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Second Conference on Pseudo-Differential Operators and Related Topics: Invited Lectures

CONTENTS

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Preface

The present issue of Rendiconti del Seminario Matematico Università e Politecnico di Torino contains the texts of the plenary talks delivered by Professors L. Cohen (City University of New York), G. Grubb (Copenhagen University), S. Pilipović (Novi Sad University, Serbia), P.R. Popivanov (Bulgarian Academy of Sciences, Sofia) and M. Ruzhansky (Imperial College, London) at the Second Conference on Pseudo-Differential Operators and Related Topics held in Växjö, Sweden, June 23-27, 2008.

Topics for the conference include Spectral Theory, Time-Frequency (Gabor) Analysis and Localization Operators, Positivity and Lower Bound Problems, Operators on Singular Manifolds, Fourier Integral Operators, Elliptic and Hyperbolic Problems.

The Växjö Conference is part of the activities of the International Society for Analysis, its Applications and Computation (ISAAC) for the year 2008. ISAAC is a non-profit organization established in 1994 to promote and advance analysis, its applications, and its interactions with computation. The current President is Prof. M.W. Wong (York University, Toronto).

During the conference the participants have honoured, on the occasion of his 60th birthday, Professor Luigi Rodino of Torino University. The event at Växjö was particularly significant in view of the fact that Professor Rodino began his long and productive scientific career in the field of pseudo-differential operators at Lund University and Mittag-Leffer Institute in 1973-74.

The present issue of Rendiconti del Seminario Matematico Università e Politecnico di Torino is dedicated to Professor Luigi Rodino.

The meeting was organized with 45-minutes plenary talks in the morning and three parallel sessions of 30-minutes communications in the afternoon. There were about 80 talks altogether the whole Conference. Contributions from other participants will be published in a second issue of Rendiconti del Seminario Matematico Università e Politecnico di Torino, the journal "CUBO" (Pernambuco University, Brazil) and "Complex Variables and Elliptic Equations" (Taylor & Francis, Oxford).

The scientific organizers thank the Växjö University, the Vetenskapsrådet (Swedish Science Council) and the Mathematics Departement "Giuseppe Peano" of Torino University for the financial support and moreover Karoline Johansson and Haidar Al-Talibi of Växjö University for the technical support as local organizers.

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Second Conf. on Pseudo-Diff. Operators: Invited Lectures

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THE WEYL TRANSFORM AND ITS GENERALIZATION [*]

Abstract. Procedures that allow one to associate operators to ordinary functions are called correspondence rules, rules of association, or just transform. There have been a number of such rules studied, among them the Weyl transform and the symmetrization rule. We present a generalization that allows one to generate an infinite number of such transforms. The advantage of the formulation is that all such transforms can be studied in a simple and consistent fashion.

1. Introduction

The concept of associating ordinary functions with operators arose in many areas of analysis but took particular importance with the discovery of quantum mechanics. For the two most fundamental quantities, position and momentum, it became clear that the operators are x and $\frac{\hbar}{i}\frac{\partial}{\partial x}$ respectively (in the position representation) where $\hbar$ is the Planck constant. Procedures to construct other operators developed into the subject now known as correspondence rules, that is, rules to associate an ordinary function with an operator. It is clear that there is an infinite number of ways to associate an ordinary function with a corresponding operator because ordinary variables commute but operators do not. Some correspondence rules that have been studied are the Weyl [4, 12, 15], normal ordering, and symmetrization rule, among others [2, 4, 11]. It is the aim of this paper to develop a methodology where all correspondence rules can be characterized and studied in a unified way.

1.1. Notation, terminology, and conventions

"Symbol", "classical function", and "c-function", are terms used in different fields to signify the same things, namely an ordinary function, $a(x,\xi)$, of two variables x and ξ. This is the common notation used in mathematics, while in physics it is position and momentum signified by $a(q,p)$ [8], and in time-frequency analysis [3,4] one generally writes $a(t,\omega)$. In this paper we will use the mathematics notation and as is standard define the conjugate operator to x by D, where

$$D = \frac{1}{i}\frac{d}{dx}$$

The commutator between x and D is denoted by $[x,D]$ and is given by

$$[x,D] = xD - Dx = i$$

[*]It is a pleasure to dedicate this paper to Prof. Luigi Rodino on the occasion of his 60th birthday.

[†]Work supported by the Air Force Office of Scientific Research.

Also, we will occasionally use $D_y = \frac{1}{i}\frac{d}{dy}$, where y is an arbitrary real variable.

The phrases "*operator transform*", "*correspondence rule*", "*rule of association*", will all mean the same thing, namely the association of an operator $\mathcal{A}_a(x,D)$ with a symbol $a(x,\xi)$. The association is symbolized by

$$\mathcal{A}_a(x,D) \leftrightarrow a(x,\xi)$$

Generally speaking operators such as $\mathcal{A}_a(x,D)$ will be denoted by script letters although there will be exceptions, D being one of them.

The word "*transform*" by itself will mean the operation of $\mathcal{A}_a(x,D)$ on an arbitrary function, say $u(x)$, and will be denoted by $A_a[u(x)]$ or just by $A_a[u]$ when it is clear what the variable is. That is,

$$A_a[u] = \mathcal{A}_a(x,D)u(x) \tag{1}$$

The Fourier transform of the symbol, $a(x,\xi)$, will be denoted by $\widehat{a}(\theta,\tau)$ and the normalization is taken so that

$$\widehat{a}(\theta,\tau) = \frac{1}{4\pi^2}\iint a(x,\xi)\, e^{-i\theta x - i\tau\xi}\, dx d\xi$$
$$a(x,\xi) = \iint \widehat{a}(\theta,\tau)\, e^{i\theta x + i\tau\xi}\, d\theta\, d\tau$$

Integrals without limits imply integration over the reals,

$$\int = \int_{\mathbb{R}}$$

2. The generalized operator transform

We define the generalized operator transform associated with the symbol $a(x,\xi)$ by

$$\mathcal{A}_a^{\Phi}(x,D) = \iint \widehat{a}(\theta,\tau)\Phi(\theta,\tau)\, e^{i\theta x + i\tau D}\, d\theta\, d\tau \tag{2}$$

where $\Phi(\theta,\tau)$ is a two dimensional function called the kernel [2]. The kernel characterizes a specific transform and its properties. Since [14]

$$e^{i\theta x + i\tau D} = e^{i\theta\tau/2}\, e^{i\theta\, x}\, e^{i\tau D} = e^{-i\theta\tau/2}\, e^{i\tau D}\, e^{i\theta x}$$

we have that

$$\mathcal{A}_a^{\Phi}(x,D) = \iint \widehat{a}(\theta,\tau)\Phi(\theta,\tau)\, e^{i\theta\tau/2}\, e^{i\theta x}\, e^{i\tau D} d\theta\, d\tau \tag{3}$$

$$= \iint \widehat{a}(\theta,\tau)\Phi(\theta,\tau)\, e^{-i\theta\tau/2}\, e^{i\tau D}\, e^{i\theta x} d\theta\, d\tau \tag{4}$$

Equivalently,

$$\mathcal{A}_a^{\Phi}(x,D) = \frac{1}{4\pi^2}\iiiint a(x,\xi)\,\Phi(\theta,\tau)\, e^{i\theta(x-x') + i\tau(D-\xi')}\, d\theta\, d\tau\, dx'\, d\xi'$$
$$= \frac{1}{4\pi^2}\iiiint a(x',\xi')\,\Phi(\theta,\tau)\; e^{i\theta\tau/2} e^{i\theta(x-x')}\; e^{i\tau(D-\xi')} d\theta\, d\tau\, dx'\, d\xi'$$

3. The generalized transform

We now consider the operation of $\mathcal{A}_a^{\Phi}(x,D)$ on an arbitrary function, $u(x)$. Using Eq. (3) we have

$$A_a^{\Phi}[u] = \mathcal{A}_a^{\Phi}(x,D)\,u(x) = \iint \widehat{a}(\theta,\tau)\Phi(\theta,\tau)\,e^{i\theta\tau/2}\,e^{i\theta x}\,u(x+\tau)d\theta\,d\tau$$

where we have used the fact that $e^{i\tau D}u(x) = u(x+\tau)$. Equivalently,

$$A_a^{\Phi}[u] = \iint \widehat{a}(\theta,\tau-x)\Phi(\theta,\tau-x)\,e^{i\theta(\tau+x)/2}\,u(\tau)d\theta\,d\tau \tag{5}$$

We call $A_a^{\Phi}[u]$ the generalized transform. Writing $A_a^{\Phi}[u]$ in terms of the symbol $a(x',\xi')$ directly one obtains

$$A_a^{\Phi}[u] = \frac{1}{4\pi^2}\iiiint a\left(q+\frac{\tau+x}{2},\xi\right)\,e^{-i\theta q-i(\tau-x)\xi}\,\Phi(\theta,\tau-x)\,u(\tau)\,d\tau\,dq\,d\xi\,d\theta \tag{6}$$

Notice $\widehat{a}(\theta,\tau-x)\Phi(\theta,\tau-x)\,e^{i\theta(\tau+x)/2}$ is a function of $\tau-x$ and $\tau+x$ and hence we write

$$A_a^{\Phi}[u] = \int k(\tau+x,\tau-x)\,u(\tau)\,d\tau \tag{7}$$

with

$$k(x,\tau) = \int \widehat{a}(\theta,\tau)\Phi(\theta,\tau)\,e^{i\theta x/2}\,d\theta \tag{8}$$

$$= \frac{1}{4\pi^2}\iiint a(q+x/2,\xi)\,e^{-i\theta q-i\tau\xi}\Phi(\theta,\tau)\,dq\,d\xi\,d\theta \tag{9}$$

3.1. From transform to symbol

We now describe how starting with the operator transform, $\mathcal{A}_a^{\Phi}$, one can obtain the corresponding symbol, $a(x,\xi)$. First we describe a notation that is helpful. For any operator $\mathcal{A}(x,D)$ we define $R_A(x,\xi)$ by the following procedure

$R_A(x,\xi) =$ the rearrangement of $\mathcal{A}(x,D)$, so that all the x factors are to the left of the D operators; then one replaces D by ξ.

The rearrangement is achieved by using $[x,D]=i$. Applying this procedure to $\mathcal{A}_a^{\Phi}(x,D)$ as given by Eq. (3) and noting that it is already in the appropriate form, (since the x factors are already to the left of the D factors), we immediately have

$$R_A^{\Phi}(x,\xi) = \iint \widehat{a}(\theta,\tau)\Phi(\theta,\tau)\,e^{i\theta\tau/2}e^{i\theta x}\,e^{i\tau\xi}d\theta\,d\tau \tag{10}$$

Inverting we have,

$$\widehat{a}(\theta,\tau)\Phi(\theta,\tau)e^{i\theta\tau/2} = \frac{1}{4\pi^2}\iint R_A^{\Phi}(x,\xi)e^{-i\theta x-i\tau\xi}dx\,d\xi$$

which gives

$$\widehat{a}(\theta,\tau)=\frac{1}{4\pi^2}\frac{e^{-i\theta\tau/2}}{\Phi(\theta,\tau)}\iint R_A^{\Phi}(x,\xi)e^{-i\theta x-i\tau\xi}dx\,d\xi$$

and from which $a(x,\xi)$ can be obtained by inversion.

We now derive an alternate form. We first state the following theorem. Suppose we have two functions $f(\theta,\tau)$ and $\widehat{a}(\theta,\tau)$, then it is easily shown that

$$\iint f(\theta,\tau)\,\widehat{a}(\theta,\tau)\;e^{i\theta x+i\tau z}d\theta\,d\tau=f\left(\frac{1}{i}\frac{\partial}{\partial x},\frac{1}{i}\frac{\partial}{\partial\xi}\right)a(x,\xi) \tag{11}$$

Taking $f(\theta,\tau)=\Phi(\theta,\tau)\,e^{i\theta\tau/2}$ in Eq. (11) we have

$$R_A^{\Phi}(x,\xi)\;=\exp\left(\frac{1}{2i}\frac{\partial}{\partial x}\frac{\partial}{\partial\xi}\right)\Phi\left(\frac{1}{i}\frac{\partial}{\partial x},\frac{1}{i}\frac{\partial}{\partial\xi}\right)a(x,\xi) \tag{12}$$

and hence

$$a(x,\xi)=\exp\left[-\frac{1}{2i}\frac{\partial}{\partial x}\frac{\partial}{\partial\xi}\right]\Phi^{-1}\left(\frac{1}{i}\frac{\partial}{\partial x},\frac{1}{i}\frac{\partial}{\partial\xi}\right)R_A^{\Phi}(x,\xi) \tag{13}$$

These can be writen in a somewhat more compact way,

$$R_A^{\Phi}(x,\xi)\;=\exp\left(\frac{i}{2}D_xD_\xi\right)\Phi\left(D_x,D_\xi\right)a(x,\xi) \tag{14}$$

$$a(x,\xi)=\exp\left[\frac{1}{2i}D_xD_\xi\right]\Phi^{-1}\left(D_x,D_\xi\right)R_A^{\Phi}(x,\xi) \tag{15}$$

4. Constraints on the kernel

The advantage of the above formulation is that one can readily obtain conditions on the kernel corresponding to properties we desire in the transform. We now list some possible properties and the constraints on the kernel to assure the requirement is met.

(i) Hermiticity. If the symbol is real and $\Phi(\theta,\tau)\;=\Phi^*(-\theta,-\tau)$ then $\mathcal{A}_a^{\Phi}(x,D)$ is a Hermitian operator. That is, for any two functions $u(x)$ and $v(x)$

$$\int v^*(x)\mathcal{A}_a^{\Phi}u(x)dx=\int u(x)\left(\mathcal{A}_a^{\Phi}v(x)\right)^*dx\qquad\text{if}\qquad\Phi(\theta,\tau)\;=\Phi^*(-\theta,-\tau)$$

(ii) Unit correspondence. If we want the correspondence between the number one and the unit operator $1\leftrightarrow\mathbf{I}$ then we must take $\Phi(0,0)=1$. That is

$$\mathbf{I}\leftrightarrow 1\qquad\text{if}\qquad\Phi(0,0)=1$$

(iii) Symbols of x or ξ only. Suppose we want to be certain that for a symbol that is a function of x or ξ only the operator should be the same function of x and ξ then the condition on the kernel is,

$$\mathcal{A}_a^{\Phi}(x,D) = a(x) \leftrightarrow a(x) \qquad \text{if} \qquad \Phi(0,\tau) = 1$$
$$\mathcal{A}_a^{\Phi}(x,D) = a(\xi) \leftrightarrow a(D) \qquad \text{if} \qquad \Phi(\theta,0) = 1$$

(iv) Translation invariance. Consider the symbol $a_0(x,\xi) = a(x-x_0,\xi-\xi_0)$, then

$$\widehat{a}_0(\theta,\tau) = e^{i\theta x_0 + i\tau\xi_0}\,\widehat{a}(\theta,\tau)$$

and substituting in Eq. (3) we have that

$$\mathcal{A}_{a_0}^{\Phi}(x,D) = \mathcal{A}_a^{\Phi}(x+x_0,D+\xi_0)$$

This is true for all kernels that are functions of only θ and τ.

5. The Fourier, polynomial, and delta function associations

The above formulation can be viewed profitably from different perspectives in ways that we now discuss.

5.1. The Fourier association

One can think of $e^{i\theta x+i\tau\xi}$ as a symbol with parameters θ and τ and associate $e^{i\theta x+i\tau\xi}$ to $\Phi(\theta,\tau)\,e^{i\theta x+i\tau D}$. That is

$$\mathcal{M}\,(\theta,\tau) = \Phi(\theta,\tau)\,e^{i\theta x+i\tau D} \leftrightarrow e^{i\theta x+i\tau\xi} \tag{16}$$

where $\mathcal{M}\,(\theta,\tau)$ is called the characteristic function operator. We call this the Fourier association. Hence, one argues, for a general symbol, expand the symbol in terms of its Fourier transform

$$a(x,\xi) = \iint \widehat{a}(\theta,\tau)\,e^{i\theta x+i\tau\xi}\,d\theta\,d\tau \tag{17}$$

and then one substitutes Eq. (16) into this to obtain Eq. (2).

We note that

$$\begin{aligned} \mathcal{M}\,(\theta,\tau)\mathcal{M}\,(\theta',\tau') &= \Phi(\theta,\tau)\,\Phi(\theta',\tau')\,e^{i\theta\tau/2}\,e^{i\theta'\tau'/2}e^{i\theta\,x}\,e^{i\tau D}e^{i\theta'\,x}\,e^{i\tau' D} \\ &= \frac{\Phi(\theta,\tau)\,\Phi(\theta',\tau')}{\Phi(\theta+\theta',\tau+\tau')}\,e^{i(\theta'\tau-\theta\tau')/2}\,\mathcal{M}\,(\theta+\theta',\tau+\tau') \end{aligned} \tag{18}$$

5.2. The Taylor series association

Suppose that there is an operator correspondence for $x^n\xi^m$ and we denote it by $A_{nm}(x,D)$,

$$P_{nm}(\,x,D) \leftrightarrow x^n\xi^m$$

We call this the polynomial association. Now, expand the symbol, $a(x,\xi)$, in a Taylor series

$$a(x,\xi)=\sum_{n,m=0}^{\infty}\frac{1}{n!m!}\left\{\frac{\partial^{n+m}}{\partial x^n\partial\xi^m}\,a(x,\xi)|_{x,\xi=0}\right\}x^n\xi^m$$

and define the operator transform by

$$\mathcal{A}_a(\,x,D)=\sum_{n,m=0}^{\infty}\frac{\left\{\frac{\partial^{n+m}}{\partial x^n\partial\xi^m}\,a(x,\xi)|_{x,\xi=0}\right\}}{n!m!}P_{nm}(x,D) \tag{19}$$

To make $\mathcal{A}_a(\,x,D)$ as given by Eq. (19) equal to $\mathcal{A}_a^{\Phi}(x,D)$ as given by Eq. (2) the same one takes

$$P_{nm}(\,x,D)\;=\;\frac{1}{i^n i^m}\frac{\partial^{n+m}}{\partial\theta^n\partial\tau^m}\Phi(\theta,\tau)\,e^{i\theta\tau/2}\,e^{i\theta\;x}\,e^{i\tau D}\bigg|_{\theta,\tau=0}\leftrightarrow x^n\xi^m \tag{20}$$

5.3. The Delta function association

Starting with the identity

$$a(x,\xi)=\iint a(x',\xi')\delta(x-x')\delta(\xi-\xi')dx'd\xi'$$

we write the correspondence between $\delta(x)\delta(\xi)$ and the corresponding operator by $A_\delta(x,D)$ as

$$A_\delta(\,x,D)\leftrightarrow\delta(x)\delta(\xi)$$

Hence, we define

$$\mathcal{A}_a(\,x,D)=\iint a(x',\xi')A_\delta(x-x',D-\xi')dx'd\xi'$$

We call this the delta function association [7, 11, 16]. From Eq. (2) we immediately have that we must take

$$A_\delta(\,x,D)\;=\;\frac{1}{4\pi^2}\iint\Phi(\theta,\tau)\,e^{i\theta\;x+i\tau D}\,d\theta\,d\tau\leftrightarrow\delta(x)\delta(\xi)$$

5.4. General association

A more general approach that encompasses the above is to consider an orthogonal complete set of functions, $v(x;\theta)$, and expand an arbitrary symbol as

$$a(x,\xi)=\iint\widehat{a}(\theta,\tau)v(x;\theta)v(x;\xi)d\theta\,d\tau \tag{21}$$

Since we assume that $v(x;\theta)$ are complete and orthogonal we have

$$\widehat{a}(\theta,\tau)=\iint a(x,\xi)v^*(x;\theta)v^*(\xi;\tau)dxd\xi$$

Now, suppose the operator association for $v(x;\theta)v(x;\tau)$ is $\mathcal{V}(\theta,\tau;x,D)$,

$$\mathcal{V}(\theta,\tau;x,D) \leftrightarrow v(x;\theta)v(x;\tau)$$

Substitting in Eq. (21) we have

$$\mathcal{A}_a^{\Phi}(x,D) = \iint \widehat{a}(\theta,\tau)\mathcal{V}(\theta,\tau;x,D)\, d\theta\, d\tau$$

This general approach will be developed in a future paper.

6. Transformation between transforms

Suppose we have two different transforms characterized by kernels $\Phi_2(\theta,\tau)$ and $\Phi_1(\theta,\tau)$

$$A_a^{\Phi_2}[u] = \int k_2(\tau+x,\tau-x)\, u(\tau)\, d\tau$$
$$A_a^{\Phi_1}[u] = \int k_1(\tau+x,\tau-x)\, u(\tau)\, d\tau$$

Simple manipulation of Eq. (8) leads to

$$k_2(x,\tau) = \frac{1}{2\pi}\iint \frac{\Phi_2(2\theta,\tau)}{\Phi_1(2\theta,\tau)} e^{i\theta(x-x')}\, k_1(x',\tau) d\theta dx'$$

Also, using Eq. (10) one can show that the corresponding operators transfroms are related by

(22) $$R_A^{\Phi_2}(x,\xi) = \frac{\Phi_2\left(\frac{1}{i}\frac{\partial}{\partial x}, \frac{1}{i}\frac{\partial}{\partial \xi}\right)}{\Phi_1\left(\frac{1}{i}\frac{\partial}{\partial x}, \frac{1}{i}\frac{\partial}{\partial \xi}\right)} R_A^{\Phi_1}(x,\xi)$$

which can be writen as

(23) $$R_A^{\Phi_2}(x,\xi) = \frac{\Phi_2\left(D_x, D_\xi\right)}{\Phi_1\left(D_x, D_\xi\right)} R_A^{\Phi_1}(x,\xi)$$

7. Relation between transforms and phase-space distributions

Shortley after the invention of quantum mechanics, Wigner [13] and Kirkwood [6] addressed the issue of quantum statical mechanics in the following way. They devised a distribution function (different ones) aimed to calculate quantum averages by way of phase space averaging. Here is the fundamental idea [2]. Suppose we have the association

$$\mathcal{A}_a^{\Phi}(x,D) \leftrightarrow a(x,\xi)$$

We want to find a distribution, $C(x,\xi)$ so that

$$\int u^*(x)\,\mathcal{A}_a^{\Phi}(x,D)\,u(x)dx = \iint a(x,\xi)\,C(x,\xi)dxd\xi \tag{24}$$

where $u(x)$ is an arbitrary function, which in quantum mechanics is called the wave function. The left hand side is the quantum mechanical way of calculating expectation values and the right hand side is the standard probabilistic method. This formulation has become known as the phase space of quantum mechanics. Moyal [10] was the first to understand the relationship between the Wigner distribution and the Weyl rule and Cohen [2] gave the general formulation for arbitrary rules and arbitrary phase space distributions.

Starting with Eq. (6) multiply it by $u^*(x)$ and integrate both sides. After some manipulation one derives that we must take [2]

$$C(x,\xi) = \frac{1}{4\pi^2}\iiint u^*(x'-\tau/2)\Phi(\theta,\tau)\,e^{i\theta x'-i\theta x-i\tau\xi}u(x'+\tau/2)d\theta\,d\tau dx'$$

to satisfy Eq. (24). Also, one can readily prove that for two arbitrary functions $v(x)$ and $u(x)$

$$\int v^*(x)\,\mathcal{A}_a^{\Phi}(\,x,D)\,u(x)\,dx = \iint a(x,\xi)\,C_{hg}(x,\xi)dxd\xi$$

if we indeed take

$$C_{vu}(x,\xi) = \frac{1}{4\pi^2}\iiint v^*(x'-\tau/2)\Phi(\theta,\tau)\,e^{i\theta x'-i\theta x-i\tau\xi}u(x'+\tau/2)d\theta\,d\tau dx'$$

We also point out that if we have two distributions characterized by Φ_1 and Φ_2 then the corresponding distributions are related by

$$C_2(x,\xi) = \frac{1}{4\pi^2}\iint \frac{\Phi_2(\theta,\tau)}{\Phi_1(\theta,\tau)}e^{-i\theta(x-x')-i\tau(\xi-\xi')}C_1(x,\xi)\,d\theta\,d\tau dx' d\xi'$$

This can be written in operational form,

$$C_2(x,\xi) = \frac{\Phi_2\left(i\frac{\partial}{\partial x}, i\frac{\partial}{\partial \xi}\right)}{\Phi_1\left(i\frac{\partial}{\partial x}, i\frac{\partial}{\partial \xi}\right)}C_1(x,\xi)$$

8. Examples

We now give a number of examples and in particular we consider some of the historical rules.

8.1. Weyl transform

The Weyl case is obtained by taking

$$\Phi_W(\theta,\tau) = 1$$

giving

$$e^{i\theta x+i\tau\xi} \leftrightarrow e^{i\theta\tau/2}e^{i\theta\, x}\, e^{i\tau D} \qquad \text{Weyl} \qquad \Phi_W(\theta,\tau) = 1$$

From Eq. (4) and (6) we have

$$\mathcal{A}_a^W\,(\,x,D) \,=\, \iint \widehat{a}(\theta,\tau)\, e^{i\theta\tau/2}\, e^{i\theta\, x}\, e^{i\tau D} d\theta\, d\tau$$

$$A_a^\Phi[u] = \frac{1}{2\pi}\iint a\left(\frac{\tau+x}{2},p\right)\; e^{ip(x-\tau)}\; u(\tau)\, d\tau\, dp$$

and also from Eq. (9) we obtain

$$k(x,\tau) = \frac{1}{2\pi}\int a(x/2,\xi)\, e^{-i\tau\xi}\; d\xi$$

We also mention that from Eq. (18) we have that

$$\mathcal{M}\;(\theta,\tau)\mathcal{M}\;(\theta',\tau') = e^{i(\theta'\tau-\theta\tau')/2}\,\mathcal{M}\;(\theta+\theta',\tau+\tau')$$

which is well known. We also point out that and using Eq. (12) we have

$$\mathcal{A}_a^W\,(\,x,D) = \frac{1}{i^n i^m}\;\frac{\partial^{n+m}}{\partial\theta^n\partial\tau^m}\, e^{i\theta\tau/2}\, e^{i\theta x}\, e^{i\tau D}\bigg|_{\theta,\tau=0} = \frac{1}{2^m}\sum_{\ell=0}^{m}\binom{m}{\ell} D^{m-\ell}\; x^n D^\ell$$

which was first derived by McCoy. Also, using Eq. (22) we have

$$R_A(x,\xi) = e^{\frac{1}{2i}\frac{\partial^2}{\partial x\partial\xi}}a(x,\xi)$$

8.2. Margenou-Hill, normal, and antinormal

Before we discuss these specific rules we consider taking the following kernel

$$\Phi_c(\theta,\tau) = e^{ic\theta\tau/2}$$

where c is a real number. Boggiatto, De Donno, and Oliaro [1] have made a careful study of this kernel and showed the relationship with other kernels. We rederive some of their results. Using Eq. (4) and Eq. (6) we obtain

$$\mathcal{A}_a\,(\,x,D) = \frac{1}{2\pi}\iint a\left(q+\frac{\tau(1+c)}{2},p\right)\, e^{-i\tau p'}\, e^{i\tau D}\, dp'\, d\tau$$

$$A_a[u] = \frac{1}{2\pi}\iint a\left(\frac{x+\tau}{2}+\frac{\tau-x}{2}c,p\right)\, e^{-i(\tau-x)p}\, u(\tau)\, dp\, d\tau$$

Consider now the case where the symbol is

$$a(x,\xi) = f(x)h(\xi)$$

then it follows that

$$\mathcal{A}_a(x,D) = \sum_{n=0}^{\infty}\left(\frac{1+c}{2i}\right)^n \frac{1}{n!}\frac{\partial^n f(x)}{\partial x^n}\frac{\partial^n h(D)}{\partial \xi^n}$$

where $\frac{\partial^n h(D)}{\partial \xi^n}$ means that after we differentiate $h(\xi)$ we set $\xi = D$ If we further take

$$f(x)h(\xi) = e^{i\tau x}e^{i\tau\xi}$$

then

$$\mathcal{A}_a(x,D) = e^{i(c+1)\theta\tau/2}e^{i\tau x}e^{i\tau D} = e^{i(c-1)\theta\tau/2}e^{i\tau D}e^{i\tau x}$$

If we take $c=-1$ and $c=1$ we obtain the socalled normal and antinormal cases,

$$e^{i\theta x+i\tau\xi} \leftrightarrow e^{i\theta x}e^{i\tau D} \qquad \text{and } x^n\xi^m \leftrightarrow x^n D^m \qquad \text{normal:} \qquad \Phi_N(\theta,\tau) = e^{-i\theta\tau/2}$$

$$e^{i\theta x+i\tau\xi} \leftrightarrow e^{i\tau D}e^{i\theta x} \qquad \text{and } x^n\xi^m \leftrightarrow D^m x^n \qquad \text{antinormal:} \qquad \Phi_A(\theta,\tau) = e^{i\theta\tau/2}$$

Now consider the case $\Phi(\theta,\tau) = \cos(c\theta\tau/2)$. We obtain

$$\mathcal{A}_a(x,D) = \frac{1}{2}\left[e^{i(c+1)\theta\tau/2}e^{i\tau x}e^{i\tau D} + e^{-i(c+1)\theta\tau/2}e^{i\tau D}e^{i\tau x}\right]$$

and if we take $c=1$ then we obtain the Margenau-Hill [9] or symmetrization rule

$$e^{i\theta x+i\tau\xi} \leftrightarrow \frac{1}{2}\left[e^{i\tau D}e^{i\theta x} + e^{i\theta x}e^{i\tau D}\right] \qquad \text{symmetrization} \qquad \Phi_{MH}(\theta,\tau) = \cos\theta\tau/2$$

and

$$x^n\xi^m \leftrightarrow x^n D^m + D^m x^n \qquad \text{symmetrization} \qquad \Phi_{MH}(\theta,\tau) = \cos\theta\tau/2$$

8.3. Born and Jordan association

Perhaps the first rule that was proposed historically was that of Born and Jordan ,

$$x^n\xi^m \leftrightarrow A_{nm}(x,D) = \frac{1}{m+1}\sum_{\ell=0}^{m} D^{m-\ell}x^n D^{\ell}$$

Using Eq. (20) on obtains the kernel [2],

$$\Phi(\theta,\tau) = \frac{\sin\theta\tau/2}{\theta\tau/2}$$

8.4. Choi-Williams kernel

Choi and Williams [5] devised a kernel that mitigates the so-called cross terms of the Winger distribution but non the less satisfies the important conditions for a representation. The kernel is

$$\Phi_{CW}(\theta,\tau) = e^{-\theta^2\tau^2/\sigma} \tag{25}$$

Where σ is a positive constant. Note that as $\sigma \to \infty$, the kernel approaches one, $\Phi(\theta,\tau) \to 1$, which is the kernel for the Weyl case. Using Eq. (12) we have

$$\begin{aligned} R_A^{\Phi_{CW}}(x,\xi) &= \exp\left[\frac{i}{2}D_xD_\xi\right]\exp\left[-\frac{1}{\sigma}D_x^2D_\xi^2\right]a(x,\xi) \\ &= \exp\left[-\frac{1}{\sigma}D_x^2D_\xi^2\right]R_A^{\Phi_W}(x,\xi) \end{aligned} \tag{26}$$

where $R_A^{\Phi_W}(x,\xi)$ is the Weyl case. As an example consider the case

$$a(x,\xi) = x\xi$$

Then, we have

$$R_A^{\Phi_W}(x,\xi)x\xi = \exp\left[\frac{i}{2}D_xD_\xi\right]x\xi = x\xi - i/2$$

and also

$$\mathcal{A}_a^{\Phi_W}(x,D) = xD - \frac{i}{2} = \frac{1}{2}[xD + Dx]$$

It is clear from Eq. (26) that

$$R_A^{\Phi_{CW}}(x,\xi) = \exp\left[-\frac{1}{\sigma}D_x^2D_\xi^2\right](x\xi - i/2) = (x\xi - i/2) \tag{27}$$

Thus the Choi-Williams association is the same as the Weyl association for $a(x,\xi) = x\xi$. Now consider the case

$$a(x,\xi) = x^2\xi^2$$

then

$$R_A^{\Phi_W}(x,\xi)x^2\xi^2 = \exp\left[\frac{i}{2}D_xD_\xi\right]x^2\xi^2 = x^2\xi^2 - 2ix\xi - 1/2$$

and

$$R_A^{\Phi_{CW}}(x,\xi) = \exp\left[-\frac{1}{\sigma}D_x^2D_\xi^2\right][x^2\xi^2 - 2ix\xi - 1/2] = x^2\xi^2 - 2ix\xi - 1/2 - \frac{4}{\sigma}$$

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KREIN RESOLVENT FORMULAS FOR ELLIPTIC BOUNDARY PROBLEMS IN NONSMOOTH DOMAINS*

Abstract. The paper reports on a recent construction of M-functions and Kreĭn resolvent formulas for general closed extensions of an adjoint pair, and their implementation to boundary value problems for second-order strongly elliptic operators on smooth domains. The results are then extended to domains with $C^{1,1}$ Hölder smoothness, by use of a recently developed calculus of pseudodifferential boundary operators with nonsmooth symbols.

1. Introduction

In the study of boundary value problems for ordinary differential equations, the Weyl-Titchmarsh m-function has played an important role for many years; it allows a reduction of questions concerning the resolvent $(\widetilde{A}-\lambda)^{-1}$ of a realisation $\widetilde{A}$ to questions concerning an associated family $M(\lambda)$ of matrices, holomorphic in $\lambda \in \rho(\widetilde{A})$. Moreover, there is a formula describing the difference between $(\widetilde{A}-\lambda)^{-1}$ and the resolvent of a well-known reference problem in terms of $M(\lambda)$, a so-called Kreĭn resolvent formula. The concepts have also been introduced in connection with the abstract theories of extensions of symmetric operators or adjoint pairs in Hilbert spaces, initiated by Kreĭn [22] and Vishik [32]. The literature on this is abundant, and we refer to e.g. Brown, Marletta, Naboko and Wood [10] and Brown, Grubb and Wood [9] for accounts of the development, and references. For elliptic partial differential equations in higher dimensions, concrete interpretations of $M(\lambda)$ have been taken up in recent years, e.g. in Amrein and Pearson [5], Behrndt and Langer [6], and in [10]; here $M(\lambda)$ is a family of operators defined over the boundary. In the present paper we report on the latest development in nonsymmetric cases worked out in [9]; it uses the early work of Grubb [14] as an important ingredient.

The interest of this in a context of pseudodifferential operators is that $M(\lambda)$ in elliptic cases, and also in some nonelliptic cases, is a pseudodifferential operator (ψdo), to which ψdo methods can be applied. The new results in the present paper are concerned with situations with a nonsmooth boundary. Our strategy here is to apply the nonsmooth pseudodifferential boundary operator (ψdbo) calculus introduced by Abels [3]. We show that when the domain is $C^{1,1}$ and the given strongly elliptic second-order operator A has smooth coefficients, then indeed the M-function can be defined as a generalized ψdo over the boundary, and a Kreĭn formula holds. Selfadjoint cases have been treated under various nonsmoothness hypotheses in Gesztesy and Mitrea [12], Posilicano and Raimondi [29], but the present study allows nonselfadjoint operators, and includes a discussion of Neumann-type boundary conditions. Besides bounded domains, we also treat exterior domains and perturbed halfspaces.

*It is a pleasure to dedicate this paper to Prof. Luigi Rodino on the occasion of his 60th birthday.

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2. Abstract results

We begin by recalling the theory of extensions and M-functions established in works of Brown, Wood and the author [9] and [14].

There is given an adjoint pair of closed, densely defined linear operators $A_{\min}$, $A'_{\min}$ in a Hilbert space H:

$$A_{\min} \subset (A'_{\min})^* = A_{\max}, \quad A'_{\min} \subset (A_{\min})^* = A'_{\max}.$$

Let $\mathcal{M}$ denote the set of linear operators lying between the minimal and maximal operator:

$$\mathcal{M} = \{\widetilde{A} \mid A_{\min} \subset \widetilde{A} \subset A_{\max}\}, \quad \mathcal{M}' = \{\widetilde{A}' \mid A'_{\min} \subset \widetilde{A}' \subset A'_{\max}\}.$$

Write $\widetilde{A}u$ as Au for any $\widetilde{A}$, and $\widetilde{A}'u$ as $A'u$ for any $\widetilde{A}'$. Assume that there exists an $A_\gamma \in \mathcal{M}$ with $0 \in \rho(A_\gamma)$; then $A_\gamma^* \in \mathcal{M}'$ with $0 \in \rho(A_\gamma^*)$. We shall define M-functions for *any* closed $\widetilde{A} \in \mathcal{M}$.

