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ON d-MAXIMAL GROUPS

Abstract. Let G be a finite p-group and let $d(G)$ denote the cardinality of a minimal generating set of G. G is said to be d-maximal if $d(H) < d(G)$ for any proper subgroup H of G. In this paper we show that if G is a d-maximal p-group where p is odd, then G has exponent p or p^2. For $p = 2$, we show that under certain conditions, a d-maximal 2-group has exponent four. We then give some classes of d-maximal p-groups. We also investigate relations between powerful p-groups and d-maximal groups.

1. Introduction

Let G be a finite p-group and let $d(G)$ denote the cardinality of a minimal generating set of G. G is said to be d-maximal if $d(H) < d(G)$ for any proper subgroup H of G. If G has order p^n and $|G/\Phi(G)| = p^{n-1}$ where $\Phi(G)$ is the Frattini subgroup of G, then it is known by the Burnside Basis Theorem that $d(G) = n-1$ and we say that G has maximal Frattini factor size. For a positive integer n, let G^n denote the subgroup of G generated by the elements x^n with $x \in G$. Following convention, let G' denote the derived subgroup of G. G is said to be a powerful p-group if either p is odd and $G' \leq G^p$, or $p = 2$ and $G' \leq G^4$.

In this paper we show that if G is a d-maximal p-group where p is odd, then G has exponent p or p^2. We also show that for $p = 2$, if the Frattini subgroup of G has order $\leq 2^3$ or is elementary abelian, or if $d(G) \leq 3$, then G has exponent four. It is also shown that if G is a non-abelian d-maximal p-group, then $d(G) \geq 3$ except when G is quaternion of order eight. We also give some explicit classes of d-maximal p-groups. In particular, we consider the d-maximal p-groups with maximal Frattini factor size and for $p = 2$, we determine completely the non-abelian d-maximal 2-groups G with $d(G) \leq 3$ and $\Phi(G)$ elementary abelian. Finally, in the last section, we study relations between powerful p-groups and d-maximal groups. It is shown that non-abelian powerful 2-groups are not d-maximal and that for p odd, there exist non-abelian powerful p-groups with exponent p^2 which are d-maximal.

2. Some preliminaries

It is known that if G is a finite p-group, then $G^pG' = \Phi(G)$ and hence, $G^p, G' \subseteq \Phi(G)$. The following result of Kahn [5] tells us when a factor group of a d-maximal group is also d-maximal.

LEMMA 1 (Kahn, [5]). *If G is a d-maximal p-group and N is a normal subgroup of G with $N \subseteq \Phi(G)$, then G/N is d-maximal.*

As a consequence of this result we have

PROPOSITION 1. *If G is a d-maximal p-group, then $\Phi(G) = G'$.*

Proof. Since G is d-maximal and $G' \subseteq \Phi(G)$, it follows by Lemma 1 that G/G' is d-maximal. But since abelian groups which are d-maximal must be elementary abelian, the inclusion $\Phi(G) \subseteq G'$ follows. Hence $\Phi(G) = G'$, as asserted. □

The following power structure of powerful p-groups has been obtained by Lubotzky and Mann [7, Theorems 1.3 and 4.1.3].

PROPOSITION 2. *Let G be a powerful p-group. Then $(G^{p^i})^{p^j} = G^{p^{i+j}}$ for any integers $i, j \geq 0$.*

We also need the following commutator identity, the proof of which is a straightforward exercise and shall be left to the reader.

PROPOSITION 3. *Let x, y be elements of a group G and let n be a positive integer. Then $(x[x, y])^n = x^n[y, x^n]^{-1}$. In particular, if G' is contained in the center of G, then $[x, y]^n = [y, x^n]^{-1} = [x^n, y]$.*

3. Main results

It is clear that any abelian d-maximal group must have exponent p. For non-abelian groups we have the following:

THEOREM 1. *Let G be a non-abelian d-maximal p-group. Then G has exponent p or p^2 if $p > 2$. If $p = 2$, then G has exponent four if any one of the following conditions is satisfied:*

(a) $|\Phi(G)| \leq 2^3$;

(b) $\Phi(G)$ is elementary abelian;

(c) $d(G) \leq 3$.

