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Second Conf. on Pseudo-Diff. Operators: Invited Lectures

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WAVE-FRONT SETS IN FOURIER LEBESGUE SPACES *

Abstract. We consider wave-front sets in the framework of weighted Fourier Lebesgue spaces, $\mathscr{F}L^q_s$. We prove that

$$(*) \qquad WF_{\mathscr{F}L^q_{s-m}}(Af) \subset WF_{\mathscr{F}L^q_s}(f) \subset WF_{\mathscr{F}L^q_{s-m}}(Af) \cup \operatorname{Char} A$$

where A is properly supported pseudo-differential operator of order m and $\operatorname{Char} A$ denotes the set of characteristic points of A. Moreover, we discuss more general class of pseudo-differential operators in the framework of modulation spaces and present $(*)$ in a more general setting.

1. Introduction

This paper is an expanded and modified version of an invited speaker's lecture given by the first author at the conference "Pseudo-differential operators with related topics II" held in Växjö, Sweden, June 23 - 27, 2008. It is a part of the authors joint research project. In order to present the main goals of the invited lecture apart from the original results (collected mainly in Section 4), we have included here some results from [19,20] without proofs.

In the present paper we study certain aspects of microlocal analysis in Fourier Lebesgue spaces. More precisely, we define wave-front sets with respect to those spaces and show that usual mapping properties for a class of pseudo-differential operators which are valid for classical wave-front sets (cf. [14, Chapter XVIII], [15, Chapter VIII]) also hold for our wave-front sets. We refer to [19,20] for the complete exposition of our definition, and results related to the wave-fronts in Fourier Lebesgue spaces. The recent study of pseudo-differential and Fourier integral operators in Fourier Lebesgue spaces as well as their connection with modulation spaces in different contexts increased the interest for such spaces, cf. [2, 3, 5, 16, 21, 23, 31].

The modulation spaces were introduced by Feichtinger in [6], and the theory was developed and generalized in [7–10]. Modulation spaces have been incorporated into the calculus of pseudo-differential operators, in the sense of the study of continuity of classical pseudo-differential operators acting on modulation spaces (cf. [4, 17, 18, 24–26]), and pseudo-differential operators for which modulation spaces are used as symbol classes, [11–13, 22, 27, 29, 30]. Microlocal analysis of modulation spaces reduces to the microlocal analysis of Fourier Lebesgue spaces. From this point of view our investigation in [19, 20] and in this paper are involved in the analysis of modulation spaces.

The paper is organized as follows. In Section 2 we fix basic notions and notation. Definitions of wave-front sets in the context of Fourier Lebesgue spaces as well

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as their basic properties are given in Propositions 1, 2 and 3 of Section 3. In Propositions 4 and 5 of Section 4 we study the continuity properties of pseudo-differential operators of the Hörmander class S^m on Fourier Lebesgue spaces. Theorems 1 and 2 of the same section are devoted to the study of microlocal properties of localized version of pseudo-differential operators. These results imply Corollary 1 where we discuss the relationship between our wave-front sets and the classical ones. In Section 5 we introduce wave-front sets in modulation spaces and discuss relations between local versions of Fourier Lebesgue and modulation spaces. Section 6 is devoted to the further study of pseudo-differential operators on a more advanced level. We present in that section the class of symbols $\mho^{s,\rho}_{(\omega)}(\mathbb{R}^{2d})$ and results on the continuity of corresponding pseudo-differential operators on weighted Fourier-Lebesgue spaces. Also, in Section 6, we present an estimate of the form (*) and discuss hypoellipticity in the same framework.

2. Notions and notation

We denote by Γ an open cone in $\mathbb{R}^d \setminus 0$ and by X an open set in $\mathbb{R}^d$. A conic neighborhood of a point $(x_0,\xi_0) \in \mathbb{R}^d \times (\mathbb{R}^d \setminus 0)$ is a product $X \times \Gamma$, where X is a neighborhood of x_0 in $\mathbb{R}^d$ and Γ is an open cone in $\mathbb{R}^d$ which contains ξ_0. Sometimes such a cone is denoted by Γ_{ξ_0} and is called a conic neighborhood of ξ_0. When $x,\xi \in \mathbb{R}^d$, their scalar product is denoted by $\langle x,\xi\rangle$. As usual, $\langle\xi\rangle = (1+|\xi|^2)^{1/2}$, $\xi \in \mathbb{R}^d$. For $q \in [1,\infty]$ we let $q' \in [1,\infty]$ denote the conjugate exponent, i. e. $1/q+1/q' = 1$.

Assume that ω and v are positive and measurable functions on $\mathbb{R}^d$. Recall that ω is called v-moderate weight if

$$\omega(x+y) \le C\omega(x)v(y) \tag{1}$$

for some constant C which is independent of $x,y \in \mathbb{R}^d$. If v in (1) can be chosen as a polynomial, then ω is called polynomially moderated. We let $\mathscr{P}(\mathbb{R}^d)$ to be the set of all polynomially moderated functions on $\mathbb{R}^d$.

For convenience we also need to consider appropriate subclasses of $\mathscr{P}$. More precisely, let $\mathscr{P}_0(\mathbb{R}^d)$ be the set of all $\omega \in \mathscr{P}(\mathbb{R}^d)\bigcap C^\infty(\mathbb{R}^d)$ such that $\partial^\alpha\omega/\omega \in L^\infty$ for all multi-indices α. By Lemma 1.2 in [29] it follows that for each $\omega \in \mathscr{P}(\mathbb{R}^d)$, there is an element $\omega_0 \in \mathscr{P}_0(\mathbb{R}^d)$ such that

$$C^{-1}\omega_0 \le \omega \le C\omega_0, \tag{2}$$

for some constant C.

Assume that $\rho \ge 0$. Then we let $\mathscr{P}_\rho(\mathbb{R}^{2d})$ to be the set of all $\omega(x,\xi)$ in $\mathscr{P}_0(\mathbb{R}^{2d})$ such that

$$\langle\xi\rangle^{\rho|\beta|}\frac{\partial_x^\alpha\partial_\xi^\beta\omega(x,\xi)}{\omega(x,\xi)} \in L^\infty(\mathbb{R}^{2d}), \tag{3}$$

for every multi-indices α and β. Note that in contrast to for $\mathscr{P}_0$ and $\mathscr{P}$, we do not have any equivalence between $\mathscr{P}_\rho$ and $\mathscr{P}$ when $\rho > 0$, in the sense of (2). If $s \in \mathbb{R}$ and $\rho \in [0,1]$, then $\mathscr{P}_\rho(\mathbb{R}^{2d})$ contains $\omega(x,\xi) = \langle\xi\rangle^s$.

The Fourier transform $\mathscr{F}$ is the linear and continuous mapping on $\mathscr{S}'(\mathbb{R}^d)$ which takes the form

$$(\mathscr{F}f)(\xi) = \widehat{f}(\xi) \equiv (2\pi)^{-d/2} \int f(x) e^{-i\langle x,\xi\rangle}\, dx, \quad \xi \in \mathbb{R}^d,$$

when $f \in L^1(\mathbb{R}^d)$. We recall that $\mathscr{F}$ is a homeomorphism on $\mathscr{S}'(\mathbb{R}^d)$ which restricts to a homeomorphism on $\mathscr{S}(\mathbb{R}^d)$ and to a unitary operator on $L^2(\mathbb{R}^d)$.

We say that a distribution $f \in \mathscr{D}'(\mathbb{R}^d)$ is *microlocally smooth* at $(x_0,\xi_0) \in \mathbb{R}^d \times (\mathbb{R}^d \setminus 0)$ if there exists $\chi \in C_0^\infty(\mathbb{R}^d)$ such that $\chi(x_0) \neq 0$, and an open cone Γ_{ξ_0} such that for every $N \in \mathbb{N}$, there exists $C_N > 0$ such that $|\mathscr{F}(\chi f)(\xi)| \le C_N \langle \xi \rangle^{-N/2}$, $\xi \in \Gamma_{\xi_0}$. The *wave-front set* of f, $WF(f)$ is the complement of the set of points (x_0,ξ_0), where f is microlocally smooth.

Assume that $a \in \mathscr{S}(\mathbb{R}^{2d})$, and that $t \in \mathbb{R}$ is fixed. Then the pseudo-differential operator $a_t(x,D)$, defined by the formula

$$\begin{aligned}(a_t(x,D)f)(x) &= (\mathrm{Op}_t(a)f)(x)\\ &= (2\pi)^{-d} \iint a((1-t)x+ty,\xi) f(y) e^{i\langle x-y,\xi\rangle}\, dy\, d\xi,\end{aligned} \tag{4}$$

is a linear and continuous operator on $\mathscr{S}(\mathbb{R}^d)$. For general $a \in \mathscr{S}'(\mathbb{R}^{2d})$, the pseudo-differential operator $a_t(x,D)$ is defined as the continuous operator from $\mathscr{S}(\mathbb{R}^d)$ to $\mathscr{S}'(\mathbb{R}^d)$ with the distribution kernel

$$K_{t,a}(x,y) = (2\pi)^{-d/2} (\mathscr{F}_2^{-1} a)((1-t)x+ty, y-x). \tag{5}$$

Here $\mathscr{F}_2 F$ is the partial Fourier transform of $F(x,y) \in \mathscr{S}'(\mathbb{R}^{2d})$ with respect to the y-variable. This definition makes sense, since the mappings $\mathscr{F}_2$ and

$$F(x,y) \mapsto F((1-t)x+ty, y-x)$$

are homeomorphisms on $\mathscr{S}'(\mathbb{R}^{2d})$. We also note that this definition of $a_t(x,D)$ agrees with the operator in (4) when $a \in \mathscr{S}(\mathbb{R}^{2d})$, and that $a_t(x,D)$ agrees with the Kohn-Nirenberg representation $a(x,D)$ when $t = 0$.