First recall some details from the treatment of extensions in [14]: Denote

$$Z = \ker A_{\max}, \quad Z' = \ker A'_{\max}.$$

Define the basic non-orthogonal decompositions

$$\begin{aligned} D(A_{\max}) &= D(A_\gamma) \dot{+} Z, \text{ denoted } u = u_\gamma + u_\zeta = \mathrm{pr}_\gamma u + \mathrm{pr}_\zeta u, \\ D(A'_{\max}) &= D(A_\gamma^*) \dot{+} Z', \text{ denoted } v = v_{\gamma'} + v_{\zeta'} = \mathrm{pr}_{\gamma'} v + \mathrm{pr}_{\zeta'} v; \end{aligned}$$

here $\mathrm{pr}_\gamma = A_\gamma^{-1} A_{\max}$, $\mathrm{pr}_\zeta = I - \mathrm{pr}_\gamma$, and $\mathrm{pr}_{\gamma'} = (A_\gamma^*)^{-1} A'_{\max}$, $\mathrm{pr}_{\zeta'} = I - \mathrm{pr}_{\gamma'}$. By $\mathrm{pr}_V u = u_V$ we denote the *orthogonal projection* of u onto V.

The following "abstract Green's formula" holds:

$$(Au, v) - (u, A'v) = ((Au)_{Z'}, v_{\zeta'}) - (u_\zeta, (A'v)_Z). \tag{1}$$

It can be used to show that when $\widetilde{A} \in \mathcal{M}$ and we set $W = \overline{\mathrm{pr}_{\zeta'} D(\widetilde{A}^*)}$, then

$$\{\{u_\zeta, (Au)_W\} \mid u \in D(\widetilde{A})\} \text{ is a graph.}$$

Denoting the operator with this graph by T, we have:

THEOREM 1. [14] *For the closed* $\widetilde{A} \in \mathcal{M}$, *there is a* 1–1 *correspondence*

$$\widetilde{A} \text{ closed} \longleftrightarrow \begin{cases} T : V \to W, \text{ closed, densely defined} \\ \text{with } V \subset Z,\ W \subset Z', \text{ closed subspaces.} \end{cases}$$

Here $D(T)=\mathrm{pr}_\zeta D(\widetilde{A})$, $V=\overline{D(T)}$, $W=\overline{\mathrm{pr}_{\zeta'} D(\widetilde{A}^*)}$, *and*

$$Tu_\zeta=(Au)_W \text{ for all } u\in D(\widetilde{A}), \text{ (the } \textbf{defining equation}).$$

In this correspondence,

(i) $\widetilde{A}^*$ *corresponds similarly to* $T^*: W\to V$.

(ii) $\ker\widetilde{A}=\ker T$; $\operatorname{ran}\widetilde{A}=\operatorname{ran} T+(H\ominus W)$.

(iii) *When* $\widetilde{A}$ *is invertible,*

$$\widetilde{A}^{-1}=A_\gamma^{-1}+\mathrm{i}_{V\to H}T^{-1}\mathrm{pr}_W .$$

Here $\mathrm{i}_{V\to H}$ indicates the injection of V into H (it is often left out).

Now provide the operators with a spectral parameter λ, then this implies, with

$$Z_\lambda=\ker(A_{\max}-\lambda),\quad Z'_{\bar\lambda}=\ker(A'_{\max}-\bar\lambda),$$
$$D(A_{\max})=D(A_\gamma)\dot{+}Z_\lambda,\quad u=u_\gamma^\lambda+u_\zeta^\lambda=\mathrm{pr}_\gamma^\lambda u+\mathrm{pr}_\zeta^\lambda u, \text{ etc.:}$$

COROLLARY 1. *Let* $\lambda\in\rho(A_\gamma)$. *For the closed* $\widetilde{A}\in\mathcal{M}$, *there is a* 1–1 *correspondence*

$$\widetilde{A}-\lambda\longleftrightarrow\begin{cases}T^\lambda: V_\lambda\to W_{\bar\lambda}, \text{ closed, densely defined}\\ \text{with } V_\lambda\subset Z_\lambda,\ W_{\bar\lambda}\subset Z'_{\bar\lambda}, \text{ closed subspaces.}\end{cases}$$

Here $D(T^\lambda)=\mathrm{pr}_\zeta^\lambda D(\widetilde{A})$, $V_\lambda=\overline{D(T^\lambda)}$, $W_{\bar\lambda}=\overline{\mathrm{pr}_{\zeta'}^{\bar\lambda} D(\widetilde{A}^*)}$, *and*

$$T^\lambda u_\zeta^\lambda=((A-\lambda)u)_{W_{\bar\lambda}} \text{ for all } u\in D(\widetilde{A}).$$

Moreover,

(i) $\ker(\widetilde{A}-\lambda)=\ker T^\lambda$; $\operatorname{ran}(\widetilde{A}-\lambda)=\operatorname{ran} T^\lambda+(H\ominus W_{\bar\lambda})$.

(ii) *When* $\lambda\in\rho(\widetilde{A})\cap\rho(A_\gamma)$,

$$(\widetilde{A}-\lambda)^{-1}=(A_\gamma-\lambda)^{-1}+\mathrm{i}_{V_\lambda\to H}(T^\lambda)^{-1}\mathrm{pr}_{W_{\bar\lambda}} .$$

This gives a Kreĭn resolvent formula for any closed $\widetilde{A}\in\mathcal{M}$.

The operators T and T^λ are related in the following way: Define

$$E^\lambda=I+\lambda(A_\gamma-\lambda)^{-1},\quad F^\lambda=I-\lambda A_\gamma^{-1},$$
$$E'^{\bar\lambda}=I+\bar\lambda(A_\gamma^*-\bar\lambda)^{-1},\quad F'^{\bar\lambda}=I-\bar\lambda(A_\gamma^*)^{-1},$$

then $E^\lambda F^\lambda=F^\lambda E^\lambda=I$, $E'^{\bar\lambda}F'^{\bar\lambda}=F'^{\bar\lambda}E'^{\bar\lambda}=I$ on H. Moreover, E^λ and $E'^{\bar\lambda}$ restrict to homeomorphisms

$$E_V^\lambda: V\xrightarrow{\sim}V_\lambda,\quad E_W'^{\bar\lambda}: W\xrightarrow{\sim}W_{\bar\lambda},$$

with inverses denoted F_V^λ resp. $F_W'^{\bar\lambda}$. In particular, $D(T^\lambda)=E_V^\lambda D(T)$.

THEOREM 2. *Let* $G^\lambda_{V,W} = -\operatorname{pr}_W \lambda E^\lambda \mathrm{i}_{V\to H}$*; then*

$$(E'^{\bar\lambda}_W)^* T^\lambda E^\lambda_V = T + G^\lambda_{V,W}. \tag{2}$$

In other words, T *and* T^λ *are related by the commutative diagram (where the horizontal maps are homeomorphisms)*

$$\begin{tikzcd} V_\lambda \arrow[d, "T^\lambda"'] & V \arrow[l, "E^\lambda_V"'] \arrow[d, "T+G^\lambda_{V,W}"] \\ W_{\bar\lambda} \arrow[r, "(E'^{\bar\lambda}_W)^*"'] & W \end{tikzcd} \qquad D(T^\lambda) = E^\lambda_V D(T).$$

This is a straightforward elaboration of [16], Prop. 2.6.

Now let us introduce boundary triplets and M-functions. The general setting is the following: There is given a pair of Hilbert spaces $\mathcal{H}$, $\mathcal{K}$ and two pairs of "boundary operators"

$$\begin{pmatrix}\Gamma_1\\ \\ \Gamma_0\end{pmatrix} : D(A_{\max}) \to \begin{matrix}\mathcal{H}\\ \times\\ \mathcal{K}\end{matrix}, \quad \begin{pmatrix}\Gamma'_1\\ \\ \Gamma'_0\end{pmatrix} : D(A'_{\max}) \to \begin{matrix}\mathcal{K}\\ \times\\ \mathcal{H}\end{matrix},$$

bounded with respect to the graph norm and surjective, such that

$$D(A_{\min}) = D(A_{\max}) \cap \ker\Gamma_1 \cap \ker\Gamma_0, \quad D(A'_{\min}) = D(A'_{\max}) \cap \ker\Gamma'_1 \cap \ker\Gamma'_0,$$

and for all $u \in D(A_{\max})$, $v \in D(A'_{\max})$,

$$(Au, v) - (u, A'v) = (\Gamma_1 u, \Gamma'_0 v)_{\mathcal{H}} - (\Gamma_0 u, \Gamma'_1 v)_{\mathcal{K}}.$$

Then the three pairs $\{\mathcal{H}, \mathcal{K}\}$, $\{\Gamma_1, \Gamma_0\}$ and $\{\Gamma'_1, \Gamma'_0\}$ are said to form a *boundary triplet*. (See [10] and [9] for references to the literature on this.)

Note that under our assumptions, the choice

$$\mathcal{H} = Z', \quad \mathcal{K} = Z, \quad \begin{pmatrix}\Gamma_1 u\\ \Gamma_0 u\end{pmatrix} = \begin{pmatrix}(Au)_{Z'}\\ u_\zeta\end{pmatrix}, \quad \begin{pmatrix}\Gamma'_1 v\\ \Gamma'_0 v\end{pmatrix} = \begin{pmatrix}(A'v)_Z\\ v_{\zeta'}\end{pmatrix}, \tag{3}$$

defines a boundary triplet, cf. (1).

Following [10], the boundary triplet is used to define operators $A_T \in \mathcal{M}$ and $A'_{T'} \in \mathcal{M}'$ for any pair of operators $T \in \mathcal{L}(\mathcal{K}, \mathcal{H})$, $T' \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ by

$$D(A_T) = \ker(\Gamma_1 - T\Gamma_0), \quad D(A'_{T'}) = \ker(\Gamma'_1 - T'\Gamma'_0). \tag{4}$$

Then they show:

PROPOSITION 1. *For $\lambda \in \rho(A_T)$, there is a well-defined M-function $M_T(\lambda)$ determined by*

$$M_T(\lambda) : \operatorname{ran}(\Gamma_1 - T\Gamma_0) \to \mathcal{K}, \quad M_T(\Gamma_1 - T\Gamma_0)u = \Gamma_0 u \text{ for all } u \in Z_\lambda.$$

Likewise, for $\lambda \in \rho(A'_{T'})$, the function $M'_{T'}(\lambda)$ is determined similarly by

$$M'_{T'}(\lambda) : \operatorname{ran}(\Gamma'_1 - T'\Gamma'_0) \to \mathcal{H}, \quad M'_{T'}(\Gamma'_1 - T'\Gamma'_0)v = \Gamma'_0 v \text{ for all } v \in Z'_\lambda.$$

Here, when $\rho(A_T) \neq \emptyset$,

$$(A_T)^* = A'_{T^*}.$$

This was set in relation to Theorem 1 in [9]: Take the boundary triplet defined in (3). Then the formula for $D(A_T)$ in (4) is the same as the defining equation (2) for $D(\widetilde{A})$. For the sake of generality, allow also unbounded, densely defined, closed operators $T : Z \to Z'$; then in fact the formulas in Proposition 1 still lead to a well-defined M-function $M_T(\lambda)$. We denote A_T by $\widetilde{A}$ and $M_T(\lambda)$ by $M_{\widetilde{A}}(\lambda)$, when they come from the special choice (3) of boundary triplet. Then we have:

THEOREM 3. *Let $\widetilde{A}$ correspond to $T : Z \to Z'$ by Theorem 1. For any $\lambda \in \rho(\widetilde{A})$, $M_{\widetilde{A}}(\lambda)$ is in $\mathcal{L}(Z', Z)$ and satisfies*

$$M_{\widetilde{A}}(\lambda) = \operatorname{pr}_\zeta (I - (\widetilde{A} - \lambda)^{-1}(A_{\max} - \lambda))A_\gamma^{-1} \operatorname{i}_{Z' \to H}.$$

Moreover, $M_{\widetilde{A}}(\lambda)$ relates to T and T^λ by:

$$M_{\widetilde{A}}(\lambda) = -(T + G^\lambda_{Z,Z'})^{-1} = -F^\lambda_Z (T^\lambda)^{-1} (F'^{\bar{\lambda}}_{Z'})^*, \text{ for } \lambda \in \rho(\widetilde{A}) \cap \rho(A_\gamma).$$

This takes care of those operators $\widetilde{A}$ for which $\operatorname{pr}_\zeta D(\widetilde{A})$ is dense in Z and $\operatorname{pr}_{\zeta'} D(\widetilde{A}^*)$ is dense in Z'. But the construction extends in a natural way to all the closed $\widetilde{A} \in \mathcal{M}$, giving the following result:

THEOREM 4. *Let $\widetilde{A}$ correspond to $T : V \to W$ by Theorem 1. For any $\lambda \in \rho(\widetilde{A})$, there is a well-defined $M_{\widetilde{A}}(\lambda) \in \mathcal{L}(W, V)$, holomorphic in λ and satisfying*

(i) $M_{\widetilde{A}}(\lambda) = \operatorname{pr}_\zeta (I - (\widetilde{A} - \lambda)^{-1}(A_{\max} - \lambda))A_\gamma^{-1} \operatorname{i}_{W \to H}$.

(ii) *When $\lambda \in \rho(\widetilde{A}) \cap \rho(A_\gamma)$,*

$$M_{\widetilde{A}}(\lambda) = -(T + G^\lambda_{V,W})^{-1}.$$

(iii) *For $\lambda \in \rho(\widetilde{A}) \cap \rho(A_\gamma)$, it enters in a Kreĭn resolvent formula*

$$(\widetilde{A} - \lambda)^{-1} = (A_\gamma - \lambda)^{-1} - \operatorname{i}_{V_\lambda \to H} E^\lambda_V M_{\widetilde{A}}(\lambda) (E'^{\bar{\lambda}}_W)^* \operatorname{pr}_{W_{\bar{\lambda}}}.$$

Other Kreĭn-type resolvent formulas in a general framework of *relations* can be found in Malamud and Mogilevskiĭ [26, Section 5.2].

3. Neumann-type conditions for second-order operators

The abstract theory can be applied to elliptic realisations by use of suitable mappings going to and from the boundary, allowing an interpretation in terms of boundary conditions. We shall demonstrate this in the strongly elliptic second-order case.

Let Ω be an open subset of $\mathbb{R}^n$ of one of the following three types: 1) Ω is bounded, 2) Ω is the complement of a bounded set (i.e., is an exterior domain), or 3) there is a ball $B(0,R)$ with center 0 and radius R such that $\Omega \setminus B(0,R) = \mathbb{R}^n_+ \setminus B(0,R)$ (we then call Ω a perturbed halfspace). More general sets or manifolds could be considered in a similar way, namely the so-called admissible manifolds as defined in the book [19].

The sets will in the present section be assumed to be C^∞; later from Section 5 on they will be taken to be $C^{k,\sigma}$, where k is an integer ≥ 0 and $\sigma \in]0,1]$. (Recall that the norm on the Hölder space $C^{k,\sigma}(V)$ is

$$\|u\|_{C^{k,\sigma}(V)} = \sup_{|\alpha|\leq k, x\in V} |D^\alpha u(x)| + \sup_{|\alpha|=k, x\neq y} |D^\alpha u(x) - D^\alpha u(y)|\,|x-y|^{-\sigma}.)$$

We then denote $k+\sigma = \tau$.

That a bounded domain Ω is $C^{k,\sigma}$ means that there is an open cover $\{U_j\}_{j=1,\dots,J}$ of $\partial\Omega$ such that by an affine coordinate change for each j, U_j is a box $\{\max_{k\leq n} |y_k| < a_j\}$, and

$$\Omega \cap U_j = \{(y', y_n) \mid \max_{k<n} |y_k| < a_j,\ f_j(y') < y_n < a_j\},$$
$$\partial\Omega \cap U_j = \{(y', y_n) \mid \max_{k<n} |y_k| < a_j, y_n = f_j(y')\},$$

with $C^{k,\sigma}$-functions f_j such that $|f_j(y')| < a_j$ for $\max_{k<n} |y_k| < a_j$. The diffeomorphism (coordinate change)

$$F_j : (y', y_n) \mapsto (y', y_n - f_j(y')) \tag{5}$$

is then also $C^{k,\sigma}$. The sets U_j must be supplied with a suitable bounded open set U_0 with closure contained in Ω, to get a full cover of $\overline{\Omega}$.

For exterior domains, we cover $\partial\Omega$ similarly, then this must be supplied with a suitable open set U_0 with closure contained in Ω to get a full cover of $\overline{\Omega}$; here U_0 contains the complement of a ball, $U_0 \supset \mathbb{R}^n \setminus B(0,R')$.

For a perturbed halfspace, we cover $\partial\Omega \cap B(0,R+1)$ as above, and supply this with $U_0 = \{x \mid x_n > -\varepsilon, |x| > R\}$ to get a full cover of $\overline{\Omega}$.

The boundary $\partial\Omega$ will be denoted Σ. We assume in the present section that Ω is C^∞; then Σ is an $(n-1)$-dimensional C^∞ manifold without boundary.

Let $A = \sum_{|\alpha|\leq 2} a_\alpha D^\alpha$ with C^∞ coefficients a_α given on a neighborhood $\widetilde{\Omega}$ of $\overline{\Omega}$ (containing U_0 in the perturbed halfspace case), and uniformly strongly elliptic:

$$\operatorname{Re} \sum_{|\alpha|=2} a_\alpha(x)\xi^\alpha \geq c_0|\xi|^2, \text{ all } x \in \widetilde{\Omega}, \xi \in \mathbb{R}^n,$$

$c_0 > 0$. The formal adjoint $A' = \sum_{|\alpha|\le 2} D^\alpha \bar{a}_\alpha = \sum_{|\alpha|\le 2} a'_\alpha D^\alpha$ likewise has C^∞ coefficients a'_α and is strongly elliptic on $\widetilde{\Omega}$. We asume that the coefficients and all their derivatives are bounded.

We denote by $A_{\max}$ resp. $A_{\min}$ the maximal resp. minimal realisations of A in $L_2(\Omega) = H$; they act like A in the distribution sense and have the domains

$$D(A_{\max}) = \{u \in L_2(\Omega) \mid Au \in L_2(\Omega)\}, \quad D(A_{\min}) = H_0^2(\Omega)$$

(using L_2 Sobolev spaces). Similarly, $A'_{\max}$ and $A'_{\min}$ denote the maximal and minimal realisations in $L_2(\Omega)$ of the formal adjoint A'; here $A_{\max} = {A'_{\min}}^*$, $A'_{\max} = {A_{\min}}^*$.

Denote $\gamma_j u = (\partial_n^j u)|_\Sigma$, where ∂_n is the derivative along the interior normal $\vec{n}$ at Σ. Let $s_0(x')$ be the coefficient of $-\partial_n^2$ when A is written in terms of normal and tangential derivatives at $x' \in \Sigma$; it is bounded with bounded inverse. Denoting

$$s_0\gamma_1 = \nu_1, \quad \bar{s}_0\gamma_1 = \nu'_1,$$

we have the Green's formula for A valid for $u, v \in H^2(\Omega)$,

$$(6) \qquad (Au, v)_{L_2(\Omega)} - (u, A'v)_{L_2(\Omega)} = (\nu_1 u, \gamma_0 v)_{L_2(\Sigma)} - (\gamma_0 u, \nu'_1 v + \mathcal{A}'_0 \gamma_0 v)_{L_2(\Sigma)},$$

where $\mathcal{A}'_0$ is a certain first-order differential operator over Σ. The formula extends e.g. to $u \in H^2(\Omega)$, $v \in D(A'_{\max})$, as

$$(7) \qquad (Au, v)_{L_2(\Omega)} - (u, A'v)_{L_2(\Omega)} = (\nu_1 u, \gamma_0 v)_{\frac12, -\frac12} - (\gamma_0 u, \nu'_1 v + \mathcal{A}'_0 \gamma_0 v)_{\frac32, -\frac32},$$

where $(\cdot,\cdot)_{s,-s}$ denotes the duality pairing between $H^s(\Sigma)$ and $H^{-s}(\Sigma)$. (Cf. Lions and Magenes [24] for this and the next results.)

The Dirichlet realisation A_γ is defined as usual by variational theory (the Lax-Milgram lemma); it is the restriction of $A_{\max}$ with domain

$$D(A_\gamma) = D(A_{\max}) \cap H_0^1(\Omega) = H^2(\Omega) \cap H_0^1(\Omega),$$

where the last equality follows by elliptic regularity theory. By addition of a constant to A if necessary, we can assume that the spectrum of A_γ is contained in $\{\lambda \in \mathbb{C} \mid \operatorname{Re}\lambda > 0\}$. For $\lambda \in \rho(A_\gamma)$, $s \in \mathbb{R}$, let

$$Z_\lambda^s(A) = \{u \in H^s(\Omega) \mid (A - \lambda)u = 0\};$$

it is a closed subspace of $H^s(\Omega)$. The trace operators γ_0, γ_1 and ν_1 extend by continuity to continuous maps

$$\gamma_0 : Z_\lambda^s(A) \to H^{s-\frac12}(\Sigma), \quad \gamma_1, \nu_1 : Z_\lambda^s(A) \to H^{s-\frac32}(\Sigma),$$

for all $s \in \mathbb{R}$. When $\lambda \in \rho(A_\gamma)$, let $K_\gamma^\lambda : \varphi \mapsto u$ denote the Poisson operator from $H^{s-\frac12}(\Sigma)$ to $H^s(\Omega)$ solving the semi-homogeneous Dirichlet problem

$$(8) \qquad (A - \lambda)u = 0 \text{ in } \Omega, \quad \gamma_0 u = \varphi \text{ on } \Sigma.$$

It is well-known that K^λ_γ maps homeomorphically

$$K^\lambda_\gamma : H^{s-\frac12}(\Sigma) \overset{\sim}{\to} Z^s_\lambda(A),$$

for all $s \in \mathbb{R}$, with γ_0 acting as an inverse there. The analogous operator for $A' - \bar\lambda$ is denoted $K'^{\bar\lambda}_\gamma$.

We shall now recall from [9, 14] how the statements in Section 2 are interpreted in terms of boundary conditions. In the rest of this section, we abbreviate $H^s(\Sigma)$ to H^s. With the notation from Section 1,

$$Z^0_0(A) = Z, \quad Z^0_0(A') = Z', \quad Z^0_\lambda(A) = Z_\lambda, \quad Z^0_\lambda(A') = Z'_\lambda.$$

We denote by γ_{Z_λ} the restriction of γ_0 to a mapping from Z_λ (closed subspace of $L_2(\Omega)$) to $H^{-\frac12}$; its adjoint $\gamma^*_{Z_\lambda}$ goes from $H^{\frac12}$ to Z_λ:

$$\gamma_{Z_\lambda} : Z_\lambda \overset{\sim}{\to} H^{-\frac12}, \text{ with adjoint } \gamma^*_{Z_\lambda} : H^{\frac12} \overset{\sim}{\to} Z_\lambda.$$

There is a similar notation for the primed operators. When $\lambda = 0$, this index is left out.

These homeomorphisms allow "translating" an operator $T : Z \to Z'$ to an operator $L : H^{-\frac12} \to H^{\frac12}$, as in the diagram

(9)
$$\begin{tikzcd} Z \arrow[r, "\gamma_Z"] \arrow[d, "T"'] & H^{-\frac12} \arrow[d, "L"] \\ Z' \arrow[r, "(\gamma^*_{Z'})^{-1}"'] & H^{\frac12} \end{tikzcd} \qquad D(L) = \gamma_0 D(T),$$

whereby $(Tz, z') = (L\gamma_0 z, \gamma_0 z')_{\frac12, -\frac12}$.

We moreover define the Dirichlet-to-Neumann operators for each $\lambda \in \rho(A_\gamma)$,

$$P^\lambda_{\gamma_0,\nu_1} = \nu_1 K^\lambda_\gamma; \quad P'^{\bar\lambda}_{\gamma_0,\nu'_1} = \nu'_1 K'^{\bar\lambda}_\gamma; \tag{10}$$

they are first-order elliptic pseudodifferential operators over Σ, continuous from $H^{s-\frac12}$ to $H^{s-\frac32}$ for all $s \in \mathbb{R}$, and Fredholm in case Σ is bounded. (Their pseudodifferential nature and ellipticity was explained e.g. in [15]).

For general trace maps β and η we write

$$P^\lambda_{\beta,\eta} : \beta u \mapsto \eta u, \quad u \in Z^s_\lambda(A), \tag{11}$$

when this operator is well-defined.

Introduce the trace operators Γ and Γ' (from [14], where they were called M and M') by

$$\Gamma = \nu_1 - P^0_{\gamma_0,\nu_1}\gamma_0 = \nu_1 A^{-1}_\gamma A_{\max}, \quad \Gamma' = \nu'_1 - P'^0_{\gamma_0,\nu'_1}\gamma_0 = \nu'_1 (A^*_\gamma)^{-1} A'_{\max}. \tag{12}$$

Here Γ and Γ' map $D(A_{\max})$ resp. $D(A'_{\max})$ continuously onto $H^{\frac{1}{2}}$. With these pseudodifferential boundary operators there is a generalized Green's formula valid *for all* $u \in D(A_{\max})$, $v \in D(A'_{\max})$:

$$(Au, v)_{L_2(\Omega)} - (u, A'v)_{L_2(\Omega)} = (\Gamma u, \gamma_0 v)_{\frac{1}{2},-\frac{1}{2}} - (\gamma_0 u, \Gamma' v)_{-\frac{1}{2},\frac{1}{2}}. \tag{13}$$

In particular,

$$(Au, w) = (\Gamma u, \gamma_0 w)_{\frac{1}{2},-\frac{1}{2}} \text{ for all } w \in Z_0^0(A') = Z'. \tag{14}$$

(Cf. [14], Th. III 1.2.) By composition with suitable isometries $\Lambda_t : H^s(\Sigma) \to H^{s-t}(\Sigma)$, (13) can be turned into a standard boundary triplet formula

$$(Au, v)_{L_2(\Omega)} - (u, A'v)_{L_2(\Omega)} = (\Gamma_1 u, \Gamma'_0 v)_{L_2(\Sigma)} - (\Gamma_0 u, \Gamma'_1 v)_{L_2(\Sigma)}, \tag{15}$$

with $\Gamma_1 = \Lambda_{\frac{1}{2}}\Gamma$, $\Gamma'_1 = \Lambda_{\frac{1}{2}}\Gamma'$, $\Gamma_0 = \Gamma'_0 = \Lambda_{-\frac{1}{2}}\gamma_0$ and $\mathcal{H} = \mathcal{K} = L_2(\Sigma)$.

There is a general "translation" of the abstract results in Section 1 to statements on closed realisations $\widetilde{A}$ of A. First let $\widetilde{A}$ correspond to $T : Z \to Z'$ (i.e., assume $V = Z$, $W = Z'$). Then in view of (9) and (14), the defining equation in Theorem 1 is turned into

$$(\Gamma u, \gamma_0 z')_{\frac{1}{2},-\frac{1}{2}} = (L\gamma_0 u, \gamma_0 z')_{\frac{1}{2},-\frac{1}{2}}, \text{ all } z' \in Z'.$$

Since $\gamma_0 z'$ runs through $H^{-\frac{1}{2}}$, this means that $\Gamma u = L\gamma_0 u$, also written

$$\nu_1 u = (L + P^0_{\gamma_0,\nu_1})\gamma_0 u.*$$

Thus $\widetilde{A}$ represents a *Neumann-type condition*

$$\nu_1 u = C\gamma_0 u, \text{ with } C = L + P^0_{\gamma_0,\nu_1}. \tag{16}$$

This allows all first-order ψdo's C to enter, namely by letting L act as $C - P^0_{\gamma_0,\nu_1}$.

The elliptic case: Consider a Neumann-type boundary condition

$$\nu_1 u = C\gamma_0 u, \tag{17}$$

where C is a first-order classical ψdo on Σ. Let $\widetilde{A}$ be the restriction of $A_{\max}$ with domain

$$D(\widetilde{A}) = \{u \in D(A_{\max}) \mid \nu_1 u = C\gamma_0 u\}.$$

Now the boundary condition satifies the Shapiro-Lopatinskiĭ condition (is *elliptic*) if and only if L is elliptic; then in fact

$$D(\widetilde{A}) = \{u \in H^2(\Omega) \mid \nu_1 u = C\gamma_0 u\}. \tag{18}$$

Then the adjoint $\widetilde{A}^*$ equals the operator that is defined similarly from A' by the boundary condition

$$\nu'_1 v = (C^* - \mathcal{A}'_0)\gamma_0 v,$$

likewise elliptic.

When we do the above considerations for $\widetilde{A}-\lambda$, we get L^λ satisfying the diagram

$$\begin{tikzcd} Z \arrow[r, "E^\lambda_Z"] \arrow[d, "T+G^\lambda_{Z,Z'}"'] & Z_\lambda \arrow[r, "\gamma_{Z_\lambda}"] \arrow[d, "T^\lambda"'] & H^{-\frac12} \arrow[d, "L^\lambda"] \\ Z' \arrow[r, "(F'^{\bar\lambda}_{Z'})^*"'] & Z'_{\bar\lambda} \arrow[r, "(\gamma^*_{Z'_{\bar\lambda}})^{-1}"'] & H^{\frac12} \end{tikzcd} \qquad D(L^\lambda)=D(L).$$

Here the horizontal maps are homeomorphisms, and they compose as $\gamma_{Z_\lambda}E^\lambda_Z=\gamma_Z$, $(\gamma^*_{Z'_{\bar\lambda}})^{-1}(F'^{\bar\lambda}_{Z'})^*=(\gamma^*_{Z'})^{-1}$, so

$$L^\lambda=\gamma_Z^{-1}(T+G^\lambda_{Z,Z'})\gamma^*_{Z'}.$$

In terms of L^λ, the boundary condition reads:

$$\nu_1 u=(L^\lambda+P^\lambda_{\gamma_0,\nu_1})\gamma_0 u.$$

Note that $L^\lambda+P^\lambda_{\gamma_0,\nu_1}=C=L+P^0_{\gamma_0,\nu_1}$, so

$$L^\lambda=L+P^0_{\gamma_0,\nu_1}-P^\lambda_{\gamma_0,\nu_1}.$$

As shown in [9], this leads to:

THEOREM 5. *Assumptions as in the start of Section* 3*, with C^∞ domain and operator. Let $\widetilde{A}$ correspond to $T:Z\to Z'$, carried over to $L:H^{-\frac12}\to H^{\frac12}$. Then $\widetilde{A}$ represents the boundary condition* (16)*. Moreover:*

(i) *For $\lambda\in\rho(A_\gamma)$, $P^0_{\gamma_0,\nu_1}-P^\lambda_{\gamma_0,\nu_1}\in\mathcal{L}(H^{-\frac12},H^{\frac12})$ and*

$$L^\lambda=L+P^0_{\gamma_0,\nu_1}-P^\lambda_{\gamma_0,\nu_1}.$$

(ii) *For $\lambda\in\rho(\widetilde{A})$, there is a related M-function $\in\mathcal{L}(H^{\frac12},H^{-\frac12})$*

$$M_L(\lambda)=\gamma_0\big(I-(\widetilde{A}-\lambda)^{-1}(A_{\max}-\lambda)\big)A_\gamma^{-1}\,\mathrm{i}_{Z'\to H}\,\gamma^*_{Z'}.$$

(iii) *For $\lambda\in\rho(\widetilde{A})\cap\rho(A_\gamma)$,*

$$M_L(\lambda)=-(L+P^0_{\gamma_0,\nu_1}-P^\lambda_{\gamma_0,\nu_1})^{-1}=-(L^\lambda)^{-1}.$$

(iv) *For $\lambda\in\rho(A_\gamma)$,*

$$\begin{aligned}\ker(\widetilde{A}-\lambda)&=K^\lambda_\gamma\ker L^\lambda,\\ \operatorname{ran}(\widetilde{A}-\lambda)&=\gamma^*_{Z'_{\bar\lambda}}\operatorname{ran}L^\lambda+\operatorname{ran}(A_{\min}-\lambda),\end{aligned}$$

so that $H \setminus (\operatorname{ran}(\widetilde{A}-\lambda)) = Z'_{\bar\lambda} \setminus (\gamma^*_{Z'_{\bar\lambda}} \operatorname{ran} L^\lambda)$.

(v) *For* $\lambda \in \rho(\widetilde{A}) \cap \rho(A_\gamma)$ *there is a Kreĭn resolvent formula:*

$$\begin{aligned}(\widetilde{A}-\lambda)^{-1} &= (A_\gamma-\lambda)^{-1} - \mathrm{i}_{Z_\lambda \to H}\, \gamma^{-1}_{Z_\lambda} M_L(\lambda) (\gamma^*_{Z'_{\bar\lambda}})^{-1} \mathrm{pr}_{Z'_{\bar\lambda}} \\ &= (A_\gamma-\lambda)^{-1} - K^\lambda_\gamma M_L(\lambda) (K'^{\bar\lambda}_\gamma)^*. \end{aligned} \tag{19}$$

(vi) *In particular, if* C *is a* ψ*do of order* 1 *such that* $C - P^0_{\gamma_0,\nu_1}$ *is elliptic, and* $\rho(\widetilde{A}) \cap \rho(A_\gamma) \neq \emptyset$*, then* $D(L) = H^{\frac32}$*, and*

$$M_L(\lambda) = -(C - P^\lambda_{\gamma_0,\nu_1})^{-1} \tag{20}$$

is elliptic of order -1 *for all* $\lambda \in \rho(\widetilde{A})$*. Here* $\widetilde{A}$ *satisfies* (18) *with* (16)*.*

Note that with the notation (11), $C - P^\lambda_{\gamma_0,\nu_1} = -P^\lambda_{\gamma_0,\nu_1 - C\gamma_0}$, and $M_L(\lambda) = P^\lambda_{\nu_1 - C\gamma_0,\gamma_0}$.

Observe the simple last formula in (19), where K^λ_γ is the Poisson operator for $A-\lambda$, the adjoint being a trace operator of class zero.