Proof. Since G is a non-abelian d-maximal p-group, it follows by [6] for the case p is odd and by [2], [5], [8] for the case $p = 2$ that the nilpotency class of G is two. Hence $G' \leq Z(G)$, the center of G. Then by Proposition 1, we have that

$$G^p \leq \Phi(G) = G' \leq Z(G). \tag{1}$$

Suppose that the exponent of G is $> p$. Let x, y be arbitrary elements of G. Since G' is contained in the center of G and $[G^p, G] = 1$ (by (1)), we have by Proposition 3 that $[x,y]^p = [x^p,y] = 1$. This tells us that G' has exponent p. Then by (1), it follows that $G^{p^2} \subseteq (G^p)^p \subseteq (G')^p = \{1\}$. Hence G has exponent p^2. If $p = 2$, G cannot have exponent 2 since any group with at least two distinct elements of order 2 contains a dihedral subgroup (see [1, Proposition 13, p. 24]) and hence, has exponent at least four. ☐

An immediate consequence of Theorem 1 is the following:

COROLLARY 1. *Let G be a non-abelian p-group with exponent $\geq p^3$. Then G is not d-maximal if any one of the following conditions is satisfied:*

(a) p is odd;

(b) $p = 2$ and $|\Phi(G)| \leq 2^3$;

(c) $p = 2$ and $\Phi(G)$ is elementary abelian;

(d) $p = 2$ and $d(G) \leq 3$.

REMARK 1. We note that the condition $d(G) \leq 3$ in part (c) of Theorem 1 is sufficient but not necessary for the nilpotency class of the group to be two. For example, the group $G = Q_8 \times C_2 \times C_2$ where Q_8 is quaternion of order eight and C_2 is cyclic of order two is d-maximal with nilpotency class two but with $d(G) = 4$.

In the following we obtain a lower bound for $d(G)$ when G is d-maximal.

PROPOSITION 4. *Let G be a non-abelian d-maximal p-group. Then $d(G) \geq 3$ except when $G = Q_8$.*

Proof. Clearly, $d(G) \neq 1$. If $d(G) = 2$, then $d(H) = 1$ for any proper subgroup H of G, which implies that every proper subgroup of G is cyclic. In particular, G has a cyclic maximal subgroup. A check through the list of p-groups with a cyclic maximal subgroup obtained in [3, Theorem 12.5.1] tells us that, other than the quaternion group of order eight, none of those groups are d-maximal. Thus $d(G) \geq 3$ except when $G = Q_8$. ☐

4. Some classes of d-maximal groups

We first consider the non-abelian d-maximal groups with maximal Frattini factor size. If G is a non-abelian p-group with maximal Frattini factor size, then $G' = \Phi(G)$ is cyclic of order p and hence, by [3, Theorem 4.3.4], $\Phi(G) \subseteq Z(G)$. It follows that there exists a subgroup E of G such that $G = EZ(G)$ and $E \cap Z(G) = \Phi(G)$. Thus G is the central product of E with $Z(G)$, and we also have that $Z(E) = E \cap Z(G) = \Phi(G)$. Since $E/\Phi(G) \leq G/\Phi(G)$, so $E/\Phi(G)$ is elementary abelian and hence, $\Phi(E) \subseteq \Phi(G)$. Then since $E' \subseteq \Phi(E) \subseteq \Phi(G) = Z(E)$, $E' \neq \{1\}$ and $\Phi(G)$ is cyclic of order p, so we must have $E' = \Phi(E) = Z(E)$. It follows by definition that E is extraspecial. Moreover, since $Z(G)/\Phi(G)$ is elementary abelian and $\Phi(G)$ is cyclic of order p, we have that $Z(G)$ is elementary abelian or abelian of type $(p, \dots, p, p^2)$. We summarize what we have just shown in the following:

PROPOSITION 5. *Let G be a non-abelian p-group with maximal Frattini factor size. Then $G \cong E * A$ where E is an extraspecial p-group and A is abelian of type $(p, \dots, p)$ or $(p, \dots, p, p^2)$.*

For p odd, let $E_{2n+1}(p)$ and $E_{2n+1}(p^2)$ denote the extraspecial p-groups of order p^{2n+1} with exponent p and p^2, respectively. Let D_8 denote the dihedral group of order eight. Denote by F_{2n+1} the central product of n copies of D_8 and by H_{2n+1} the central product of $n-1$ copies of D_8 with one copy of Q_8. Then F_{2n+1} and H_{2n+1} are extraspecial 2-groups of order 2^{2n+1}.

It is not difficult to deduce via Proposition 5 that for p odd, the smallest example of a d-maximal p-group with maximal Frattini factor size is

$$\begin{cases} E_5(p), & \text{for exponent } p \\ E_3(p) * C_{p^2} \cong E_3(p^2) * C_{p^2}, & \text{for exponent } p^2 \end{cases}$$

where C_{p^2} is the cyclic group of order p^2. In the following we give some classes of d-maximal p-groups (p odd) with maximal Frattini factor size. The notation $C_p^{(r)}$ is used to denote the elementary abelian p-group of rank r.

EXAMPLE 1. **(Exponent p (p odd))**

$$\begin{aligned} G_1 &= \langle g_1, \dots, g_n \mid g_1^p = \cdots = g_n^p = 1, [g_2, g_1] = [g_3, g_2] = [g_4, g_1] = g_n, \\ &\qquad [g_i, g_j] = 1 \text{ for all other } i > j \geq 1 \rangle \\ &\cong E_5(p) * C_p^{(n-4)}, \qquad n \geq 5. \end{aligned}$$

EXAMPLE 2. **(Exponent p^2 (p odd))**

$$\begin{aligned} G_2 &= \langle g_1, \dots, g_n \mid g_i^p = 1 \text{ for } i \in \{1, \dots, n\} \setminus \{3\}, g_3^p = g_n, [g_2, g_1] = g_n, \\ &\qquad [g_i, g_j] = 1 \text{ for all other } i > j \geq 1 \rangle \\ &\cong E_3(p) * (C_{p^2} \times C_p^{(n-4)}), \qquad n \geq 4. \end{aligned}$$

EXAMPLE 3. **(Exponent p^2 (p odd))**

$$\begin{aligned} G_3 &= \langle g_1, \dots, g_n \mid g_1^p = g_n, g_2^p = \cdots = g_n^p = 1, [g_2, g_1] = [g_3, g_2] = [g_4, g_1] \\ &\qquad = g_n, [g_i, g_j] = 1 \text{ for all other } i > j \geq 1 \rangle \\ &\cong E_5(p^2) * C_p^{(n-4)}, \qquad n \geq 5. \end{aligned}$$

For $p = 2$, the smallest example of a d-maximal 2-group with maximal Frattini factor size is Q_8. Some classes of d-maximal 2-groups with maximal Frattini factor size are as follows:

EXAMPLE 4.

$$\begin{aligned} G_4 &= \langle g_1, \ldots, g_n \mid g_1^2 = \cdots = g_n^2 = 1, [g_2, g_1] = [g_3, g_2] = [g_4, g_1] = g_n, \\ &\qquad [g_i, g_j] = 1 \text{ for all other } i > j \geq 1 \rangle \\ &\cong F_5 * C_2^{(n-4)}, \qquad n \geq 5. \end{aligned}$$

EXAMPLE 5.

$$\begin{aligned} G_5 &= \langle g_1, \ldots, g_n \mid g_2^2 = g_3^2 = g_n, g_i^2 = 1 \text{ for } i \in \{1, \ldots, n\} \setminus \{2, 3\}, [g_2, g_1] \\ &\qquad = [g_3, g_2] = [g_4, g_1] = g_n, [g_i, g_j] = 1 \text{ for all other} \\ &\qquad i > j \geq 1 \rangle \\ &\cong H_5 * C_p^{(n-4)}, \qquad n \geq 5. \end{aligned}$$