Furthermore, any linear and continuous operator T from $\mathscr{S}(\mathbb{R}^d)$ to $\mathscr{S}'(\mathbb{R}^d)$ has a distribution kernel K in $\mathscr{S}'(\mathbb{R}^{2d})$ in view of kernel theorem of Schwartz. By Fourier's inversion formula we may then find a unique $a \in \mathscr{S}'(\mathbb{R}^{2d})$ such that (5) is fulfilled with $K = K_{t,a}$. Consequently, for every fixed $t \in \mathbb{R}$, there is a one to one correspondence between linear and continuous operators from $\mathscr{S}(\mathbb{R}^d)$ to $\mathscr{S}'(\mathbb{R}^d)$, and $\mathrm{Op}_t(\mathscr{S}'(\mathbb{R}^{2d}))$, the set of all $a_t(x,D)$ such that $a \in \mathscr{S}'(\mathbb{R}^{2d})$.

In particular, if $a \in \mathscr{S}'(\mathbb{R}^{2d})$ and $s,t \in \mathbb{R}$, then there is a unique $b \in \mathscr{S}'(\mathbb{R}^{2d})$ such that $a_s(x,D) = b_t(x,D)$. By straight-forward applications of Fourier's inversion formula, it follows that

$$a_s(x,D) = b_t(x,D) \quad \Longleftrightarrow \quad b(x,\xi) = e^{i(t-s)\langle D_x,D_\xi\rangle} a(x,\xi). \tag{6}$$

(Cf. Section 18.5 in [14].)

3. Wave-front sets in Fourier Lebesgue spaces

In this section we define wave-front sets with respect to Fourier Lebesgue spaces, and recall some general properties from [19, 20].

Let $\omega \in \mathscr{P}(\mathbb{R}^{2d})$ and let $q \in [1,\infty]$. The (weighted) Fourier-Lebesgue space $\mathscr{F}L^q_{(\omega)}(\mathbb{R}^d)$ is the Banach space which consists of all $f \in \mathscr{S}'(\mathbb{R}^d)$ such that

$$\|f\|_{\mathscr{F}L^q_{(\omega)}} = \|f\|_{\mathscr{F}L^q_{(\omega),x}} \equiv \|\widehat{f}\cdot\omega(x,\cdot)\|_{L^q} < \infty. \tag{7}$$

The weight $\omega(x,\xi)$ in (7) depends on both x and ξ, although $\widehat{f}(\xi)$ only depends on ξ. However, since ω is v-moderate for some $v \in \mathscr{P}(\mathbb{R}^{2d})$, different choices of x give rise to equivalent norms. Therefore, the condition $\|f\|_{\mathscr{F}L^q_{(\omega),x}} < \infty$ is independent of x and for different $x_1, x_2 \in \mathbb{R}^d$ there exists a constant $C_{x_1,x_2} > 0$ such that

$$C^{-1}_{x_1,x_2}\|f\|_{\mathscr{F}L^q_{(\omega),x_2}} \le \|f\|_{\mathscr{F}L^q_{(\omega),x_1}} \le C_{x_1,x_2}\|f\|_{\mathscr{F}L^q_{(\omega),x_2}}.$$

We say that $f \in \mathscr{D}'(\mathbb{R}^d)$ is locally in $\mathscr{F}L^q_{(\omega)}(\mathbb{R}^d)$, if $\chi f \in \mathscr{F}L^q_{(\omega)}(\mathbb{R}^d)$ for every $\chi \in C_0^\infty(\mathbb{R}^d)$ and in that case we use the notation $f \in \mathscr{F}L^q_{(\omega),loc}(\mathbb{R}^d)$. It is said that $f \in \mathscr{F}L^q_{(\omega),loc}(\mathbb{R}^d)$ at x_0 if there exists a function $\chi \in C_0^\infty(\mathbb{R}^d)$, $\chi(x_0) \neq 0$, such that $\chi f \in \mathscr{F}L^q_{(\omega)}(\mathbb{R}^d)$.

In the remaining part of the paper we study weighted Fourier-Lebesgue spaces with weights which depend on ξ. Thus, with $\omega_0(\xi) = \omega(0,\xi) \in \mathscr{P}(\mathbb{R}^d)$,

$$f \in \mathscr{F}L^q_{(\omega)}(\mathbb{R}^d) = \mathscr{F}L^q_{(\omega_0)}(\mathbb{R}^d) \quad \Longleftrightarrow \quad \|f\|_{\mathscr{F}L^q_{(\omega_0)}} \equiv \|\widehat{f}w_0\|_{L^q} < \infty.$$

We usually assume that the involved weight functions $\omega_0(\xi)$ is given by $\omega_0(\xi) = \omega(x_0,\xi) = \langle\xi\rangle^s$, for some $x_0 \in \mathbb{R}^d$ and $s \in \mathbb{R}$. In this case we use the notation $\mathscr{F}L^q_s$ instead of $\mathscr{F}L^q_{(\omega_0)}$. If $\omega = 1$, then the notation $\mathscr{F}L^q$ is used instead of $\mathscr{F}L^q_{(\omega)}$.

Let $\omega_0 \in \mathscr{P}(\mathbb{R}^d)$, $\Gamma \subseteq \mathbb{R}^d \setminus 0$ be open cone and $q \in [1,\infty]$ be fixed. For any $f \in \mathscr{S}'(\mathbb{R}^d)$, let

$$|f|_{\mathscr{F}L^{q,\Gamma}_{(\omega_0)}} \equiv \Big(\int_\Gamma |\widehat{f}(\xi)\omega_0(\xi)|^q\,d\xi\Big)^{1/q}$$

(with obvious interpretation when $q = \infty$). We note that $|\cdot|_{\mathscr{F}L^{q,\Gamma}_{(\omega_0)}}$ defines a semi-norm on $\mathscr{S}'(\mathbb{R}^d)$ which might attain the value $+\infty$. If $\Gamma = \mathbb{R}^d \setminus 0$, $f \in \mathscr{F}L^q_{(\omega)}(\mathbb{R}^d)$ and $q < \infty$, then $|f|_{\mathscr{F}L^{q,\Gamma}_{(\omega)}}$ agrees with the Fourier Lebesgue norm $\|f\|_{\mathscr{F}L^q_{(\omega),x}}$ of f.

We let $\Theta_{\mathscr{F}L^q_{(\omega_0)}}(f)$ to be the set of all $\xi \in \mathbb{R}^d \setminus 0$ such that $|f|_{\mathscr{F}L^{q,\Gamma}_{(\omega_0)}} < \infty$, for some $\Gamma = \Gamma_\xi$. Its complement in $\mathbb{R}^d \setminus 0$ is denoted by $\Sigma_{\mathscr{F}L^q_{(\omega_0)}}(f)$.

We have now the following result.

PROPOSITION 1. *Assume that* $q \in [1,\infty]$, $\chi \in \mathscr{S}(\mathbb{R}^d)$, *and that* $\omega_0 \in \mathscr{P}(\mathbb{R}^d)$. *Also assume that* $f \in \mathscr{E}'(\mathbb{R}^d)$. *Then*

$$\Sigma_{\mathscr{F}L^q_{(\omega_0)}}(\chi f) \subseteq \Sigma_{\mathscr{F}L^q_{(\omega_0)}}(f). \tag{8}$$

Proof. Assume that $\xi_0 \in \Theta_{\mathscr{F}L^q_{(\omega_0)}}(f)$, and choose open cones Γ_1 and Γ_2 in $\mathbb{R}^d$ such that $\overline{\Gamma_2} \subseteq \Gamma_1$. Since f has a compact support, it follows that $|\widehat{f}(\xi)\omega_0(\xi)| \le C\langle\xi\rangle^{N_0}$ for some positive constants C and N_0. The idea of the proof is to show that for each N, there are constants C_N such that

$$|\chi f|_{\mathscr{F}L^{q_2,\Gamma_2}_{(\omega_0)}} \le C_N\Big(|f|_{\mathscr{F}L^{q_1,\Gamma_1}_{(\omega_0)}} + \sup_{\xi\in\mathbb{R}^d}\big(|\widehat{f}(\xi)\omega_0(\xi)|\langle\xi\rangle^{-N}\big)\Big) \tag{9}$$

$$\text{when}\quad q_1 \le q_2,\quad \overline{\Gamma}_2 \subseteq \Gamma_1 \quad\text{and}\quad N = 1,2,\ldots.$$

The result then follows by taking $q_1 = q_2 = q$ and $N \ge N_0$. We refer to [19] for details of the proof of (9). □

Now we are ready to define wave-front sets in the framework of Fourier Lebesgue spaces.

DEFINITION 1. *Assume that* $q \in [1,\infty]$, $f \in \mathscr{D}'(\mathbb{R}^d)$ *and* $\omega_0 \in \mathscr{P}(\mathbb{R}^d)$. *The wave-front set* $WF_{\mathscr{F}L^q_{(\omega_0)}}(f)$ *with respect to* $\mathscr{F}L^q_{(\omega_0)}(\mathbb{R}^d)$ *consists of all pairs* (x_0,ξ_0) *in* $\mathbb{R}^d \times (\mathbb{R}^d \setminus 0)$ *such that* $\xi_0 \in \Sigma_{\mathscr{F}L^q_{(\omega_0)}}(\chi f)$, *holds for each* $\chi \in C_0^\infty(\mathbb{R}^d)$ *such that* $\chi(x_0) \neq 0$.

The following proposition shows that the wave-front set $WF_{\mathscr{F}L^q_{(\omega_0)}}(f)$ decreases with respect to the parameter q and increases with respect to the weight function ω, when $f \in \mathscr{D}'(\mathbb{R}^d)$ is fixed.