The Kreĭn formula is consistent with formulas found for selfadjoint cases with Robin-type conditions in other works, such as Posilicano [28], Posilicano and Raimondi [29], Gesztesy and Mitrea [12], when one observes that

$$(K'^{\bar\lambda}_\gamma)^* = \nu_1 (A_\gamma - \lambda)^{-1}; \tag{21}$$

this follows from the fact that for $\varphi \in H^{-\frac12}(\Sigma)$ and $v = K'^{\bar\lambda}_\gamma \varphi$, $f \in L_2(\Omega)$ and $u = (A_\gamma-\lambda)^{-1} f$, one has using Green's formula (7):

$$\begin{aligned}(f, K'^{\bar\lambda}_\gamma \varphi)_{L_2(\Omega)} &= ((A-\lambda)u, v)_{L_2(\Omega)} - (u, (A'-\bar\lambda)v)_{L_2(\Omega)} \\ &= (\nu_1 u, \gamma_0 v)_{\frac12,-\frac12} - (\gamma_0 u, \nu'_1 v + \mathcal{A}'_0 \gamma_0 v)_{\frac32,-\frac32} = (\nu_1 (A_\gamma - \lambda)^{-1} f, \varphi)_{\frac12,-\frac12}.\end{aligned}$$

For the general case of $\widetilde{A}$ corresponding to $T : V \to W$ with subspaces $V \subset Z$, $W \subset Z'$, there is a related "translation" to boundary conditions. Details are given in [9], let us here just mention some ingredients:

We use the notation in (15) ff. Set

$$X_1 = \overline{\Gamma_0 D(\widetilde{A})} = \Lambda_{-\frac12} \gamma_0 V \subset L_2(\Sigma), \quad Y_1 = \overline{\Gamma_0 D(\widetilde{A}^*)} = \Lambda_{-\frac12} \gamma_0 W \subset L_2(\Sigma),$$

where Γ_0 restricts to homeomorphisms

$$\Gamma_{0,V} : V \xrightarrow{\sim} X_1, \quad \Gamma_{0,W} : W \xrightarrow{\sim} Y_1.$$

Then $T : V \to W$ is carried over to $L_1 : X_1 \to Y_1$ by

$$\begin{array}{ccc} V & \xrightarrow{\Gamma_{0,V}} & X_1 \\ {\scriptstyle T}\downarrow & & \downarrow{\scriptstyle L_1} \\ W & \xrightarrow[(\Gamma^*_{0,W})^{-1}]{} & Y_1 \end{array} \qquad D(L_1) = \Gamma_0 D(T),$$

The boundary condition is:

$$\Gamma_0 u \in D(L_1), \quad L_1 \Gamma_0 u = \mathrm{pr}_{Y_1} \Gamma_1 u.$$

There is a similar reduction for $\widetilde{A} - \lambda$ when $\lambda \in \rho(A_\gamma)$, and we find that

$$L_1^\lambda = L_1 + \mathrm{pr}_{Y_1} \Lambda_{\frac{1}{2}} (P^0_{\gamma_0,\nu_1} - P^\lambda_{\gamma_0,\nu_1}) \Lambda_{\frac{1}{2}} \mathrm{i}_{X_1 \to L_2(\Sigma)} \,.$$

There is an M-function $M_{L_1}(\lambda) : Y_1 \to X_1$ defined for $\lambda \in \rho(\widetilde{A})$. It equals $-(L_1^\lambda)^{-1}$ when $\lambda \in \rho(\widetilde{A}) \cap \rho(A_\gamma)$, and there is then a Kreĭn resolvent formula

$$\begin{aligned} (\widetilde{A} - \lambda)^{-1} &= (A_\gamma - \lambda)^{-1} - \mathrm{i}_{V_\lambda \to H} \Gamma^{-1}_{0,V_\lambda} M_{L_1}(\lambda) (\Gamma^*_{0,W_{\bar\lambda}})^{-1} \mathrm{pr}_{W_{\bar\lambda}} \\ &= (A_\gamma - \lambda)^{-1} - K^\lambda_{\gamma,X_1} M_{L_1}(\lambda) (K'^{\bar\lambda}_{\gamma,Y_1})^*; \end{aligned}$$

here $K^\lambda_{\gamma,X_1} : X_1 \subset L_2(\Sigma) \xrightarrow{\Lambda_{\frac{1}{2}}} H^{-\frac{1}{2}}(\Sigma) \xrightarrow{K^\lambda_\gamma} L_2(\Omega)$.

For higher order elliptic operators, and systems, there are similar results on M-functions and Kreĭn resolvent formulas, see [9]. In such cases there occur interesting subspace situations where X and Y are (homeomorphic to) full products of Sobolev spaces over Σ.

4. The nonsmooth ψdbo calculus

The study of the smooth case was formulated in [9] in terms of the pseudodifferential boundary operator (ψdbo) calculus, which was initiated by Boutet de Monvel [8] and further developed e.g. in Grubb [17], [19] (we refer to these works or to [20] for details on the calculus). The ψdbo theory has been adapted to nonsmooth situations by Abels in [3], by use of ideas from the adaptation of ψdo's to nonsmooth cases by Kumano-go and Nagase [23], Taylor [30]. The operators considered by Abels have symbols that satisfy the usual estimates in the conormal variables ξ', ξ, η_n, pointwise in the space variable x, but are only of class $C^{k,\sigma}$ in x (so that the symbol estimates hold with respect to $C^{k,\sigma}$-norm in x). (For $\tau = k + \sigma$ integer, one could replace $C^{k,\sigma}$ by the so-called Zygmund space $C^\tau = B^\tau_{\infty,\infty}$, which is slightly larger, and gives the scale of spaces slightly better interpolation properties, cf. Abels [1, 2], but we shall let that aspect lie.) We call (k,σ) the Hölder smoothness of the operator and its symbol.

The theory allows the operators to act beween L_p-based Besov and Bessel-potential spaces ($1 < p < \infty$), but we shall here just use it in the case $p = 2$ (although an extension to $p \neq 2$ would also be interesting). Some important results of [3] are:

THEOREM 6. 1° *One has that the continuous mapping property*

$$\mathcal{A} = \begin{pmatrix} P_+ + G & K \\ T & S \end{pmatrix} : \begin{matrix} H^{s+m}(\mathbb{R}^n_+)^N \\ \times \\ H^{s+m-\frac{1}{2}}(\mathbb{R}^{n-1})^M \end{matrix} \to \begin{matrix} H^{s}(\mathbb{R}^n_+)^{N'} \\ \times \\ H^{s-\frac{1}{2}}(\mathbb{R}^{n-1})^{M'} \end{matrix}$$

holds when $\mathcal{A}$ is a Green operator on $\mathbb{R}^n_+$ of order $m \in \mathbb{Z}$ and class r, with Hölder smoothness (k,σ), provided that (with $\tau = k+\sigma$)

1. $|s| < \tau$ *if* $N' \neq 0$,
2. $|s - \frac{1}{2}| < \tau$ *if* $M' \neq 0$,
3. $s + m > r - \frac{1}{2}$ *if* $N \neq 0$ *(class restriction).*

2° *Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be as in 1°, with symbols a_1 resp. a_2 and constants $k_1, \sigma_1, \tau_1, m_1, N_1, \dots$ resp. $k_2, \sigma_2, \tau_2, m_2, N_2, \dots$. Assume that $N_2' = N_1, M_2' = M_1$, so that the operators can be composed. Let $k_3 = \min\{k_1,k_2\}$, $\sigma_3 = \min\{\sigma_1,\sigma_2\}$, $\tau_3 = \min\{\tau_1,\tau_2\}$, $0 < \theta < \min\{1,\tau_2\}$. The boundary symbol composition $a_1 \circ_n a_2$ is a Green symbol a_3 of order $m_3 = m_1 + m_2$, class $r_3 = \max\{r_1 + m_2, r_2\}$ and Hölder smoothness (k_3,σ_3), defining a Green operator $\mathcal{A}_3$. The remainder is continuous:*

$$\mathcal{A}_1\mathcal{A}_2 - \mathcal{A}_3 : \begin{matrix} H^{s+m_3-\theta}(\mathbb{R}^n_+)^{N_2} \\ \times \\ H^{s+m_3-\frac{1}{2}-\theta}(\mathbb{R}^{n-1})^{M_2} \end{matrix} \to \begin{matrix} H^{s}(\mathbb{R}^n_+)^{N_1'} \\ \times \\ H^{s-\frac{1}{2}}(\mathbb{R}^{n-1})^{M_1'} \end{matrix},$$

if the following conditions are satisfied:

1. $|s| < \tau_3$ *and* $s - \theta > -\tau_2$ *if* $N_1' > 0$, $|s - \frac{1}{2}| < \tau_3$ *and* $s - \frac{1}{2} - \theta > -\tau_2$ *if* $M_1' > 0$;
2. $-\tau_2 + \theta < s + m_1 < \tau_2$ *if* $N_1 > 0$, $-\tau_2 + \theta < s + m_1 - \frac{1}{2} < \tau_2$ *if* $M_1 > 0$;
3. $s + m_1 > r_1 - \frac{1}{2}$ *if* $N_1 > 0$, $s + m_3 - \theta > r_2 - \frac{1}{2}$ *if* $N_2 > 0$ *(class restrictions).*

3° *Let $\mathcal{A}$ be as in 1°, and polyhomogeneous and uniformly elliptic with principal symbol a^0 (here $N = N' > 0$). Then there is a Green operator $\mathcal{B}^0$ (the operator with symbol $(a^0)^{-1}$ if $m = 0$) of order $-m$, class $r - m$ and Hölder smoothness (k,σ), continuous in the opposite direction of $\mathcal{A}$, such that $\mathcal{R} = \mathcal{A}\mathcal{B}^0 - I$ is continuous:*

$$\mathcal{R} : \begin{matrix} H^{s-\theta}(\mathbb{R}^n_+)^{N} \\ \times \\ H^{s-\theta-\frac{1}{2}}(\mathbb{R}^{n-1})^{M'} \end{matrix} \to \begin{matrix} H^{s}(\mathbb{R}^n_+)^{N} \\ \times \\ H^{s-\frac{1}{2}}(\mathbb{R}^{n-1})^{M} \end{matrix},$$

if, with $\tau = k + \sigma$,

1. $-\tau+\theta<s<\tau$;

2. $s-\frac{1}{2}>-\tau+\theta$ *if* M *or* $M'>0$;

3. $s-\theta>r-m-\frac{1}{2}$ *(class restriction).*

See [3] (Theorems 1.1, 1.2 and 6.4). For integer τ, the results are worked out there for symbols in Zygmund spaces, but they imply the results with Hölder spaces, see also [1, 2]. The class restrictions are imposed even when the operators have C^∞ coefficients. $\mathcal{B}^0$ is called a parametrix of $\mathcal{A}$.

Abels has also generalized the calculus of [19] for symbols depending on a parameter μ to nonsmooth coefficients; again the estimates in the cotangent variables ξ',ξ,η_n,μ are the usual ones, but valid in x w.r.t. Hölder norms.

We recall from the theory of ψdo's that P is said to be "in x-form" resp. "in y-form", when it is defined from a symbol p by

$$Pu=c\int e^{ix\cdot\xi}p(x,\xi)\hat{u}(\xi)\,d\xi,\ \text{resp. } Pu=c\int e^{i(x-y)\cdot\xi}p(y,\xi)u(y)\,dyd\xi,$$

$c=(2\pi)^{-n}$; the concept extends to ψdbo's. In Theorem 6, all the operators labeled with $\mathcal{A}$ are in x-form. So is $\mathcal{B}^0$ when $m=0$; otherwise it is a composition of an operator in x-form with an order-reducing operator system to the left, see Remark 1 below. The adjoints of operators in x-form are operators in y-form. [3] does not discuss the reduction from y-form to x-form; some indications may be inferred from Taylor [31], Ch. 1 §9. For operators in y-form one has at least the results that can be derived from the above results by transposition.

REMARK 1. An important tool in the calculus is "order-reducing operators". There are two types, one acting over the domain and one acting over the boundary:

$$\begin{aligned}&\Lambda^r_{-,+}=\mathrm{OP}(\lambda^r_-(\xi))_+ : H^t(\mathbb{R}^n_+)\xrightarrow{\sim}H^{t-r}(\mathbb{R}^n_+),\\&\Lambda^r_0=\mathrm{OP}'(\langle\xi'\rangle^r): H^t(\mathbb{R}^{n-1})\xrightarrow{\sim}H^{t-r}(\mathbb{R}^{n-1}),\ \text{all } t\in\mathbb{R},\end{aligned}$$

with inverses $\Lambda^{-r}_{-,+}$ resp. Λ^{-r}_0. Here λ^r_- is the "minus-symbol" defined in [18] Prop. 4.2 as a refinement of $(\langle\xi'\rangle-i\xi_n)^r$. In Theorem 6 3°, whereas $\mathcal{B}^0$ is the operator with symbol $(a^0)^{-1}$ when $m=0$, one applies the zero-order construction to $\mathcal{A}_1=\mathcal{A}\begin{pmatrix}\Lambda^{-m}_{-,+}&0\\0&\Lambda^{-m}_0\end{pmatrix}$ to define $\mathcal{B}^0=\begin{pmatrix}\Lambda^{-m}_{-,+}&0\\0&\Lambda^{-m}_0\end{pmatrix}\mathcal{B}^0_1$ when $m\neq0$.

It should be noted that when e.g. P_+ is as in Theorem 6 1°, then

$$\Lambda^r_{-,+}P_+ : H^{s+m}(\mathbb{R}^n_+)\to H^{s-r}(\mathbb{R}^n_+)\ \text{for } -\tau<s<\tau, \tag{22}$$

whereas the composition rule Theorem 6 2° shows that $\Lambda^r_{-,+}P_+$ can be written as the sum of an operator in the calculus $\mathrm{OP}'(\lambda^r_{-,+}\circ_n p(x,\xi)_+)$ in x-form and a remainder, such that the sum maps $H^{s'+m+r}(\mathbb{R}^n_+)\to H^{s'}(\mathbb{R}^n_+)$ for $-\tau<s'<\tau$; this gives a mapping property like in (22) but with $-\tau+r<s<\tau+r$. This apparently extends the range, but the decompositions into a primary part and a remainder are not the same; $\Lambda^r_{-,+}P_+$ is not in x-form but is an operator in x-form composed to the left with $\Lambda^r_{-,+}$, not equal to $\mathrm{OP}'(\lambda^r_{-,+}\circ_n p(x,\xi)_+)$. Compositions to the right with $\Lambda^r_{-,+}$ are simpler and preserve x-form directly. We shall say that operators formed by composing an operator in x-form with an order-reducing operator to the left are "in order-reduced x-form".

Coordinate changes give some inconveniences in the nonsmooth calculus because, in a $C^{k,\sigma}$-setting, the action of D_j after a $C^{k,\sigma}$-coordinate change gets Jacobian factors that are $C^{k-1,\sigma}$, and higher powers D^α get coefficients in $C^{k-|\alpha|,\sigma}$ (when $k-|\alpha|\ge 0$).

We say that an operator is a generalized Green operator (of one of the respective types) if it is the sum of an operator defined from symbols in the calculus and a remainder of lower order (for s in an interval, specified in each case or understood from the context).

5. Resolvent formulas in the case of non-smooth domains

To treat one difficulty at a time, we consider in the following the case where the domain is nonsmooth, but the operator A is given with smooth coefficients (this includes of course constant coefficients).

Let Ω be an open set in $\mathbb{R}^n$ of one of the three types described in Section 3, of class $C^{k,\sigma}$. We still take A with C^∞-coefficients on a neighborhood $\widetilde{\Omega}$ of $\overline{\Omega}$, as described in Section 2.

Recall from Grisvard [13] (Th. 1.3.3.1, 1.5.1.2, 1.4.1.1, 1.5.3.4):

THEOREM 7. *Let Ω be bounded and $C^{k,\sigma}$, let $\tau=k+\sigma$.*

1° *When Φ is a $C^{k,\sigma}$-diffeomorphism, τ integer, then $u\in H^s_{\mathrm{loc}} \implies u\circ\Phi\in H^s_{\mathrm{loc}}$ for $|s|\le\tau$.*

2° *One can for $|s|\le\tau$, integer, define $H^s(\Sigma)$ to be the space of distributions u on Σ such that for each j, $u\circ F_j^{-1}$ is in H^s on $\{y' \mid \max|y_k|\le a_j\}$. The trace map $\gamma_0: H^s(\Omega)\to H^{s-\frac12}(\Sigma)$ is well-defined for $\frac12<s\le\tau$, and the trace map $\gamma_1: H^s(\Omega)\to H^{s-\frac32}(\Sigma)$ is well-defined for $\frac32<s\le\tau$. There is a continous right inverse of each map, and of the two maps jointly for $\frac32<s\le\tau$.*

3° *Let φ be C^{k_1,σ_1}, $\tau_1=k_1+\sigma_1$, then $u\mapsto\varphi u$ is continuous in $H^s(\mathbb{R}^n)$ for $|s|\le\tau_1$ if τ_1 is integer, $|s|<\tau_1$ if τ_1 is non-integer.*

4° *When $\tau\ge 2$ and A is a second-order differential operator on Ω in a divergence form ($A=-\sum_{j,k}\partial_j a_{jk}\partial_k+\sum_k a_k\partial_k+a_0$) with $C^{0,1}$-coefficients, and we define the associated oblique Neumann trace operators by*

$$\nu_A=\sum_{j,k} n_j a_{jk}\gamma_0\partial_k,\quad \nu_{A'}=\sum_{j,k} n_k\bar a_{jk}\gamma_0\partial_j, \tag{23}$$

there holds a Green's formula

$$(Au,v)_{L_2(\Omega)}-(u,A'v)_{L_2(\Omega)}=(\nu_A u,\gamma_0 v)_{L_2(\Sigma)}-(\gamma_0 u,\nu_{A'}v-\textstyle\sum_k n_k\bar a_k\gamma_0 v)_{L_2(\Sigma)}, \tag{24}$$

for $u,v\in H^2(\Omega)$.

The Green's formula (24) can be reorganized as (6); for our A with smooth coefficients, ν_1, ν_1' and $\mathcal{A}_0'$ get $C^{k-1,\sigma}$-coefficients when Ω is $C^{k,\sigma}$.

We define the Dirichlet realisation A_γ of A, with domain $D(A_\gamma)=D(A_{\max})\cap H_0^1(\Omega)$ by the usual variational construction, and we shall assume that A_γ is invertible. Its adjoint is the analogous operator for A'.

By the difference quotient method of Nirenberg [27] one has that $D(A_\gamma)=H^2(\Omega)\cap H_0^1(\Omega)$ when $\tau\geq 2$ (this fact is also derived below); detailed proofs are e.g. found in the textbooks of Evans [11] (for C^2-domains) or McLean [25] (for $C^{1,1}$-domains).

Also the extended Green's formula (7) is valid when $\tau\geq 2$; this follows by an extension of the proof in Lions and Magenes [24], as mentioned in [13] Remark 1.5.3.5. It follows that the generalized Green's formula (13) holds, when Γ and Γ' are defined by

$$\Gamma=\nu_1 A_\gamma^{-1}A_{\max},\quad \Gamma'=\nu_1'(A_\gamma^*)^{-1}A'_{\max}. \tag{25}$$

The local coordinates (cf. (5)) are used to reduce the curved situation to the flat situation; then the boundary becomes straight but nonsmoothness is imposed on the symbols.

In the following we work out what the nonsmooth ψdbo method can give for the Dirichlet problem; this can be regarded as a basic exercise in the calculus (some other cases appear in works of Abels and coauthors).

First we consider the case of a uniformly strongly elliptic second-order operator on $\mathbb{R}^n_+$ — which we for simplicity of notation also call A — with Hölder smoothness (k_1,σ_1) and $\tau_1=k_1+\sigma_1$, together with a Dirichlet trace operator,

$$\mathcal{A}=\begin{pmatrix}A\\ \gamma_0\end{pmatrix}:H^{s+2}(\mathbb{R}^n_+)\to\begin{matrix}H^s(\mathbb{R}^n_+)\\ \times\\ H^{s+\frac32}(\mathbb{R}^{n-1})\end{matrix};$$

it is continuous for

$$-\tau_1<s<\tau_1,\quad s>-\tfrac32, \tag{26}$$

extended to $|s|\leq\tau_1$ if integer (cf. Theorem 7 3°). To prepare for an application of Theorem 6, we apply order-reducing operators (cf. Remark 1) to reduce to order 0, introducing

$$\mathcal{A}_1=\begin{pmatrix}I&0\\0&\Lambda_0^2\end{pmatrix}\mathcal{A}\Lambda_{-,+}^{-2}=\begin{pmatrix}A\Lambda_{-,+}^{-2}\\ \\ \Lambda_0^2\gamma_0\Lambda_{-,+}^{-2}\end{pmatrix}:H^s(\mathbb{R}^n_+)\to\begin{matrix}H^s(\mathbb{R}^n_+)\\ \times\\ H^{s-\frac12}(\mathbb{R}^{n-1})\end{matrix}, \tag{27}$$

for s as in (26) ff. By Theorem 6 3° it has a parametrix $\mathcal{B}_1^0$ of order 0 and class -1 defined from the principal symbols,

$$\mathcal{B}_1^0 = \begin{pmatrix} R_1^0 & K_1^0 \end{pmatrix} : \begin{matrix} H^s(\mathbb{R}^n_+) \\ \times \\ H^{s-\frac12}(\mathbb{R}^{n-1}) \end{matrix} \to H^s(\mathbb{R}^n_+), \tag{28}$$

for s satisfying

$$-\tau_1 + \tfrac12 < s < \tau_1, \quad s > -\tfrac32; \tag{29}$$

here the remainder $\mathcal{R}_1 = \mathcal{A}_1\mathcal{B}_1^0 - I$ satisfies

$$\mathcal{R}_1 : \begin{matrix} H^{s-\theta}(\mathbb{R}^n_+) \\ \times \\ H^{s-\theta-\frac12}(\mathbb{R}^{n-1}) \end{matrix} \to \begin{matrix} H^s(\mathbb{R}^n_+) \\ \times \\ H^{s-\frac12}(\mathbb{R}^{n-1}) \end{matrix}, \tag{30}$$

when $0 < \theta < \min\{1, \tau_1\}$,

$$-\tau_1 + \tfrac12 + \theta < s < \tau_1, \quad s > -\tfrac32 + \theta. \tag{31}$$

Then the equation $\mathcal{A}_1\mathcal{B}_1^0 = I + \mathcal{R}_1$, also written

$$\begin{pmatrix} I & 0 \\ 0 & \Lambda_0^2 \end{pmatrix} \mathcal{A}\Lambda_{-,+}^{-2}\mathcal{B}_1^0 = I + \mathcal{R}_1,$$

implies by composition to the left with $\begin{pmatrix} I & 0 \\ 0 & \Lambda_0^{-2} \end{pmatrix}$ and to the right with $\begin{pmatrix} I & 0 \\ 0 & \Lambda_0^2 \end{pmatrix}$:

$$\mathcal{A}\Lambda_{-,+}^{-2}\mathcal{B}_1^0 \begin{pmatrix} I & 0 \\ 0 & \Lambda_0^2 \end{pmatrix} = I + \mathcal{R}, \text{ with } \mathcal{R} = \begin{pmatrix} I & 0 \\ 0 & \Lambda_0^{-2} \end{pmatrix} \mathcal{R}_1 \begin{pmatrix} I & 0 \\ 0 & \Lambda_0^2 \end{pmatrix}.$$

Hence

$$\mathcal{B}^0 = \Lambda_{-,+}^{-2}\mathcal{B}_1^0 \begin{pmatrix} I & 0 \\ 0 & \Lambda_0^2 \end{pmatrix} = \begin{pmatrix} R^0 & K^0 \end{pmatrix}$$

is a parametrix of $\mathcal{A}$, with

$$\mathcal{A}\mathcal{B}^0 = I + \mathcal{R}, \tag{32}$$

$$\mathcal{B}^0 : \begin{matrix} H^s(\mathbb{R}^n_+) \\ \times \\ H^{s+\frac32}(\mathbb{R}^{n-1}) \end{matrix} \to H^{s+2}(\mathbb{R}^n_+), \quad \mathcal{R} : \begin{matrix} H^{s-\theta}(\mathbb{R}^n_+) \\ \times \\ H^{s-\theta+\frac32}(\mathbb{R}^{n-1}) \end{matrix} \to \begin{matrix} H^s(\mathbb{R}^n_+) \\ \times \\ H^{s+\frac32}(\mathbb{R}^{n-1}) \end{matrix}, \tag{33}$$

for s as in (29) resp. (31). With the notation from Remark 1, $\mathcal{B}^0$ is in order-reduced x-form.

Now consider the situation where A has smooth coefficients and the domain is nonsmooth. We shall go through the parametrix and inverse construction in the case where the Hölder smoothness of the domain is $(1,1)$ so that $\tau = 2$. We have the direct operator

$$\mathcal{A} = \begin{pmatrix} A \\ \gamma_0 \end{pmatrix} : H^{s+2}(\Omega) \to \begin{matrix} H^s(\Omega) \\ \times \\ H^{s+\frac{3}{2}}(\Sigma) \end{matrix}, \tag{34}$$

it is continuous for $-\frac{3}{2} < s \leq 0$ (recall the restriction $s+2 \leq 2$ coming from Theorem 7 2°).

For each $i = 1, \dots, J$, the diffeomorhism (5) carries $\Omega \cap U_j$ over to $V_j = \{(y', y_n) \mid \max_{k<n} |y_k| < a_j, 0 < y_n < a_j - f_j(y')\}$, such that $\partial\Omega \cap U_j$ is mapped to $\{(y', y_n) \mid \max_{k<n} |y_k| < a_j, y_n = 0\}$. When the smooth differential operator A is transformed to local coordinates in this way, the principal part of the resulting operator $\underline{A}$ has Hölder smoothness $(0,1)$, so here $\tau_1 = 1$. In each of these charts one constructs a parametrix $\underline{\mathcal{B}}^0$ for $\begin{pmatrix} \underline{A} \\ \gamma_0 \end{pmatrix}$ as above (the coefficients of $\underline{A}$ can be assumed to be extended to $\overline{\mathbb{R}}^n_+$). When Ω is bounded or is an exterior domain, one uses for the set U_0 a parametrix of A without changing coordinates. In the perturbed halfspace case, for the set U_0 one extends A smoothly to $\overline{\mathbb{R}}^n_+$ and uses a smooth version of the above construction. These parametrices are carried back to the curved situation and pieced together using a partition of unity subordinate to the cover $\{U_0, U_1, \dots, U_J\}$, as indicated in [19], p. 228 (the first factor φ_i in each term in (2.4.77) should be replaced by a function $\eta_i \in C_0^\infty(U_i)$ such that $\eta_i\varphi_i = \varphi_i$, to get preservation of the principal symbol after summation). Here the coordinate changes allow the smoothness to remain at $(0,1)$; cf. [2], in particular Section 5.3 there. The sum over i is then a parametrix of (34); its composition with $\mathcal{A}$ gives the identity plus a remainder of lower order, for values s as indicated above.

In the subsequent compositions below, it will always be understood that they take place in local coordinates (after decomposing the operators in pieces supported in the U_i by use of suitable partitions of unity) and are taken back to the curved situation afterwards.

In the present construction, we shall actually carry a spectral parameter along that will be useful for discussions of invertibility. So we now replace the originally given A by $A - \lambda$, to be studied for large negative λ.

The parametrix will be of the form

$$\mathcal{B}^0(\lambda) = \begin{pmatrix} R^0(\lambda) & K^0(\lambda) \end{pmatrix} : \begin{matrix} H^s(\Omega) \\ \times \\ H^{s+\frac{3}{2}}(\Sigma) \end{matrix} \to H^{s+2}(\Omega); \tag{35}$$

with $(k_1, \sigma_1) = (0,1)$ the condition (29) means that $-\frac{1}{2} < s < 1$, so that, along with the restriction coming from Theorem 7, we have altogether that

$$-\tfrac{1}{2} < s \leq 0 \tag{36}$$

is allowed. The remainder maps as follows:

(37) $$\mathcal{R}(\lambda)=\mathcal{A}(\lambda)\mathcal{B}^0(\lambda)-I:\begin{matrix}H^{s-\theta}(\Omega)\\ \times\\ H^{s-\theta+\frac32}(\Sigma)\end{matrix}\to\begin{matrix}H^{s}(\Omega)\\ \times\\ H^{s+\frac32}(\Sigma)\end{matrix}$$

for

(38) $$-\tfrac12+\theta<s\le 0.$$

In order to get hold of the exact inverse, we shall use an old trick of Agmon [4], which implies a useful λ-dependent estimate of the remainder: Write $-\lambda=\mu^2$ ($\mu>0$), introduce an extra variable $t\in S^1$, and replace μ by $D_t=-i\partial_t$; let

(39) $$\widehat{A}=A+D_t^2 \text{ on } \Omega\times S^1.$$

Then $\widehat{A}$ is strongly elliptic on $\Omega\times S^1$, and by the preceding construction (carried out with local coordinates respecting the product structure),

$$\widehat{\mathcal{A}}=\begin{pmatrix}\widehat{A}\\ \gamma_0\end{pmatrix} \text{ has a parametrix } \widehat{\mathcal{B}}^0,$$

with mapping properties of $\widehat{\mathcal{B}}^0$ and the remainder $\widehat{\mathcal{R}}=\widehat{\mathcal{A}}\widehat{\mathcal{B}}^0-I$ as in (35) and (37) with Ω,Σ replaced by $\widehat{\Omega}=\Omega\times S^1$, $\widehat{\Sigma}=\Sigma\times S^1$.

For functions w of the form $w(x,t)=u(x)e^{i\mu t}$,

$$\widehat{\mathcal{A}}w=\begin{pmatrix}(A+\mu^2)w\\ \gamma_0 w\end{pmatrix},$$

and similarly, the parametrix $\widehat{\mathcal{B}}^0$ and the remainder $\widehat{\mathcal{R}}$ act on such functions like $\mathcal{B}^0(\lambda)$ and $\mathcal{R}(\lambda)$ applied in the x-coordinate.

Moreover, for $w(x,t)=u(x)e^{i\mu t}$, $u\in\mathcal{S}(\mathbb{R}^n)$,

$$\|w\|_{H^s(\mathbb{R}^n\times S^1)}\simeq\|(1-\Delta+\mu^2)^s u(x)\|_{L_2(\mathbb{R}^n)}\simeq\|(1+|\xi|^2+\mu^2)^{s/2}\hat{u}(\xi)\|_{L_2},$$

with similar relations for Sobolev spaces over other sets. Norms as in the right-hand side are called $H^{s,\mu}$-norms; they were extensively used [19], see the Appendix there for the definition on subsets. The important observation is now that when $s'<s$ and $w(x,t)=u(x)e^{i\mu t}$, then

$$\begin{aligned}\|w\|_{H^{s'}(\mathbb{R}^n\times S^1)}&\simeq\|(1+|\xi|^2+\mu^2)^{s'/2}\hat{u}(\xi)\|_{L_2}\\ &\le\langle\mu\rangle^{s'-s}\|(1+|\xi|^2+\mu^2)^{s/2}\hat{u}(\xi)\|_{L_2}\simeq\langle\mu\rangle^{s-s'}\|w\|_{H^s(\mathbb{R}^n\times S^1)},\end{aligned}$$

with constants independent of u and μ. Analogous estimates hold with $\mathbb{R}^n$ replaced by Ω or Σ.

Applying this principle to the estimates of the remainder $\widehat{\mathcal{R}}$, we find that

$$\|\mathcal{R}(\lambda)u\|_{H^{s,\mu}(\Omega)\times H^{s+\frac{3}{2},\mu}(\Sigma)} \le c_s\|u\|_{H^{s-\theta,\mu}(\Omega)\times H^{s-\theta+\frac{3}{2},\mu}(\Sigma)} \le c_s'\langle\mu\rangle^{-\theta}\|u\|_{H^{s,\mu}(\Omega)\times H^{s+\frac{3}{2},\mu}(\Sigma)}$$

for s as in (38).

For each s, take a fixed λ with $|\lambda|$ so large that $c_s'\langle\mu\rangle^{-\theta} \le \frac{1}{2}$. Then $I+\mathcal{R}(\lambda)$ has the inverse $I+\mathcal{R}'(\lambda) = I+\sum_{k\ge 1}(-\mathcal{R}(\lambda))^k$ (converging in the operator norm for operators on $H^{s,\mu}(\Omega)\times H^{s+\frac{3}{2},\mu}(\Sigma)$), and

$$\mathcal{A}(\lambda)\mathcal{B}^0(\lambda)(I+\mathcal{R}'(\lambda)) = I.$$

This gives a right inverse

$$\mathcal{B}(\lambda) = \mathcal{B}^0(\lambda)+\mathcal{B}^0(\lambda)\mathcal{R}'(\lambda) = \begin{pmatrix} R(\lambda) & K(\lambda)\end{pmatrix},$$

with the same Sobolev space continuity as $\mathcal{B}^0(\lambda)$, and $\mathcal{B}^0(\lambda)\mathcal{R}'(\lambda)$ of lower order. Since

$$\mathcal{A}(\lambda)\mathcal{B}(\lambda) = \begin{pmatrix}(A-\lambda)R(\lambda) & (A-\lambda)K(\lambda)\\ \gamma_0 R(\lambda) & \gamma_0 K(\lambda)\end{pmatrix} = \begin{pmatrix} I & 0\\ 0 & I\end{pmatrix}, \tag{40}$$

$R(\lambda)$ solves

$$(A-\lambda)u = f,\quad \gamma_0 u = 0, \tag{41}$$

and $K(\lambda)$ solves

$$(A-\lambda)u = 0,\quad \gamma_0 u = \psi. \tag{42}$$

For such large λ, $R(\lambda)$ coincides with the resolvent of A_γ defined by variational theory, and $K(\lambda)$ is the Poisson-type operator we called K_γ^λ in Section 3;

$$(A_\gamma-\lambda)^{-1} : H^s(\Omega)\to H^{s+2}(\Omega),\quad K_\gamma^\lambda : H^{s+\frac{3}{2}}(\Sigma)\to H^{s+2}(\Omega), \tag{43}$$

for s satisfying (36).

The mapping properties extend to all the λ for which the operators are well-defined, especially to $\lambda = 0$. For A_γ^{-1}, this goes as follows: When $u\in H^1(\Omega)$ and $f\in H^s(\Omega)$ with $s<1$, $f+\lambda u$ is likewise in $H^s(\Omega)$. Then $A_\gamma u = f+\lambda u$ allows the conclusion $u\in H^{s+2}(\Omega)$. The argument works for all s satisfying (36) (for each such s, there is room to take $\theta>0$ so small that (38) is satified. Moreover, since $A_\gamma^{-1} - (A_\gamma-\lambda)^{-1} = -\lambda A_\gamma^{-1}(A_\gamma-\lambda)^{-1}$ is of lower order than A_γ^{-1}, A_γ^{-1} equals a nonsmooth ψdbo plus a lower-order remainder.

The Poisson operator solving (42) can be further described as follows (for all $\lambda\in\rho(A_\gamma)$): There is a right inverse $\mathcal{K} : H^{s+\frac{3}{2}}(\Sigma)\to H^{s+2}(\Omega)$ of γ_0 for $-\frac{3}{2}<s\le 0$ (cf. Theorem 7 2°). When we set $v = u-\mathcal{K}\varphi$, we find that v should solve

$$(A-\lambda)v = -(A-\lambda)\mathcal{K}\varphi,\quad \gamma_0 v = 0,$$

to which we apply the preceding results; then when $\lambda \in \rho(A_\gamma)$,

$$K_\gamma^\lambda = \mathcal{K} - (A_\gamma - \lambda)^{-1}(A-\lambda)\mathcal{K}; \tag{44}$$

solves (42) uniquely. It maps $H^{s+\frac{3}{2}}(\Sigma) \to H^{s+2}(\Omega)$ for s satisfying (36).

Since our original operator had C^∞ coefficients, the same construction works for the adjoint Dirichlet problem, so we also here get the mapping properties

$$(A'_\gamma - \bar{\lambda})^{-1} : H^s(\Omega) \to H^{s+2}(\Omega), \quad K'^{\bar{\lambda}}_\gamma : H^{s+\frac{3}{2}}(\Sigma) \to H^{s+2}(\Omega), \tag{45}$$

for s satisfying (36).

The condition $s > -\frac{1}{2}$ prevents the Poisson operator from starting from $H^{-\frac{1}{2}}(\Sigma)$, which would be needed for an analysis as in Section 3. Fortunately, it is possible to get supplementing information in other ways.

By (7) we have, analogously to (21), that K_γ^λ is the adjoint of a trace operator of class 0 as follows:

$$K_\gamma^\lambda = (\nu'_1(A'_\gamma - \bar{\lambda})^{-1})^*; \tag{46}$$

(it is used here that $\mathcal{A}'_0\gamma_0(A'_\gamma - \bar{\lambda})^{-1} = 0$).

Now use the mapping property in (45). The resolvent can be composed with ν'_1 for $s > -\frac{1}{2}$, so

$$\nu'_1(A'_\gamma - \lambda)^{-1} = (K_\gamma^\lambda)^* : H^s(\Omega) \to H^{s+\frac{1}{2}}(\Sigma) \text{ for } -\tfrac{1}{2} < s \le 0.$$

It follows that

$$K_\gamma^\lambda : H^{s'-\frac{1}{2}}(\Sigma) \to H^{s'}(\Omega), \tag{47}$$

when $0 \le s' < \frac{1}{2}$. In particular, $s' = 0$ is allowed.

Taking this together with the larger values that were covered by (43), we find that (47) holds for

$$0 \le s' \le 2; \tag{48}$$

the intermediate values are included by interpolation. We denote s' by s from here on.

One can analyze the structure of K_γ^λ for the low values of s further, decomposing it into terms belonging to the calculus and lower-order remainders. There is a difficulty here in the fact that order-reducing operators as well as operators in y-form enter, and both types affect the s-values for which the decompositions and mapping properties are valid (cf. Remark 1). We refrain from including a deeper analysis.

