E^i_{p,n}=E^{\infty}_{p,n}$ for i sufficiently large and $E^i_{p,n}=\ker d^i_{p,n}$ for all $i\geq 2$ and $p=0,1$, we have that $E^i_{p,n}$ and $\ker d^i_{p,n}$ are Artinian for $p=0,1$ and all large i. Fix i and suppose $E^{i+1}_{p,n}$ is Artinian for $p=0,1$. From the exact sequence (2) with conjunction $\ker d^i_{p,n}=E^i_{p,n}$ for $p=0,1$ we obtain that $E^i_{p,n}$ is Artinian for $p=0,1$. Continuing in this fashion we see that $E^i_{p,n}$ is Artinian for all $i\geq 2$ and $p=0,1$. In particular, $\operatorname{Tor}^R_i(R/\mathfrak{a},H^{\mathfrak{a}}_n(A))$ is Artinian for for $i=0,1$ and also by Lemma 3.5 the set $\operatorname{Coass}(H^{\mathfrak{a}}_n(A))$ is finite. □

COROLLARY 4. *Let* $(R,\mathfrak{m})$ *be a complete local ring and that* A *be an Artinian* R*-module of Noetherian dimension at most three. Then* $\operatorname{Coass}(H^{\mathfrak{a}}_i(A))$ *is finite for all* i *and all ideal* $\mathfrak{a}$*.*

Proof. This is immediate by Theorem 1, Lemma 4 and Theorem 2. □

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