PROPOSITION 2. *Assume that* $f \in \mathscr{D}'(\mathbb{R}^d)$, $q_j \in [1,\infty]$ *and* $\omega_j \in \mathscr{P}(\mathbb{R}^d)$ *for* $j = 1,2$ *satisfy*

$$q_1 \le q_2, \quad \textit{and} \quad \omega_2(\xi) \le C\omega_1(\xi), \tag{10}$$

for some constant C *which is independent of* $\xi \in \mathbb{R}^d$. *Then*

$$WF_{\mathscr{F}L^{q_2}_{(\omega_2)}}(f) \subseteq WF_{\mathscr{F}L^{q_1}_{(\omega_1)}}(f).$$

Proof. It is no restriction to assume that f has a compact support, and that $\omega_1(\xi) = \omega_2(\xi) = \omega_0(\xi)$. This implies that

$$\sup_{\xi\in\mathbb{R}^d} |\langle\xi\rangle^{-N_0}\widehat{f}(\xi)\omega_0(\xi)| < \infty$$

provided N_0 is chosen large enough. Hence (9) implies that $\Theta_{\mathscr{F}L^{q_1}_{(\omega_1)}}(f) \subseteq \Theta_{\mathscr{F}L^{q_2}_{(\omega_2)}}(f)$, and the assertion follows. □

PROPOSITION 3. *Assume that* $q \in [1,\infty]$, $f \in \mathscr{D}'(X)$, $\omega_0 \in \mathscr{P}(\mathbb{R}^d)$ *and* $(x_0,\xi_0) \in X \times (\mathbb{R}^d \setminus 0)$. *The following conditions are equivalent:*

(1) *there exist* $g \in \mathscr{F}L^q_{(\omega_0)}(\mathbb{R}^d)$ *such that* $(x_0,\xi_0) \notin WF(f-g)$*;*

(2) $(x_0,\xi_0) \notin WF_{\mathscr{F}L^q_{(\omega_0)}}(f)$.

Proof. First we show that $\varphi g \in \mathscr{F}L^q_{(\omega_0)}(\mathbb{R}^d)$ if $\varphi \in \mathscr{S}(\mathbb{R}^d)$ and $g \in \mathscr{F}L^q_{(\omega_0)}(\mathbb{R}^d)$.

Let $\varphi \in \mathscr{S}(\mathbb{R}^d)$ and let ψ be defined by $\widehat{\psi} = \varphi$. Then

$$
\begin{aligned}
\|g\varphi\|_{\mathscr{F}L^q_{(\omega_0)}} &= (2\pi)^{-d/2}\Big(\int_{\mathbb{R}^d}\Big|\Big(\int_{\mathbb{R}^d}\widehat{g}(\xi-\eta)\psi(\eta)\,d\eta\Big)\omega_0(\xi)\Big|^q d\xi\Big)^{1/q} \\
&\le (2\pi)^{-d/2}\Big(\int_{\mathbb{R}^d}\Big(\int_{\mathbb{R}^d}|\widehat{g}(\xi-\eta)||\psi(\eta)|\,d\eta|\omega_0(\xi)|\Big)^q d\xi\Big)^{1/q} \\
&\le (2\pi)^{-d/2}\int_{\mathbb{R}^d}\Big(\int_{\mathbb{R}^d}|\widehat{g}(\xi-\eta)\psi(\eta)\omega_0(\xi)|^q\,d\xi\Big)^{1/q}d\eta \\
&\le (2\pi)^{-d/2}\int_{\mathbb{R}^d}\Big(\int_{\mathbb{R}^d}|\widehat{g}(\xi-\eta)\omega_0(\xi-\eta)|^q\,d\xi\Big)^{1/q}|\psi(\eta)v(\eta)|\,d\eta \\
&= C\|g\|_{\mathscr{F}L^q_{(\omega_0)}},
\end{aligned}
$$

where $C = (2\pi)^{-d/2}\|\psi v\|_{L^1}$ is finite, since v is of polynomial growth.

By the assumption, $f-g$ is microlocally smooth at (x_0,ξ_0). Hence there exists an open cone $\Gamma = \Gamma_{\xi_0}$ such that

$$\int_\Gamma |\mathscr{F}(\varphi(f-g))(\xi)\omega_0(\xi)|^q\,d\xi < \infty,$$

where $\varphi(x_0) \neq 0$ and the support of φ can be chosen to be sufficiently close to x_0. This, together with the decomposition $\mathscr{F}(\varphi f) = \mathscr{F}(\varphi g) + \mathscr{F}(\varphi(f-g))$ implies that

$$\int_\Gamma |\mathscr{F}(\varphi f)(\xi)\omega_0(\xi)|^q\,d\xi < \infty, \tag{11}$$

i.e. $(x_0,\xi_0) \notin WF_{\mathscr{F}L^q_{(\omega_0)}}(f)$.

Conversely, if $(x_0,\xi_0) \notin WF_{\mathscr{F}L^q_{(\omega_0)}}(f)$, then there exist $\varphi \in C_0^\infty$ such that $\varphi(x_0) \neq 0$ and a conic neighborhood Γ of ξ_0 such that (11) holds.

Let $g \in \mathscr{F}L^q_{(\omega_0)}(\mathbb{R}^d)$ be defined by

$$\widehat{g}(\xi) = \begin{cases} \mathscr{F}(\varphi f)(\xi), & \text{if} \quad \xi \in \Gamma \\ 0, & \text{if} \quad \xi \notin \Gamma. \end{cases}$$

Then $\widehat{h} = \mathscr{F}(\varphi f) - \widehat{g}$ vanishes in Γ and $\widehat{h}$ has a polynomial bound. Therefore $(x_0,\xi_0) \notin WF(h)$. Choose $\psi \in C_0^\infty$ so that $\psi\varphi = 1$ in a neighborhood of x_0. Note $(x_0,\xi_0) \notin$

$WF(\psi h)$. Now, since

$$f - \psi g = (1 - \psi\varphi) f + \psi h,$$

we conclude that $(x_0, \xi_0) \notin WF(f - \psi g)$. Since $\psi g \in \mathscr{F}L^q_{(\omega_0)}(\mathbb{R}^d)$ the proof is complete. $\square$

4. Pseudo-differential operators with classical symbols

In this section we prove mapping properties of pseudo-differential operators in the background of wave-front sets of Fourier Lebesgue types.

Assume that $m \in \mathbb{R}$. Then we recall that the Hörmander symbol class

$$S^m = S^m(\mathbb{R}^d \times \mathbb{R}^d) = S^m(\mathbb{R}^{2d})$$

consists of all smooth functions a such that for each pair of multi-indices α, β there are constants $C_{\alpha,\beta}$ such that

$$|\partial_\xi^\alpha \partial_x^\beta a(x,\xi)| \le C_{\alpha,\beta} \langle \xi \rangle^{m-|\alpha|}, \quad x, \xi \in \mathbb{R}^d.$$

We also set $S^{-\infty} = \cap_{m \in \mathbb{R}} S^m$, and

$$\mathrm{Op}(S^m) = \{\, a(x,D) \,;\, a \in S^m(\mathbb{R}^d \times \mathbb{R}^d) \,\}.$$

The following result is needed in the proof of Theorem 1 below.

PROPOSITION 4. *Assume that $q \in [1,\infty]$ and $\chi \in C_0^\infty(\mathbb{R}^d)$. Then the following is true:*

(1) *if $a \in S^0$ then the mapping $\chi(x)a(x,D) : \mathscr{F}L^q(\mathbb{R}^d) \to \mathscr{F}L^q(\mathbb{R}^d)$ is continuous. In particular, if $a(x,\xi) = a(\xi) \in S^0$, then the mapping $a(D) : \mathscr{F}L^q(\mathbb{R}^d) \to \mathscr{F}L^q(\mathbb{R}^d)$ is continuous;*

(2) *if $a \in S^m$ and $s \in \mathbb{R}$ then $\chi(x)a(x,D) : \mathscr{F}L^q_s(\mathbb{R}^d) \to \mathscr{F}L^q_{s-m}(\mathbb{R}^d)$. In particular, if $a(x,\xi) = a(\xi) \in S^m$, $a(D) : \mathscr{F}L^q_s(\mathbb{R}^d) \to \mathscr{F}L^q_{s-m}(\mathbb{R}^d)$.*

Proof. For the proof it is convenient to put $E_s(\xi) = \langle \xi \rangle^s$ when $s \in \mathbb{R}$. We only prove assertions for $1 \le q < \infty$. The case $q = \infty$ follows by similar arguments and is left to the reader.

(1) Let $a \in S^0$ and $\chi \in C_0^\infty(\mathbb{R}^d)$. We denote the support of χ by K. The oscillatory integral (4) is well defined for the symbol χa, $t = 0$ and $f \in \mathscr{F}L^q(\mathbb{R}^d)$. Namely, after $2s$ times integration by parts we obtain

$$\chi(x)a(x,D)f(x) = (2\pi)^{-d/2} \int e^{i\langle x,\eta \rangle} \big(E_{2s}(D_x)(\chi(x)a(x,\eta))\big) \langle \eta \rangle^{-2s} \widehat{f}(\eta)\, d\eta, \tag{12}$$

which, by the Hölder inequality, gives

$$|\chi(x)a(x,D)f(x)| \le (2\pi)^{-d/2} \|\big(E_{2s}(D_x)(\chi(x)a(x,\cdot))\big)\langle \cdot \rangle^{-2s}\|_{L^{q'}} \|\widehat{f}\|_{L^q} \le C\|f\|_{\mathscr{F}L^q}$$

for $2s > d/q'$.