There is a similar result for $K'^{\bar{\lambda}}_\gamma$. The adjoints also extend, e.g.

$$(K'^{\bar{\lambda}}_\gamma)^* : H_0^s(\overline{\Omega}) \to H^{s+\frac{1}{2}}(\Sigma), \text{ for } -2 \le s \le 0; \tag{49}$$

recall that $H_0^s(\overline{\Omega}) = H^s(\Omega)$ when $|s| < \frac{1}{2}$. To sum up, we have shown:

THEOREM 8. *When Ω is $C^{1,1}$ and A has C^∞-coefficients, the solution operators K_γ^λ and $K_\gamma'^{\bar\lambda}$ for* (8) *and its primed version map $H^{s-\frac12}(\Sigma)$ to $H^s(\Omega)$ for $0 \le s \le 2$. They are generalized Poisson operators in the sense that for $s \in]\frac32, 2]$, they can be written as the sum of a Poisson operator of Hölder smoothness $(0,1)$, in order-reduced x-form, and a lower order operator.*

The next step is to study $P_{\gamma_0,\nu_1}^\lambda = \nu_1 K_\gamma^\lambda$ and $P_{\gamma_0,\nu_1'}'^{\bar\lambda} = \nu_1' K_\gamma'^{\bar\lambda}$, cf. (10) ff.

We have immediately from the mapping properties established above, that

$$P_{\gamma_0,\nu_1}^\lambda, P_{\gamma_0,\nu_1'}'^{\bar\lambda} : H^{s-\frac12}(\Sigma) \to H^{s-\frac32}(\Sigma), \tag{50}$$

when $\frac32 < s \le 2$. Let us also introduce the operator $\nu_1'' = \nu_1' + \mathcal{A}_0'\gamma_0$, then Green's formula (7) takes the form

$$(Au, v)_{L_2(\Omega)} - (u, A'v)_{L_2(\Omega)} = (\nu_1 u, \gamma_0 v)_{\frac12,-\frac12} - (\gamma_0 u, \nu_1'' v)_{\frac32,-\frac32}, \tag{51}$$

for $u \in H^2(\Omega)$, $v \in D(A'_{\max})$, and $P_{\gamma_0,\nu_1''}'^{\bar\lambda}$ (cf. (11)) likewise maps as in (50) ff. Applying (51) to functions u, v with $Au = 0$, $A'v = 0$, we see that $P_{\gamma_0,\nu_1}^\lambda$ and $P_{\gamma_0,\nu_1''}'^{\bar\lambda}$ are contained in each other's adjoints. Therefore $P_{\gamma_0,\nu_1}^\lambda$ considered in (50) has the extension $(P_{\gamma_0,\nu_1''}'^{\bar\lambda})^*$, which is continuous from $H^{s'+\frac32}(\Sigma)$ to $H^{s'+\frac12}(\Sigma)$ for $-2 \le s' < -\frac32$. This extends the statement in (50) to the values $0 \le s < \frac12$, and by interpolation we obtain the validity of (50) for $0 \le s \le 2$.

$P_{\gamma_0,\nu_1}^\lambda$ can in the localizations to $\mathbb{R}^n_+$ be described as the composition of the operator $\nu_1 = s_0\gamma_1$ (with $s_0 \in C^{0,1}$) and a generalized Poisson operator consisting of an operator in order-reduced x-form having $C^{0,1}$-smoothness plus a remainder of lower order. For $s \in]\frac32, 2]$ we can apply Theorem 6 2° to the compositions, using that K_γ^λ is locally the sum of a composition $\Lambda_{-,+}^{-2} K_1^0(\lambda)\Lambda_0^2$ (multiplied with smooth cut-off functions) where $K_1^0(\lambda)$ is in x-form, and a remainder of lower order. This implies that $P_{\gamma_0,\nu_1}^\lambda$, apart from the remainder term coming from K_γ^λ, is the sum of a first-order ψdo in x-form with $C^{0,1}$-smoothness and a remainder term, mapping $H^{t+1}(\Sigma)$ to $H^t(\Sigma)$ for $|t| < 1$, resp. $H^{t+1-\theta}(\Sigma)$ to $H^t(\Sigma)$ for $-1+\theta < t < 1$. With $s - \frac12 = t+1$, s runs in $]\frac12, \frac52[$ resp. $]\frac12 + \theta, \frac52[$ here, which covers the interval $s \in]\frac32, 2]$ allowed by the other remainder.

For low values of s there is again the difficulty that we are dealing with a composition with ingredients of order-reducing operators and x- or y-form operators, which each have different rules for the spaces in which the decompositions and mapping properties are valid, and we refrain from a further discussion here.

Observe moreover that $P_{\gamma_0,\nu_1}^\lambda$ is elliptic (the principal symbol is invertible) — since this is known for P_{γ_0,γ_1}^0 ([4], [15]).

This shows:

THEOREM 9. *Assumptions as in Theorem* 8. $P^{\lambda}_{\gamma_0,\nu_1}$ *and* $P'^{\bar\lambda}_{\gamma_0,\nu'_1}$ *map* $H^{s-\frac{1}{2}}(\Sigma)$ *to* $H^{s-\frac{3}{2}}(\Sigma)$ *for* $s\in[0,2]$. *They are generalized elliptic* ψ*do's of order* 1*, in the sense that for* $s\in]\frac{3}{2},2]$*, they have the form of an elliptic principal part in* x*-form of Hölder smoothness* $(0,1)$ *plus a lower order part.*

With these mapping properties it is straightforward to verify that Γ and Γ' defined in (25) satisfy the full statement in (12).

When more smoothness of Ω is assumed, the representation of $P^{\lambda}_{\gamma_0,\nu_1}$ as the sum of a principal part in x-form and a lower-order term can of course be extended to larger intervals than found above.

6. Interpretation of realisations

We now have all the ingredients to interpret the abstract characterisation of closed realisations $\widetilde{A}$ in terms of operators $T:V\to W$ recalled in Section 2, to boundary conditions. In fact, we have the mappings defined from the trace operator γ_0

$$\gamma_{Z_\lambda}:Z_\lambda\xrightarrow{\sim}H^{-\frac{1}{2}}(\Sigma),\quad \gamma^*_{Z_\lambda}:H^{\frac{1}{2}}(\Sigma)\xrightarrow{\sim}Z_\lambda,$$

and the mappings defined from Poisson-type operators

$$K^{\lambda}_{\gamma}:H^{-\frac{1}{2}}(\Sigma)\to H^0(\Omega),\quad (K^{\lambda}_{\gamma})^*:H^0(\Omega)\to H^{\frac{1}{2}}(\Sigma),$$

as well as the versions with primes. Then the various definitions recalled in Section 3 for the smooth case, carrying $T^{\lambda}:V_\lambda\to W_{\bar\lambda}$ over to $L^{\lambda}:H^{-\frac{1}{2}}(\Sigma)\to H^{\frac{1}{2}}(\Sigma)$ if $V=Z$, $W=Z'$, resp. to $L^{\lambda}_1:X_1\to Y_1$ in general, are effective in exactly the same way, and all the diagrams are valid in this situation.

In this way, $\widetilde{A}$ is determined by a Neumann-type boundary condition

$$\nu_1 u=(L+P^0_{\gamma_0,\nu_1})\gamma_0 u$$

in the case $V=Z$, $W=Z'$, and by a condition involving projections in the general case.

The adjoint $\widetilde{A}$ is determined by the boundary condition

$$\nu'_1 u=(L^*+P'^0_{\gamma_0,\nu'_1})\gamma_0 u$$

in the case $V=Z$, $W=Z'$ (resp. by a condition involving projections in the general case), where L^* is the adjoint of L, considered as a generally unbounded operator from $H^{-\frac{1}{2}}(\Sigma)$ to $H^{\frac{1}{2}}(\Sigma)$.

There is a well-defined M-function $M_L(\lambda)$, which coincides with $-(L^{\lambda})^{-1}$ for $\lambda\in\rho(A_\gamma)\cap\rho(\widetilde{A})$; here (20) and (19) hold. Suitably modified results hold in cases of general V,W.

For the case $V=Z$, $W=Z'$, we have obtained:

THEOREM 10. *When Ω is $C^{1,1}$ and A has C^∞ coefficients, bounded with bounded derivatives on a neighborhood of Ω, and is uniformly strongly elliptic, then Theorem* 5 (i)–(v) *and* (20) *are valid.*

Gesztesy and Mitrea have in [12] established Kreĭn resolvent formulas for the Laplacian under a weaker smoothness hypothesis, namely that Ω is $C^{1,\sigma}$ with $\sigma > \frac{1}{2}$. Here they treat *selfadjoint* realisations determined by Robin-type boundary conditions

$$\gamma_1 u = B\gamma_0 u, \tag{52}$$

with B compact from H^1 to H^0 (assured if B is of order < 1). Posilicano and Raimondi [29] describe results for *selfadjoint* realisations in case Ω is $C^{1,1}$ and the coefficients of A, when it is written in symmetric divergence form, are $C^{0,1}$ satisfying various hypotheses. They remark that their treatment works for boundary conditions (52) with γ_1 replaced by the oblique Neumann trace operator ν_A (23) connected with the divergence form. Here B is taken of order < 1, so it is a Robin-type perturbation of the natural Neumann condition.

It is an important point in the present treatment, besides that it deals with nonselfadjoint situations, that Neumann-type conditions (17) with general ψdo's C of order 1 are included in the detailed discussion.

Furthermore, our pseudodifferential strategy allows the application of ellipticity concepts:

When C is a generalized pseudodifferential operator of order 1 and Hölder smoothness $(0,1)$, $L = C - P^0_{\gamma_0,\nu_1}$ is a generalized pseudodifferential operator of order 1 and Hölder smoothness $(0,1)$, and vice versa. L is elliptic precisely when the model boundary value problem for A with the boundary condition (17) is uniquely solvable at all (x',ξ') with $\xi' \neq 0$ in the boundary cotangent space (this is the Shapiro-Lopatinskiĭ condition). L^λ is then also elliptic at each $\lambda \in \rho(A_\gamma)$ (since $P^\lambda_{\gamma_0,\nu_1} - P^0_{\gamma_0,\nu_1}$ is of order < 1).

Moreover, there is then a parametrix of L, and this can be used to investigate the regularity of the domain of L. Likewise, each L^λ has a parametrix then. However, we want to set the true inverse $-M_L(\lambda)$ in relation to such a parametrix.

Restrict the attention to the case where C is a first-order *differential* operator on Σ with $C^{0,1}$-coefficients; then we can say more about $M_L(\lambda)$ with the present methods.

Assume a little more, namely that there is a ray $\lambda = -\mu^2 e^{i\theta}$, $\mu \in \mathbb{R}$, such that when we include λ in the principal symbol of $P^\lambda_{\gamma_0,\nu_1}$, then the principal symbol of $L^\lambda = C - P^\lambda_{\gamma_0,\nu_1}$ is invertible for $|\xi'|^2 + |\mu|^2 \geq 1$ ("parameter-ellipticity"). Let $s \in]\frac{3}{2}, 2]$. As in Section 5, we can invoke the system for $\widehat{A}$ on $\widehat{\Omega} = \Omega \times S^1$ (39) coupled with the same boundary operator (constant in the t-direction)

$$\widehat{\mathcal{A}} = \begin{pmatrix} \widehat{A} \\ \nu_1 - C\gamma_0 \end{pmatrix} : H^s(\widehat{\Omega}) \to \begin{matrix} H^{s-2}(\widehat{\Omega}) \\ \times \\ H^{s-\frac{3}{2}}(\widehat{\Sigma}) \end{matrix};$$

it is elliptic and has a parametrix $\widehat{\mathcal{B}}^0$. For the functions $u(x,t) = w(x)e^{i\mu t}$, this gives a

λ-dependent parametrix family for $\mathcal{A}(\lambda) = \big(A - \lambda\nu_1 - C\gamma_0\big)$ (when $|\lambda| \geq 1$) such that the remainder in the composition with $\mathcal{A}(\lambda)$ is $O(\langle\mu\rangle^{-\theta})$ for $\lambda \to \infty$ on the ray. Then there is a true inverse of $\mathcal{A}(\lambda)$, hence of L^λ, for sufficiently large λ on the ray. We can follow this up for the operator $\widehat{L} = C - \widehat{P}_{\gamma_0,\nu_1}$ over $\widehat{\Sigma}$, which gives L^λ when applied to functions $\varphi(x')e^{i\mu t}$. Here $\widehat{L}$ has a parametrix $\widehat{\widehat{L}}$ such that $\widehat{L}\widehat{\widehat{L}} - I$ is of negative order; this gives a parametrix $\widetilde{L}^\lambda$ of L^λ such that $L^\lambda\widetilde{L}^\lambda - I$ has an $O(\langle\mu\rangle^{-\theta})$ estimate. For sufficiently large λ on the ray this allows us to write $M_L(\lambda) = -(L^\lambda)^{-1}$ as $-\widetilde{L}^\lambda + \mathcal{R}$ with $\mathcal{R}$ of lower order. More precisely, $\widetilde{L}^\lambda$ is obtained as a composition of an operator in x-form with an order-reducing operator to the left; it maps from $H^{s-\frac{3}{2}}$ to $H^{s-\frac{1}{2}}$, and the remainder maps from $H^{s-\frac{3}{2}-\theta}$ to $H^{s-\frac{1}{2}}$. (The $s \in]\frac{3}{2}, 2]$ run inside the interval where the parametrix construction for elliptic first-order ψdo's of Hölder smoothness $(0,1)$ works, as in Theorem 6 3° and Remark 1.) In this sense, $M_L(\lambda)$ is a generalized ψdo of order -1.

Using this information for $s = 2$, we see that $M_L(\lambda)$ map $H^{\frac{1}{2}}$ not just to $H^{-\frac{1}{2}}$, but to $H^{\frac{3}{2}}$. Then $D(L) = D(L^\lambda) = H^{\frac{3}{2}}$ and $D(\widetilde{A})$ is in $H^2(\Omega)$.

If, moreover, C^* has Hölder smoothness $C^{0,1}$, the adjoint $\widetilde{A}^*$ is of the same type. In particular, there is selfadjointness if A and L are formally selfadjoint. This gives a very satisfactory version of the Kreĭn formula.

THEOREM 11. *If, in addition to the hypotheses of Theorem* 10, *C is a first-order differential operator with Hölder smoothness* $(0,1)$ *and the principal symbol of* $L^\lambda = C - P^\lambda_{\gamma_0,\nu_1}$ *is parameter-elliptic on a ray* $\lambda = -\mu^2 e^{i\theta}$, $\mu \in \mathbb{R}$, *then* $D(L) = H^{\frac{3}{2}}(\Sigma)$, *and* $M_L(\lambda)$ *is for large* λ *on the ray the sum of an elliptic* ψ*do of order* -1 *and Hölder smoothness* $(0,1)$, *in order-reduced x-form, and a lower-order term. Then* $D(\widetilde{A}) \subset H^2(\Omega)$.

If, moreover, C^* *has Hölder smoothness* $(0,1)$, *the adjoint* $\widetilde{A}^*$ *is defined similarly from of* L^* *with* $D(L^*) = H^{\frac{3}{2}}$, $D(\widetilde{A}^*) \subset H^2(\Omega)$. *In particular,* $\widetilde{A}$ *is selfadjoint if A and L are formally selfadjoint.*

From the point of view of the systematic parameter-dependent calculus of [19], the symbols of C and $P^\lambda_{\gamma_0,\nu_1}$ have "regularity $\nu = +\infty$" when C is a differential operator, so there is a parametrix with the same "regularity $+\infty$".

Pseudodifferential operators C can be included in the discussion if the symbol classes in [19] are used in a more definitive way (here when C is of order 1, it has "regularity 1", and the same will hold for the resulting principal symbols of L^λ and $M_L(\lambda)$). Considerations with finite positive "regularity" play an important role in [1,2]. We hope to return to such cases in future works, but here just wanted to show what can be done using Agmon's principle.

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Second Conf. on Pseudo-Diff. Operators: Invited Lectures

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WAVE-FRONT SETS IN FOURIER LEBESGUE SPACES [*]

Abstract. We consider wave-front sets in the framework of weighted Fourier Lebesgue spaces, $\mathscr{F}L^q_s$. We prove that

$$(*) \qquad WF_{\mathscr{F}L^q_{s-m}}(Af) \subset WF_{\mathscr{F}L^q_s}(f) \subset WF_{\mathscr{F}L^q_{s-m}}(Af) \cup \operatorname{Char} A$$

where A is properly supported pseudo-differential operator of order m and $\operatorname{Char} A$ denotes the set of characteristic points of A. Moreover, we discuss more general class of pseudo-differential operators in the framework of modulation spaces and present $(*)$ in a more general setting.

1. Introduction

This paper is an expanded and modified version of an invited speaker's lecture given by the first author at the conference "Pseudo-differential operators with related topics II" held in Växjö, Sweden, June 23 - 27, 2008. It is a part of the authors joint research project. In order to present the main goals of the invited lecture apart from the original results (collected mainly in Section 4), we have included here some results from [19,20] without proofs.

In the present paper we study certain aspects of microlocal analysis in Fourier Lebesgue spaces. More precisely, we define wave-front sets with respect to those spaces and show that usual mapping properties for a class of pseudo-differential operators which are valid for classical wave-front sets (cf. [14, Chapter XVIII], [15, Chapter VIII]) also hold for our wave-front sets. We refer to [19,20] for the complete exposition of our definition, and results related to the wave-fronts in Fourier Lebesgue spaces. The recent study of pseudo-differential and Fourier integral operators in Fourier Lebesgue spaces as well as their connection with modulation spaces in different contexts increased the interest for such spaces, cf. [2, 3, 5, 16, 21, 23, 31].

The modulation spaces were introduced by Feichtinger in [6], and the theory was developed and generalized in [7–10]. Modulation spaces have been incorporated into the calculus of pseudo-differential operators, in the sense of the study of continuity of classical pseudo-differential operators acting on modulation spaces (cf. [4, 17, 18, 24–26]), and pseudo-differential operators for which modulation spaces are used as symbol classes, [11–13, 22, 27, 29, 30]. Microlocal analysis of modulation spaces reduces to the microlocal analysis of Fourier Lebesgue spaces. From this point of view our investigation in [19, 20] and in this paper are involved in the analysis of modulation spaces.

The paper is organized as follows. In Section 2 we fix basic notions and notation. Definitions of wave-front sets in the context of Fourier Lebesgue spaces as well

[*]It is a pleasure to dedicate this paper to Prof. Luigi Rodino on the occasion of his 60th birthday.
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as their basic properties are given in Propositions 1, 2 and 3 of Section 3. In Propositions 4 and 5 of Section 4 we study the continuity properties of pseudo-differential operators of the Hörmander class S^m on Fourier Lebesgue spaces. Theorems 1 and 2 of the same section are devoted to the study of microlocal properties of localized version of pseudo-differential operators. These results imply Corollary 1 where we discuss the relationship between our wave-front sets and the classical ones. In Section 5 we introduce wave-front sets in modulation spaces and discuss relations between local versions of Fourier Lebesgue and modulation spaces. Section 6 is devoted to the further study of pseudo-differential operators on a more advanced level. We present in that section the class of symbols $\mho^{s,\rho}_{(\omega)}(\mathbb{R}^{2d})$ and results on the continuity of corresponding pseudo-differential operators on weighted Fourier-Lebesgue spaces. Also, in Section 6, we present an estimate of the form (*) and discuss hypoellipticity in the same framework.

2. Notions and notation

We denote by Γ an open cone in $\mathbb{R}^d \setminus 0$ and by X an open set in $\mathbb{R}^d$. A conic neighborhood of a point $(x_0, \xi_0) \in \mathbb{R}^d \times (\mathbb{R}^d \setminus 0)$ is a product $X \times \Gamma$, where X is a neighborhood of x_0 in $\mathbb{R}^d$ and Γ is an open cone in $\mathbb{R}^d$ which contains ξ_0. Sometimes such a cone is denoted by Γ_{ξ_0} and is called a conic neighborhood of ξ_0. When $x, \xi \in \mathbb{R}^d$, their scalar product is denoted by $\langle x, \xi \rangle$. As usual, $\langle \xi \rangle = (1 + |\xi|^2)^{1/2}$, $\xi \in \mathbb{R}^d$. For $q \in [1, \infty]$ we let $q' \in [1, \infty]$ denote the conjugate exponent, i. e. $1/q + 1/q' = 1$.

Assume that ω and v are positive and measurable functions on $\mathbb{R}^d$. Recall that ω is called v-moderate weight if

$$\omega(x+y) \le C\omega(x)v(y) \tag{1}$$

for some constant C which is independent of $x, y \in \mathbb{R}^d$. If v in (1) can be chosen as a polynomial, then ω is called polynomially moderated. We let $\mathscr{P}(\mathbb{R}^d)$ to be the set of all polynomially moderated functions on $\mathbb{R}^d$.

For convenience we also need to consider appropriate subclasses of $\mathscr{P}$. More precisely, let $\mathscr{P}_0(\mathbb{R}^d)$ be the set of all $\omega \in \mathscr{P}(\mathbb{R}^d) \bigcap C^\infty(\mathbb{R}^d)$ such that $\partial^\alpha \omega / \omega \in L^\infty$ for all multi-indices α. By Lemma 1.2 in [29] it follows that for each $\omega \in \mathscr{P}(\mathbb{R}^d)$, there is an element $\omega_0 \in \mathscr{P}_0(\mathbb{R}^d)$ such that

$$C^{-1}\omega_0 \le \omega \le C\omega_0, \tag{2}$$

for some constant C.

Assume that $\rho \ge 0$. Then we let $\mathscr{P}_\rho(\mathbb{R}^{2d})$ to be the set of all $\omega(x,\xi)$ in $\mathscr{P}_0(\mathbb{R}^{2d})$ such that

$$\langle \xi \rangle^{\rho|\beta|} \frac{\partial_x^\alpha \partial_\xi^\beta \omega(x,\xi)}{\omega(x,\xi)} \in L^\infty(\mathbb{R}^{2d}), \tag{3}$$

for every multi-indices α and β. Note that in contrast to for $\mathscr{P}_0$ and $\mathscr{P}$, we do not have any equivalence between $\mathscr{P}_\rho$ and $\mathscr{P}$ when $\rho > 0$, in the sense of (2). If $s \in \mathbb{R}$ and $\rho \in [0,1]$, then $\mathscr{P}_\rho(\mathbb{R}^{2d})$ contains $\omega(x,\xi) = \langle \xi \rangle^s$.

The Fourier transform $\mathscr{F}$ is the linear and continuous mapping on $\mathscr{S}'(\mathbb{R}^d)$ which takes the form

$$(\mathscr{F}f)(\xi) = \widehat{f}(\xi) \equiv (2\pi)^{-d/2} \int f(x) e^{-i\langle x,\xi\rangle}\, dx, \quad \xi \in \mathbb{R}^d,$$

when $f \in L^1(\mathbb{R}^d)$. We recall that $\mathscr{F}$ is a homeomorphism on $\mathscr{S}'(\mathbb{R}^d)$ which restricts to a homeomorphism on $\mathscr{S}(\mathbb{R}^d)$ and to a unitary operator on $L^2(\mathbb{R}^d)$.

We say that a distribution $f \in \mathscr{D}'(\mathbb{R}^d)$ is *microlocally smooth* at $(x_0, \xi_0) \in \mathbb{R}^d \times (\mathbb{R}^d \setminus 0)$ if there exists $\chi \in C_0^\infty(\mathbb{R}^d)$ such that $\chi(x_0) \neq 0$, and an open cone Γ_{ξ_0} such that for every $N \in \mathbb{N}$, there exists $C_N > 0$ such that $|\mathscr{F}(\chi f)(\xi)| \le C_N \langle \xi \rangle^{-N/2}$, $\xi \in \Gamma_{\xi_0}$. The *wave-front set* of f, $WF(f)$ is the complement of the set of points (x_0, ξ_0), where f is microlocally smooth.

Assume that $a \in \mathscr{S}(\mathbb{R}^{2d})$, and that $t \in \mathbb{R}$ is fixed. Then the pseudo-differential operator $a_t(x,D)$, defined by the formula

$$\begin{aligned} (a_t(x,D)f)(x) &= (\mathrm{Op}_t(a)f)(x) \\ &= (2\pi)^{-d} \iint a((1-t)x + ty, \xi) f(y) e^{i\langle x-y,\xi\rangle}\, dy\, d\xi, \end{aligned} \tag{4}$$

is a linear and continuous operator on $\mathscr{S}(\mathbb{R}^d)$. For general $a \in \mathscr{S}'(\mathbb{R}^{2d})$, the pseudo-differential operator $a_t(x,D)$ is defined as the continuous operator from $\mathscr{S}(\mathbb{R}^d)$ to $\mathscr{S}'(\mathbb{R}^d)$ with the distribution kernel

$$K_{t,a}(x,y) = (2\pi)^{-d/2} (\mathscr{F}_2^{-1} a)((1-t)x + ty, y - x). \tag{5}$$

Here $\mathscr{F}_2 F$ is the partial Fourier transform of $F(x,y) \in \mathscr{S}'(\mathbb{R}^{2d})$ with respect to the y-variable. This definition makes sense, since the mappings $\mathscr{F}_2$ and

$$F(x,y) \mapsto F((1-t)x + ty, y - x)$$

are homeomorphisms on $\mathscr{S}'(\mathbb{R}^{2d})$. We also note that this definition of $a_t(x,D)$ agrees with the operator in (4) when $a \in \mathscr{S}(\mathbb{R}^{2d})$, and that $a_t(x,D)$ agrees with the Kohn-Nirenberg representation $a(x,D)$ when $t = 0$.

Furthermore, any linear and continuous operator T from $\mathscr{S}(\mathbb{R}^d)$ to $\mathscr{S}'(\mathbb{R}^d)$ has a distribution kernel K in $\mathscr{S}'(\mathbb{R}^{2d})$ in view of kernel theorem of Schwartz. By Fourier's inversion formula we may then find a unique $a \in \mathscr{S}'(\mathbb{R}^{2d})$ such that (5) is fulfilled with $K = K_{t,a}$. Consequently, for every fixed $t \in \mathbb{R}$, there is a one to one correspondence between linear and continuous operators from $\mathscr{S}(\mathbb{R}^d)$ to $\mathscr{S}'(\mathbb{R}^d)$, and $\mathrm{Op}_t(\mathscr{S}'(\mathbb{R}^{2d}))$, the set of all $a_t(x,D)$ such that $a \in \mathscr{S}'(\mathbb{R}^{2d})$.

In particular, if $a \in \mathscr{S}'(\mathbb{R}^{2d})$ and $s,t \in \mathbb{R}$, then there is a unique $b \in \mathscr{S}'(\mathbb{R}^{2d})$ such that $a_s(x,D) = b_t(x,D)$. By straight-forward applications of Fourier's inversion formula, it follows that

$$a_s(x,D) = b_t(x,D) \quad \Longleftrightarrow \quad b(x,\xi) = e^{i(t-s)\langle D_x, D_\xi\rangle} a(x,\xi). \tag{6}$$

(Cf. Section 18.5 in [14].)

3. Wave-front sets in Fourier Lebesgue spaces

In this section we define wave-front sets with respect to Fourier Lebesgue spaces, and recall some general properties from [19, 20].

Let $\omega \in \mathscr{P}(\mathbb{R}^{2d})$ and let $q \in [1,\infty]$. The (weighted) Fourier-Lebesgue space $\mathscr{F}L^q_{(\omega)}(\mathbb{R}^d)$ is the Banach space which consists of all $f \in \mathscr{S}'(\mathbb{R}^d)$ such that

$$\|f\|_{\mathscr{F}L^q_{(\omega)}} = \|f\|_{\mathscr{F}L^q_{(\omega),x}} \equiv \|\widehat{f}\cdot\omega(x,\cdot)\|_{L^q} < \infty. \tag{7}$$

The weight $\omega(x,\xi)$ in (7) depends on both x and ξ, although $\widehat{f}(\xi)$ only depends on ξ. However, since ω is v-moderate for some $v \in \mathscr{P}(\mathbb{R}^{2d})$, different choices of x give rise to equivalent norms. Therefore, the condition $\|f\|_{\mathscr{F}L^q_{(\omega),x}} < \infty$ is independent of x and for different $x_1, x_2 \in \mathbb{R}^d$ there exists a constant $C_{x_1,x_2} > 0$ such that

$$C^{-1}_{x_1,x_2}\|f\|_{\mathscr{F}L^q_{(\omega),x_2}} \le \|f\|_{\mathscr{F}L^q_{(\omega),x_1}} \le C_{x_1,x_2}\|f\|_{\mathscr{F}L^q_{(\omega),x_2}}.$$

We say that $f \in \mathscr{D}'(\mathbb{R}^d)$ is locally in $\mathscr{F}L^q_{(\omega)}(\mathbb{R}^d)$, if $\chi f \in \mathscr{F}L^q_{(\omega)}(\mathbb{R}^d)$ for every $\chi \in C_0^\infty(\mathbb{R}^d)$ and in that case we use the notation $f \in \mathscr{F}L^q_{(\omega),loc}(\mathbb{R}^d)$. It is said that $f \in \mathscr{F}L^q_{(\omega),loc}(\mathbb{R}^d)$ at x_0 if there exists a function $\chi \in C_0^\infty(\mathbb{R}^d)$, $\chi(x_0) \neq 0$, such that $\chi f \in \mathscr{F}L^q_{(\omega)}(\mathbb{R}^d)$.

In the remaining part of the paper we study weighted Fourier-Lebesgue spaces with weights which depend on ξ. Thus, with $\omega_0(\xi) = \omega(0,\xi) \in \mathscr{P}(\mathbb{R}^d)$,

$$f \in \mathscr{F}L^q_{(\omega)}(\mathbb{R}^d) = \mathscr{F}L^q_{(\omega_0)}(\mathbb{R}^d) \quad \Longleftrightarrow \quad \|f\|_{\mathscr{F}L^q_{(\omega_0)}} \equiv \|\widehat{f}w_0\|_{L^q} < \infty.$$

We usually assume that the involved weight functions $\omega_0(\xi)$ is given by $\omega_0(\xi) = \omega(x_0,\xi) = \langle\xi\rangle^s$, for some $x_0 \in \mathbb{R}^d$ and $s \in \mathbb{R}$. In this case we use the notation $\mathscr{F}L^q_s$ instead of $\mathscr{F}L^q_{(\omega_0)}$. If $\omega = 1$, then the notation $\mathscr{F}L^q$ is used instead of $\mathscr{F}L^q_{(\omega)}$.

Let $\omega_0 \in \mathscr{P}(\mathbb{R}^d)$, $\Gamma \subseteq \mathbb{R}^d \setminus 0$ be open cone and $q \in [1,\infty]$ be fixed. For any $f \in \mathscr{S}'(\mathbb{R}^d)$, let

$$|f|_{\mathscr{F}L^{q,\Gamma}_{(\omega_0)}} \equiv \Big(\int_\Gamma |\widehat{f}(\xi)\omega_0(\xi)|^q\,d\xi\Big)^{1/q}$$

(with obvious interpretation when $q = \infty$). We note that $|\cdot|_{\mathscr{F}L^{q,\Gamma}_{(\omega_0)}}$ defines a semi-norm on $\mathscr{S}'(\mathbb{R}^d)$ which might attain the value $+\infty$. If $\Gamma = \mathbb{R}^d \setminus 0$, $f \in \mathscr{F}L^q_{(\omega)}(\mathbb{R}^d)$ and $q < \infty$, then $|f|_{\mathscr{F}L^{q,\Gamma}_{(\omega)}}$ agrees with the Fourier Lebesgue norm $\|f\|_{\mathscr{F}L^q_{(\omega),x}}$ of f.

We let $\Theta_{\mathscr{F}L^q_{(\omega_0)}}(f)$ to be the set of all $\xi \in \mathbb{R}^d \setminus 0$ such that $|f|_{\mathscr{F}L^{q,\Gamma}_{(\omega_0)}} < \infty$, for some $\Gamma = \Gamma_\xi$. Its complement in $\mathbb{R}^d \setminus 0$ is denoted by $\Sigma_{\mathscr{F}L^q_{(\omega_0)}}(f)$.

We have now the following result.

PROPOSITION 1. *Assume that* $q \in [1,\infty]$, $\chi \in \mathscr{S}(\mathbb{R}^d)$, *and that* $\omega_0 \in \mathscr{P}(\mathbb{R}^d)$. *Also assume that* $f \in \mathscr{E}'(\mathbb{R}^d)$. *Then*

$$\Sigma_{\mathscr{F}L^q_{(\omega_0)}}(\chi f) \subseteq \Sigma_{\mathscr{F}L^q_{(\omega_0)}}(f). \tag{8}$$

Proof. Assume that $\xi_0 \in \Theta_{\mathscr{F}L^q_{(\omega_0)}}(f)$, and choose open cones Γ_1 and Γ_2 in $\mathbb{R}^d$ such that $\overline{\Gamma_2} \subseteq \Gamma_1$. Since f has a compact support, it follows that $|\widehat{f}(\xi)\omega_0(\xi)| \le C\langle\xi\rangle^{N_0}$ for some positive constants C and N_0. The idea of the proof is to show that for each N, there are constants C_N such that

$$|\chi f|_{\mathscr{F}L^{q_2,\Gamma_2}_{(\omega_0)}} \le C_N\Big(|f|_{\mathscr{F}L^{q_1,\Gamma_1}_{(\omega_0)}} + \sup_{\xi\in\mathbb{R}^d}\big(|\widehat{f}(\xi)\omega_0(\xi)|\langle\xi\rangle^{-N}\big)\Big) \tag{9}$$

$$\text{when}\quad q_1 \le q_2, \quad \overline{\Gamma}_2 \subseteq \Gamma_1 \quad \text{and} \quad N = 1,2,\ldots.$$

The result then follows by taking $q_1 = q_2 = q$ and $N \ge N_0$. We refer to [19] for details of the proof of (9). □

Now we are ready to define wave-front sets in the framework of Fourier Lebesgue spaces.

DEFINITION 1. *Assume that* $q \in [1,\infty]$, $f \in \mathscr{D}'(\mathbb{R}^d)$ *and* $\omega_0 \in \mathscr{P}(\mathbb{R}^d)$. *The wave-front set* $WF_{\mathscr{F}L^q_{(\omega_0)}}(f)$ *with respect to* $\mathscr{F}L^q_{(\omega_0)}(\mathbb{R}^d)$ *consists of all pairs* (x_0,ξ_0) *in* $\mathbb{R}^d \times (\mathbb{R}^d \setminus 0)$ *such that* $\xi_0 \in \Sigma_{\mathscr{F}L^q_{(\omega_0)}}(\chi f)$, *holds for each* $\chi \in C_0^\infty(\mathbb{R}^d)$ *such that* $\chi(x_0) \neq 0$.

The following proposition shows that the wave-front set $WF_{\mathscr{F}L^q_{(\omega_0)}}(f)$ decreases with respect to the parameter q and increases with respect to the weight function ω, when $f \in \mathscr{D}'(\mathbb{R}^d)$ is fixed.

PROPOSITION 2. *Assume that* $f \in \mathscr{D}'(\mathbb{R}^d)$, $q_j \in [1,\infty]$ *and* $\omega_j \in \mathscr{P}(\mathbb{R}^d)$ *for* $j = 1,2$ *satisfy*

$$q_1 \le q_2, \quad \textit{and} \quad \omega_2(\xi) \le C\omega_1(\xi), \tag{10}$$

for some constant C *which is independent of* $\xi \in \mathbb{R}^d$. *Then*

$$WF_{\mathscr{F}L^{q_2}_{(\omega_2)}}(f) \subseteq WF_{\mathscr{F}L^{q_1}_{(\omega_1)}}(f).$$

Proof. It is no restriction to assume that f has a compact support, and that $\omega_1(\xi) = \omega_2(\xi) = \omega_0(\xi)$. This implies that

$$\sup_{\xi\in\mathbb{R}^d} |\langle\xi\rangle^{-N_0}\widehat{f}(\xi)\omega_0(\xi)| < \infty$$

provided N_0 is chosen large enough. Hence (9) implies that $\Theta_{\mathscr{F}L^{q_1}_{(\omega_1)}}(f) \subseteq \Theta_{\mathscr{F}L^{q_2}_{(\omega_2)}}(f)$, and the assertion follows. □

PROPOSITION 3. *Assume that* $q \in [1,\infty]$, $f \in \mathscr{D}'(X)$, $\omega_0 \in \mathscr{P}(\mathbb{R}^d)$ *and* $(x_0,\xi_0) \in X \times (\mathbb{R}^d \setminus 0)$. *The following conditions are equivalent:*

(1) *there exist* $g \in \mathscr{F}L^q_{(\omega_0)}(\mathbb{R}^d)$ *such that* $(x_0,\xi_0) \notin WF(f-g)$;

(2) $(x_0,\xi_0) \notin WF_{\mathscr{F}L^q_{(\omega_0)}}(f)$.