We now determine completely the non-abelian d-maximal 2-groups G with $d(G) \leq 3$ and $\Phi(G)$ elementary abelian. For $d(G) = 2$, the only possibility according to Proposition 4 is when G is quaternion of order eight. For $d(G) = 3$, we have by the Burnside Basis Theorem that $|G/\Phi(G)| = 2^3$ and since G is d-maximal, $d(\Phi(G)) = 1$ or 2. Thus, $|\Phi(G)| = 2$ or 2^2 (because $\Phi(G)$ is elementary abelian). It follows that $|G| = 2^4$ or 2^5. With the help of GAP [9], we then obtain the non-abelian d-maximal 2-groups G with $d(G) = 3$ and $\Phi(G)$ elementary abelian as follows. We have adopted the Hall-Senior notations (see [4]) to identify these groups.

(a) $|G| = 2^4$:

$$\begin{aligned} \text{(i) } \Gamma_2 a_2 &= \langle a, b, c \mid a^4 = c^2 = 1, a^2 = b^2, ba = a^3 b, ca = ac, cb = bc \rangle \\ &\cong Q_8 \times C_2; \\ \text{(ii) } \Gamma_2 b &= \langle a, b, c \mid a^4 = b^2 = c^2 = 1, ab = ba, ac = ca, bc = ca^2 b \rangle \\ &\cong (C_4 \times C_2) \rtimes C_2. \end{aligned}$$

(b) $|G| = 2^5$:

(iii) $\Gamma_4 a_3 = \langle a, b, c \mid a^4 = b^4 = 1, c^2 = a^2, ab = ba, ac = ca^{-1}, bc = cb^{-1} \rangle$;

(iv) $\Gamma_4 c_3 = \langle a, b, c \mid a^4 = b^4 = 1, c^2 = a^2 b^2, ab = ba, ac = ca^{-1}, bc = ca^2 b^{-1} \rangle$.

5. Powerful p-groups

In this section we consider connections between powerful p-groups and d-maximal groups. First we show that non-abelian powerful 2-groups are not d-maximal and state the result as follows:

THEOREM 2. *Let G be a non-abelian powerful 2-group. Then there exists a proper subgroup H of G such that $d(H) = d(G)$.*

Proof. By a result of Lubotzky and Mann [7], $d(H) \leq d(G)$ for any subgroup H of G. Suppose that G is d-maximal, that is, $d(H) < d(G)$ for every proper subgroup H of G. Then by Proposition 1 we have that $\Phi(G) = G'$. Since $\Phi(G) = G^2G'$, the inclusion $G^2 \subseteq G'$ follows. From the fact that G is a powerful 2-group, we get $G' \subseteq G^4$. Then $G^2 \subseteq G' \subseteq G^4 \subseteq G^2$ which gives us $G' = G^2 = G^4$. By Proposition 2 and induction, we obtain $G' = G^2 = G^4 = \cdots = G^{2^n} = \ldots$. Since G has exponent 2^k for some k, it follows that $G^{2^k} = \{1\}$ and therefore, $G' = \{1\}$. This tells us that G is abelian which is a contradiction. Hence, there exists a proper subgroup H of G such that $d(H) = d(G)$. □

For the case p is odd, it is clear by definition that a non-abelian powerful p-group must have exponent $\geq p^2$. By Corollary 1 we know that any non-abelian p-group of exponent $\geq p^3$ cannot be d-maximal. This leaves us with exponent p^2 to consider and we note that there does in fact exist non-abelian powerful p-groups of exponent p^2 which are d-maximal. Indeed, the groups G_2 and G_3 described in Examples 2 and 3 are examples of such groups. Note that G_2 is powerful because $G_2' = \langle g_n \rangle = G_2^p$ and similarly for G_3.

Thus the only possible non-abelian powerful p-groups which are d-maximal are the ones with exponent p^2 where p is odd.

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