Let $\mathscr{F}_1 a_\chi(\xi,\eta)$ be the partial Fourier transform of $a_\chi(x,\eta) = \chi(x)a(x,\eta)$ with respect to the x variable. Then it follows from the assumptions that for each $N \in \mathbb{N}$, there is a constant $C_N > 0$ such that

$$|\mathscr{F}_1 a_\chi(\xi,\eta)| \le C_N \langle\xi\rangle^{-N}, \qquad \xi,\eta \in \mathbb{R}^d. \tag{13}$$

Now,

$$\begin{aligned}
&|\mathscr{F}(\chi(x)a(x,D)f(x))(\xi)| \\
&\quad = (2\pi)^{-d}\Big|\int_K e^{-i\langle x,\xi\rangle}\Big(\int_{\mathbb{R}^d} e^{i\langle x,\eta\rangle} E_{2s}(D_x)a_\chi(x,\eta)\langle\eta\rangle^{-2s}\widehat{f}(\eta)\,d\eta\Big)\,dx\Big| \\
&\quad = (2\pi)^{-d}\Big|\int_{\mathbb{R}^d}\widehat{f}(\eta)\langle\eta\rangle^{-2s}\Big(\int_K e^{-i\langle x,\xi-\eta\rangle}E_{2s}(D_x)a_\chi(x,\eta)\,dx\Big)\,d\eta\Big| \\
&\quad \le (2\pi)^{-d}\Big|\int_{\mathbb{R}^d}\widehat{f}(\eta)\langle\eta\rangle^{-2s}\langle\xi-\eta\rangle^{2s}\mathscr{F}_1 a_\chi(\xi-\eta,\eta)\,d\eta\Big|
\end{aligned}$$

By taking ξ and $\xi-\eta$ as new variables of integrations, and assuming that $N > s+\frac{d}{2}$, by (13) and Minkowski's inequality we obtain

$$\begin{aligned}
&\|\mathscr{F}(\chi(x)a(x,D)f(x))(\xi)\|_{L^q} \\
&\le (2\pi)^{-d}\Big(\int_{\mathbb{R}^d}\Big|\int_{\mathbb{R}^d}\widehat{f}(\xi-\eta)\langle\eta\rangle^{2s}\mathscr{F}_1 a_\chi(\eta,\xi-\eta)\langle\xi-\eta\rangle^{-2s}\,d\eta\Big|^q d\xi\Big)^{1/q} \\
&\le (2\pi)^{-d}\int_{\mathbb{R}^d}\langle\eta\rangle^{2s}\Big(\int_{\mathbb{R}^d}|\widehat{f}(\xi-\eta)\mathscr{F}_1 a_\chi(\eta,\xi-\eta)|^q\,d\xi\Big)^{1/q}d\eta \\
&\le (2\pi)^{-d}C_N\int_{\mathbb{R}^d}\langle\eta\rangle^{2(s-N)}\Big(\int_{\mathbb{R}^d}|\widehat{f}(\xi-\eta)|^q\,d\xi\Big)^{1/q}d\eta \\
&\le C\|\widehat{f}\|_{L^q}.
\end{aligned}$$

If we instead have that the symbol is a Fourier multiplier $a = a(\xi) \in S^0$, then it is obvious that

$$\|\mathscr{F}(a(D)f)\|_{L^q} \le C\|\mathscr{F}f\|_{L^q},$$

and (1) follows in this case as well.

The assertion (2) follows from (1), the fact that the map

$$a(x,\xi) \mapsto a(x,\xi)\langle\xi\rangle^m$$

is a homeomorphism from S^0 to S^m, and the fact that the map

$$f \mapsto \langle D\rangle^{-s}f$$

is a homeomorphism from $\mathscr{F}L^q$ to $\mathscr{F}L^q_s$. The proof is complete. □

REMARK 1. It is known that an operator $a(x,D)$ whose symbol belongs to the Hörmander class S^0 is continuous from L^q to L^q, $1<q<\infty$, see [32]. In order to prove the continuity in $\mathscr{F}L^q(\mathbb{R}^d)$ it is not sufficient to assume the boundedness of the corresponding symbol a with respect to the x variable because for such symbol and $f\in\mathscr{F}L^q(\mathbb{R}^d)$ the convolution $\widehat{a}*\widehat{f}$ does not belong to L^q, in general. For that reason we observe the operators of the form $\chi(x)a(x,D)$. Alternatively, we could impose a decay condition on a with respect to the x variable. For example, one can prove that $a(x,D):\mathscr{F}L^q(\mathbb{R}^d)\to\mathscr{F}L^q(\mathbb{R}^d)$ is continuous if the symbol a satisfies

$$|\partial_\xi^\alpha\partial_x^\beta a(x,\xi)|\le C_{\alpha,\beta}\langle x\rangle^{k-|\beta|}\langle\xi\rangle^{m-|\alpha|},$$

for $k<-d$. More details on this topic can be found in [19, 20].

Next we recall the definition of characteristic sets and elliptic pseudo-differential operators. The symbol $a\in S^m(\mathbb{R}^{2d})$ is called *non-characteristic* at $(x_0,\xi_0)\in\mathbb{R}^d\times(\mathbb{R}^d\setminus 0)$ if there is a neighborhood U of x_0, a conical neighborhood Γ of ξ_0 and constants c and R such that

$$|a(x,\xi)|>c|\xi|^m,\quad \text{if}\quad |\xi|>R, \tag{14}$$

and $\xi\in\Gamma$. Then one can find $b\in S^{-m}(\mathbb{R}^{2d})$ such that

$$a(x,D)b(x,D)-Id\in \mathrm{Op}(S^{-\infty})\quad\text{and}\quad b(x,D)a(x,D)-Id\in\mathrm{Op}(S^{-\infty})$$

in a conical neighborhood of (x_0,ξ_0) (cf. [14, 19]). The point $(x_0,\xi_0)\in\mathbb{R}^d\times(\mathbb{R}^d\setminus 0)$ is called characteristic for a if it is not non-characteristic point of $a(x,D)$. The set of characteristic points (the characteristic set) of $a(x,D)$ is denoted by $\mathrm{Char}\, a(x,D)$. We shall identify operators with their symbols when discussing characteristic sets.

The operator $a(x,D)\in\mathrm{Op}(S^m)$ is called *elliptic* if the set of characteristic points is empty. This means that for each bounded neighbourhood U of x_0, there are constants $c,R>0$ such that (14) holds when $x\in U$.

PROPOSITION 5. *Let $a\in S^m$ be elliptic and assume that $f\in\mathscr{F}L^q_{t,loc}(\mathbb{R}^d)$ for some $q\in[1,\infty]$ and for some $t\in\mathbb{R}$. If $a(x,D)f\in\mathscr{F}L^q_{s,loc}(\mathbb{R}^d)$, then $f\in\mathscr{F}L^q_{s+m,loc}(\mathbb{R}^d)$ and for every $\chi\in C_0^\infty(\mathbb{R}^d)$ we have*

$$||\chi f||_{\mathscr{F}L^q_{s+m}}\le C_{s,t}(||\chi a(x,D)f||_{\mathscr{F}L^q_s}+||\chi f||_{\mathscr{F}L^q_t}). \tag{15}$$

In particular, if $a(x,D)=a(D)\in S^m$ is elliptic, then (15) *holds without* χ.

Proof. The ellipticity condition implies that there is an operator $b(x,D)\in\mathrm{Op}(S^{-m})$ such that

$$\chi f=b(x,D)a(x,D)\chi f+r(x,D)\chi f,$$

for some $r\in S^{-\infty}$. Hence $b(x,D)$ is continuous from $\mathscr{F}L^q_s(\mathbb{R}^d)$ to $\mathscr{F}L^q_{s+m}(\mathbb{R}^d)$ and $r(x,D)$ is continuous from $\mathscr{F}L^q_t(\mathbb{R}^d)$ to $\mathscr{F}L^q_{s+m}(\mathbb{R}^d)$. This implies

$$\begin{aligned}||\chi f||_{\mathscr{F}L^q_{s+m}}&\le||b(x,D)a(x,D)\chi f||_{\mathscr{F}L^q_{s+m}}+||r(x,D)\chi f||_{\mathscr{F}L^q_{s+m}}\\&\le C_{s,t}(||\chi a(x,D)f||_{\mathscr{F}L^q_s}+||\chi f||_{\mathscr{F}L^q_t}).\end{aligned}$$

□

REMARK 2. The above propositions can be reformulated in the language of the symbol class $S^m_{loc}(X\times\mathbb{R}^d)$, where X is an open set in $\mathbb{R}^d$. This class is introduced in [14] as the starting point in the study of pseudo-differential operators on manifolds.

We say that a continuous linear map $A: C_0^\infty(X)\to C^\infty(X)$ is a pseudo-differential operator of order m in X, $A\in\Psi^m(X)$, if for arbitrary $\phi,\psi\in C_0^\infty(X)$ the operator $f\mapsto \phi A(\psi f)$ is in $\mathrm{Op}(S^m)$. For example, the restriction of $a(x,D)\in\mathrm{Op}(S^m)$ to X belongs to $\Psi^m(X)$.

According to [14, Proposition 18.1.22], every $A\in\Psi^m(X)$ can be decomposed as $A=A_0+A_1$ where $A_1\in\Psi^m(X)$ is properly supported and the kernel of A_0 is in C^∞. In that sense it is no essential restriction to require proper supports in the following statements.

THEOREM 1. *Assume that* $q\in[1,\infty]$, $s\in\mathbb{R}$, $f\in\mathscr{D}'(\mathbb{R}^d)$ *and* $(x_0,\xi_0)\in\mathbb{R}^d\times(\mathbb{R}^d\setminus 0)$. *Then the following conditions are equivalent:*

(1) $(x_0,\xi_0)\notin WF_{\mathscr{F}L^q_s}(f)$;

(2) $(x_0,\xi_0)\notin WF_{\mathscr{F}L^q}(Af)$ *for some properly supported* $A\in\Psi^s(X)$ *which is non-characteristic at* (x_0,ξ_0);

(3) *there is a conic neighborhood* $U\times\Gamma_0$ *of* (x_0,ξ_0) *in* $X\times(\mathbb{R}^d\setminus 0)$ *such that* Bf *in* $\mathscr{F}L^q_{s-m,loc}(X)$ *for every properly supported pseudo-differential operator* $B\in\Psi^m(X)$ *with the symbol of class* $-\infty$ *outside* $U\times\Gamma_0$.