Proof. First we show that $\varphi g \in \mathscr{F}L^q_{(\omega_0)}(\mathbb{R}^d)$ if $\varphi \in \mathscr{S}(\mathbb{R}^d)$ and $g \in \mathscr{F}L^q_{(\omega_0)}(\mathbb{R}^d)$.

Let $\varphi \in \mathscr{S}(\mathbb{R}^d)$ and let ψ be defined by $\widehat{\psi} = \varphi$. Then

$$
\begin{aligned}
\|g\varphi\|_{\mathscr{F}L^q_{(\omega_0)}} &= (2\pi)^{-d/2}\Big(\int_{\mathbb{R}^d}\Big|\Big(\int_{\mathbb{R}^d}\widehat{g}(\xi-\eta)\psi(\eta)\,d\eta\Big)\omega_0(\xi)\Big|^q d\xi\Big)^{1/q}\\
&\le (2\pi)^{-d/2}\Big(\int_{\mathbb{R}^d}\Big(\int_{\mathbb{R}^d}|\widehat{g}(\xi-\eta)||\psi(\eta)|\,d\eta|\omega_0(\xi)|\Big)^q d\xi\Big)^{1/q}\\
&\le (2\pi)^{-d/2}\int_{\mathbb{R}^d}\Big(\int_{\mathbb{R}^d}|\widehat{g}(\xi-\eta)\psi(\eta)\omega_0(\xi)|^q\,d\xi\Big)^{1/q}d\eta\\
&\le (2\pi)^{-d/2}\int_{\mathbb{R}^d}\Big(\int_{\mathbb{R}^d}|\widehat{g}(\xi-\eta)\omega_0(\xi-\eta)|^q\,d\xi\Big)^{1/q}|\psi(\eta)v(\eta)|\,d\eta\\
&= C\|g\|_{\mathscr{F}L^q_{(\omega_0)}},
\end{aligned}
$$

where $C = (2\pi)^{-d/2}\|\psi v\|_{L^1}$ is finite, since v is of polynomial growth.

By the assumption, $f-g$ is microlocally smooth at (x_0,ξ_0). Hence there exists an open cone $\Gamma = \Gamma_{\xi_0}$ such that

$$\int_\Gamma |\mathscr{F}(\varphi(f-g))(\xi)\omega_0(\xi)|^q\,d\xi < \infty,$$

where $\varphi(x_0) \neq 0$ and the support of φ can be chosen to be sufficiently close to x_0. This, together with the decomposition $\mathscr{F}(\varphi f) = \mathscr{F}(\varphi g) + \mathscr{F}(\varphi(f-g))$ implies that

$$\int_\Gamma |\mathscr{F}(\varphi f)(\xi)\omega_0(\xi)|^q\,d\xi < \infty, \tag{11}$$

i.e. $(x_0,\xi_0) \notin WF_{\mathscr{F}L^q_{(\omega_0)}}(f)$.

Conversely, if $(x_0,\xi_0) \notin WF_{\mathscr{F}L^q_{(\omega_0)}}(f)$, then there exist $\varphi \in C_0^\infty$ such that $\varphi(x_0) \neq 0$ and a conic neighborhood Γ of ξ_0 such that (11) holds.

Let $g \in \mathscr{F}L^q_{(\omega_0)}(\mathbb{R}^d)$ be defined by

$$\widehat{g}(\xi) = \begin{cases} \mathscr{F}(\varphi f)(\xi), & \text{if} \quad \xi \in \Gamma\\ 0, & \text{if} \quad \xi \notin \Gamma. \end{cases}$$

Then $\widehat{h} = \mathscr{F}(\varphi f) - \widehat{g}$ vanishes in Γ and $\widehat{h}$ has a polynomial bound. Therefore $(x_0,\xi_0) \notin WF(h)$. Choose $\psi \in C_0^\infty$ so that $\psi\varphi = 1$ in a neighborhood of x_0. Note $(x_0,\xi_0) \notin$

$WF(\psi h)$. Now, since

$$f - \psi g = (1 - \psi\varphi) f + \psi h,$$

we conclude that $(x_0, \xi_0) \notin WF(f - \psi g)$. Since $\psi g \in \mathscr{F}L^q_{(\omega_0)}(\mathbb{R}^d)$ the proof is complete. $\square$

4. Pseudo-differential operators with classical symbols

In this section we prove mapping properties of pseudo-differential operators in the background of wave-front sets of Fourier Lebesgue types.

Assume that $m \in \mathbb{R}$. Then we recall that the Hörmander symbol class

$$S^m = S^m(\mathbb{R}^d \times \mathbb{R}^d) = S^m(\mathbb{R}^{2d})$$

consists of all smooth functions a such that for each pair of multi-indices α, β there are constants $C_{\alpha,\beta}$ such that

$$|\partial_\xi^\alpha \partial_x^\beta a(x,\xi)| \le C_{\alpha,\beta} \langle \xi \rangle^{m-|\alpha|}, \quad x, \xi \in \mathbb{R}^d.$$

We also set $S^{-\infty} = \cap_{m \in \mathbb{R}} S^m$, and

$$\operatorname{Op}(S^m) = \{\, a(x,D)\, ;\, a \in S^m(\mathbb{R}^d \times \mathbb{R}^d)\,\}.$$

The following result is needed in the proof of Theorem 1 below.

PROPOSITION 4. *Assume that $q \in [1, \infty]$ and $\chi \in C_0^\infty(\mathbb{R}^d)$. Then the following is true:*

(1) *if $a \in S^0$ then the mapping $\chi(x)a(x,D) : \mathscr{F}L^q(\mathbb{R}^d) \to \mathscr{F}L^q(\mathbb{R}^d)$ is continuous. In particular, if $a(x,\xi) = a(\xi) \in S^0$, then the mapping $a(D) : \mathscr{F}L^q(\mathbb{R}^d) \to \mathscr{F}L^q(\mathbb{R}^d)$ is continuous;*

(2) *if $a \in S^m$ and $s \in \mathbb{R}$ then $\chi(x)a(x,D) : \mathscr{F}L^q_s(\mathbb{R}^d) \to \mathscr{F}L^q_{s-m}(\mathbb{R}^d)$. In particular, if $a(x,\xi) = a(\xi) \in S^m$, $a(D) : \mathscr{F}L^q_s(\mathbb{R}^d) \to \mathscr{F}L^q_{s-m}(\mathbb{R}^d)$.*

Proof. For the proof it is convenient to put $E_s(\xi) = \langle \xi \rangle^s$ when $s \in \mathbb{R}$. We only prove assertions for $1 \le q < \infty$. The case $q = \infty$ follows by similar arguments and is left to the reader.

(1) Let $a \in S^0$ and $\chi \in C_0^\infty(\mathbb{R}^d)$. We denote the support of χ by K. The oscillatory integral (4) is well defined for the symbol χa, $t = 0$ and $f \in \mathscr{F}L^q(\mathbb{R}^d)$. Namely, after $2s$ times integration by parts we obtain

$$\chi(x)a(x,D)f(x) = (2\pi)^{-d/2} \int e^{i\langle x,\eta\rangle} \big(E_{2s}(D_x)(\chi(x)a(x,\eta))\big) \langle \eta \rangle^{-2s} \widehat{f}(\eta)\, d\eta, \tag{12}$$

which, by the Hölder inequality, gives

$$|\chi(x)a(x,D)f(x)| \le (2\pi)^{-d/2} \|\big(E_{2s}(D_x)(\chi(x)a(x,\cdot))\big)\langle \cdot \rangle^{-2s}\|_{L^{q'}} \|\widehat{f}\|_{L^q} \le C\|f\|_{\mathscr{F}L^q}$$

for $2s > d/q'$.

Let $\mathscr{F}_1 a_\chi(\xi,\eta)$ be the partial Fourier transform of $a_\chi(x,\eta) = \chi(x)a(x,\eta)$ with respect to the x variable. Then it follows from the assumptions that for each $N \in \mathbb{N}$, there is a constant $C_N > 0$ such that

$$|\mathscr{F}_1 a_\chi(\xi,\eta)| \le C_N \langle \xi \rangle^{-N}, \qquad \xi,\eta \in \mathbb{R}^d. \tag{13}$$

Now,

$$\begin{aligned}
&|\mathscr{F}(\chi(x)a(x,D)f(x))(\xi)| \\
&\quad = (2\pi)^{-d}\Big|\int_K e^{-i\langle x,\xi\rangle}\Big(\int_{\mathbb{R}^d} e^{i\langle x,\eta\rangle} E_{2s}(D_x)a_\chi(x,\eta)\langle\eta\rangle^{-2s}\widehat{f}(\eta)\,d\eta\Big)\,dx\Big| \\
&\quad = (2\pi)^{-d}\Big|\int_{\mathbb{R}^d} \widehat{f}(\eta)\langle\eta\rangle^{-2s}\Big(\int_K e^{-i\langle x,\xi-\eta\rangle} E_{2s}(D_x)a_\chi(x,\eta)\,dx\Big)\,d\eta\Big| \\
&\quad \le (2\pi)^{-d}\Big|\int_{\mathbb{R}^d} \widehat{f}(\eta)\langle\eta\rangle^{-2s}\langle\xi-\eta\rangle^{2s}\mathscr{F}_1 a_\chi(\xi-\eta,\eta)\,d\eta\Big|
\end{aligned}$$

By taking ξ and $\xi-\eta$ as new variables of integrations, and assuming that $N > s + \frac{d}{2}$, by (13) and Minkowski's inequality we obtain

$$\begin{aligned}
&\|\mathscr{F}(\chi(x)a(x,D)f(x))(\xi)\|_{L^q} \\
&\le (2\pi)^{-d}\Big(\int_{\mathbb{R}^d}\Big|\int_{\mathbb{R}^d}\widehat{f}(\xi-\eta)\langle\eta\rangle^{2s}\mathscr{F}_1 a_\chi(\eta,\xi-\eta)\langle\xi-\eta\rangle^{-2s}\,d\eta\Big|^q d\xi\Big)^{1/q} \\
&\le (2\pi)^{-d}\int_{\mathbb{R}^d}\langle\eta\rangle^{2s}\Big(\int_{\mathbb{R}^d}|\widehat{f}(\xi-\eta)\mathscr{F}_1 a_\chi(\eta,\xi-\eta)|^q\,d\xi\Big)^{1/q} d\eta \\
&\le (2\pi)^{-d} C_N\int_{\mathbb{R}^d}\langle\eta\rangle^{2(s-N)}\Big(\int_{\mathbb{R}^d}|\widehat{f}(\xi-\eta)|^q\,d\xi\Big)^{1/q} d\eta \\
&\le C\|\widehat{f}\|_{L^q}.
\end{aligned}$$

If we instead have that the symbol is a Fourier multiplier $a = a(\xi) \in S^0$, then it is obvious that

$$\|\mathscr{F}(a(D)f)\|_{L^q} \le C\|\mathscr{F}f\|_{L^q},$$

and (1) follows in this case as well.

The assertion (2) follows from (1), the fact that the map

$$a(x,\xi) \mapsto a(x,\xi)\langle\xi\rangle^m$$

is a homeomorphism from S^0 to S^m, and the fact that the map

$$f \mapsto \langle D\rangle^{-s} f$$

is a homeomorphism from $\mathscr{F}L^q$ to $\mathscr{F}L^q_s$. The proof is complete. $\square$

REMARK 1. It is known that an operator $a(x,D)$ whose symbol belongs to the Hörmander class S^0 is continuous from L^q to L^q, $1<q<\infty$, see [32]. In order to prove the continuity in $\mathscr{F}L^q(\mathbb{R}^d)$ it is not sufficient to assume the boundedness of the corresponding symbol a with respect to the x variable because for such symbol and $f\in\mathscr{F}L^q(\mathbb{R}^d)$ the convolution $\widehat{a}*\widehat{f}$ does not belong to L^q, in general. For that reason we observe the operators of the form $\chi(x)a(x,D)$. Alternatively, we could impose a decay condition on a with respect to the x variable. For example, one can prove that $a(x,D):\mathscr{F}L^q(\mathbb{R}^d)\to\mathscr{F}L^q(\mathbb{R}^d)$ is continuous if the symbol a satisfies

$$|\partial_\xi^\alpha\partial_x^\beta a(x,\xi)|\le C_{\alpha,\beta}\langle x\rangle^{k-|\beta|}\langle\xi\rangle^{m-|\alpha|},$$

for $k<-d$. More details on this topic can be found in [19, 20].

Next we recall the definition of characteristic sets and elliptic pseudo-differential operators. The symbol $a\in S^m(\mathbb{R}^{2d})$ is called *non-characteristic* at $(x_0,\xi_0)\in\mathbb{R}^d\times(\mathbb{R}^d\setminus 0)$ if there is a neighborhood U of x_0, a conical neighborhood Γ of ξ_0 and constants c and R such that

$$|a(x,\xi)|>c|\xi|^m,\quad \text{if}\quad |\xi|>R, \tag{14}$$

and $\xi\in\Gamma$. Then one can find $b\in S^{-m}(\mathbb{R}^{2d})$ such that

$$a(x,D)b(x,D)-Id\in\mathrm{Op}(S^{-\infty})\quad\text{and}\quad b(x,D)a(x,D)-Id\in\mathrm{Op}(S^{-\infty})$$

in a conical neighborhood of (x_0,ξ_0) (cf. [14, 19]). The point $(x_0,\xi_0)\in\mathbb{R}^d\times(\mathbb{R}^d\setminus 0)$ is called characteristic for a if it is not non-characteristic point of $a(x,D)$. The set of characteristic points (the characteristic set) of $a(x,D)$ is denoted by $\mathrm{Char}\,a(x,D)$. We shall identify operators with their symbols when discussing characteristic sets.

The operator $a(x,D)\in\mathrm{Op}(S^m)$ is called *elliptic* if the set of characteristic points is empty. This means that for each bounded neighbourhood U of x_0, there are constants $c,R>0$ such that (14) holds when $x\in U$.

PROPOSITION 5. *Let $a\in S^m$ be elliptic and assume that $f\in\mathscr{F}L^q_{t,loc}(\mathbb{R}^d)$ for some $q\in[1,\infty]$ and for some $t\in\mathbb{R}$. If $a(x,D)f\in\mathscr{F}L^q_{s,loc}(\mathbb{R}^d)$, then $f\in\mathscr{F}L^q_{s+m,loc}(\mathbb{R}^d)$ and for every $\chi\in C_0^\infty(\mathbb{R}^d)$ we have*

$$||\chi f||_{\mathscr{F}L^q_{s+m}}\le C_{s,t}(||\chi a(x,D)f||_{\mathscr{F}L^q_s}+||\chi f||_{\mathscr{F}L^q_t}). \tag{15}$$

In particular, if $a(x,D)=a(D)\in S^m$ is elliptic, then (15) *holds without* χ.

Proof. The ellipticity condition implies that there is an operator $b(x,D)\in\mathrm{Op}(S^{-m})$ such that

$$\chi f=b(x,D)a(x,D)\chi f+r(x,D)\chi f,$$

for some $r\in S^{-\infty}$. Hence $b(x,D)$ is continuous from $\mathscr{F}L^q_s(\mathbb{R}^d)$ to $\mathscr{F}L^q_{s+m}(\mathbb{R}^d)$ and $r(x,D)$ is continuous from $\mathscr{F}L^q_t(\mathbb{R}^d)$ to $\mathscr{F}L^q_{s+m}(\mathbb{R}^d)$. This implies

$$\begin{aligned}||\chi f||_{\mathscr{F}L^q_{s+m}}&\le ||b(x,D)a(x,D)\chi f||_{\mathscr{F}L^q_{s+m}}+||r(x,D)\chi f||_{\mathscr{F}L^q_{s+m}}\\&\le C_{s,t}(||\chi a(x,D)f||_{\mathscr{F}L^q_s}+||\chi f||_{\mathscr{F}L^q_t}).\end{aligned}$$

□

REMARK 2. The above propositions can be reformulated in the language of the symbol class $S^m_{loc}(X\times\mathbb{R}^d)$, where X is an open set in $\mathbb{R}^d$. This class is introduced in [14] as the starting point in the study of pseudo-differential operators on manifolds.

We say that a continuous linear map $A: C_0^\infty(X)\to C^\infty(X)$ is a pseudo-differential operator of order m in X, $A\in\Psi^m(X)$, if for arbitrary $\phi,\psi\in C_0^\infty(X)$ the operator $f\mapsto \phi A(\psi f)$ is in $\mathrm{Op}(S^m)$. For example, the restriction of $a(x,D)\in\mathrm{Op}(S^m)$ to X belongs to $\Psi^m(X)$.

According to [14, Proposition 18.1.22], every $A\in\Psi^m(X)$ can be decomposed as $A=A_0+A_1$ where $A_1\in\Psi^m(X)$ is properly supported and the kernel of A_0 is in C^∞. In that sense it is no essential restriction to require proper supports in the following statements.

THEOREM 1. *Assume that* $q\in[1,\infty]$, $s\in\mathbb{R}$, $f\in\mathscr{D}'(\mathbb{R}^d)$ *and* $(x_0,\xi_0)\in\mathbb{R}^d\times(\mathbb{R}^d\setminus 0)$. *Then the following conditions are equivalent:*

(1) $(x_0,\xi_0)\notin WF_{\mathscr{F}L^q_s}(f)$*;*

(2) $(x_0,\xi_0)\notin WF_{\mathscr{F}L^q}(Af)$ *for some properly supported* $A\in\Psi^s(X)$ *which is non-characteristic at* (x_0,ξ_0)*;*

(3) *there is a conic neighborhood* $U\times\Gamma_0$ *of* (x_0,ξ_0) *in* $X\times(\mathbb{R}^d\setminus 0)$ *such that* Bf *in* $\mathscr{F}L^q_{s-m,loc}(X)$ *for every properly supported pseudo-differential operator* $B\in\Psi^m(X)$ *with the symbol of class* $-\infty$ *outside* $U\times\Gamma_0$.

Proof. We follow the proof of Theorem 8.4.8 in [15] which concerns the classical wave-front set.

Assume that (1) holds. Then (11) holds with $\omega_0(\xi)=\langle\xi\rangle^s$ and for some conic neighborhood Γ of ξ_0 and for some $\varphi\in C_0^\infty(X)$ such that $\varphi(x_0)\neq 0$. Let $q(\xi)\in C^\infty$ be a homogeneous function of degree s for $|\xi|\geq 1$, with support in Γ. We define $A=\varphi q(D)\varphi$, where φ is from (11). Then $A\in\Psi^s(X)$ and (11) give

$$\|Af\|_{\mathscr{F}L^q}\leq C\|q(D)\varphi f\|_{\mathscr{F}L^q}<\infty.$$

Moreover, the symbol of A is $q(\xi)$ (mod $S^{-\infty}$) near x_0, which proves (2).

Now, assume that (3) holds. To prove (1) it is sufficient to find $\varphi\in C_0^\infty$ such that supp φ is sufficiently close to x_0 and a conic neighborhood Γ of ξ_0 such that (11) holds. Let $B\in\Psi^m(X)$ be fixed. By (3) we may choose $\phi,\psi\in C_0^\infty(X)$ and $q(\xi)\in C^\infty$ a homogeneous of degree m for $|\xi|\geq 1$, with $\psi=1$ in a neighborhood of supp ϕ and supp $\phi\times$ supp $q\subset U\times\Gamma_0$, so that $B=\psi q(D)\phi$.

By using the fact that

$$\mathscr{F}(q(D)\phi f)=\mathscr{F}(\psi q(D)\phi f)+\mathscr{F}((1-\psi)q(D)\phi f),$$

it follows from (3) that $\psi q(D)\phi f \in \mathscr{F}L^q_{s-m}$ and $(1-\psi)q(D)\phi$ is of order $-\infty$. Therefore (11) holds with $\omega_0(\xi) = \langle\xi\rangle^s$, $\Gamma = \Gamma_0$ and $\varphi = q(D)\phi$.

Finally, assume that (2) holds and choose a closed conic neighborhood $U \times \Gamma_0$ of (x_0,ξ_0) such that the symbol $a(x,\xi)$ of A satisfies $|a(x,\xi)| > c|\xi|^m$, if $\xi \in \Gamma_0$, $|\xi| > C$, for some constants $c, C > 0$. Therefore, for every B as in condition (3), we can find a properly supported $\tilde{B} \in \Psi^{m-s}(X)$ such that

$$B - \tilde{B}A \in \Psi^{-\infty}(X).$$

Hence $Bf - \tilde{B}Af \in C^\infty(X)$. By the assumption and Proposition 4 (2) it follows that $\tilde{B}Af \in \mathscr{F}L^q_{s-m,loc}(X)$. Therefore, $Bf \in \mathscr{F}L^q_{s-m,loc}(X)$ which completes the proof. □

COROLLARY 1. *Assume that $q \in [1,\infty]$, $s \in \mathbb{R}$, $f \in \mathscr{D}'(\mathbb{R}^d)$ and $(x_0,\xi_0) \in \mathbb{R}^d \times (\mathbb{R}^d \setminus 0)$. Then the following is true:*

(1) *If $(x_0,\xi_0) \notin WF_{\mathscr{F}L^q_s}(f)$ then $(x_0,\xi_0) \notin WF_{\mathscr{F}L^q_{s-m}}(Af)$ for every properly supported $A \in \Psi^m(X)$;*

(2) *if $(x_0,\xi_0) \notin WF_{\mathscr{F}L^q_{s-m}}(Af)$ for some properly supported $A \in \Psi^m(X)$ which is non-characteristic at (x_0,ξ_0), then $(x_0,\xi_0) \notin WF_{\mathscr{F}L^q_s}(f)$;*

(3) *there is a conical neighborhood $U \times \Gamma$ of (x_0,ξ_0) such that $(x,\xi) \notin WF_{\mathscr{F}L^q_s}(f)$ for every $(x,\xi) \in U \times \Gamma$ and for every $s \in \mathbb{R}$ if and only if $(x_0,\xi_0) \notin WF(f)$.*

Proof. We use the same idea as in the proof of [15, Theorem 8.4.8]. Here, we only present the proof of (3) and remark that this property is discussed in [19] via the so called superposition type wave-front sets.

Assume that U is a bounded neighborhood of x_0 and $\Gamma = \Gamma_{\xi_0}$, and that $(x,\xi) \notin WF_{\mathscr{F}L^q_s}(f)$ for each $s \in \mathbb{R}$ when $x \in U$ and $\xi \in \Gamma$. By compactness it follows that $|\chi f|_{\mathscr{F}L^{q,\Gamma}_s} < \infty$ for every $\chi \in C_0^\infty(U)$ such that $\chi(x_0) \neq 0$, and for every s. Let $\varphi \in C_0^\infty(U)$ and $0 \leq \psi \in C^\infty(\mathbb{R}^d)$ be such that

$$\psi(t\xi) = \psi(\xi), \qquad \text{when} \quad |\xi| \geq 1,\ t \geq 1,$$
$$\varphi(x_0) = \psi(\xi_0/|\xi_0|) = 1, \quad \text{and} \quad \operatorname{supp}\psi \subseteq \Gamma.$$

Then $\psi(D)\varphi(x)f \in \mathscr{F}L^q_s$ for every s by Theorem 1 (3). Hence $\psi(D)\varphi(x)f \in C^\infty$.

From the assumptions it follows that if $K(x-y) = (2\pi)^{-d/2}\widehat{\psi}(y-x)$ is the Schwartz kernel of $\psi(D)$, then $K \in \mathscr{S}'(\mathbb{R}^d) \cap C^\infty(\mathbb{R}^d \setminus 0)$. Furthermore, K turns rapidly to zero at infinity. It follows that $\psi(D)\varphi(x)f \in \mathscr{S}(\mathbb{R}^d)$. Hence $\psi(\xi)\mathscr{F}(\varphi f)(\xi)$ turns rapidly to zero at infinity. Consequently, $(x_0,\xi_0) \notin WF(f)$.

On the other hand, if $(x_0,\xi_0) \notin WF(f)$ then, by the definition, it immediately follows that there is a conic neighborhood $U \times \Gamma$ of (x_0,ξ_0) such that $(x,\xi) \notin WF_{\mathscr{F}L^q_s}(f)$ for every $(x,\xi) \in U \times \Gamma$ and for every $s \in \mathbb{R}$. The proof is complete. □

THEOREM 2. *Let $f \in \mathscr{D}'(\mathbb{R}^d)$ and let $A \in \Psi^m(X)$ be properly supported. Then we have the microlocal property*

$$WF_{\mathscr{F}L^q_{s-m}}(Af) \subset WF_{\mathscr{F}L^q_s}(f) \subset WF_{\mathscr{F}L^q_{s-m}}(Af) \cup \mathrm{Char} A,$$

where Char$A$ *denotes the set of characteristic points of A.*

Proof. The statement follows directly from Corollary 1. □

5. Modulation spaces

Assume that $\varphi \in \mathscr{S}'(\mathbb{R}^d)$ is fixed. Then the short-time Fourier transform of $f \in \mathscr{S}'(\mathbb{R}^d)$ with respect to φ is defined by

$$(V_\varphi f)(x,\xi) = \mathscr{F}(f \cdot \overline{\varphi(\,\cdot\, - x)})(\xi).$$

We note that the left-hand side makes sense, since it is the partial Fourier transform of the tempered distribution $F(x,y) = (f \otimes \overline{\varphi})(y, y-x)$ with respect to the y-variable.

We usually assume that $\varphi \in \mathscr{S}(\mathbb{R}^d)$, and in this case $V_\varphi f$ takes the form

$$V_\varphi f(x,\xi) = (2\pi)^{-d/2} \int f(y)\overline{\varphi(y-x)} e^{-i\langle y,\xi\rangle}\, dy. \tag{16}$$

Assume that $\omega \in \mathscr{P}(\mathbb{R}^{2d})$, $p,q \in [1,\infty]$, and that $\varphi \in \mathscr{S}(\mathbb{R}^d) \setminus 0$. Then the *modulation space* $M^{p,q}_{(\omega)}(\mathbb{R}^d)$ consists of all $f \in \mathscr{S}'(\mathbb{R}^d)$ such that

$$\begin{aligned} \|f\|_{M^{p,q}_{(\omega)}} &= \|f\|_{M^{p,q,\varphi}_{(\omega)}} \\ &\equiv \Big(\int \Big(\int |V_\varphi f(x,\xi)\omega(x,\xi)|^p\, dx \Big)^{q/p} d\xi \Big)^{1/q} < \infty \end{aligned} \tag{17}$$

(with obvious interpretation when $p = \infty$ or $q = \infty$).

If $\omega = 1$, then the notation $M^{p,q}(\mathbb{R}^d)$ is used instead of $M^{p,q}_{(\omega)}(\mathbb{R}^d)$. Moreover we set $M^p_{(\omega)}(\mathbb{R}^d) = M^{p,p}_{(\omega)}(\mathbb{R}^d)$ and $M^p(\mathbb{R}^d) = M^{p,p}(\mathbb{R}^d)$.

Locally, the spaces $\mathscr{F}L^q_{(\omega)}(\mathbb{R}^d)$ and $M^{p,q}_{(\omega)}(\mathbb{R}^d)$ coincide, in the sense that

$$\mathscr{F}L^q_{(\omega)}(\mathbb{R}^d) \cap \mathscr{E}'(\mathbb{R}^d) = M^{p,q}_{(\omega)}(\mathbb{R}^d) \cap \mathscr{E}'(\mathbb{R}^d)$$

(see Theorem 2.1 and Remark 4.4 in [21]). This result is extended in [19] in the context of wave-front sets.

Now we define wave-front sets with respect to modulation spaces, and claim that they coincide with wave-front sets of Fourier Lebesgue types. In particular, any property valid for wave-front set of Fourier Lebesgue type carry over to wave-front set of modulation space type (cf. [19, 20]).

Assume that $\varphi \in \mathscr{S}(\mathbb{R}^d)\setminus 0$, $\omega \in \mathscr{P}(\mathbb{R}^{2d})$, $\Gamma \subseteq \mathbb{R}^d\setminus 0$ is an open cone and $p,q\in[1,\infty]$ are fixed. For any $f\in\mathscr{S}'(\mathbb{R}^d)$, let

$$
\begin{aligned}
|f|_{M^{p,q,\Gamma}_{(\omega)}} &= |f|_{M^{p,q,\Gamma,\varphi}_{(\omega)}}\\
&\equiv \Big(\int_\Gamma\Big(\int_{\mathbb{R}^d}|V_\varphi f(x,\xi)\omega(x,\xi)|^p\,dx\Big)^{q/p}d\xi\Big)^{1/q}
\end{aligned}
\tag{18}
$$

(with obvious interpretation when $p=\infty$ or $q=\infty$). We note that $|\cdot|_{M^{p,q,\Gamma}_{(\omega)}}$ defines a semi-norm on $\mathscr{S}'$ which might attain the value $+\infty$. If $\Gamma=\mathbb{R}^d\setminus 0$ and $q<\infty$, then $|f|_{M^{p,q,\Gamma}_{(\omega)}}$ agrees with the modulation space norm $\|f\|_{M^{p,q}_{(\omega)}}$ of f.

We let $\Theta_{M^{p,q}_{(\omega)}}(f)=\Theta_{M^{p,q,\varphi}_{(\omega)}}(f)$ be the sets of all $\xi\in\mathbb{R}^d\setminus 0$ such that $|f|_{M^{p,q,\Gamma,\varphi}_{(\omega)}}<\infty$, for some $\Gamma=\Gamma_\xi$. We also let $\Sigma_{M^{p,q,\varphi}_{(\omega)}}(f)$ be the complement of $\Theta_{M^{p,q,\varphi}_{(\omega)}}(f)$ in $\mathbb{R}^d\setminus 0$. Then $\Theta_{M^{p,q,\varphi}_{(\omega)}}(f)$ and $\Sigma_{M^{p,q,\varphi}_{(\omega)}}(f)$ are open respectively closed subsets in $\mathbb{R}^d\setminus 0$.

THEOREM 3. *[19] Assume that $p,q\in[1,\infty]$, $\varphi\in C_0^\infty(\mathbb{R}^d)\setminus 0$, $\chi\in C^\infty(\mathbb{R}^d)$, and that $\omega\in\mathscr{P}(\mathbb{R}^{2d})$. Also assume that $f\in\mathscr{E}'(\mathbb{R}^d)$. Then*

$$
\Theta_{M^{p,q,\varphi}_{(\omega)}}(f)=\Theta_{\mathscr{F}L^q_{(\omega)}}(f),\qquad \Sigma_{M^{p,q,\varphi}_{(\omega)}}(f)=\Sigma_{\mathscr{F}L^q_{(\omega)}}(f),
\tag{19}
$$

and

$$
\Sigma_{M^{p,q,\varphi}_{(\omega)}}(\chi f)\subseteq\Sigma_{M^{p,q,\varphi}_{(\omega)}}(f),\qquad \Sigma_{\mathscr{F}L^q_{(\omega)}}(\chi f)\subseteq\Sigma_{\mathscr{F}L^q_{(\omega)}}(f).
\tag{20}
$$

COROLLARY 2. *[19] Assume that $p,q\in[1,\infty]$, $f\in\mathscr{D}'(\mathbb{R}^d)$, $\varphi\in C_0^\infty(\mathbb{R}^d)\setminus 0$, $x_0,y_0\in\mathbb{R}^d$, $\xi_0\in\mathbb{R}^d\setminus 0$ and that $\omega\in\mathscr{P}(\mathbb{R}^{2d})$. Also let $\omega_0(\xi)=\omega(y_0,\xi)$. Then the following conditions are equivalent:*

(1) *there exists an open cone $\Gamma=\Gamma_{\xi_0}$ and $\chi\in C_0^\infty(\mathbb{R}^d)$ such that $\chi(x_0)\neq 0$, and $|\chi f|_{M^{p,q,\Gamma,\varphi}_{(\omega)}}<\infty$ (i. e. $\xi_0\in\Theta_{M^{p,q}_{(\omega)}}(\chi f)$);*

(2) *there exists an open cone $\Gamma=\Gamma_{\xi_0}$ and $\chi\in C_0^\infty(\mathbb{R}^d)$ such that $\chi(x_0)\neq 0$, and $|\chi f|_{\mathscr{F}L^q_{(\omega)}}<\infty$ (i. e. $\xi_0\in\Theta_{\mathscr{F}L^q_{(\omega)}}(\chi f)$);*

(3) *there exists an open cone $\Gamma=\Gamma_{\xi_0}$ and $\chi\in C_0^\infty(\mathbb{R}^d)$ such that $\chi(x_0)\neq 0$, and $|\chi f|_{\mathscr{F}L^q_{(\omega_0)}}<\infty$ (i. e. $\xi_0\in\Theta_{\mathscr{F}L^q_{(\omega_0)}}(\chi f)$).*

The following definition makes sense in view of Corollary 2.

DEFINITION 2. *[19] Assume that $p,q\in[1,\infty]$, $f\in\mathscr{D}'(\mathbb{R}^d)$ and $\omega\in\mathscr{P}(\mathbb{R}^{2d})$. The wave-front set $WF_{M^{p,q}_{(\omega)}}(f)$ with respect to $M^{p,q}_{(\omega)}(\mathbb{R}^d)$ consists of all pairs (x_0,ξ_0) in $\mathbb{R}^d\times(\mathbb{R}^d\setminus 0)$ such that $\xi_0\in\Sigma_{M^{p,q,\varphi}_{(\omega)}}(\chi f)$ holds for each $\chi\in C_0^\infty(\mathbb{R}^d)$ such that $\chi(x_0)\neq 0$.*

By Corollary 2 it follows that

$$WF_{M^{p_1,q}_{(\omega_1)}}(f) = WF_{M^{p_2,q}_{(\omega_2)}}(f)$$

when $p_1, p_2 \in [1,\infty]$ and

$$C^{-1}\omega_2(x,\xi)\langle x\rangle^{-N} \le \omega_1(x,\xi) \le C\omega_2(x,\xi)\langle x\rangle^{N},$$

for some constants C and N. By the same corollary it follows that the following holds.

PROPOSITION 6. *[19] Assume that $p,q \in [1,\infty]$, $f \in \mathscr{D}'(\mathbb{R}^d)$, $\omega_0 \in \mathscr{P}(\mathbb{R}^d)$ and $\omega \in \mathscr{P}(\mathbb{R}^{2d})$ are such that $\omega_0(\xi) = \omega(y_0,\xi)$ for some $y_0 \in \mathbb{R}^d$. Then*

$$WF_{\mathscr{F}L^q_{(\omega_0)}}(f) = WF_{\mathscr{F}L^q_{(\omega)}}(f) = WF_{M^{p,q}_{(\omega)}}(f).$$

We also note that if $f \in \mathscr{E}'(\mathbb{R}^d)$, then it follows from Corollary 2 that

$$f \in \mathscr{F}L^q_{(\omega_0)} \quad \Longleftrightarrow \quad f \in M^{p,q}_{(\omega)} \Longleftrightarrow WF_{\mathscr{F}L^q_{(\omega_0)}}(f) = WF_{M^{p,q}_{(\omega)}}(f) = \emptyset.$$

In particular, we recover Theorem 2.1 and Remark 4.4 in [21].

6. Pseudo-differential operators, an extenstion

In this section we present a part of our results from [19] related to the action of more general classes of pseudo-differential operators. The presentation of this section follows the first author's lecture given at the conference "Pseudo-differential operators with related topics II".

Assume that $\rho, m \in \mathbb{R}$ are fixed. Recall, $S^m_{\rho,0}(\mathbb{R}^{2d})$ is the set of all $a \in C^\infty(\mathbb{R}^{2d})$ such that for each pairs of multi-indices α and β, there is a constant $C_{\alpha,\beta}$ such that $|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \le C_{\alpha,\beta}\langle\xi\rangle^{m-\rho|\beta|}$. Usually we assume that $0 < \rho \le 1$. Clearly, $S^m_{1,0} = S^m$ of Section 5.

More generally, assume that $\omega_0 \in \mathscr{P}_\rho(\mathbb{R}^{2d})$. Then we recall from [19] that $S^\rho_{(\omega_0)}(\mathbb{R}^{2d})$ consists of all $a \in C^\infty(\mathbb{R}^{2d})$ such that

$$|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \le C_{\alpha,\beta}\omega_0(x,\xi)\langle\xi\rangle^{-\rho|\beta|}.$$

(Cf. Section 18.4–18.6 in [14].) Clearly, $S^\rho_{(\omega_0)} = S^m_{\rho,0}(\mathbb{R}^{2d})$ when $\omega_0(x,\xi) = \langle\xi\rangle^m$.

The next result is a special case of Theorem 4.2 in [30].

PROPOSITION 7. *Assume that $p,q,p_j,q_j \in [1,\infty]$ for $j = 1,2$, satisfy*

$$1/p_1 - 1/p_2 = 1/q_1 - 1/q_2 = 1 - 1/p - 1/q, \quad q \le p_2, q_2 \le p.$$

Also assume that $\omega \in \mathscr{P}(\mathbb{R}^{2d} \oplus \mathbb{R}^{2d})$ and $\omega_1, \omega_2 \in \mathscr{P}(\mathbb{R}^{2d})$ satisfy

$$\frac{\omega_2(x,\xi+\eta)}{\omega_1(x+z,\xi)} \le C\omega(x,\xi,\eta,z) \tag{21}$$

for some constant C. If $a \in M^{p,q}_{(\omega)}(\mathbb{R}^{2d})$*, then* $a(x,D)$ *from* $\mathscr{S}(\mathbb{R}^d)$ *to* $\mathscr{S}'(\mathbb{R}^d)$ *extends uniquely to a continuous mapping from* $M^{p_1,q_1}_{(\omega_1)}(\mathbb{R}^d)$ *to* $M^{p_2,q_2}_{(\omega_2)}(\mathbb{R}^d)$*.*

For the later convenience we set

$$\omega_{s,\rho}(x,\xi,\eta,z) = \omega(x,\xi,\eta,z)\langle x\rangle^{s_4}\langle\eta\rangle^{s_3}\langle\xi\rangle^{\rho s_2}\langle z\rangle^{s_1}, \tag{22}$$

when $\rho \in \mathbb{R}$ and $s \in \mathbb{R}^4$.

DEFINITION 3. *[19] Assume that* $s \in \mathbb{R}^4$ *is such that* $s_2 \geq 0$*,* $\rho \in \mathbb{R}$*,* $\omega \in \mathscr{P}(\mathbb{R}^{4d})$*, and that* $\omega_{s,\rho}$ *is given by* (22)*. Then the symbol class* $\mho^{s,\rho}_{(\omega)}(\mathbb{R}^{2d})$ *is the set of all* $a \in \mathscr{S}'(\mathbb{R}^{2d})$ *which satisfy*

$$\partial^\alpha_\xi a \in M^{\infty,1}_{(\omega_{u(s,\alpha),\rho})}(\mathbb{R}^{2d}), \qquad u(s,\alpha) = (s_1,|\alpha|,s_3,s_4),$$

for each multi-indices α *such that* $|\alpha| \leq 2s_2$*.*

It follows from the following lemma that the symbol classes $\mho^{s,\rho}_{(\omega)}(\mathbb{R}^{2d})$ are interesting also in the classical theory.

LEMMA 1. *[19] Assume that* $\rho \in [0,1]$*,* $\omega \in \mathscr{P}_0(\mathbb{R}^{4d})$ *and* $\omega_0 \in \mathscr{P}_\rho(\mathbb{R}^{2d})$ *satisfy*

$$\omega_0(x,\xi) = \omega(x,\xi,0,0).$$

Then the following conditions are equivalent:

(1) $a \in S^\rho_{(\omega_0)}(\mathbb{R}^{2d})$*;*

(2) $\omega_0^{-1} a \in S^0_{\rho,0}(\mathbb{R}^{2d})$*;*

(3) $\langle x\rangle^{-s_4} a \in \bigcap_{s_1,s_2,s_3 \geq 0} \mho^{s,\rho}_{(\omega)}(\mathbb{R}^{2d})$.

REMARK 3. Let $H^\infty_s(\mathbb{R}^d)$ be the Sobolev space of distributions with $s \in \mathbb{R}$ derivatives in $L^\infty(\mathbb{R}^d)$, i. e. $H^\infty_s(\mathbb{R}^d)$ consists of all $f \in \mathscr{S}'$ such that $\mathscr{F}^{-1}(\langle\cdot\rangle^s \hat f)$ belongs to $L^\infty(\mathbb{R}^d)$. Then it is easily seen that $\bigcap_{s\geq 0} H^\infty_s(\mathbb{R}^{2d}) = S^0_{0,0}(\mathbb{R}^{2d})$, which is the set of all smooth functions on $\mathbb{R}^{2d}$ which are bounded together with all their derivatives. Hence, (3.2) in [27] and Theorem 4.4 in [28] imply that

$$\bigcap_{s\geq 0} M^{\infty,1}_{(v_s)}(\mathbb{R}^{2d}) = S^0_{0,0}(\mathbb{R}^{2d}), \qquad v_s(x,\xi,\eta,z) = \langle\eta\rangle^s\langle z\rangle^s.$$

By Theorem 2.2 in [29] it follows more generally that

$$\bigcap_{s\geq 0} M^{\infty,1}_{(v_{\rho,s})}(\mathbb{R}^{2d}) = S^{-\rho}_{0,0}(\mathbb{R}^{2d}), \quad v_{\rho,s}(x,\xi,\eta,z) = \langle\xi\rangle^\rho\langle\eta\rangle^s\langle z\rangle^s. \tag{23}$$

The following definition of the characteristic set is different from that given in Section 5 and in [14, Section 18.1]. Here, it is defined for symbols which are not polyhomogeneous while in the case of polyhomogeneous symbols, our sets of characteristic points are smaller than the set of characteristic points in Section 5 and [14].

DEFINITION 4. *[19] Assume that $\rho \in (0,1]$ and $\omega \in \mathscr{P}_\rho(\mathbb{R}^{2d})$. For each open cone $\Gamma \subseteq \mathbb{R}^d \setminus 0$, open set $U \subseteq \mathbb{R}^d$ and real number $R > 0$, let*

$$\Omega_{U,\Gamma,R} \equiv \{ (x,\xi) \, ; \, x \in U, \; \xi \in \Gamma, \; |\xi| > R \}.$$

Also let $\Xi_{U,\Gamma,R,\rho}$ be the set of all $c \in S^0_{\rho,0}(\mathbb{R}^{2d})$ such that $c = 1$ on $\Omega_{U,\Gamma,R}$.

The pair $(x_0,\xi_0) \in \mathbb{R}^d \times (\mathbb{R}^d \setminus 0)$ is called non-characteristic *for $a \in S^\rho_{(\omega)}(\mathbb{R}^{2d})$ (with respect to ω), if there is a conical neighborhood Γ of ξ_0, a neighborhood U of x_0, a real number $R > 0$, and elements $b \in S^\rho_{(\omega^{-1})}(\mathbb{R}^{2d})$, $c \in \Xi_{U,\Gamma,R,\rho}$ and $h \in S^{-\rho}_{\rho,0}(\mathbb{R}^{2d})$ such that*

$$b(x,\xi)a(x,\xi) = c(x,\xi) + h(x,\xi), \qquad (x,\xi) \in \mathbb{R}^{2d}.$$

The pair (x_0,ξ_0) in $\mathbb{R}^d \times (\mathbb{R}^d \setminus 0)$ is called characteristic *for a (with respect to $\omega \in \mathscr{P}_\rho(\mathbb{R}^{2d})$), if it is* not *non-characteristic for a with respect to $\omega \in \mathscr{P}_\rho(\mathbb{R}^{2d})$. The* set of characteristic points *(the characteristic set), for $a \in S^\rho_{(\omega)}(\mathbb{R}^{2d})$ with respect to ω, is denoted by* $\operatorname{Char}(a) = \operatorname{Char}_{(\omega)}(a)$.

In order to state the results we use the convention

$$(\vartheta_1,\vartheta_2) \lesssim (\omega_1,\omega_2) \tag{24}$$

when $\omega_j, \vartheta_j \in \mathscr{P}(\mathbb{R}^d)$ for $j = 1,2$ satisfy $\vartheta_j \le C\omega_j$ for some constant C.

If instead $\omega_j \in \mathscr{P}(\mathbb{R}^{2d})$, then it follows that

$$\omega_j(x,\xi_1+\xi_2) \le C\omega_j(x,\xi_1)\langle\xi_2\rangle^{t_j}, \tag{25}$$

for some constants $C > 0$ and t_j, $j = 1,2$, independent of $x,\xi_1,\xi_2 \in \mathbb{R}^d$. Then it is necessary that t_1 and t_2 are non-negative. Here we let $\omega_{s,\rho}$ to be as in (22) and we use the notation $(\omega_1,\omega_2) \preccurlyeq \omega$ when (21) holds for some constant C.

THEOREM 4. *[19] Assume that $0 < \rho \le 1$, $\omega_j, \vartheta_j \in \mathscr{P}(\mathbb{R}^{2d})$ for $j = 1,2$, $\omega \in \mathscr{P}_\rho(\mathbb{R}^{4d})$ satisfy* (24)*, and that $(\omega_1,\omega_2) \preccurlyeq \omega_{s,\rho}$ and $(\vartheta_1,\vartheta_2) \preccurlyeq \omega_{s,\rho}$, for some $s \in \mathbb{R}^4$, are such that*

$$s_1 \ge 0, \quad s_2 \in \mathbb{N}, \quad s_3 > t_1 + t_2 + 2d,$$

where t_1 and t_2 are chosen such that (25) *holds. Also assume that $a \in \mho^{s,\rho}_{(\omega)}(\mathbb{R}^{2d})$ and that $f \in M^\infty_{(\vartheta_1)}(\mathbb{R}^d)$. Then*

$$WF_{\mathscr{F}L^q_{(\omega_2)}}(a(x,D)f) \subseteq WF_{\mathscr{F}L^q_{(\omega_1)}}(f).$$

We refer to [19] for a detailed proof and give here only a hint.

The first part of the proof concerns the contribution to the wave-front set of $a(x,D)f$ at a particular point x_0, when the support of f is far away from x_0. It can be proved that this contribution is limited. The precise formulation and the proof can be found in [19].

To finish the proof of Theorem 4 it remains to describe properties of the wave-front set of $a(x,D)f$ at a fixed point when f is concentrated to that point. In these considerations it is natural to assume that involved weight functions satisfy

$$\begin{aligned} &\omega_j(x,\xi) = \omega_j(\xi), && \vartheta_j(x,\xi) = \vartheta_j(\xi), \quad j = 1,2, \\ &\frac{\omega_2(\xi+\eta)}{\omega_1(\xi)} \le C\omega(\xi,\eta,z), && \frac{\vartheta_2(\xi+\eta)}{\vartheta_1(\xi)} \le C\omega(\xi,\eta,z) \end{aligned} \tag{26}$$

and we set

$$\omega_s(x,\xi,\eta,z) = \langle x\rangle^{s_4}\langle\eta\rangle^{s_3}\omega(\xi,\eta,z), \quad s \in \mathbb{R}^4. \tag{27}$$

We also note that

$$\omega_j(\xi_1+\xi_2) \le C\omega_j(\xi_1)\langle\xi_2\rangle^{t_j}, \quad j = 1,2, \tag{28}$$

for some real numbers t_1 and t_2.

The precise result which we need is the following.

PROPOSITION 8. *[19] Assume that* $q \in [1,\infty]$, $s \in \mathbb{R}^4$, $t_j \in \mathbb{R}$, $\omega \in \mathscr{P}(\mathbb{R}^{3d})$, $\omega_j, \vartheta_j \in \mathscr{P}(\mathbb{R}^d)$ *for* $j = 1,2$ *and* $\omega_s \in \mathscr{P}(\mathbb{R}^{4d})$ *fulfill* $(\vartheta_1,\vartheta_2) \lesssim (\omega_1,\omega_2)$, (26)–(28), $s_4 > d$ *and*

$$s_3 > t_1 + t_2 + 2d.$$

Also assume that $a \in M^{\infty,1}_{(\omega_s)}(\mathbb{R}^{2d})$ *and* $f \in M^{\infty}_{(\vartheta_1)}(\mathbb{R}^d) \bigcap \mathscr{E}'(\mathbb{R}^d)$. *Then the following is true:*

(1) *if* Γ_1 *is an open conical neighborhood of* $\eta_0 \in \mathbb{R}^d \setminus 0$, *then there is an open conical neighborhood* Γ_2 *of* η_0 *which only depends on* Γ_1 *such that*

$$|a(x,D)f|_{\mathscr{F}L^{q,\Gamma_2}_{(\omega_2)}} \le C\|a\|_{M^{\infty,1}_{(\omega_s)}} |f|_{\mathscr{F}L^{q,\Gamma_1}_{(\omega_1)}},$$

for some constant C *which is independent of* $a \in M^{\infty,1}_{(\omega_s)}(\mathbb{R}^{2d})$ *and* $f \in M^{\infty}_{(\omega_1)}(\mathbb{R}^d)$;

(2) $WF_{\mathscr{F}L^q_{(\omega_2)}}(a(x,D)f) \subseteq WF_{\mathscr{F}L^q_{(\omega_1)}}(f)$.

We note that by Proposition 7 it follows that $a(x,D)f$ in Proposition 8 makes sense as an element in $M^{\infty}_{(\vartheta_2)}$. This space contains each space $M^{p,q}_{(\omega_2)}$.

Let $t \in \mathbb{R}$, and let $\mho^{s,\rho,t}_{(\omega)}(\mathbb{R}^{2d})$ be as $\mho^{s,\rho}_{(\omega)}(\mathbb{R}^{2d})$, after $\omega_{s,\rho}(x,\xi,\eta,z)$ has been replaced by

$$\omega_{s,t,\rho}(x,\xi,\eta,z) = \omega_{s,\rho}(x+tz,\xi+t\eta,\eta,z),$$

in the definition of $\mho^{s,\rho}_{(\omega)}(\mathbb{R}^{2d})$. Then it follows from Proposition 1.7 in [30] that if $a \in \mho^{s,\rho}_{(\omega)}(\mathbb{R}^{2d})$, then Theorem 4 remains valid after $\omega(x,\xi,\eta,z)$ has been replaced by $\omega(x+tz,\xi+t\eta,\eta,z)$ and $a(x,D)$ has been replaced by $a_t(x,D)$.

We present a counter result of Theorem 4 for pseudo-differential operators with smooth symbols.

Assume now that the involved weight functions satisfy

$$\frac{\omega_2(x,\xi)}{\omega_1(x,\xi)} \le C\omega_0(x,\xi), \tag{29}$$

for some constant C.

THEOREM 5. *[19] Assume that* $0 < \rho \le 1$, $\omega_1, \omega_2 \in \mathscr{P}(\mathbb{R}^{2d})$ *and* $\omega_0 \in \mathscr{P}_\rho(\mathbb{R}^{2d})$ *satisfy* (29). *If* $a \in S^{\rho}_{(\omega_0)}$ *and* $q \in [1,\infty]$, *then*

$$WF_{\mathscr{F}L^q_{(\omega_2)}}(a(x,D)f) \subseteq WF_{\mathscr{F}L^q_{(\omega_1)}}(f), \qquad f \in \mathscr{S}'(\mathbb{R}^d).$$

We also have the following counter result to Theorem 5. Here it is natural to assume that the involved weight functions satisfy

$$C^{-1}\omega_0(x,\xi) \le \frac{\omega_2(x,\xi)}{\omega_1(x,\xi)}, \tag{30}$$

for some constant C, instead of (29).

THEOREM 6. *[19] Assume that* $0 < \rho \le 1$, $\omega_1, \omega_2 \in \mathscr{P}(\mathbb{R}^{2d})$ *and* $\omega_0 \in \mathscr{P}_\rho(\mathbb{R}^{2d})$ *satisfy* (30). *If* $a \in S^{\rho}_{(\omega_0)}$ *and* $q \in [1,\infty]$, *then*

$$WF_{\mathscr{F}L^q_{(\omega_1)}}(f) \subseteq WF_{\mathscr{F}L^q_{(\omega_2)}}(a(x,D)f) \bigcup \mathrm{Char}_{(\omega_0)}(a), \qquad f \in \mathscr{S}'(\mathbb{R}^d).$$

REMARK 4. We note that the statements in Theorems 5 and 6 are not true if the assumption $\rho > 0$ is replaced by $\rho = 0$. In fact, we only prove this in the case $\omega_0 = 1$ and $\omega_1 = \omega_2$. The general case is left for the reader.

Let $a(x,\xi) = e^{-i\langle x_0,\xi\rangle}$ for some fixed $x_0 \in \mathbb{R}^d$ and choose α in such way that $f_\alpha(x) = \delta_0^{(\alpha)}$ does not belong to $\mathscr{F}L^q_{(\omega_1)}$. Since

$$(a(x,D)f_\alpha)(x) = f_\alpha(x-x_0),$$

straight-forward computations implies that, for some closed cone $\Gamma \in \mathbb{R}^d \setminus 0$,

$$WF_{\mathscr{F}L^q_{(\omega_1)}}(f) = \{\,(0,\xi)\,;\,\xi \in \Gamma\,\};$$

$$WF_{\mathscr{F}L^q_{(\omega_1)}}(a(x,D)f) = \{\,(x_0,\xi)\,;\,\xi \in \Gamma\,\},$$

which are not overlapping when $x_0 \neq 0$.

Next we apply Theorems 5 and 6 on hypoelliptic operators. Assume that $a \in C^\infty(\mathbb{R}^{2d})$ is bounded by a polynomial. Then $a(x,D)$ is called *hypoelliptic*, if there are positive constants C, $C_{\alpha,\beta}$, N, ρ and R such that

$$
\begin{gathered}
|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \le C_{\alpha,\beta} |a(x,\xi)| \langle \xi \rangle^{-\rho|\beta|}, \quad \text{and} \\
C\langle \xi \rangle^{-N} \le |a(x,\xi)| \quad \text{when} \quad x \in \mathbb{R}^d, \quad \text{and} \quad |\xi| > R.
\end{gathered} \tag{31}
$$

(See e. g. [1,14].) We note that if $a(x,D)$ is hypoelliptic, $\chi \in C_0^\infty(\mathbb{R}^d)$ and if (31) is fulfilled, then $\chi(x)a(x,\xi) \in S^\rho_{(\omega)}(\mathbb{R}^{2d})$, where

$$
\omega(x,\xi) = \omega_a(x,\xi) = (\langle \xi \rangle^{-2N} + |a(x,\xi)|^2)^{1/2} \in \mathscr{P}_\rho(\mathbb{R}^{2d}).
$$

Furthermore, since $\mathrm{Char}_{(\omega_a)}(a) = \emptyset$, by definitions, the following result is an immediate consequence of Theorems 5 and 6.

THEOREM 7. *Assume that $a \in C^\infty(\mathbb{R}^{2d})$ is such that $a(x,D)$ is hypoelliptic, $q \in [1,\infty]$, and that $\omega_1, \omega_2 \in \mathscr{P}(\mathbb{R}^{2d})$ satisfy*

$$
C^{-1}\frac{\omega_2(x,\xi)}{\omega_1(x,\xi)} \le \omega_a(x,\xi) \le C\frac{\omega_2(x,\xi)}{\omega_1(x,\xi)}, \tag{32}
$$

for some constant C which is independent of $(x,\xi) \in \mathbb{R}^{2d}$. If $f \in \mathscr{S}'(\mathbb{R}^d)$, then

$$
WF_{\mathscr{F}L^q_{(\omega_2)}}(a(x,D)f) = WF_{\mathscr{F}L^q_{(\omega_1)}}(f).
$$

Note that for any hypoelliptic operator, we may choose the symbol class which contains the symbol of the operator in such way that the corresponding set of characteristic points is empty. Consequently, in the view of Theorem 7, it follows that hypoelliptic operators preserve the wave-front sets, as it should.

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HYPOELLIPTICITY, SOLVABILITY AND CONSTRUCTION OF SOLUTIONS WITH PRESCRIBED SINGULARITIES FOR SEVERAL CLASSES OF PDE HAVING SYMPLECTIC CHARACTERISTICS*

Abstract. This paper deals with the hypoellipticity, local solvability and construction of solutions with prescribed singularities for several classes of PDE having double symplectic characteristics. We not only propose a short survey but we investigate several instructive model examples as well. As our results are obtained in the C^∞ category, it is interesting to study the same operators in the Gevrey category too.

1. Several definitions and formulation of the main results

1. In the paper under consideration we denote by $L^m(X)$ the set of all classical scalar properly supported pseudodifferential operators of order m and $D'(X)$ stands for the set of all Schwartz distributions on the smooth manifold X. As usual the closed conic in ξ set $WF(u)$, $u \in D'(X)$ (wave front set of u) is defined by

$$WF(u) = \left\{\rho \in T^*X\backslash 0 : a \in L^0(X),\, a(x,D)u \in C^\infty(X) \Rightarrow a^0(\rho) = 0\right\}.$$

We have denoted by $a^0(\rho) = \tau(a)$ the principal symbol of the operator $a(x,D) \in L^0(X)$.

The s-wave front set of $u \in D'(X)$, $s \in \mathbb{R}^1$ is given by

$$WF_s(u) = \left\{\rho \in T^*(X)\backslash 0 : a \in L^0(X),\, a(x,D)u \in H^s(X) \Rightarrow a^0(\rho) = 0\right\}.$$

Certainly, $\rho = (x,\xi)$, $\xi \neq 0$ and $WF_s(u)$ is a closed conical in ξ set.

Evidently, $s' < s \Rightarrow WF_s(u) \subset WF_{s'}(u)$.

Let $V \subset T^*(X)\backslash 0$ be an open conical in ξ set and N is a closed cone in ξ contained in $T^*(X)\backslash 0$, $N \subset V$.

THEOREM 1. *[11] Assume that the operator $P \in L^m(X)$, $s' < s$. Suppose that it does not exist a function $u \in H^{s'}(X)$ such that*

$$(*) \qquad V \cap WF(Pu) = \emptyset, \quad V \cap WF_s(u) = V \cap WF(u) = N.$$

Then there exists $\rho^0 \in N$, pseudodifferential operators ψ, φ, $\varphi' \in L^0(X)$, cone supp $\varphi \subset V\backslash N$, cone supp $\varphi' \subset V$, $\psi(\rho) \equiv 1$ in a tiny neighborhood of ρ^0, $C = const > 0$, $\mu \in \mathbb{Z}_+$ and such that

$$(1) \qquad \|\psi w\|_s \leq C\left[\|\varphi' P w\|_\mu + \|\varphi w\|_\mu + \|w\|_{s'}\right], \quad \forall w \in C_0^\infty(X).$$

*It is a pleasure to dedicate this paper to Prof. Luigi Rodino on the occasion of his 60th birthday.

REMARK 1. Instead of (1) we can write

$$\|w\|_s \le C\left[\|Pw\|_\mu + \|Aw\|_0 + \|w\|_{s'}\right], \quad \forall w \in C_0^\infty(X), \tag{2}$$

where the full symbol of A is identically 0 near ρ^0.

It is evident that $N \subset \operatorname{Char} P = \{\rho \in T^*X\backslash 0 : \ p_m^0(\rho) = 0\}$ is the non-trivial case in Theorem 1. In fact, if $\rho^0 \in N$, $\rho^0 \notin \operatorname{Char} P$ then $p_m^0(\rho^0) \neq 0$ and consequently $\rho^0 \notin WF(u)$, i. e. $(*)$ does not hold.

This way we conclude that the problem of existence of solution of the equation $Pu = f$ with given (prescribed singularity) $(*)$ is reduced to the violation of the a-priori estimate (1) /(2)/ for some $w \in C_0^\infty(X)$.

We shall illustrate Theorem 1 by the following example.

EXAMPLE 1. Let $P \in L^m(X)$, $p_m^0(\rho^0) = \nabla_{x,\xi} p_m^0(\rho^0) = 0$, $\rho^0 \in T^*X\backslash 0$ and let

$$C_{2m-1}^0(\rho) \le -\alpha|\rho - \rho^0|^2, \quad \alpha = \text{const} > 0, \quad \forall \rho^0 \in V, \tag{3}$$

where V is an open conical neighborhood of ρ^0, while

$$C_{2m-1}^0 = \tau[P^*, P] = \frac{1}{i}\left\{\overline{p_m^0},\ p_m^0\right\} = 2\Im \sum_{j=1}^n \frac{\partial \overline{p_m^0}}{\partial \xi_j}\frac{\partial p_m^0}{\partial x_j}.$$

Then it is proved in [13] that the L_2 adjoint operator P^* of P is locally and even microlocally non solvable at $x_0(\rho^0)$. Applying Theorem 1 we conclude that for each closed cone $N \subset \operatorname{Char} P \cap V$ and for each $s' < s$ one can find a distribution $u \in H^{s'}(X)$ for which $WF(Pu) \cap V = \emptyset$, $V \cap WF(u) = V \cap WF_s(u) = N$.

Assume now that again $p_m^0(\rho^0) = \nabla_{x,\xi} p_m^0(\rho^0) = 0$ but contrary to (3) the following inequality holds:

$$C_{2m-1}^0(\rho) \ge C|\nabla_{x,\xi} p_m^0(\rho)|^2, \quad \forall \rho \in V,\ C = \text{const} > 0.$$

More precisely, $\left|\nabla_{x,\xi} p_m^0(\rho)\right|^2 = \left|\nabla_x p_m^0\right|^2 |\xi|^{-1} + \left|\nabla_\xi p_m^0\right|^2 |\xi|$. Suppose also that the spectrum of the Hamilton map (fundamental matrix) $F_{C_{2m-1}^0}(\rho^0)$ is non trivial.

Then the operator P is microlocally hypoelliptic at ρ^0 with sharp loss of regularity 1 and without any importance of the lower order terms. Thus, $Pu \in H^s_{\mathrm{mcl}}(\rho^0) \Rightarrow u \in H^{s+m-1}_{\mathrm{mcl}}(\rho^0)$ $\left(u \in H^s_{\mathrm{mcl}}(\rho^0) \Leftrightarrow \rho^0 \notin WF_s(\rho^0)\right)$. The proof can be found in [10]. Below we shall discuss the fundamental matrix and its spectrum.

2. Consider now the symplectic manifold Σ, $\operatorname{codim}\Sigma = 2\nu < 2n$, written in canonical coordinates:

$$\Sigma = \{(x,\xi), \xi \neq 0 :\ x_1 = \cdots = x_\nu = 0,\ \xi_1 = \cdots = \xi_\nu = 0\}, 1 \le \nu < n, \tag{i}$$

and suppose that for $\nu \ge 2$ the principal symbol p_m^0 of $P \in L^m(X)$ vanishes of sharp order 2 on Σ and $p_m^0(\rho) \in \Gamma = \left\{z \in \mathbb{C}^1 : |\Im z| \le \gamma \Re z\right\}, \gamma = \text{const} > 0$, i. e. $p_m^0(\rho)$ describes a closed angle in $\mathbb{C}^1$ with vertex at 0 and an opening strictly less than π.

This is the definition of the subprincipal symbol:

$$p'_{m-1}(\rho) = p_{m-1}(\rho) + \frac{i}{2}\sum_{j=1}^{n}\frac{\partial^2 p_m^0(\rho)}{\partial x_j \partial \xi_j}.$$

It is well known that p'_{m-1} is symplectic invariant on Σ.

From geometrical reasons it is clear that one can define winding number of $p_m^0(\rho)$ on Σ.

(ii) We shall suppose that in the special case $\nu = 1$ the winding number of p_m^0 is 0. Then with some constant $c \neq 0$ we have $cp_m^0 \in \Gamma$. The Hamilton map (fundamental matrix) $F_{p_m^0}$ is symplectic invariant on Σ and is defined by

$$F_{p_m^0} = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{2}p^{0''}_{m,\xi\xi} & p^{0''}_{m,x\xi} \\ {}^t p^{0''}_{m,x\xi} & \frac{1}{2}p^{0''}_{m,xx} \end{pmatrix} = \begin{pmatrix} {}^t p^{0''}_{m,x\xi} & \frac{1}{2}p^{0''}_{m,xx} \\ -\frac{1}{2}p^{0''}_{m,\xi\xi} & -p^{0''}_{m,x\xi} \end{pmatrix},$$

I_n being the unit matrix in $\mathbb{C}^n$.

The eigenvalues of $F_{p_m^0(\rho)}$, $\rho \in \Sigma$ are denoted by $\mu_j(\rho)$, $1 \leq j \leq 2\nu$; $\mu_j \in i\Gamma$, $1 \leq j \leq \nu$, i.e. $\mu_j = i\lambda_j$, $\lambda_j \in \Gamma$. In the special case $p_m^0(\rho) \geq 0$: $\operatorname{spec} F_{p_m^0(\rho)} = \left\{i\lambda_1, \ldots, i\lambda_\nu,\ \lambda_j \geq 0, 1 \leq j \leq \nu\right\} \cup \{-i\lambda_1, \ldots, -i\lambda_\nu\}$.

THEOREM 2. *(Grušin [3, 4], B. de Monvel, Trèves [5, 6], Hörmander [14]). Under the assumptions (i), (ii) and*

$$(iii) \qquad p'_{2m-1}(\rho) + \sum_{j=1}^{\nu}(2\alpha_j + 1)\lambda_j(\rho) \neq 0,\ \forall \rho \in \Sigma,\ \forall \alpha_j \in \mathbb{Z}_+$$

the operator $P(x, D)$ is hypoelliptic and even microhypoelliptic with sharp loss of regularity 1.

According to Sjöstrand [7] if (iii) is violated at some point $\rho^0 \in \Sigma$ and for some $\alpha_j \in \mathbb{Z}_+$, $\lambda_j(\rho^0)$ then the loss of regularity r of the operator P is $\geq 3/2$. We remind of the reader that $r \geq 0$ is called loss of regularity of P if $u \in D'(\omega)$, $Pu \in H^s_{\mathrm{loc}}(\omega) \Rightarrow u \in H^{s+m-r}_{\mathrm{loc}}(\omega)$. Certainly, s is arbitrary and r is fixed.

A rather interesting question is to study the hypoellipticity of the operator P, (i), (ii), when (iii) is violated.

THEOREM 3. *(Helffer [8]). Consider the operator P under the conditions (i), (ii) and in the special case $\nu = 1$. Suppose that there exists $\rho^0 \in \Sigma$, $\exists j \in \mathbb{Z}_+$ such that*

$$(iv) \qquad p'_{m-1}(\rho^0) + (2j+1)\lambda_1(\rho^0) = 0.$$

Define now on Σ the function

$$\widetilde{p}_{m-1}(\rho) = p'_{m-1}(\rho) + (2j+1)\lambda_1(\rho).$$

Then $P(x,D)$ is microlocally hypoelliptic at ρ^0 with sharp loss of regularity $r = 3/2$ iff

$$\frac{1}{i}\left\{\widetilde{p}_{m-1},\overline{\widetilde{p}}_{m-1}\right\}_\Sigma < 0 \text{ at } \rho^0.$$

REMARK 2. It is well known that $\lambda_1(\rho) \in C^\infty(\Sigma)$ and both $p'_{m-1}(\rho)$ and $\lambda_1(\rho)$ are symplectic invariant on Σ, i. e. $\widetilde{p}_{m-1}(\rho)$ is well defined on Σ. As it concerns the Poisson bracket $\{a,b\}_\Sigma$ it is better to write $\{a,b\}_{T(\Sigma)}$, i. e. the Poisson bracket is taken along the canonical vector fields tangential to Σ and consequently belonging to $T(\Sigma)$.

EXAMPLE 2. Let

$$P = D_1^2 + x_1^2 D_2^2 + \lambda D_2 + D_1 + ix_1 D_2 \tag{4}$$

be a differential operator in $\mathbb{R}^2$. Then $\Sigma = \{x_1 = \xi_1 = 0, \xi_2 \neq 0\}$, $m = 2$, $\nu = 1$, the winding number of $p_2^0 \geq 0$ is 0, $p'_1 = \lambda\xi_2 + \xi_1 + ix_1\xi_2$, spec$F_{p_2^0} = \{\pm i\xi_2\} \neq 0$, $\lambda_1 = \xi_2$ if $\xi_2 > 0$. Let $\xi_2^0 > 0$, i. e. $\rho^0 = (0,0;0,\xi_2^0 > 0)$. Then $(iv) \Leftrightarrow \lambda = -(2j+1)$ for some $j \in \mathbb{Z}_+$. Thus $\widetilde{p}_1 = \xi_1 + ix_1\xi_2 \Rightarrow \widetilde{p}_1|_\Sigma = 0$, $\left\{\widetilde{p}_1,\overline{\widetilde{p}}_1\right\}_\Sigma = 0$ and therefore P is not microlocally hypoelliptic for $\lambda = -(2j+1)$ with a loss of regularity $r \leq 3/2$.

In Helffer [8] nothing is mentioned about the local (non) solvability of the operator P (4) at the origin, about the existence of solution of the equation $P^*u = f \in C^\infty$ with fixed singularity $WFu = \{\rho^0\}$ etc.

PROPOSITION 1. *The operator* (4) *and with $\lambda = -(2j+1)$ for some $j \in \mathbb{Z}_+$ is locally nonsolvable at the origin and in D'. Moreover, P^* is not microlocally hypoelliptic and possesses a distribution solution with fixed singularity at $\rho^0 = (0,0;0,\xi_2^0 > 0) \in \mathrm{Char}\,P$. More precisely, let $t < s$ and s be fixed. Then one can find $u \in D'$ and such that $P^*u = f \in C^\infty$, $WF(u) = \{\rho^0\}$, $u \in H^t_{\mathrm{mcl}}(\rho^0)$ but $u \notin H^s_{\mathrm{mcl}}(\rho^0)$.*

EXAMPLE 3. As we saw in Example 2 the operator (4) does not enter in the frames of Theorem 3. Because of this reason we study in $\mathbb{R}^3$ the operator

$$P = D_1^2 + x_1^2 D_2^2 + \lambda D_2 + D_3 + ix_3 D_2,\ \lambda = -(2j+1),\ j \in \mathbb{Z}_+. \tag{5}$$

and its L_2 adjoint operator $S = P^* = D_1^2 + x_1^2 D_2^2 + \lambda D_2 + D_3 - ix_3 D_2$, $\lambda = -(2j+1)$.

Our investigation will be microlocally near the point $\rho^0 = (0,0,0;\xi_1 = 0,\xi_2^0 > 0,\xi_3 = 0)$. Evidently, $p'_1 = \lambda\xi_2 + \xi_3 + ix_3\xi_2$, spec $F_{p^0} = \{\pm i\xi_2\}$; we take $\lambda_1 = \xi_2 > 0$ and therefore $\widetilde{p}_1 = \xi_3 + ix_3\xi_2$, $\dfrac{1}{i}\left\{\widetilde{p}_1,\overline{\widetilde{p}}_1\right\}_\Sigma = -2\xi_2 < 0$. Thus the operator P (5) is microhypoelliptic at ρ^0 with sharp loss of regularity $r = 3/2$. Assume that $\lambda \neq 2j+1$, $\forall j \in \mathbb{Z}_+$, $\lambda \in \mathbb{R}$. Then in a conical neighborhood of ρ^0 we have that $p'_1(\rho) + (2j+1)\xi_2 = (\lambda + 2j + 1)\xi_2 + \xi_3 + ix_3\xi_2 \neq 0$ for each $j \in \mathbb{Z}_+$ and according to Theorem 2 ((iii) is satisfied) we have that the operator P (5) is microhypoelliptic at ρ^0 with sharp loss of regularity 1.