Proof. We follow the proof of Theorem 8.4.8 in [15] which concerns the classical wave-front set.

Assume that (1) holds. Then (11) holds with $\omega_0(\xi)=\langle\xi\rangle^s$ and for some conic neighborhood Γ of ξ_0 and for some $\varphi\in C_0^\infty(X)$ such that $\varphi(x_0)\neq 0$. Let $q(\xi)\in C^\infty$ be a homogeneous function of degree s for $|\xi|\geq 1$, with support in Γ. We define $A=\varphi q(D)\varphi$, where φ is from (11). Then $A\in\Psi^s(X)$ and (11) give

$$\|Af\|_{\mathscr{F}L^q}\leq C\|q(D)\varphi f\|_{\mathscr{F}L^q}<\infty.$$

Moreover, the symbol of A is $q(\xi)$ (mod $S^{-\infty}$) near x_0, which proves (2).

Now, assume that (3) holds. To prove (1) it is sufficient to find $\varphi\in C_0^\infty$ such that supp φ is sufficiently close to x_0 and a conic neighborhood Γ of ξ_0 such that (11) holds. Let $B\in\Psi^m(X)$ be fixed. By (3) we may choose $\phi,\psi\in C_0^\infty(X)$ and $q(\xi)\in C^\infty$ a homogeneous of degree m for $|\xi|\geq 1$, with $\psi=1$ in a neighborhood of supp ϕ and supp $\phi\times$ supp $q\subset U\times\Gamma_0$, so that $B=\psi q(D)\phi$.

By using the fact that

$$\mathscr{F}(q(D)\phi f)=\mathscr{F}(\psi q(D)\phi f)+\mathscr{F}((1-\psi)q(D)\phi f),$$

it follows from (3) that $\psi q(D)\phi f \in \mathscr{F}L^q_{s-m}$ and $(1-\psi)q(D)\phi$ is of order $-\infty$. Therefore (11) holds with $\omega_0(\xi) = \langle \xi \rangle^s$, $\Gamma = \Gamma_0$ and $\varphi = q(D)\phi$.

Finally, assume that (2) holds and choose a closed conic neighborhood $U \times \Gamma_0$ of (x_0,ξ_0) such that the symbol $a(x,\xi)$ of A satisfies $|a(x,\xi)| > c|\xi|^m, \quad \text{if} \quad \xi \in \Gamma_0, \quad |\xi| > C$, for some constants $c, C > 0$. Therefore, for every B as in condition (3), we can find a properly supported $\tilde{B} \in \Psi^{m-s}(X)$ such that

$$B - \tilde{B}A \in \Psi^{-\infty}(X).$$

Hence $Bf - \tilde{B}Af \in C^\infty(X)$. By the assumption and Proposition 4 (2) it follows that $\tilde{B}Af \in \mathscr{F}L^q_{s-m,loc}(X)$. Therefore, $Bf \in \mathscr{F}L^q_{s-m,loc}(X)$ which completes the proof. □

COROLLARY 1. *Assume that $q \in [1,\infty]$, $s \in \mathbb{R}$, $f \in \mathscr{D}'(\mathbb{R}^d)$ and $(x_0,\xi_0) \in \mathbb{R}^d \times (\mathbb{R}^d \setminus 0)$. Then the following is true:*

(1) *If $(x_0,\xi_0) \notin WF_{\mathscr{F}L^q_s}(f)$ then $(x_0,\xi_0) \notin WF_{\mathscr{F}L^q_{s-m}}(Af)$ for every properly supported $A \in \Psi^m(X)$;*

(2) *if $(x_0,\xi_0) \notin WF_{\mathscr{F}L^q_{s-m}}(Af)$ for some properly supported $A \in \Psi^m(X)$ which is non-characteristic at (x_0,ξ_0), then $(x_0,\xi_0) \notin WF_{\mathscr{F}L^q_s}(f)$;*

(3) *there is a conical neighborhood $U \times \Gamma$ of (x_0,ξ_0) such that $(x,\xi) \notin WF_{\mathscr{F}L^q_s}(f)$ for every $(x,\xi) \in U \times \Gamma$ and for every $s \in \mathbb{R}$ if and only if $(x_0,\xi_0) \notin WF(f)$.*

Proof. We use the same idea as in the proof of [15, Theorem 8.4.8]. Here, we only present the proof of (3) and remark that this property is discussed in [19] via the so called superposition type wave-front sets.

Assume that U is a bounded neighborhood of x_0 and $\Gamma = \Gamma_{\xi_0}$, and that $(x,\xi) \notin WF_{\mathscr{F}L^q_s}(f)$ for each $s \in \mathbb{R}$ when $x \in U$ and $\xi \in \Gamma$. By compactness it follows that $|\chi f|_{\mathscr{F}L^{q,\Gamma}_s} < \infty$ for every $\chi \in C_0^\infty(U)$ such that $\chi(x_0) \neq 0$, and for every s. Let $\varphi \in C_0^\infty(U)$ and $0 \le \psi \in C^\infty(\mathbb{R}^d)$ be such that

$$\psi(t\xi) = \psi(\xi), \qquad \text{when} \quad |\xi| \ge 1,\ t \ge 1,$$

$$\varphi(x_0) = \psi(\xi_0/|\xi_0|) = 1, \quad \text{and} \quad \operatorname{supp}\psi \subseteq \Gamma.$$

Then $\psi(D)\varphi(x)f \in \mathscr{F}L^q_s$ for every s by Theorem 1 (3). Hence $\psi(D)\varphi(x)f \in C^\infty$.

From the assumptions it follows that if $K(x-y) = (2\pi)^{-d/2}\widehat{\psi}(y-x)$ is the Schwartz kernel of $\psi(D)$, then $K \in \mathscr{S}'(\mathbb{R}^d) \cap C^\infty(\mathbb{R}^d \setminus 0)$. Furthermore, K turns rapidly to zero at infinity. It follows that $\psi(D)\varphi(x)f \in \mathscr{S}(\mathbb{R}^d)$. Hence $\psi(\xi)\mathscr{F}(\varphi f)(\xi)$ turns rapidly to zero at infinity. Consequently, $(x_0,\xi_0) \notin WF(f)$.

On the other hand, if $(x_0,\xi_0) \notin WF(f)$ then, by the definition, it immediately follows that there is a conic neighborhood $U \times \Gamma$ of (x_0,ξ_0) such that $(x,\xi) \notin WF_{\mathscr{F}L^q_s}(f)$ for every $(x,\xi) \in U \times \Gamma$ and for every $s \in \mathbb{R}$. The proof is complete. □

THEOREM 2. *Let $f \in \mathscr{D}'(\mathbb{R}^d)$ and let $A \in \Psi^m(X)$ be properly supported. Then we have the microlocal property*

$$WF_{\mathscr{F}L^q_{s-m}}(Af) \subset WF_{\mathscr{F}L^q_s}(f) \subset WF_{\mathscr{F}L^q_{s-m}}(Af) \cup \mathrm{Char}A,$$

where Char$A$ *denotes the set of characteristic points of A.*

Proof. The statement follows directly from Corollary 1. □

5. Modulation spaces

Assume that $\varphi \in \mathscr{S}'(\mathbb{R}^d)$ is fixed. Then the short-time Fourier transform of $f \in \mathscr{S}'(\mathbb{R}^d)$ with respect to φ is defined by

$$(V_\varphi f)(x,\xi) = \mathscr{F}(f \cdot \overline{\varphi(\,\cdot\, - x)})(\xi).$$

We note that the left-hand side makes sense, since it is the partial Fourier transform of the tempered distribution $F(x,y) = (f \otimes \overline{\varphi})(y, y-x)$ with respect to the y-variable.

We usually assume that $\varphi \in \mathscr{S}(\mathbb{R}^d)$, and in this case $V_\varphi f$ takes the form

$$V_\varphi f(x,\xi) = (2\pi)^{-d/2} \int f(y)\overline{\varphi(y-x)}e^{-i\langle y,\xi\rangle}\,dy. \tag{16}$$

Assume that $\omega \in \mathscr{P}(\mathbb{R}^{2d})$, $p,q \in [1,\infty]$, and that $\varphi \in \mathscr{S}(\mathbb{R}^d) \setminus 0$. Then the *modulation space* $M^{p,q}_{(\omega)}(\mathbb{R}^d)$ consists of all $f \in \mathscr{S}'(\mathbb{R}^d)$ such that

$$\begin{aligned} \|f\|_{M^{p,q}_{(\omega)}} &= \|f\|_{M^{p,q,\varphi}_{(\omega)}} \\ &\equiv \Big(\int \Big(\int |V_\varphi f(x,\xi)\omega(x,\xi)|^p\,dx \Big)^{q/p} d\xi \Big)^{1/q} < \infty \end{aligned} \tag{17}$$

(with obvious interpretation when $p = \infty$ or $q = \infty$).

If $\omega = 1$, then the notation $M^{p,q}(\mathbb{R}^d)$ is used instead of $M^{p,q}_{(\omega)}(\mathbb{R}^d)$. Moreover we set $M^p_{(\omega)}(\mathbb{R}^d) = M^{p,p}_{(\omega)}(\mathbb{R}^d)$ and $M^p(\mathbb{R}^d) = M^{p,p}(\mathbb{R}^d)$.

Locally, the spaces $\mathscr{F}L^q_{(\omega)}(\mathbb{R}^d)$ and $M^{p,q}_{(\omega)}(\mathbb{R}^d)$ coincide, in the sense that

$$\mathscr{F}L^q_{(\omega)}(\mathbb{R}^d) \cap \mathscr{E}'(\mathbb{R}^d) = M^{p,q}_{(\omega)}(\mathbb{R}^d) \cap \mathscr{E}'(\mathbb{R}^d)$$

(see Theorem 2.1 and Remark 4.4 in [21]). This result is extended in [19] in the context of wave-front sets.

Now we define wave-front sets with respect to modulation spaces, and claim that they coincide with wave-front sets of Fourier Lebesgue types. In particular, any property valid for wave-front set of Fourier Lebesgue type carry over to wave-front set of modulation space type (cf. [19, 20]).

Assume that $\varphi \in \mathscr{S}(\mathbb{R}^d)\setminus 0$, $\omega \in \mathscr{P}(\mathbb{R}^{2d})$, $\Gamma \subseteq \mathbb{R}^d\setminus 0$ is an open cone and $p,q\in[1,\infty]$ are fixed. For any $f\in\mathscr{S}'(\mathbb{R}^d)$, let

$$
\begin{aligned}
|f|_{M^{p,q,\Gamma}_{(\omega)}} &= |f|_{M^{p,q,\Gamma,\varphi}_{(\omega)}} \\
&\equiv \Big(\int_\Gamma\Big(\int_{\mathbb{R}^d}|V_\varphi f(x,\xi)\omega(x,\xi)|^p\,dx\Big)^{q/p}d\xi\Big)^{1/q}
\end{aligned}
\tag{18}
$$

(with obvious interpretation when $p=\infty$ or $q=\infty$). We note that $|\cdot|_{M^{p,q,\Gamma}_{(\omega)}}$ defines a semi-norm on $\mathscr{S}'$ which might attain the value $+\infty$. If $\Gamma=\mathbb{R}^d\setminus 0$ and $q<\infty$, then $|f|_{M^{p,q,\Gamma}_{(\omega)}}$ agrees with the modulation space norm $\|f\|_{M^{p,q}_{(\omega)}}$ of f.

We let $\Theta_{M^{p,q}_{(\omega)}}(f)=\Theta_{M^{p,q,\varphi}_{(\omega)}}(f)$ be the sets of all $\xi\in\mathbb{R}^d\setminus 0$ such that $|f|_{M^{p,q,\Gamma,\varphi}_{(\omega)}}<\infty$, for some $\Gamma=\Gamma_\xi$. We also let $\Sigma_{M^{p,q,\varphi}_{(\omega)}}(f)$ be the complement of $\Theta_{M^{p,q,\varphi}_{(\omega)}}(f)$ in $\mathbb{R}^d\setminus 0$. Then $\Theta_{M^{p,q,\varphi}_{(\omega)}}(f)$ and $\Sigma_{M^{p,q,\varphi}_{(\omega)}}(f)$ are open respectively closed subsets in $\mathbb{R}^d\setminus 0$.

THEOREM 3. *[19] Assume that $p,q\in[1,\infty]$, $\varphi\in C_0^\infty(\mathbb{R}^d)\setminus 0$, $\chi\in C^\infty(\mathbb{R}^d)$, and that $\omega\in\mathscr{P}(\mathbb{R}^{2d})$. Also assume that $f\in\mathscr{E}'(\mathbb{R}^d)$. Then*

$$
\Theta_{M^{p,q,\varphi}_{(\omega)}}(f)=\Theta_{\mathscr{F}L^q_{(\omega)}}(f),\qquad \Sigma_{M^{p,q,\varphi}_{(\omega)}}(f)=\Sigma_{\mathscr{F}L^q_{(\omega)}}(f),
\tag{19}
$$

and

$$
\Sigma_{M^{p,q,\varphi}_{(\omega)}}(\chi f)\subseteq\Sigma_{M^{p,q,\varphi}_{(\omega)}}(f),\qquad \Sigma_{\mathscr{F}L^q_{(\omega)}}(\chi f)\subseteq\Sigma_{\mathscr{F}L^q_{(\omega)}}(f).
\tag{20}
$$

COROLLARY 2. *[19] Assume that $p,q\in[1,\infty]$, $f\in\mathscr{D}'(\mathbb{R}^d)$, $\varphi\in C_0^\infty(\mathbb{R}^d)\setminus 0$, $x_0,y_0\in\mathbb{R}^d$, $\xi_0\in\mathbb{R}^d\setminus 0$ and that $\omega\in\mathscr{P}(\mathbb{R}^{2d})$. Also let $\omega_0(\xi)=\omega(y_0,\xi)$. Then the following conditions are equivalent:*

(1) *there exists an open cone $\Gamma=\Gamma_{\xi_0}$ and $\chi\in C_0^\infty(\mathbb{R}^d)$ such that $\chi(x_0)\neq 0$, and $|\chi f|_{M^{p,q,\Gamma,\varphi}_{(\omega)}}<\infty$ (i. e. $\xi_0\in\Theta_{M^{p,q}_{(\omega)}}(\chi f)$);*

(2) *there exists an open cone $\Gamma=\Gamma_{\xi_0}$ and $\chi\in C_0^\infty(\mathbb{R}^d)$ such that $\chi(x_0)\neq 0$, and $|\chi f|_{\mathscr{F}L^q_{(\omega)}}<\infty$ (i. e. $\xi_0\in\Theta_{\mathscr{F}L^q_{(\omega)}}(\chi f)$);*

(3) *there exists an open cone $\Gamma=\Gamma_{\xi_0}$ and $\chi\in C_0^\infty(\mathbb{R}^d)$ such that $\chi(x_0)\neq 0$, and $|\chi f|_{\mathscr{F}L^q_{(\omega_0)}}<\infty$ (i. e. $\xi_0\in\Theta_{\mathscr{F}L^q_{(\omega_0)}}(\chi f)$).*

The following definition makes sense in view of Corollary 2.

DEFINITION 2. *[19] Assume that $p,q\in[1,\infty]$, $f\in\mathscr{D}'(\mathbb{R}^d)$ and $\omega\in\mathscr{P}(\mathbb{R}^{2d})$. The wave-front set $WF_{M^{p,q}_{(\omega)}}(f)$ with respect to $M^{p,q}_{(\omega)}(\mathbb{R}^d)$ consists of all pairs (x_0,ξ_0) in $\mathbb{R}^d\times(\mathbb{R}^d\setminus 0)$ such that $\xi_0\in\Sigma_{M^{p,q,\varphi}_{(\omega)}}(\chi f)$ holds for each $\chi\in C_0^\infty(\mathbb{R}^d)$ such that $\chi(x_0)\neq 0$.*

By Corollary 2 it follows that

$$WF_{M^{p_1,q}_{(\omega_1)}}(f) = WF_{M^{p_2,q}_{(\omega_2)}}(f)$$

when $p_1, p_2 \in [1,\infty]$ and

$$C^{-1}\omega_2(x,\xi)\langle x\rangle^{-N} \le \omega_1(x,\xi) \le C\omega_2(x,\xi)\langle x\rangle^{N},$$

for some constants C and N. By the same corollary it follows that the following holds.

PROPOSITION 6. *[19] Assume that $p,q \in [1,\infty]$, $f \in \mathscr{D}'(\mathbb{R}^d)$, $\omega_0 \in \mathscr{P}(\mathbb{R}^d)$ and $\omega \in \mathscr{P}(\mathbb{R}^{2d})$ are such that $\omega_0(\xi) = \omega(y_0,\xi)$ for some $y_0 \in \mathbb{R}^d$. Then*

$$WF_{\mathscr{F}L^q_{(\omega_0)}}(f) = WF_{\mathscr{F}L^q_{(\omega)}}(f) = WF_{M^{p,q}_{(\omega)}}(f).$$

We also note that if $f \in \mathscr{E}'(\mathbb{R}^d)$, then it follows from Corollary 2 that

$$f \in \mathscr{F}L^q_{(\omega_0)} \quad \Longleftrightarrow \quad f \in M^{p,q}_{(\omega)} \Longleftrightarrow WF_{\mathscr{F}L^q_{(\omega_0)}}(f) = WF_{M^{p,q}_{(\omega)}}(f) = \emptyset.$$

In particular, we recover Theorem 2.1 and Remark 4.4 in [21].

6. Pseudo-differential operators, an extenstion

In this section we present a part of our results from [19] related to the action of more general classes of pseudo-differential operators. The presentation of this section follows the first author's lecture given at the conference "Pseudo-differential operators with related topics II".

Assume that $\rho, m \in \mathbb{R}$ are fixed. Recall, $S^m_{\rho,0}(\mathbb{R}^{2d})$ is the set of all $a \in C^\infty(\mathbb{R}^{2d})$ such that for each pairs of multi-indices α and β, there is a constant $C_{\alpha,\beta}$ such that $|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \le C_{\alpha,\beta}\langle\xi\rangle^{m-\rho|\beta|}$. Usually we assume that $0 < \rho \le 1$. Clearly, $S^m_{1,0} = S^m$ of Section 5.

More generally, assume that $\omega_0 \in \mathscr{P}_\rho(\mathbb{R}^{2d})$. Then we recall from [19] that $S^\rho_{(\omega_0)}(\mathbb{R}^{2d})$ consists of all $a \in C^\infty(\mathbb{R}^{2d})$ such that

$$|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \le C_{\alpha,\beta}\omega_0(x,\xi)\langle\xi\rangle^{-\rho|\beta|}.$$

(Cf. Section 18.4–18.6 in [14].) Clearly, $S^\rho_{(\omega_0)} = S^m_{\rho,0}(\mathbb{R}^{2d})$ when $\omega_0(x,\xi) = \langle\xi\rangle^m$.

The next result is a special case of Theorem 4.2 in [30].