PROPOSITION 2. *The operator* (5), $\lambda = -(2j+1)$, $j \in \mathbb{Z}_+$ *is not locally solvable at the origin, while $S = P^*$ is not (micro)locally hypoelliptic (at ρ^0) at the origin* *in $\mathbb{R}^3$. More precisely, let $t < s$ and s be fixed. Then there exists $u \in D'$ such that $Su = f \in C^\infty$, $WF(u) = \{\rho^0\}$, $u \in H^t_{\mathrm{mcl}}(\rho^0)$, while $u \notin H^s_{\mathrm{mcl}}(\rho^0)$.*

We point out that there are possible generalizations of the results formulated in Propositions 1, 2. To do this we must use the definitions of symplectic manifold, the properties of Fourier integral operators and impose such conditions on the full symbol $p(x,\xi)$ that the corresponding microlocal form of p will be of the type (4), (5). Other possible generalizations are in the case $\nu \geq 2$. We omit the corresponding results as they are purely technical and we prefer to fix the ideas on the level of instructive examples that can be considered as appropriate microlocal forms of operators with double symplectic characteristics.

The paper is organized as follows. In Section 2 we propose some useful results about Hermite polynomials and Hermite functions. In Section 3 we construct distribution solution of the equation: $P^*u = f \in C^\infty$ with prescribed singularity of u: $WF(u) \cap V = WF_s(u) \cap V = N$. Here P is defined by (4) or (5). In Section 4 it is proved the local nonsolvability of (4), (5). “Grosso modo” the proof of Proposition 2 imitates the proof of Proposition 1. As it concerns Theorem 3, we give in a small Appendix a short sketch of the proof, avoiding the use of Hermite pseudodifferential operators. Our proof is elementary as it is based on a simple identity in L_2.

At the end of this section we shall mention that there is a renewed interest to the problems of hypoellipticity and subellipticity for operators with double symplectic characteristics (see for example [17], [18]).

2. Some useful results about Hermite polynomials and Hermite functions

1. We shall begin with several definitions.

DEFINITION 1. *Hermite polynomials are defined by the formula*

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n}\left(e^{-x^2}\right),\ n \in \mathbb{Z}_+,\ \deg H_n = n.$$

Evidently, $H_n(-x) = (-1)^n H_n(x)$, i. e. $H_{2n+1}(x)$ is an odd function $\Rightarrow H_{2n+1}(0) = 0$, while $H_{2n}(x)$ is even. Moreover, $H_{2n}(0) \neq 0$. One can see that $H_0(x) = 1$, $H_1(x) = 2x$, $H_2(x) = 4x^2 - 2$. It is known that the following recurrent formulas hold:

$$H_n'(x) = 2nH_{n-1}(x),\ n \geq 1,\ H_{n+1}(x) - 2xH_n(x) + 2nH_{n-1}(x) = 0. \tag{6}$$

Evidently, $H_3(x) = 8x^3 - 12x$ (see (6) for $n = 2$).

The Hermite polynomials satisfy the ODE:

$$H_n'' - 2xH_n' + 2nH_n = 0,\ n \geq 0,$$

i. e. $\dfrac{d}{dx}\left(e^{-x^2}\dfrac{dH_n}{dx}\right) + 2ne^{-x^2}H_n = 0.$

Integrating by parts one obtains:

$$\int_{-\infty}^{\infty} H_n(x)H_m(x)e^{-x^2}dx = \begin{cases} 0, \ n \neq m \\ 2^n n!\sqrt{\pi}, \ n = m. \end{cases}$$

DEFINITION 2. *Hermite functions are defined by*

(7) $$\psi_n(x) = h_n(x)e^{-\frac{x^2}{2}},$$

where $h_n(x) = \dfrac{H_n(x)}{\|H_n(x)\|}$, $\|H_n\| = \sqrt{2^n n!\sqrt{\pi}}$.

Therefore, $\displaystyle\int_{-\infty}^{\infty} h_n(x)h_m(x) = \delta_{n,m}$ *and* $\delta_{n,m}$ *is the Kronecker symbol.*

2. It is well known that $\{\psi_n\}$ form an orthonormal basis in $L_2(\mathbb{R}^1)$ and bases in the Schwartz spaces $\mathcal{S}(\mathbb{R}^1)$, $\mathcal{S}'(\mathbb{R}^1)$ ([1,2]).

Consider now the Hermite series $u = \sum_{n=0}^{\infty} c_n\psi_n$, where $c_n = \langle u, \psi_n\rangle$.

Then $u \in \mathcal{S}(\mathbb{R}^1) \Leftrightarrow \forall m \in \mathbb{Z}_+$ there exists a constant $\widetilde{c}_m > 0$ and such that $|c_n| \leq \widetilde{c}_m(1+n)^{-m}$, for each $n \in \mathbb{Z}_+$ ([2]).

Similarly, $u \in \mathcal{S}'(\mathbb{R}^1) \Leftrightarrow \exists m_0$ and $\widetilde{c}_0 > 0$ such that $|c_n| \leq \widetilde{c}_0(1+n)^{m_0}$ for each $n \in \mathbb{Z}_+$.

One can easily see that for each fixed $\xi_2 > 0$ the system $\left\{\psi_n\left(x\xi_2^{\frac{1}{2}}\right)\xi_2^{\frac{1}{2}}\right\}$ forms an orthonormal basis in $L_2(\mathbb{R}^1)$.

Below we propose the very important inequality of Cramer (Cramer – Charlier):

$$|H_n(x)|\, e^{-\frac{x^2}{2}} \leq k\sqrt{2^n n!},$$

where the constant $k = 1{,}086435\ldots$ (see [1]).

Define now the following differential operators:

(8) $$M_1 = \frac{d}{dx} + x, \ M_2 = \frac{d}{dx} - x.$$

The one guesses that

(9) $$M_1\psi_n = \sqrt{2n}\psi_{n-1}, \ \ n \geq 1.$$

Let $n = 0$. Then $M_1\psi_0 = 0$, i. e. (9) holds for each $n \geq 0$ if we define $\psi_{-1} = 0$.

In a similar way we obtain

(10) $$M_2\psi_n = -\sqrt{2(n+1)}\psi_{n+1}, \ \ n \geq 0.$$

Combining (9), (10) we get:

$$M_1M_2\psi_n = -2(n+1)\psi_n \Rightarrow$$

$$\left(\frac{d^2}{dx^2} - x^2\right)\psi_n = -(2n+1)\psi_n, \ \forall n \geq 0. \tag{11}$$

Iterating the formula $M_2\psi_0 = -\sqrt{2}\psi_1$ we obtain $M_2^n\psi_0 = (-1)^n 2^{\frac{n}{2}}\sqrt{n!}\psi_n$, $n \geq 0$, i. e.

$$\psi_n = (-1)^n \frac{M_2^n\psi_0}{\sqrt{2^n n!}} = \frac{(-1)^n}{\sqrt{2^n n!}}\left(\frac{d}{dx} - x\right)^n \psi_0, \ n \geq 0.$$

Having in mind that $\psi_0 = \dfrac{e^{-\frac{x^2}{2}}}{\sqrt[4]{\pi}}$ we have

$$\psi_n = \frac{(-1)^n}{\sqrt{2^n n! \sqrt{\pi}}}\left(\frac{d}{dx} - x\right)^n e^{-\frac{x^2}{2}}, \ n \geq 0. \tag{12}$$

Combining (12) and the fact that $e^{-\frac{x^2}{2}} \in \mathcal{S}(\mathbb{R}^1)$ we conclude that the Fourier transform $\widehat{\psi}_n$ is given by

$$\widehat{\psi}_n(\xi) = \sqrt{2\pi}(-i)^n\psi_n(\xi). \tag{13}$$

3. Construction of solutions of the equations $Qu = f_1$, $Sv = f_2 \in C^\infty$ with prescribed singularity

1. We shall deal at first with the operator (4) and its L_2 adjoint operator $Q = P^* = D_1^2 + x_1^2 D_2 + \lambda D_2 + D_1 - ix_1 D_2$, $\lambda \in \mathbb{R}^1$. We shall denote by $\widehat{u}(x_1, \xi_2)$ the partial Fourier transformation of $u \in \mathcal{S}'(\mathbb{R}^2)$ with respect to x_2 $(x_2 \to \xi_2)$. Then $Qu = 0$ implies

$$\widehat{\widehat{Q}}\,\widehat{u} = \left(D_1^2 + x_1^2\xi_2^2 + \lambda\xi_2 + D_1 - ix_1\xi_2\right)\widehat{u} = 0 \tag{14}$$

and for $\xi_2 > 0$ we make the following change of the variable x_1 in the homogeneous ODE (14): $y_1 = x_1\xi_2^{\frac{1}{2}}$. Then (14) takes the form:

$$-\xi_2\left(\frac{d^2}{dy_1^2} - y_1^2 - \lambda + i\xi_2^{-\frac{1}{2}}\left(\frac{d}{dy_1} + y_1\right)\right)\widehat{u}(y_1, \xi_2) = 0. \tag{15}$$

Certainly, $M_1 = \dfrac{d}{dy_1} + y_1$ according to (8).

In a similar way we obtain:

$$\widehat{\widehat{P}}\,\widehat{u} = 0 \Leftrightarrow -\xi_2\left(\frac{d^2}{dy_1^2} - y_1^2 - \lambda + i\xi_2^{-\frac{1}{2}} M_2\right)\widehat{u} = 0, \ \xi_2 > 0, \ M_2 = \frac{d}{dy_1} - y_1.$$

In the case $\lambda = -(2j+1)$, $j \in \mathbb{Z}_+$ we are looking for the kernel of (14) in $\mathcal{S}'(\mathbb{R}^1)$ and for $\xi_2 > 0$ being a fixed parameter. As we know from Section 2, $\{\psi_n(y_1)\}$ form bases in $L_2(\mathbb{R}^1)$ and $\mathcal{S}'(\mathbb{R}^1)$. Therefore,

$$\widehat{u} = \sum_{n=0}^{\infty} c_n\psi_n(y_1)$$

and according to (15) $\widehat{\widehat{Q}}\,\widehat{u} = 0$, i. e.

$$\sum_{n=0}^{\infty} c_n \left[-(2n+1)\psi_n + (2j+1)\psi_n + i\xi_2^{-\frac{1}{2}}\sqrt{2n}\psi_{n-1} \right] = 0,\ \psi_{-1} \equiv 0.$$

This way we obtain the following infinite linear system for the unknown coefficients c_n:

$$\begin{array}{ll}
(0): & 2c_0 j + ic_1\xi_2^{-\frac{1}{2}}\sqrt{2\cdot 1} = 0 \\
(1): & 2(j-1)c_1 + ic_2\xi_2^{-\frac{1}{2}}\sqrt{2\cdot 2} = 0 \\
(2): & 2(j-2)c_2 + ic_3\xi_2^{-\frac{1}{2}}\sqrt{2\cdot 3} = 0 \\
 & \dots \\
(n-1): & 2(j-n+1)c_{n-1} + ic_n\xi_2^{-\frac{1}{2}}\sqrt{2\cdot n} = 0 \\
(n): & 2(j-n)c_n + ic_{n+1}\xi_2^{-\frac{1}{2}}\sqrt{2(n+1)} = 0 \\
(n+1): & 2(j-n-1)c_{n+1} + ic_{n+2}\xi_2^{-\frac{1}{2}}\sqrt{2(n+2)} = 0 \\
 & \dots
\end{array}$$

If $j = n$ (see equation (n)) the constant $c_n = c_j$ is arbitrary but $c_{n+1} = 0$. Then the $(n+1)$ equation implies that $c_{n+2} = 0$, etc. Therefore, $c_{j+k} = 0$ for each $k \geq 1$. We conclude that for each fixed $\xi_2 > 0$, $\dim \operatorname{Ker} \widehat{\widehat{Q}} = 1$, $\operatorname{Ker} \widehat{\widehat{Q}} \subset \mathcal{S}(\mathbb{R}^1)$ and

$$\widehat{\widehat{u}}(y_1) = \sum_{k=0}^{j} c_k \psi_k(y_1),$$

where

$$\begin{aligned}
c_{j-1} &= -\frac{ic_j\xi_2^{-\frac{1}{2}}\sqrt{2j}}{2\cdot 1}, \\
c_{j-2} &= -\frac{ic_{j-1}\xi_2^{-\frac{1}{2}}\sqrt{2(j-1)}}{2\cdot 2} = \left(-i\xi_2^{-\frac{1}{2}}\right)^2 \frac{\sqrt{2j}\sqrt{2(j-1)}}{2^2\cdot 1\cdot 2} c_j, \\
&\dots \\
c_{j-l} &= \left(-i\xi_2^{-\frac{1}{2}}\right)^l \frac{\sqrt{2j}\sqrt{2(j-1)}\dots\sqrt{2(j-l+1)}}{2^l l!} c_j,\ j \geq l \geq 1, \\
&\dots \\
c_0 &= \left(-i\xi_2^{-\frac{1}{2}}\right)^j \frac{1}{2^{\frac{j}{2}}\sqrt{j!}} c_j.
\end{aligned}$$

We shall take $c_j = 1$. Then for $k = j-l$, $1 \leq l \leq j$ we have:

$$c_k = \left(-i\xi_2^{-\frac{1}{2}}\right)^{j-k} \frac{\sqrt{j(j-1)\dots(j-(j-k-1))}}{2^{\frac{j-k}{2}}(j-k)!},\ k = 0, 1, \dots, j-1.$$

To simplify the notations we write

$$c_k = \left(-i\xi_2^{-\frac{1}{2}}\right)^{j-k} \widetilde{c}_k,\ \widetilde{c}_k \neq 0,\ 0 \leq k \leq j-1,\ \widetilde{c}_j = c_j = 1.$$

Going back to the old coordinate x_1 we have:

$$\begin{aligned}\widehat{u}(x_1,\xi_2) &= \sum_{k=0}^{j} c_k\psi_k(y_1) = \sum_{k=0}^{j}\widetilde{c}_k\left(-i\xi_2^{-\frac{1}{2}}\right)^{j-k}\psi_k\left(x_1\xi_2^{\frac{1}{2}}\right)\\ &= \Sigma_{k=0}^{j}\widetilde{c}_k\left(-i\xi_2^{-\frac{1}{2}}\right)^{j-k}\frac{H_k\left(x_1\xi_2^{\frac{1}{2}}\right)}{\|H_k\|}e^{-\frac{1}{2}x_1^2\xi_2},\ \xi_2>0.\end{aligned}$$

Let $\psi(\xi_2)\in C^\infty(\mathbb{R}^1)$, $\psi=0$ for $\xi_2\in(-\infty,1]$, $\psi(\xi_2)=1$ for $\xi_2\geq 2$, $0\leq\psi(\xi_2)\leq 1$ for $\xi_2\in[1,2]$.

One can easily see that for each constant $a\in\mathbb{R}^1$ the function

$$\begin{aligned}&u_a(x_1,x_2)=\int_{-\infty}^{\infty}e^{ix_2\xi_2}\psi(\xi_2)\xi_2^a\widehat{u}(x_1,\xi_2)d\xi_2=\\ &=\sum_{k=0}^{j}\widetilde{c}_k\int_{-\infty}^{\infty}e^{ix_2\xi_2-\frac{1}{2}x_1^2\xi_2}\psi(\xi_2)\xi_2^a\left(-i\xi_2^{-\frac{1}{2}}\right)^{j-k}\frac{H_k\left(x_1\xi_2^{\frac{1}{2}}\right)}{\|H_k\|}d\xi_2.\end{aligned}\tag{16}$$

satisfies the equation $Qu=0$.

The integral (16) is rapidly oscillating and it enters in the Hörmander scheme from Vol. I of his monograph [14] (see Chapter VII, Section 7.8, Th. 7.8.2, Th. 7.8.3 and Section 8.1, Th. 8.1.9). In fact, the amplitudes are of the type

$$\psi(\xi_2)\xi_2^a\xi_2^{\frac{k-j}{2}}H_k\left(x_1\xi_2^{\frac{1}{2}}\right)\in S_{1,0}^{a-\frac{j}{2}+k},\ \ 0\leq k\leq j.$$

The phase function of (16)

$$i\phi=ix_2\xi_2-\frac{x_1^2}{2}\xi_2=i\xi_2\left(x_2+\frac{i}{2}x_1^2\right),\ \ \Im\phi=\frac{\xi_2x_1^2}{2}\geq 0,$$

as $\phi=\xi_2\left(x_2+\frac{i}{2}x_1^2\right)$ and evidently, $d_{x,\xi}\phi=(ix_1\xi_2,\ \xi_2\neq 0;\ 0, x_2+\frac{i}{2}x_1^2)\neq 0$, as $\xi_2\geq 1$. Certainly, $\phi(x,t\xi)=t\phi(x,\xi)$, $\forall t>0$. Then

$$WF(u_a)\subset\left\{(x,\phi_x):\ \phi_{\xi_2}=0\right\}=\left\{(x_1=0,\ x_2=0;\ \xi_1=0,\ \xi_2>0)\right\}.$$

Thus, $u_a\in C^\infty(\mathbb{R}^2\backslash(0,0))$ as sing supp $u_a\subset\{(0,0)\}$.

2. We shall study now the behavior of $x^\alpha D^\beta u_a(x)$ for $|x|\geq\varepsilon_0>0$ and ε_0 is arbitrary small.

Evidently,

$$\begin{aligned}&x^\beta D_1^{\alpha_1}D_2^{\alpha_2}u_a(x)=x^\beta\sum_{k=0}^{j}\int_{-\infty}^{\infty}e^{ix_2\xi_2}\xi_2^{a+\frac{\alpha_1}{2}+\alpha_2}\times\\ &\times\psi(\xi_2)\widetilde{c}_k\left(-i\xi_2^{-\frac{1}{2}}\right)^{j-k}\left(D_1^{\alpha_1}\psi_k\right)\left(x_1\xi_2^{\frac{1}{2}}\right)d\xi_2.\end{aligned}\tag{17}$$

On the other hand,

$$e^{i\phi} = \partial^N_{\xi_2}\left(e^{i\phi}\right) \Big/ i^N \left(x_2 + \frac{i}{2}x_1^2\right)^N \text{ for } |x| \geq \varepsilon_0$$

and for arbitrary $N \in \mathbb{N}$.

According to the theory of Fourier integral operators with complex phase function, we can integrate by parts with respect to ξ_2 in (17) as its phase function is $i\phi$ and its amplitude belongs to some class $S^m_{1,0}$.

This way we conclude that

$$\left|x^\beta D^\alpha u_a(x)\right| \leq C_{\alpha\beta} \text{ for } |x| \geq \varepsilon_0,\ \forall(\alpha,\beta) \in \mathbb{Z}^2_+. \tag{18}$$

Of course, $C_{\alpha\beta} > 0$ are appropriate constants.

Introduce now the cut off function $\eta(x) \in C_0^\infty(\mathbb{R}^2)$, $\eta \equiv 1$ near $(0,0)$, $0 \leq \eta \leq 1$. According to (18), $u = \eta u + (1-\eta)u$ and $(1-\eta)u \in \mathcal{S}(\mathbb{R}^2)$.

Having in mind that $\rho^0 = (0,0;\, 0, \xi_2^0 > 0) \in WF(u_a)$ we conclude that $\widehat{\eta u_a}(\xi)$ is rapidly decreasing in the angle $\Gamma_1 = \{(\xi_1,\xi_2):$ either $\xi_2 < 0$ or $0 \leq \varepsilon_0\xi_2 \leq |\xi_1|\}$, $0 < \varepsilon_0 \ll 1$ and is not decreasing in the angle $\Gamma_2 = \{(\xi_1,\xi_2):\ \varepsilon_0\xi_2 \geq |\xi_1|\}$.

The above mentioned words enable us to conclude that $u_a \in H^t(\mathbb{R}^2) \Leftrightarrow u_a \in H^t_{\text{mcl}}(\rho^0)$, i. e.

$$\iint \left(1+|\xi|^2\right)^t |\widehat{u}_a(\xi)|^2\, d\xi < \infty \Leftrightarrow \iint_{\Gamma_2} \left(1+|\xi|^2\right)^t |\widehat{u}_a(\xi)|^2\, d\xi < \infty.$$

But in Γ_2 we have that $\xi_2^2 \leq |\xi|^2 \leq (1+\varepsilon_0^2)\xi_2^2$ and consequently $(1+|\xi|)^t \sim (1+\xi_2^2)^t$ in Γ_2. So

$$u_a \in H^t(\mathbb{R}^2) \Leftrightarrow \iint_{\Gamma_2} (1+\xi_2^2)^t\, |\widehat{u}_a(\xi)|^2\, d\xi < \infty. \tag{19}$$

Then the definition (16) of u_a gives us that

$$\widehat{u}_a(\xi_1,\xi_2) = \int e^{-ix_2(\xi_2-\theta)} \sum_{k=0}^{j} \theta^{a-\frac{1}{2}} \psi(\theta) \widetilde{c}_k \left(-i\theta^{-\frac{1}{2}}\right)^{j-k} \widehat{\psi}_k\left(\frac{\xi_1}{\theta^{\frac{1}{2}}}\right) dx_2 d\theta.$$

Applying (13) to the previous integral we get

$$\begin{aligned}\widehat{u}_a(\xi_1,\xi_2) = \quad & \sqrt{2\pi}(-i)^j \textstyle\int \theta^{a-\frac{1}{2}} \psi(\theta) \sum_{k=0}^{j} \widetilde{c}_k \theta^{\frac{k-j}{2}} \psi_k\left(\frac{\xi_1}{\theta^{\frac{1}{2}}}\right) \times \\ & \times \left[\int e^{-ix_2(\xi_2-\theta)} dx_2\right] d\theta = (-i)^j \sqrt{2\pi}\xi_2^{a-\frac{1}{2}} \psi(\xi_2) \times \\ & \times \sum_{k=0}^{j} \widetilde{c}_k \xi_2^{\frac{k-j}{2}} \psi_k\left(\frac{\xi_1}{\xi_2^{\frac{1}{2}}}\right).\end{aligned}$$

In fact, $\int e^{-ix_2(\xi_2-\theta)} dx_2 = \delta(\xi_2-\theta)$, δ being the Dirac delta function.

Applying (19) we get that

$$u_a \in H^t(\mathbb{R}^2) \Leftrightarrow \int_{-\infty}^{\infty} (1+\xi_2^2)^{t+a-\frac{1}{2}} \left[\int_{-\infty}^{\infty} \left| \sum_{k=0}^{j} \widetilde{c}_k \xi_2^{\frac{k-j}{2}} \psi_k \left(\frac{\xi_1}{\xi_2^{\frac{1}{2}}} \right) \right|^2 d\xi_1 \right] d\xi_2 < \infty.$$

On the other hand, $\{\psi_k(y_1)\}$ form an orthonormal basis in $L_2(\mathbb{R}^1)$ and

$$\int_{-\infty}^{\infty} \left| \sum_{k=0}^{j} \widetilde{c}_k \xi_2^{\frac{k-j}{2}} \psi_k \left(\frac{\xi_1}{\xi_2^{\frac{1}{2}}} \right) \right|^2 d\xi_1 = \xi_2^{\frac{1}{2}} \int_{-\infty}^{\infty} \left| \sum_{k=0}^{j} \widetilde{c}_k \xi_2^{\frac{k-j}{2}} \psi_k(y) \right|^2 dy =$$
$$= \xi_2^{\frac{1}{2}} \sum_{k=0}^{j} |\widetilde{c}_k|^2 \xi_2^{k-j}, \text{ i. e.}$$

$$u_a \in H^t(\mathbb{R}^2) \Leftrightarrow \sum_{k=0}^{j} \int_{-\infty}^{\infty} (1+\xi_2^2)^{t+a-\frac{1}{4}} \psi^2(\xi_2) \, |\widetilde{c}_k|^2 \, \xi_2^{k-j} d\xi_2 < \infty. \tag{20}$$

We have taken $\widetilde{c}_j = 1$ and therefore the integral participating in (20) is $< \infty$ iff $2t + 2a - \dfrac{1}{2} < -1 \Leftrightarrow t + a < -\dfrac{1}{4} \Leftrightarrow t < -a - \dfrac{1}{4}$. Put $s = -a - \dfrac{1}{4}$. Evidently, the integral (20) is divergent for $t + a = -\dfrac{1}{4}$, i. e. for $t = s$.

CONCLUSION. $u_a \in H^t_{\mathrm{mcl}}(\rho^0) \Leftrightarrow t < s$, while $u_a \notin H^s_{\mathrm{mcl}}(\rho^0)$. This way we have constructed the solution of $Qu = 0$ (4) with $WF(u_a) = \{\rho^0\}$, $u_a \in H^t_{\mathrm{mcl}}(\rho^0)$ for each $t < s$, $u_a \notin H^s_{\mathrm{mcl}}(\rho^0)$ and s is arbitrary real number.

3. To prove the existence of a solution with prescribed singularity of the equation $P^*u = f \in C^\infty$, where P is given by (5), we make the partial Fourier transformation with respect to x_2 in $Su = 0$, $u = u(x_1, x_2, x_3)$, $S = P^*$. Thus,

$$\widehat{\widehat{S}}\, \widehat{\widehat{u}} = \left(-\partial_{x_1}^2 + x_1^2\xi_2^2 + \lambda\xi_2 - i\,(\partial_3 + x_3\xi_2)\right) \widehat{\widehat{u}} = 0,$$

where $\widehat{\widehat{u}} = \widehat{\widehat{u}}(x_1, \xi_2, x_3)$, $x_2 \to \xi_2$, $-\lambda = 2j+1$. Our investigation will be microlocal near the point $\rho^0 = (0,0,0;\, \xi_1 = 0, \xi_2^0 > 0, \xi_3 = 0) \in \Sigma$ where

$$\Sigma = \left\{ (x,\xi) : \; x_1 = \xi_1 = 0,\; \xi \neq 0,\; (x,\xi) \in \mathbb{R}^6 \right\}.$$

The change $\left| \begin{array}{l} y_1 = x_1 \xi_2^{\frac{1}{2}} \\ y_3 = x_3 \xi_2^{\frac{1}{2}} \end{array} \right.$, $\xi_2 > 0$ in the equation $\widehat{\widehat{S}}\,\widehat{\widehat{u}} = 0$ leads to the following PDE:

$$\left[\partial_{y_1}^2 - y_1^2 - \lambda + i\xi_2^{-\frac{1}{2}} (\partial_{y_3} + y_3) \right] \widehat{\widehat{u}}(y_1, \xi_2, y_3) = 0 \tag{21}$$

and (21) is an equation with separate variables. We are looking for $\widehat{\widehat{u}}(y_1, \xi_2, y_3) = C\psi_j(y_1)\psi_0(y_3)$ with $C = \mathrm{const} > 0$ and $\psi_j(y_1)$, $\psi_0(y_3)$ are the corresponding Hermite functions defined in Section 2 (see (7), (11)):

$$\left(\partial_{y_1}^2 - y_1^2 + 2j+1\right)\psi_j(y_1) = 0,\ \psi_0(y_3) = \frac{1}{\sqrt[4]{\pi}} e^{-\frac{1}{2}y_3^2},\ (\partial_{y_3} + y_3)\,\psi_0 = 0.$$

Taking $C = 1$ we have that the function

$$\begin{aligned}\widehat{\widehat{u}}(x_1,\xi_2,x_3) &= \psi_j\left(x_1\xi_2^{\frac{1}{2}}\right)\psi_0\left(x_3\xi_2^{\frac{1}{2}}\right) = \\ &= \frac{H_j\left(x_1\xi_2^{\frac{1}{2}}\right)}{\sqrt[4]{\pi}\|H_j\|} e^{-\frac{1}{2}\left(x_1^2+x_3^2\right)\xi_2} \in \mathcal{S}\left(\mathbb{R}^2_{x_1,x_3}\right)\end{aligned}$$

belongs to the kernel of the operator $\widehat{\widehat{S}}$ for each fixed value $\xi_2 > 0$ of the parameter ξ_2. To prove the existence of a solution with prescribed singularity of $Su = f \in C^\infty$ we use instead of the function (16) the following function

$$v_a(x_1,x_2,x_3) = \int_{-\infty}^{\infty} e^{ix_2\xi_2 - \frac{\xi_2}{2}\left(x_1^2+x_3^2\right)}\psi(\xi_2)\xi_2^a \frac{H_j\left(x_1\xi_2^{\frac{1}{2}}\right)}{\sqrt[4]{\pi}\|H_j\|} d\xi_2.$$

Then $Sv_a = 0$, $WF(v_a) = \{\rho^0\}$, $v_a \in H^t_{\mathrm{mcl}}(\rho^0) \Leftrightarrow t < s = -a - \dfrac{1}{4}$; $v_a \notin H^s_{\mathrm{mcl}}(\rho^0)$. The proof of these facts is the same as in the case of the operator Q and we omit the details.

The proof of the nonsolvability of the operators (4), (5), $\lambda = -(2j+1)$, $j \in \mathbb{Z}_+$ will be given in Section 4.

4. Local nonsolvability in D' of the operators (4), (5) in the case $\lambda = -(2j+1)$, $j \in \mathbb{Z}_+$

1. As is well known if the PDO $P(x,D)$ with C^∞ coefficients is locally solvable at the origin $0 \in \mathbb{R}^n$ and in D' then the following a-priori estimate holds.

There exists a neighborhood $\omega \ni 0$, an integer $N \in \mathbb{N}$ and a constant $C_N > 0$ such that

$$\left|\int f(x)v(x)dx\right| \le C_N \sum_{|\alpha|\le N} \sup|D^\alpha f| \sum_{|\alpha|\le N} \sup|D^\alpha P^* v|,\ \ \forall f, v \in C_0^\infty(\omega). \tag{22}$$

Therefore, in order to prove the local nonsolvability of the operator P (4), (5), $P^* = Q$ or $P^* = S$ we must violate (22) for arbitrary but fixed ω, N, C_N. The proof here repeats with some changes the proof of Theorem 1 from [12]. Because of this reason we shall not give everywhere the details. Moreover, we shall concentrate on the case (4), i. e. $P^* = Q$ in (4.1).

2. Introduce now the function $\eta(\rho) \in C_0^\infty(\mathbb{R}^1)$, $\eta \ge 0$, $\int \eta(\rho)d\rho = 1$, $\operatorname{supp}\eta \subset [1,2]$, $0 < \eta(\rho) < 1$ for $\rho \in (1,2)$. We consider the function $F \in C_0^\infty(\mathbb{R}^2)$, such that $\iint F(x_1,x_2)dx_1dx_2 = 1$ and define

$$f_\lambda(x_1,x_2) = F(\lambda^2 x_1, \lambda^2 x_2),\ \lambda \ge 1,\ \lambda - \text{parameter},$$

in the case j – even.

In the case j – odd we define

$$f_\lambda(x) = \frac{\partial}{\partial x_1}\left(F\left(\lambda^2 x_1, \lambda^2 x_2\right)\right). \tag{23}$$

Evidently, supp $f_\lambda \Subset \omega$ for $\lambda \geq \lambda_0 \gg 1$.

From the considerations in Section 3 we know that

$$u(x_1,x_2) = \int_{-\infty}^{\infty} \eta(\xi_2) e^{ix_2\xi_2} \sum_{k=0}^{j} \widetilde{c}_k \left(-i\xi_2^{-\frac{1}{2}}\right)^{j-k} \psi_k\left(x_1 \xi_2^{\frac{1}{2}}\right) d\xi_2 \in C^\infty(\mathbb{R}^2)$$

satisfies the equation $Qu = 0$.

Put

$$u_\lambda(x_1,x_2) = \int_{-\infty}^{\infty} \eta(\xi_2) e^{i\lambda x_2\xi_2} \sum_{k=0}^{j} \widetilde{\widetilde{c}}_k \left(\lambda\xi_2\right)^{-\frac{j-k}{2}} \psi_k\left(x_1\sqrt{\lambda\xi_2}\right) d\xi_2,$$

where $\widetilde{\widetilde{c}}_k = (-i)^{j-k}\widetilde{c}_k$.

Then $u_\lambda(x) \in C^\infty(\mathbb{R}^2)$ and $Qu_\lambda = 0$, $\forall x \in \mathbb{R}^2$, $\lambda \geq 1$. Let $\omega' \Subset \omega'' \Subset \omega$ be neighborhoods of the origin and the function $\varphi \in C_0^\infty(\mathbb{R}^2)$ be equal to 1 on ω', $0 \leq \varphi(x) \leq 1$ for $x \in \omega$ and $\varphi = 0$ outside ω''. Evidently, $\sum_{|\alpha|\leq N} \sup |D^\alpha f_\lambda| \leq \text{const}\, \lambda^{N+2}$.

Define now

$$v_\lambda(x) = \varphi(x) u_\lambda \Rightarrow \text{supp}\, v_\lambda \Subset \omega.$$

Then $Qv_\lambda = \varphi Q u_\lambda + [Q,\varphi]u_\lambda = [Q,\varphi]u_\lambda$ and $\text{supp}\,[Q,\varphi] \subset \omega''\backslash\omega'$.

As we mentioned before there are two cases to be studied: a) j – even $\Rightarrow$ $\psi_j(0) \neq 0$, b) j – odd $\Rightarrow H_j(0) = 0 \Rightarrow \psi_j(0) = 0$. In the case b) we have that according to (6)

$$\partial_{x_1}\psi_j(0) = \frac{H_j'(0)}{\|H_j\|} = \frac{2jH_{j-1}(0)}{\|H_j\|} \neq 0,\ j \geq 1.$$

We shall investigate the case a) only. Let us estimate the left hand side of (22), namely $I_\lambda = \iint f_\lambda(x) v_\lambda(x) dx$. The standard change of the variables $y_1 = \lambda^2 x_1$, $y_2 = \lambda^2 x_2$ in the previous integral gives us that

$$\lim_{\lambda\to\infty} \lambda^4 I_\lambda = \varphi(0,0) \iiint F(y_1,y_2)\eta(\xi_2)\widetilde{\widetilde{c}}_j \psi_j(0) dy_1 dy_2 d\xi_2 = \psi_j(0) \neq 0$$

as $\widetilde{\widetilde{c}}_j = \widetilde{c}_j = 1$, i. e.

$$I_\lambda = \lambda^{-4}\left(\psi_j(0) + o(1)\right),\ \lambda \to \infty. \tag{24}$$

In estimating $\sup |D^\alpha(Qv_\lambda)|$, $|\alpha| \leq N$ we have in mind that $Qv_\lambda = [Q,\varphi]u_\lambda = \varphi_1 u_\lambda$ and the function $\varphi_1 \in C_0^\infty(\mathbb{R}^2)$, $\text{supp}\,\varphi_1 \subset \omega''\backslash\omega'$, does not depend on λ.

The Leibnitz rule gives us the typical term participating in $D^{\alpha}(Qv_{\lambda}) = D^{\alpha}(\varphi_1 u_{\lambda})$:

$$D^{\alpha_1}_{x_1} D^{\alpha_2}_{x_2} u_{\lambda} = \int_{-\infty}^{\infty} \eta(\xi_2) e^{i\lambda x_2 \xi_2} \xi_2^{\alpha_2 + \frac{\alpha_1}{2}} \sum_{k=0}^{j} \widetilde{\widetilde{c}}_k (\lambda\xi_2)^{-\frac{j-k}{2}} \times \\ \times \left(D^{\alpha_1}_{x_1} \psi_k\right) \left(x_1 \sqrt{\lambda\xi_2}\right) d\xi_2 \cdot \lambda^{\alpha_2 + \frac{\alpha_1}{2}}. \tag{25}$$

Certainly, $\lambda^{\alpha_2 + \frac{\alpha_1}{2}} \leq \lambda^N$.

We shall consider two different cases in estimating $\sup |D^{\alpha}(\varphi_1 u_{\lambda})|$:
I) $|x_1| \geq \varepsilon_0 > 0$ and II) $|x_2| \geq \varepsilon_0$, $0 < \varepsilon_0 \ll 1$. In fact, $\varphi_1 \equiv 0$ near the origin.

Case I. The Hermite function $\psi_k \in \mathcal{S}(\mathbb{R}^1) \Rightarrow D^{\alpha_1}_{x_1} \psi_k \in \mathcal{S}(\mathbb{R}^1)$ and therefore for each integer $M \geq 1$ there exists a constant $C_M > 0$ and such that

$$\left| \left(D^{\alpha_1}_{x_1} \psi_k\right) \left(x_1 \sqrt{\lambda\xi_2}\right) \right| \leq \frac{C_M}{\left(1 + \left|x_1 \sqrt{\lambda\xi_2}\right|\right)^{2M}} \leq \varepsilon_0^{-2M} \frac{C_M}{\lambda^M}$$

as $\left|x_1 \sqrt{\lambda\xi_2}\right| \geq \varepsilon_0 \sqrt{\lambda}$ for $\xi_2 \geq 1$.