PROPOSITION 7. *Assume that $p,q,p_j,q_j \in [1,\infty]$ for $j = 1,2$, satisfy*

$$1/p_1 - 1/p_2 = 1/q_1 - 1/q_2 = 1 - 1/p - 1/q, \quad q \le p_2, q_2 \le p.$$

Also assume that $\omega \in \mathscr{P}(\mathbb{R}^{2d} \oplus \mathbb{R}^{2d})$ and $\omega_1, \omega_2 \in \mathscr{P}(\mathbb{R}^{2d})$ satisfy

$$\frac{\omega_2(x,\xi+\eta)}{\omega_1(x+z,\xi)} \le C\omega(x,\xi,\eta,z) \tag{21}$$

for some constant C. If $a \in M^{p,q}_{(\omega)}(\mathbb{R}^{2d})$, then $a(x,D)$ from $\mathscr{S}(\mathbb{R}^d)$ to $\mathscr{S}'(\mathbb{R}^d)$ extends uniquely to a continuous mapping from $M^{p_1,q_1}_{(\omega_1)}(\mathbb{R}^d)$ to $M^{p_2,q_2}_{(\omega_2)}(\mathbb{R}^d)$.

For the later convenience we set

$$\omega_{s,\rho}(x,\xi,\eta,z) = \omega(x,\xi,\eta,z)\langle x\rangle^{s_4}\langle\eta\rangle^{s_3}\langle\xi\rangle^{\rho s_2}\langle z\rangle^{s_1}, \tag{22}$$

when $\rho \in \mathbb{R}$ and $s \in \mathbb{R}^4$.

DEFINITION 3. *[19] Assume that $s \in \mathbb{R}^4$ is such that $s_2 \geq 0$, $\rho \in \mathbb{R}$, $\omega \in \mathscr{P}(\mathbb{R}^{4d})$, and that $\omega_{s,\rho}$ is given by (22). Then the symbol class $\mho^{s,\rho}_{(\omega)}(\mathbb{R}^{2d})$ is the set of all $a \in \mathscr{S}'(\mathbb{R}^{2d})$ which satisfy*

$$\partial_\xi^\alpha a \in M^{\infty,1}_{(\omega_{u(s,\alpha),\rho})}(\mathbb{R}^{2d}), \qquad u(s,\alpha) = (s_1,|\alpha|,s_3,s_4),$$

for each multi-indices α such that $|\alpha| \leq 2s_2$.

It follows from the following lemma that the symbol classes $\mho^{s,\rho}_{(\omega)}(\mathbb{R}^{2d})$ are interesting also in the classical theory.

LEMMA 1. *[19] Assume that $\rho \in [0,1]$, $\omega \in \mathscr{P}_0(\mathbb{R}^{4d})$ and $\omega_0 \in \mathscr{P}_\rho(\mathbb{R}^{2d})$ satisfy*

$$\omega_0(x,\xi) = \omega(x,\xi,0,0).$$

Then the following conditions are equivalent:

(1) $a \in S^\rho_{(\omega_0)}(\mathbb{R}^{2d})$*;*

(2) $\omega_0^{-1} a \in S^0_{\rho,0}(\mathbb{R}^{2d})$*;*

(3) $\langle x\rangle^{-s_4} a \in \bigcap_{s_1,s_2,s_3\geq 0} \mho^{s,\rho}_{(\omega)}(\mathbb{R}^{2d})$*.*

REMARK 3. Let $H^\infty_s(\mathbb{R}^d)$ be the Sobolev space of distributions with $s \in \mathbb{R}$ derivatives in $L^\infty(\mathbb{R}^d)$, i. e. $H^\infty_s(\mathbb{R}^d)$ consists of all $f \in \mathscr{S}'$ such that $\mathscr{F}^{-1}(\langle\cdot\rangle^s \hat{f})$ belongs to $L^\infty(\mathbb{R}^d)$. Then it is easily seen that $\bigcap_{s\geq 0} H^\infty_s(\mathbb{R}^{2d}) = S^0_{0,0}(\mathbb{R}^{2d})$, which is the set of all smooth functions on $\mathbb{R}^{2d}$ which are bounded together with all their derivatives. Hence, (3.2) in [27] and Theorem 4.4 in [28] imply that

$$\bigcap_{s\geq 0} M^{\infty,1}_{(v_s)}(\mathbb{R}^{2d}) = S^0_{0,0}(\mathbb{R}^{2d}), \qquad v_s(x,\xi,\eta,z) = \langle\eta\rangle^s\langle z\rangle^s.$$

By Theorem 2.2 in [29] it follows more generally that

$$\bigcap_{s\geq 0} M^{\infty,1}_{(v_{\rho,s})}(\mathbb{R}^{2d}) = S^{-\rho}_{0,0}(\mathbb{R}^{2d}), \quad v_{\rho,s}(x,\xi,\eta,z) = \langle\xi\rangle^\rho\langle\eta\rangle^s\langle z\rangle^s. \tag{23}$$

The following definition of the characteristic set is different from that given in Section 5 and in [14, Section 18.1]. Here, it is defined for symbols which are not polyhomogeneous while in the case of polyhomogeneous symbols, our sets of characteristic points are smaller than the set of characteristic points in Section 5 and [14].

DEFINITION 4. *[19] Assume that* $\rho \in (0,1]$ *and* $\omega \in \mathscr{P}_\rho(\mathbb{R}^{2d})$. *For each open cone* $\Gamma \subseteq \mathbb{R}^d \setminus 0$, *open set* $U \subseteq \mathbb{R}^d$ *and real number* $R > 0$, *let*

$$\Omega_{U,\Gamma,R} \equiv \{\, (x,\xi)\, ;\, x \in U,\ \xi \in \Gamma,\ |\xi| > R \,\}.$$

Also let $\Xi_{U,\Gamma,R,\rho}$ *be the set of all* $c \in S^0_{\rho,0}(\mathbb{R}^{2d})$ *such that* $c = 1$ *on* $\Omega_{U,\Gamma,R}$.

The pair $(x_0,\xi_0) \in \mathbb{R}^d \times (\mathbb{R}^d \setminus 0)$ *is called* non-characteristic *for* $a \in S^\rho_{(\omega)}(\mathbb{R}^{2d})$ *(with respect to* ω*), if there is a conical neighborhood* Γ *of* ξ_0, *a neighborhood* U *of* x_0, *a real number* $R > 0$, *and elements* $b \in S^\rho_{(\omega^{-1})}(\mathbb{R}^{2d})$, $c \in \Xi_{U,\Gamma,R,\rho}$ *and* $h \in S^{-\rho}_{\rho,0}(\mathbb{R}^{2d})$ *such that*

$$b(x,\xi)a(x,\xi) = c(x,\xi) + h(x,\xi), \qquad (x,\xi) \in \mathbb{R}^{2d}.$$

The pair (x_0,ξ_0) *in* $\mathbb{R}^d \times (\mathbb{R}^d \setminus 0)$ *is called* characteristic *for* a *(with respect to* $\omega \in \mathscr{P}_\rho(\mathbb{R}^{2d})$*), if it is* not *non-characteristic for* a *with respect to* $\omega \in \mathscr{P}_\rho(\mathbb{R}^{2d})$. *The* set of characteristic points *(the characteristic set), for* $a \in S^\rho_{(\omega)}(\mathbb{R}^{2d})$ *with respect to* ω, *is denoted by* $\operatorname{Char}(a) = \operatorname{Char}_{(\omega)}(a)$.

In order to state the results we use the convention

$$(\vartheta_1,\vartheta_2) \lesssim (\omega_1,\omega_2) \tag{24}$$

when $\omega_j, \vartheta_j \in \mathscr{P}(\mathbb{R}^d)$ for $j = 1,2$ satisfy $\vartheta_j \le C\omega_j$ for some constant C.

If instead $\omega_j \in \mathscr{P}(\mathbb{R}^{2d})$, then it follows that

$$\omega_j(x,\xi_1+\xi_2) \le C\omega_j(x,\xi_1)\langle \xi_2\rangle^{t_j}, \tag{25}$$

for some constants $C > 0$ and t_j, $j = 1,2$, independent of $x,\xi_1,\xi_2 \in \mathbb{R}^d$. Then it is necessary that t_1 and t_2 are non-negative. Here we let $\omega_{s,\rho}$ to be as in (22) and we use the notation $(\omega_1,\omega_2) \preccurlyeq \omega$ when (21) holds for some constant C.

THEOREM 4. *[19] Assume that* $0 < \rho \le 1$, $\omega_j, \vartheta_j \in \mathscr{P}(\mathbb{R}^{2d})$ *for* $j = 1,2$, $\omega \in \mathscr{P}_\rho(\mathbb{R}^{4d})$ *satisfy* (24), *and that* $(\omega_1,\omega_2) \preccurlyeq \omega_{s,\rho}$ *and* $(\vartheta_1,\vartheta_2) \preccurlyeq \omega_{s,\rho}$, *for some* $s \in \mathbb{R}^4$, *are such that*

$$s_1 \ge 0, \quad s_2 \in \mathbb{N}, \quad s_3 > t_1 + t_2 + 2d,$$

where t_1 *and* t_2 *are chosen such that* (25) *holds. Also assume that* $a \in \mho^{s,\rho}_{(\omega)}(\mathbb{R}^{2d})$ *and that* $f \in M^\infty_{(\vartheta_1)}(\mathbb{R}^d)$. *Then*

$$WF_{\mathscr{F}L^q_{(\omega_2)}}(a(x,D)f) \subseteq WF_{\mathscr{F}L^q_{(\omega_1)}}(f).$$

We refer to [19] for a detailed proof and give here only a hint.

The first part of the proof concerns the contribution to the wave-front set of $a(x,D)f$ at a particular point x_0, when the support of f is far away from x_0. It can be proved that this contribution is limited. The precise formulation and the proof can be found in [19].

To finish the proof of Theorem 4 it remains to describe properties of the wave-front set of $a(x,D)f$ at a fixed point when f is concentrated to that point. In these considerations it is natural to assume that involved weight functions satisfy

$$\begin{aligned} &\omega_j(x,\xi) = \omega_j(\xi), \qquad && \vartheta_j(x,\xi) = \vartheta_j(\xi), \quad j = 1,2, \\ &\frac{\omega_2(\xi+\eta)}{\omega_1(\xi)} \le C\omega(\xi,\eta,z), && \frac{\vartheta_2(\xi+\eta)}{\vartheta_1(\xi)} \le C\omega(\xi,\eta,z) \end{aligned} \tag{26}$$

and we set

$$\omega_s(x,\xi,\eta,z) = \langle x\rangle^{s_4}\langle \eta\rangle^{s_3}\omega(\xi,\eta,z), \quad s \in \mathbb{R}^4. \tag{27}$$

We also note that

$$\omega_j(\xi_1+\xi_2) \le C\omega_j(\xi_1)\langle \xi_2\rangle^{t_j}, \quad j = 1,2, \tag{28}$$

for some real numbers t_1 and t_2.

The precise result which we need is the following.

PROPOSITION 8. *[19] Assume that $q \in [1,\infty]$, $s \in \mathbb{R}^4$, $t_j \in \mathbb{R}$, $\omega \in \mathscr{P}(\mathbb{R}^{3d})$, $\omega_j, \vartheta_j \in \mathscr{P}(\mathbb{R}^d)$ for $j = 1,2$ and $\omega_s \in \mathscr{P}(\mathbb{R}^{4d})$ fulfill $(\vartheta_1,\vartheta_2) \lesssim (\omega_1,\omega_2)$,* (26)–(28), *$s_4 > d$ and*

$$s_3 > t_1 + t_2 + 2d.$$

Also assume that $a \in M^{\infty,1}_{(\omega_s)}(\mathbb{R}^{2d})$ and $f \in M^{\infty}_{(\vartheta_1)}(\mathbb{R}^d) \bigcap \mathscr{E}'(\mathbb{R}^d)$. Then the following is true:

(1) *if Γ_1 is an open conical neighborhood of $\eta_0 \in \mathbb{R}^d \setminus 0$, then there is an open conical neighborhood Γ_2 of η_0 which only depends on Γ_1 such that*

$$|a(x,D)f|_{\mathscr{F}L^{q,\Gamma_2}_{(\omega_2)}} \le C\|a\|_{M^{\infty,1}_{(\omega_s)}} |f|_{\mathscr{F}L^{q,\Gamma_1}_{(\omega_1)}},$$

for some constant C which is independent of $a \in M^{\infty,1}_{(\omega_s)}(\mathbb{R}^{2d})$ and $f \in M^{\infty}_{(\omega_1)}(\mathbb{R}^d)$;

(2) $WF_{\mathscr{F}L^q_{(\omega_2)}}(a(x,D)f) \subseteq WF_{\mathscr{F}L^q_{(\omega_1)}}(f)$.

We note that by Proposition 7 it follows that $a(x,D)f$ in Proposition 8 makes sense as an element in $M^{\infty}_{(\vartheta_2)}$. This space contains each space $M^{p,q}_{(\omega_2)}$.

Let $t \in \mathbb{R}$, and let $\mho^{s,\rho,t}_{(\omega)}(\mathbb{R}^{2d})$ be as $\mho^{s,\rho}_{(\omega)}(\mathbb{R}^{2d})$, after $\omega_{s,\rho}(x,\xi,\eta,z)$ has been replaced by

$$\omega_{s,t,\rho}(x,\xi,\eta,z) = \omega_{s,\rho}(x+tz,\xi+t\eta,\eta,z),$$

in the definition of $\mho^{s,\rho}_{(\omega)}(\mathbb{R}^{2d})$. Then it follows from Proposition 1.7 in [30] that if $a \in \mho^{s,\rho}_{(\omega)}(\mathbb{R}^{2d})$, then Theorem 4 remains valid after $\omega(x,\xi,\eta,z)$ has been replaced by $\omega(x+tz,\xi+t\eta,\eta,z)$ and $a(x,D)$ has been replaced by $a_t(x,D)$.

We present a counter result of Theorem 4 for pseudo-differential operators with smooth symbols.

Assume now that the involved weight functions satisfy

$$\frac{\omega_2(x,\xi)}{\omega_1(x,\xi)} \le C\omega_0(x,\xi), \tag{29}$$

for some constant C.

THEOREM 5. *[19] Assume that* $0<\rho\le 1$, $\omega_1,\omega_2 \in \mathscr{P}(\mathbb{R}^{2d})$ *and* $\omega_0 \in \mathscr{P}_\rho(\mathbb{R}^{2d})$ *satisfy* (29)*. If* $a \in S^{\rho}_{(\omega_0)}$ *and* $q \in [1,\infty]$*, then*

$$WF_{\mathscr{F}L^q_{(\omega_2)}}(a(x,D)f) \subseteq WF_{\mathscr{F}L^q_{(\omega_1)}}(f), \qquad f \in \mathscr{S}'(\mathbb{R}^d).$$

We also have the following counter result to Theorem 5. Here it is natural to assume that the involved weight functions satisfy

$$C^{-1}\omega_0(x,\xi) \le \frac{\omega_2(x,\xi)}{\omega_1(x,\xi)}, \tag{30}$$

for some constant C, instead of (29).

THEOREM 6. *[19] Assume that* $0<\rho\le 1$, $\omega_1,\omega_2 \in \mathscr{P}(\mathbb{R}^{2d})$ *and* $\omega_0 \in \mathscr{P}_\rho(\mathbb{R}^{2d})$ *satisfy* (30)*. If* $a \in S^{\rho}_{(\omega_0)}$ *and* $q \in [1,\infty]$*, then*

$$WF_{\mathscr{F}L^q_{(\omega_1)}}(f) \subseteq WF_{\mathscr{F}L^q_{(\omega_2)}}(a(x,D)f) \bigcup \mathrm{Char}_{(\omega_0)}(a), \qquad f \in \mathscr{S}'(\mathbb{R}^d).$$

REMARK 4. We note that the statements in Theorems 5 and 6 are not true if the assumption $\rho>0$ is replaced by $\rho=0$. In fact, we only prove this in the case $\omega_0=1$ and $\omega_1=\omega_2$. The general case is left for the reader.

Let $a(x,\xi)=e^{-i\langle x_0,\xi\rangle}$ for some fixed $x_0 \in \mathbb{R}^d$ and choose α in such way that $f_\alpha(x)=\delta_0^{(\alpha)}$ does not belong to $\mathscr{F}L^q_{(\omega_1)}$. Since

$$(a(x,D)f_\alpha)(x) = f_\alpha(x-x_0),$$

straight-forward computations implies that, for some closed cone $\Gamma \in \mathbb{R}^d \setminus 0$,

$$WF_{\mathscr{F}L^q_{(\omega_1)}}(f) = \{\,(0,\xi)\,;\,\xi\in\Gamma\,\};$$

$$WF_{\mathscr{F}L^q_{(\omega_1)}}(a(x,D)f) = \{\,(x_0,\xi)\,;\,\xi\in\Gamma\,\},$$

which are not overlapping when $x_0 \neq 0$.

Next we apply Theorems 5 and 6 on hypoelliptic operators. Assume that $a \in C^\infty(\mathbb{R}^{2d})$ is bounded by a polynomial. Then $a(x,D)$ is called *hypoelliptic*, if there are positive constants C, $C_{\alpha,\beta}$, N, ρ and R such that

$$
\begin{gathered}
|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \le C_{\alpha,\beta}|a(x,\xi)|\langle\xi\rangle^{-\rho|\beta|}, \quad \text{and} \\
C\langle\xi\rangle^{-N} \le |a(x,\xi)| \quad \text{when} \quad x \in \mathbb{R}^d, \quad \text{and} \quad |\xi| > R.
\end{gathered} \tag{31}
$$

(See e. g. [1,14].) We note that if $a(x,D)$ is hypoelliptic, $\chi \in C_0^\infty(\mathbb{R}^d)$ and if (31) is fulfilled, then $\chi(x)a(x,\xi) \in S^\rho_{(\omega)}(\mathbb{R}^{2d})$, where

$$
\omega(x,\xi) = \omega_a(x,\xi) = (\langle\xi\rangle^{-2N} + |a(x,\xi)|^2)^{1/2} \in \mathscr{P}_\rho(\mathbb{R}^{2d}).
$$

Furthermore, since $\mathrm{Char}_{(\omega_a)}(a) = \emptyset$, by definitions, the following result is an immediate consequence of Theorems 5 and 6.

THEOREM 7. *Assume that $a \in C^\infty(\mathbb{R}^{2d})$ is such that $a(x,D)$ is hypoelliptic, $q \in [1,\infty]$, and that $\omega_1, \omega_2 \in \mathscr{P}(\mathbb{R}^{2d})$ satisfy*

$$
C^{-1}\frac{\omega_2(x,\xi)}{\omega_1(x,\xi)} \le \omega_a(x,\xi) \le C\frac{\omega_2(x,\xi)}{\omega_1(x,\xi)}, \tag{32}
$$

for some constant C which is independent of $(x,\xi) \in \mathbb{R}^{2d}$. If $f \in \mathscr{S}'(\mathbb{R}^d)$, then

$$
WF_{\mathscr{F}L^q_{(\omega_2)}}(a(x,D)f) = WF_{\mathscr{F}L^q_{(\omega_1)}}(f).
$$

Note that for any hypoelliptic operator, we may choose the symbol class which contains the symbol of the operator in such way that the corresponding set of characteristic points is empty. Consequently, in the view of Theorem 7, it follows that hypoelliptic operators preserve the wave-front sets, as it should.

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