From (25) we obtain in the case I and for $|\alpha| \leq N$ that

$$\left|D^{\alpha_1}_{x_1} D^{\alpha_2}_{x_2} u_{\lambda}\right| \leq \widetilde{C}_M \lambda^{N-M} \int_{-\infty}^{\infty} \eta(\xi_2) d\xi_2 = \widetilde{C}_M \lambda^{N-M} \tag{26}$$

as $j \geq k$, $\lambda \geq 1$, $1 \leq \xi_2 \leq 2 \Rightarrow (\lambda\xi_2)^{-\frac{j-k}{2}} \leq 1$. Assume now that $|x_2| \geq \varepsilon_0 > 0$. Evidently, for each integer $M \geq 1$ we have

$$\frac{\partial^M}{\partial \xi_2^M} \left(e^{i\lambda x_2 \xi_2}\right) = (i\lambda x_2)^M e^{i\lambda x_2 \xi_2}$$

and we can integrate by parts in (25) with respect to ξ_2.

Thus, in the case II we get:

$$D^{\alpha_1}_{x_1} D^{\alpha_2}_{x_2} u_{\lambda} = \frac{\lambda^{\frac{\alpha_1}{2} + \alpha_2} (-1)^M}{(i\lambda x_2)^M} \int_{-\infty}^{\infty} e^{i\lambda x_2 \xi_2} \frac{\partial^M}{\partial \xi_2^M} \times \\ \times \left[\eta(\xi_2) \xi_2^{\frac{\alpha_1}{2} + \alpha_2} \sum_{k=0}^{j} \widetilde{\widetilde{c}}_k \lambda^{\frac{k-j}{2}} \xi_2^{\frac{k-j}{2}} \left(D^{\alpha_1}_{x_1} \psi_k\right) \left(x_1 \sqrt{\lambda\xi_2}\right) \right] d\xi_2, \tag{27}$$

$|i\lambda x_2|^M \geq \lambda^M \varepsilon_0^M$.

Using the fact that we are integrating in the interval $1 \leq \xi_2 \leq 2$ we conclude that the "most dangerous term" in the previous integral (i. e. the term containing the highest power of λ for $0 \leq k \leq j$) is: $\frac{\partial^M}{\partial \xi_2^M} \left[\left(D^{\alpha_1}_{x_1} \psi_j\right) \left(x_1 \sqrt{\lambda\xi_2}\right) \right]$. Having in mind that $\left| \left(\frac{\partial^l}{\partial \xi_2^l} D^{\alpha_1}_{x_1} \psi_j \right) \left(x_1 \sqrt{\lambda\xi_2}\right) \right| \leq C_{l,\alpha_1} = \text{const}$ we obtain that the highest power of

λ arising in the integral of (27) is $\lambda^{\frac{M}{2}}$ as $x_1\sqrt{\lambda\xi_2} = x_1\sqrt{\lambda}\sqrt{\xi_2}$. More precisely, "the most dangerous term" is:

$$(28) \qquad (-1)^M \left(\frac{x_1\sqrt{\lambda}}{2}\right)^M \xi_2^{-\frac{M}{2}} \left(\frac{\partial^M}{\partial\xi_2^M} D_{x_1}^{\alpha_1}\psi_j\right)\left(x_1\sqrt{\lambda\xi_2}\right).$$

Consequently, in the case II and for $|\alpha| \leq N$ (28) $|D^\alpha u_\lambda| \leq D_M \lambda^{N-M/2}$, $D_M = \text{const} > 0$. Combining (24), (26) and (28) we violate (22) for $\lambda \to \infty$ as M is arbitrary integer. This way we complete the proof of the nonsolvability of the operator (4), $-\lambda = 2j+1$ in the case j – even.

3. The case j – odd is studied in a similar way. In fact, then f_λ is given by (23) and therefore $|I_\lambda| = \left|\int f_\lambda v_\lambda\right| = \left|\int F(\lambda^2 x)\frac{\partial v_\lambda}{\partial x_1}\right|$, $\frac{\partial\psi_j}{\partial x_1}(0) \neq 0$, etc.

To prove the local nonsolvability of the operator (5) at the origin we violate (22) by using the following functions: $f_\lambda(x) = F(\lambda^2 x)$, $x \in \mathbb{R}^3$ and $w_\lambda = \varphi(x)v_\lambda$, where $\varphi \in C_0^\infty(\mathbb{R}^3)$, $\varphi \equiv 1$ near the origin and

$$v_\lambda = \int_{-\infty}^{\infty} \eta(\xi_2) e^{i\lambda x_2\xi_2} \frac{H_j\left(x_1\sqrt{\lambda\xi_2}\right)}{\sqrt[4]{\pi}\|H_j\|} e^{-\frac{\lambda}{2}\left(x_1^2+x_3^2\right)\xi_2} d\xi_2.$$

We assume j – even and as in the previous case we estimate $D^\alpha v_\lambda$ in two situations: I) $|x_2| \geq \varepsilon_0 > 0$ and II) $|x_2| \leq \varepsilon_0 (\Rightarrow x_1^2 + x_3^2 \geq \varepsilon_0^2$ on $\operatorname{supp} D\varphi)$, etc.

5. Appendix

Short sketch of the proof of the microhypoellipticity of the operator (5), $\lambda = -(2j+1)$ will be given here.

1. We are working in a conical neighborhood of the point $\rho^0 = (0,0,0;0,1,0)$, i.e. in the cone $\Gamma = \left\{\xi \in \mathbb{R}^3 \backslash 0 : \ \xi_2 \geq \varepsilon_0\sqrt{\xi_1^2+\xi_3^2}\right\}$, $\varepsilon_0 > 0$. Consider now the identity $\|Pu\|_0^2 = \|P^*u\|_0^2 + ([P^*, P]u, u)$. In our case $P = D_1^2 + x_1^2 D_2^2 - (2j+1)D_2 + D_3 + ix_3D_2$. Put $Q = Q^* = D_1^2 + x_1^2D_2^2 - (2j+1)D_2$; $R = D_3 + ix_3D_2$, $R^* = D_3 - ix_3D_2$.

Thus, $[P^*, P] = [Q+R^*, Q+R] = [Q, R-R^*] + [R^*, R] = 2i[Q, x_3D_2] + 2D_2 = 2D_2$, as $[Q, D_2] = 0$. Therefore

$$(5.1) \qquad \|Pu\|_0^2 \geq c\|u\|^2_{H^{\frac{1}{2}}_{\text{mcl}}(\rho^0)}, \ \ c = \text{const} > 0,$$

as $u \in H^{\frac{1}{2}}_{\text{mcl}}(\rho^0) \Leftrightarrow \int_\Gamma \left(1+\xi_2^2\right)^2 |\hat{u}(\xi)|^2 d\xi < \infty$.

Assume now that $Pu = f \in H^s_{\text{mcl}}(\rho^0)$. Then $|D_2|^s f \in L_2$, $|D_2|^s f = |D_2|^s Pu = P(|D_2|^s u) \Rightarrow |D_2|^s u \in H^{\frac{1}{2}}_{\text{mcl}}(\rho^0)$ according to (5.1) and consequently $u \in H^{s+\frac{1}{2}}_{\text{mcl}}(\rho^0)$. The estimate (5.1) holds for each $\lambda \in \mathbb{R}^1$ in (5) too.

2. It is interesting to study the operator $P = D_1^2 + x_1^2 D_2^2 + \lambda D_2 + D_3 + i x_3^{2k+1} D_2$. By using the repeated Poisson brackets technique one can expect to prove microlocal hypoellipticity of P at ρ^0 with loss of regularity $r = 1 + \frac{2k+1}{2k+2}$. For $k = 0$ this is (5.1).

We mentioned above that the examples here proposed can be generalized in the frames of the C^∞ category. On the other hand side, it is very interesting to investigate the same operators in the Gevrey spaces. A precise microlocal analysis of several classes of ψdo with multiple characteristics and in Gevrey spaces is given in the Rodino's monographs [15], [16]. We hope that a combination of the approach there and the technique of the Hermite operators will enlarge the scope of the microlocal analysis and its applications.

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DISPERSIVE ESTIMATES FOR T-DEPENDENT HYPERBOLIC SYSTEMS*

Abstract. This note is devoted to the study of time-dependent symmetric hyperbolic systems and the derivation of dispersive estimates for their solutions. It is based on a diagonalisation of the full symbol within adapted symbol classes.

We are going to consider the hyperbolic system

$$\mathrm{D}_t U = A(t,\mathrm{D})U, \qquad U(0,\cdot) = U_0, \tag{1}$$

where $A(t,\mathrm{D})$ denotes a smoothly time-dependent matrix Fourier multiplier with first order symbol

$$A(t,\xi) \in C^\infty(\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{C}^{m\times m})$$

subject to certain (natural) assumptions which are described later on in detail. As usual we denote $\mathrm{D}_t = -\mathrm{i}\partial_t$.

Our approach is based on diagonalising the (full) symbol of the operator in order to get a representation of solutions in terms of Fourier integrals and later on to use these representations to deduce dispersive estimates for solutions.

1. Prerequisites and basic assumptions

1.1. Hyperbolic symbol classes

We make use of the implicitly defined function t_ξ from

$$(1+t_\xi)|\xi| = N \tag{2}$$

with a suitable constant N and define the zones

$$Z_{hyp}(N) = \{(t,\xi) | t \ge t_\xi\}, \qquad Z_{pd}(N) = \{(t,\xi) | 0 \le t \le t_\xi\}. \tag{3}$$

In $Z_{hyp}(N,)$ we apply a diagonalisation procedure to the full symbol. The basic idea of this diagonalisation scheme comes from the treatment of degenerate hyperbolic problems and is closely related to the approach of [3].

DEFINITION 1. *The time-dependent Fourier multiplier $a(t,\xi)$ belongs to the hyperbolic symbol class $\mathcal{S}^{\ell_1,\ell_2}\{m_1,m_2\}$ if it satisfies the symbol estimates*

$$\left|\mathrm{D}_t^k \mathrm{D}_\xi^\alpha a(t,\xi)\right| \le C_{k,\alpha} |\xi|_{N,t}^{m_1-|\alpha|} \left(\frac{1}{1+t}\right)^{m_2+k} \tag{4}$$

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*It is a pleasure to dedicate this paper to Prof. Luigi Rodino on the occasion of his 60th birthday.

for all multi-indices $\alpha \in \mathbb{N}^n$ *with* $|\alpha| \le \ell_1$ *and all natural numbers* $k \le \ell_2$ *and with* $|\xi|_{N,t} = \max(|\xi|, N/(1+t))$. *We say it belongs to* $\mathcal{S}_N^{\ell_1,\ell_2}\{m_1,m_2\}$ *if the estimates are true within the hyperbolic zone* $Z_{hyp}(N)$.

EXAMPLE 1. A polynomial $p(t,\xi) = \sum_{|\alpha|=m} h_\alpha(t)\xi^\alpha$ with $t^k h_\alpha^{(k)}(t) \in L^\infty(\mathbb{R})$ for $k \le \ell$ belongs to $\mathcal{S}^{\infty,\ell}\{m,0\}$.

If the symbol estimates hold for all derivatives we write $\mathcal{S}_{(N)}\{m_1,m_2\}$ for $\mathcal{S}_{(N)}^{\infty,\infty}\{m_1,m_2\}$. Furthermore, the definition extents immediately to matrix-valued Fourier multiplier. The rules of the corresponding symbolic calculus are simple consequences of Definition 1 together with (2), (3) and collected in the following proposition.

PROPOSITION 1. *1.* $\mathcal{S}_{(N)}^{\ell_1,\ell_2}\{m_1,m_2\}$ *is a vector space.*

2. $\mathcal{S}_{(N')}^{\ell_1',\ell_2'}\{m_1-k,m_2+\ell\} \hookrightarrow \mathcal{S}_{(N)}^{\ell_1,\ell_2}\{m_1,m_2\}$ *for all* $\ell \ge k \ge 0$, $\ell_1' \ge \ell_1$, $\ell_2' \ge \ell_2$ *(and* $N' \le N$*).*

3. $\mathcal{S}_{(N)}^{\ell_1,\ell_2}\{m_1,m_2\} \cdot \mathcal{S}_{(N)}^{\ell_1,\ell_2}\{m_1',m_2'\} \hookrightarrow \mathcal{S}_{(N)}^{\ell_1,\ell_2}\{m_1+m_1',m_2+m_2'\}$.

4. $\mathrm{D}_t^k \mathrm{D}_\xi^\alpha \mathcal{S}_{(N)}^{\ell_1,\ell_2}\{m_1,m_2\} \hookrightarrow \mathcal{S}_{(N)}^{\ell_1-|\alpha|,\ell_2-k}\{m_1-|\alpha|,m_2+k\}$.

5. $\mathcal{S}_{(N)}^{0,0}\{-1,2\} \hookrightarrow L_\xi^\infty L_t^1(Z_{hyp}(N))$.

Of particular importance are the embedding relations of point 2 with $k = \ell$. They constitute a symbolic hierarchy, which is used in the diagonalisation scheme, cf. Section 2.1. We define the residual symbol classes

$$\mathcal{H}_{(N)}^{\ell_1,\ell_2}\{m\} = \bigcap_{k\in\mathbb{Z}} \mathcal{S}_{(N)}^{\ell_1,\ell_2}\{m-k,k\}.$$

1.2. Basic assumptions

We collect our assumptions on the symbol $A(t,\xi)$. Throughout this note we require

(A1)$_{\ell_1,\ell_2}$ *Operator of first order with bounded coefficients.* We assume that the matrix operator $A(t,\mathrm{D})$ has a smooth symbol satisfying

$$A(t,\xi) \in \mathcal{S}^{\ell_1,\ell_2}\{1,0\}.$$

Furthermore, we assume that there exists a ξ-homogeneous matrix $A_0(t,\xi)$ with $A(t,\xi)$ $-A_0(t,\xi) \in \mathcal{S}_N^{\ell_1,\ell_2}\{0,1\}$. We will always denote $\omega = \xi/|\xi| \in \mathbb{S}^{n-1}$.The symbol $A_0(t,\xi)$ is determined by its values $A_0(t,\omega)$ on the cylinder $\mathbb{R}_+ \times \mathbb{S}^{n-1}$.

(A2) *Uniform strict hyperbolicity up to* $t = \infty$. We assume that the characterisitic roots (eigenvalues) of the symbol $A_0(t,\xi)$ are real and distinct for all t and $\xi \ne 0$. In ascending order we denote them as $\lambda_1(t,\xi),\dots,\lambda_m(t,\xi)$. Furthermore, we assume that

$$\liminf_{t\to\infty}\min_{\omega\in\mathbb{S}^{n-1}}|\lambda_i(t,\omega)-\lambda_j(t,\omega)|>0$$

for all $i\neq j$.

PROPOSITION 2. *Assume (A1)$_{\ell_1,\ell_2}$ and (A2). For all $j=1,\dots,m$ the characteristic roots satisfy $\lambda_j(t,\xi)\in\mathcal{S}_N^{\infty,\ell_2}\{1,0\}$ and for all $i\neq j$ their difference satisfies $(\lambda_i(t,\xi)-\lambda_j(t,\xi))^{-1}\in\mathcal{S}_N^{\infty,\ell_2}\{-1,0\}$. Furthermore, the eigenprojection $P_j(t,\xi)$ corresponding to $\lambda_j(t,\xi)$ satisfies $P_j(t,\xi)\in\mathcal{S}_N^{\infty,\ell_2}\{0,0\}$.*

Sketch of proof. The properties of the characteristic roots follow from the spectral estimate $|\lambda_j(t,\omega)|\leq||A(t,\omega)||$ together with the obvious symbol properties of the coefficients of the characteristic polynomial and the uniform strict hyperbolicity. The eigenprojections can be expressed in terms of the characteristic roots

$$P_j(t,\xi)=\prod_{i\neq j}\frac{A(t,\xi)-\lambda_i(t,\xi)}{\lambda_j(t,\xi)-\lambda_i(t,\xi)}$$

and again the symbolic calculus yields the desired result. $\square$

PROPOSITION 3. *Assume (A1)$_{\ell_1,\ell_2}$ and (A2). There exists an invertible matrix $M(t,\omega)\in\mathcal{S}_N\{0,0\}$ which diagonalises the symbol $A(t,\omega)$,*

$$A(t,\omega)M(t,\omega)=M(t,\omega)\mathcal{D}(t,\omega),\qquad \mathcal{D}(t,\omega)=\operatorname{diag}\big(\lambda_1(t,\omega),\dots,\lambda_m(t,\omega)\big).$$

Furthermore, its inverse satisfies $M^{-1}(t,\omega)\in\mathcal{S}_N^{\infty,\ell_2}\{0,0\}$.

We require two more assumptions.

(A3) The matrix $F^{(0)}=\operatorname{diag}\big((\mathrm{D}_tM^{-1})M+M^{-1}(A-A_0)M\big)$ satisfies

$$\sup_{(s,\xi),(t,\xi)\in Z_{hyp}(N)}\left\|\int_s^t \operatorname{Im}F^{(0)}(\theta,\xi)\mathrm{d}\theta\right\|<\infty. \tag{5}$$

This assumption is independent of the choice of the diagonaliser $M(t,\xi)$ in Proposition 3 and trivially satisfied when $A(t,\xi)$ is symmetric and homogeneous.

(A4) The imaginary part $\operatorname{Im}A(t,\xi)=\frac{1}{2\mathrm{i}}(A(t,\xi)-A^*(t,\xi)$ satisfies the estimate

$$\operatorname{Im}A(t,\xi)+c|\xi|\mathrm{I}\geq 0$$

within $Z_{pd}(N)$ for sufficiently large N and some constant c.

2. Representation of solutions

Using the partial Fourier transform $\mathscr{F}$ with respect to the spatial variables we can reduce the system (1) into a system of ordinary differential equations. Our first objective is to represent its fundamental solution

$$\mathrm{D}_t \mathcal{E}(t,s,\xi) = A(t,\xi)\mathcal{E}(t,s,\xi), \qquad \mathcal{E}(s,s,\xi) = \mathrm{I} \tag{6}$$

within the hyperbolic zone $(t,\xi),(s,\xi) \in Z_{hyp}(N)$.

2.1. Diagonalisation scheme

We follow the treatment of [3] to construct the fundamental solution to (6). To avoid unnecessary repetitions we just give the corresponding statements.

LEMMA 1. *Let $M(t,\xi)$ be the diagonaliser from Proposition 3. Then $\mathcal{E}_0(t,s,\xi) = M^{-1}(t,\xi)\mathcal{E}(t,s,\xi)M(s,\xi)$ satisfies*

$$\mathrm{D}_t \mathcal{E}_0(t,s,\xi) = \big(\mathcal{D}(t,\xi) + R_0(t,\xi)\big)\mathcal{E}_0(t,s,\xi), \qquad \mathcal{E}_0(s,s,\xi) = \mathrm{I} \tag{7}$$

with $R_0(t,\xi) = \big(\mathrm{D}_t M^{-1}\big)M + M^{-1}(A - A_0)M \in \mathcal{S}_N^{\ell_1,\ell_2-1}\{0,1\}$.

LEMMA 2. *For each $1 \le k \le \ell_2 - 1$ there exists a zone constant N and matrix valued symbols*

- $N_k(t,\xi) = \mathrm{I} + \sum_{\mu=1}^{k} N^{(\mu)}(t,\xi)$, $N^{(\mu)}(t,\xi) \in \mathcal{S}_N^{\ell_1,\ell_2-\mu}\{-\mu,\mu\}$, *invertible for all $(t,\xi) \in Z_{hyp}(N)$ and with inverse satisfying $N_k^{-1}(t,\xi) \in \mathcal{S}_N\{0,0\}$*
- $F_{k-1}(t,\xi) = \sum_{\mu=0}^{k-1} F^{(\mu)}(t,\xi)$, $F^{(\mu)}(t,\xi) \in \mathcal{S}_N^{\ell_1,\ell_2-\mu-1}\{-\mu,\mu+1\}$, *diagonal,*
- $R_k(t,\xi) \in \mathcal{S}_N^{\ell_1,\ell_2-k-1}\{-k,k+1\}$,

such that $\mathcal{E}_k(t,s,\xi) = N_k^{-1}(t,\xi)\mathcal{E}_0(t,s,\xi)N_k(s,\xi)$ satisfies

$$\mathrm{D}_t \mathcal{E}_k(t,s,\xi) = \big(\mathcal{D}(t,\xi) + F_{k-1}(t,\xi) + R_k(t,\xi)\big)\mathcal{E}_k(t,s,\xi), \qquad \mathcal{E}_k(s,s,\xi) = \mathrm{I} \tag{8}$$

for all $(t,\xi),(s,\xi) \in Z_{hyp}(N)$.

REMARK 1. For $k = 1$ we have in particular $F^{(0)}(t,\xi) = \operatorname{diag} R_0(t,\xi)$.

REMARK 2. The proof of this statement is analogous to the corresponding statement from [3] and applies the standard diagonalisation scheme from [11], [4], etc. Under $(\mathrm{A1})_{\ell_1,\infty}$ we can form the asymptotic sums $N(t,\xi) \sim \sum N^{(\mu)}(t,\xi) \in \mathcal{S}_N^{\ell_1,\infty}\{0,0\}$ and $F(t,\xi) \sim \sum F^{(\mu)}(t,\xi) \in \mathcal{S}_N^{\ell_1,\infty}\{0,1\}$ and the statement can be understood as perfect diagonalisation modulo $\mathcal{H}_N^{\ell_1,\infty}\{1\}$,

$$\big(\mathrm{D}_t - \mathcal{D}(t,\xi) - R_0(t,\xi)\big)N(t,\xi) = N(t,\xi)\big(\mathrm{D}_t - F(t,\xi)\big) \mod \mathcal{H}_N^{\ell_1,\infty}\{1\}.$$

2.2. Estimates of the fundamental solution

We construct the fundamental solution $\mathcal{E}_k(t,s,\xi)$ within $Z_{hyp}(N)$.

THEOREM 1. *Assume (A1)$_{k-1,2k}$ for some $k \geq 1$. There exists a matrix family $\mathcal{Q}_k(t,s,\xi)$, uniformly bounded and invertible and satisfying*

$$\|\mathrm{D}_\xi^\alpha \mathcal{Q}_k(t,s,\xi)\| \leq C|\xi|^{-|\alpha|}, \tag{9}$$

$$\|\mathrm{D}_\xi^\alpha \mathcal{Q}_k(t,t_\xi,\xi)\| \leq C|\xi|^{-|\alpha|}, \qquad ,|\xi| \leq N, \tag{10}$$

for all $|\alpha| \leq k-1$, such that for all $(t,\xi),(s,\xi) \in Z_{hyp}(N)$

$$\mathcal{E}_k(t,s,\xi) = \exp\left(\mathrm{i}\int_s^t \big(\mathcal{D}(\tau,\xi)+F_{k-1}(\tau,\xi)\big)\mathrm{d}\tau\right)\mathcal{Q}_k(t,s,\xi). \tag{11}$$

Proof. We sketch the main steps of the proof. We denote the exponential in (11) by $\widetilde{\mathcal{E}}_k(t,s,\xi)$. Assumption (A3) implies

$$\|\widetilde{\mathcal{E}}_k(t,s,\xi)\| \lesssim 1 \tag{12}$$

uniformly in $(t,\xi),(s,\xi) \in Z_{hyp}(N)$ regardless of the order of s and t, because $F_{k-1}(t,\xi)$ $-F^{(0)}(t,\xi) \in \mathcal{S}_N^{0,0}\{-1,2\}$ and $\mathcal{D}(t,\xi)$ is real. Furthermore, the transformed equation (8) implies for $\mathcal{Q}_k(t,s,\xi)$ the system

$$\mathrm{D}_t\mathcal{Q}_k(t,s,\xi) = \mathcal{R}_k(t,s,\xi)\mathcal{Q}_k(t,s,\xi), \qquad \mathcal{Q}_k(s,s,\xi) = \mathrm{I}$$

with $\mathcal{R}_k(t,s,\xi) = \widetilde{\mathcal{E}}_k(s,t,\xi)R_k(t,\xi)\widetilde{\mathcal{E}}_k(t,s,\xi)$. This system can be solved by means of the Peano-Baker series

$$\mathcal{Q}_k(t,s,\xi) = \mathrm{I} + \sum_{j=1}^\infty \mathrm{i}^j \int_s^t \mathcal{R}_k(t_1,s,\xi)\int_s^{t_2}\mathcal{R}_k(t_2,s,\xi) \cdots \int_s^{t_{j-1}}\mathcal{R}_k(t_j,s,\xi)\mathrm{d}t_j\cdots\mathrm{d}t_2\mathrm{d}t_1. \tag{13}$$

Using (12) it follows that $\mathcal{R}_k(t,s,\xi)$ satisfies uniform in s the same bounds as $R_k(t,\xi)$ and hence for $k \geq 1$ all integrands are uniformly integrable over the hyperbolic zone. This implies that $\mathcal{Q}_k(t,s,\xi)$ is uniformly bounded,

$$\|\mathcal{Q}_k(t,s,\xi)\| \lesssim \exp\left(\int_s^t R_k(\tau,\xi)\mathrm{d}\tau\right) \lesssim 1,$$

and converges locally uniform in $(s,\xi) \in Z_{hyp}(N)$ to a limit $\mathcal{Q}_k(\infty,s,\xi)$. Furthermore by Liouville theorem,

$$\det\mathcal{Q}_k(t,s,\xi) = \exp\left(\int_s^t \operatorname{trace} R_k(\tau,\xi)\mathrm{d}\tau\right) \simeq 1,$$

and all matrices $\mathcal{Q}_k(t,s,\xi)$ are uniformly invertible over the $Z_{hyp}(N)$.

It remains to obtain symbol type estimates for derivatives of $\mathcal{Q}_k(t,s,\xi)$ with respect to ξ. They are achieved by differentiating (13) term by term using the symbol estimate of $R_k(t,\xi) \in \mathcal{S}_N^{k-1,k-1}\{-k,k+1\}$ in combination with

$$\widetilde{\mathcal{E}}_k(t,s,\xi)R_k(t,\xi)\widetilde{\mathcal{E}}_k(t,s,\xi) \in \mathcal{S}_N^{k-1,k-1}\{-1,2\} \quad \text{uniform in } s$$

and $|\mathrm{D}_\xi^\alpha t_\xi| \le C_\alpha |\xi|^{-1-|\alpha|}$. See [3], [11] or [10] for a more detailed argument. □

REMARK 3. The benefit of applying k steps of diagonalisation is that we obtain symbol type estimates for $k-1$ derivatives of the amplitude $\mathcal{Q}_k(t,s,\xi)$ (provided that we assume sufficient smoothness of $A(t,\xi)$ in t and ξ). If we are satisfied with uniform bounds—which are enough to prove energy estimates—, one step of diagonalisation (i.e., $k=1$ and $(A1)_{0,2}$) is enough.

The following theorem clarifies the rôle of assumption (A3), provided we have knowledge about arbitrary many derivatives.

THEOREM 2. *Assume $(A1)_{0,\infty}$ and (A2). Then assumption (A3) is equivalent to the existence of constants c and C such that*

$$c\|V\| \le \|\mathcal{E}(t,s,\xi)V\| \le C\|V\|, \qquad V \in \mathbb{C}^m,$$

holds true uniformly in $(t,\xi),(s,\xi) \in Z_{hyp}(N)$ for a sufficiently big N.

Sketch of proof. Theorem 1 gives the uniform bound under (A3). Without (A3) equation (12) has to be replaced by a polynomial bound

$$\|\widetilde{\mathcal{E}}_k(t,s,\xi)\|, \|\widetilde{\mathcal{E}}_k(s,t,\xi)\| \le C_k \left(\frac{1+t}{1+s}\right)^K, \qquad t \ge s,$$

where the constant K is independent of k. Similarly, we obtain with the same exponent

$$\|\mathcal{E}_k(t,s,\xi)\| \le \exp\left(\int_s^t \|\mathrm{Im}\,(F_{k-1}(\tau,\xi)+R_k(\tau,\xi))\|\mathrm{d}\tau\right) \le C_k'\left(\frac{1+t}{1+s}\right)^K$$

for all $t \ge s$. Choosing k big enough, the polynomial decay of the remainder $R_k(t,\xi)$ becomes strong enough to compensate all increasing terms and we obtain

$$\mathcal{E}_k(t,s,\xi) = \widetilde{\mathcal{E}}_k(t,s,\xi)\mathcal{Z}_k(s,\xi) - \mathrm{i}\int_t^\infty \widetilde{\mathcal{E}}_k(t,\theta,\xi)R_k(\theta,\xi)\mathcal{E}_k(\theta,s,\xi)\mathrm{d}\theta \tag{14}$$

with

$$\mathcal{Z}_k(s,\xi) = \mathrm{I} + \mathrm{i}\int_s^\infty \widetilde{\mathcal{E}}_k(t,\theta,\xi)R_k(\theta,\xi)\mathcal{E}_k(\theta,s,\xi)\mathrm{d}\theta \lesssim 1.$$

The integral in (14) is bounded by $(1+s)^{K-1}(1+t)^{-K}$, while the first term has the lower bound $(1+s)^K(1+t)^{-K}$. Chosing s big enough implies that $\mathcal{E}_k(t,s,\xi)$ is a small perturbation of $\widetilde{\mathcal{E}}_k(t,s,\xi)$.

Assume now that (A3) is violated. Then we find sequences $t_\mu \to \infty$, s_μ, and ξ_μ such that one matrix entry of the integral in (5) tends to either ∞ or $-\infty$. We consider the $+\infty$ case, and assume w.l.o.g. that $s_\mu > s$ for sufficiently big s and that the matrix entry corresponds to the first diagonal element. Then $\widetilde{\mathcal{E}}_k(t_\mu, s_\mu, \xi_\mu)e_1 \to \infty$ and therefore also $\mathcal{E}(t_\mu, s_\mu, \xi_\mu)N_k(s_\mu, \xi_\mu)M(s_\mu, \xi_\mu)e_1 \to \infty$ which contradicts to the uniform upper bound. Similarly, the $-\infty$ case contradicts to the lower bound and the statement is proven. $\square$

The estimate in the pseudo-differential zone is based on (A4).

LEMMA 3. *Assume (A4). Then the fundamental solution to* (6) *satisfies*

$$\|\mathcal{E}(t,0,\xi)\| \lesssim 1$$

uniform in $(t,\xi) \in Z_{pd}(N)$.

Proof. We fix ξ. Let $V(t)$ be the solution to $\mathrm{D}_t V = A(t,\xi)V$, $V(0) = V_0$. Then with $(\cdot,\cdot)$ the Euclidean inner product on $\mathbb{C}^m$ we obtain from (A4)

$$\frac{\mathrm{d}}{\mathrm{d}t}\|V(t)\|^2 = -2(\operatorname{Im} AV, V) \le 2c|\xi|\|V(t)\|^2$$

for all t with $(t,\xi) \in Z_{pd}(N)$. Hence, by applying Gronwall inequality we obtain

$$\|V(t)\|^2 \le C\|V_0\|^2 \exp(2ct|\xi|) \lesssim \|V_0\|^2.$$

$\square$

Symbol-like estimates for derivatives follow by an inductive argument as used in [3], [11] or [10].

LEMMA 4. *Assume (A1)$_{\ell_1,\ell_2}$, (A4). Then the estimate*

$$\|\mathrm{D}_\xi^\alpha \mathcal{E}(t_\xi, 0, \xi)\| \le C|\xi|^{-|\alpha|}, \qquad |\xi| \le N$$

holds true for any $|\alpha| \le \min(\ell_1, \ell_2 + 1)$.

3. Generalised energy conservation

The results of the previous section with $k = 1$ allow to conclude upper and lower bounds for the energy. We only state the result.

THEOREM 3. *Assume (A1)$_{0,2}$—(A4). Then the solution $U = U(t,x)$ of* (1) *satisfies*

$$\|U(t,\cdot)\|_{L^2(\mathbb{R}^n)} \le C\|U_0\|_{L^2(\mathbb{R}^n)}.$$

Furthermore, $\lim_{t\to\infty} \|U(t,\cdot)\|_{L^2(\mathbb{R}^n)} = 0$ *implies* $U_0 = 0$.

4. Dispersive estimates

We want to explain how to use the information derived in Section 2 to derive dispersive estimates for solutions. We note first, that interesting estimates depend only on the hyperbolic zone. Let for this $\chi \in C_0^\infty(\mathbb{R}^n)$ be a cut-off function, $\chi(\xi) = 1$ for $|\xi| \le 1$, and denote $\chi_{pd}(t,\xi) = \chi((1+t)|\xi|/N)$ and $\chi_{hyp}(t,\xi) = 1 - \chi_{pd}(t,\xi)$.

LEMMA 5. *Assume (A4). Then solution $U = U(t,x)$ to* (1) *satisfies*

$$\|\mathscr{F}^{-1}[\chi_{pd}(t,\xi)\hat{U}(t,\xi)]\|_{L^\infty(\mathbb{R}^n)} \le C(1+t)^{-n}\|U_0\|_{L^1(\mathbb{R}^n)}$$

localised to the pseudo-differential zone $Z_{pd}(N)$ (for any choice of N).

Proof. Based on $\mathscr{F} : L^1(\mathbb{R}^n) \to L^\infty(\mathbb{R}^n)$ and Hölder inequality it is sufficient to estimate $\|\mathcal{E}(t,0,\xi)\chi_{pd}(t,\xi)\|_{L^1(\mathbb{R}^n)} \le \|\mathcal{E}(t,0,\xi)\|_{L^\infty(|\xi|\le\xi_t)}\|\chi_{pd}\|_{L^1(\mathbb{R}^n)}$ and the estimate follows from Lemma 3 and the geometry of the zone. $\square$

This estimate is much stronger than any estimate we could expect for the solution $U(t) = \mathscr{F}^{-1}[\mathcal{E}(t,0,\xi)\mathscr{F}U_0]$ itself. Therefore, we concentrate on the remaining hyperbolic zone. By Theorem 1 we know that solutions are represented as Fourier integrals of a particular form,

$$\mathscr{F}^{-1}[\chi_{hyp}(t,\xi)\hat{U}(t,\xi)] = \sum_{j=1}^{m}\int \mathrm{e}^{\mathrm{i}(x\cdot\xi+t\vartheta_j(t,\xi))}B_j(t,\xi)\hat{U}_0(\xi)\mathrm{d}\xi, \tag{15}$$

where the matrix-valued symbol $B_j(t,\xi)$ contains all contributions from the matrices $\mathcal{Q}_k(t,t_\xi,\xi)$, $\mathcal{E}(t_\xi,0,\xi)$, $N_k(t_\xi,\xi)M(t_\xi,\xi)$, $M^{-1}(t,\xi)N^{-1}(t,\xi)$ and $F_{k-1}(t,\xi)$ and is supported within $Z_{hyp}(N)$. Under $(\mathrm{A1})_{k-1,2k}$–(A4) it satisfies

$$\|\mathrm{D}_\xi^\alpha B_j(t,\xi)\| \le C|\xi|^{-|\alpha|}, \qquad |\alpha| \le k-1,$$

k the number of diagonalisation steps used in the construction. The phase function is real, homogeneous in ξ and given by

$$\vartheta_j(t,\xi) = \frac{1}{t}\int_0^t \lambda_j(\theta,\xi)\mathrm{d}\theta.$$

Fourier integrals of this type can be estimated generalising ideas of Sugimoto, [8], [9]. He introduced for a closed surface Σ two indices

$$\gamma_0(\Sigma) = \sup_{p\in\Sigma}\inf_{\eta\in\mathrm{T}_p\Sigma}\gamma(\Sigma;p,\eta), \qquad \gamma(\Sigma) = \sup_{p\in\Sigma}\sup_{\eta\in\mathrm{T}_p\Sigma}\gamma(\Sigma;p,\eta),$$

where for any tangent vector η on the surface the number $\gamma(\Sigma;p,\eta)$ denotes the order of contact between the tangent $p+\eta\mathbb{R}$ and $\Sigma\cap(p+\eta\mathbb{R}\oplus\mathrm{N}_p\Sigma)$. We will give two estimates related to the statements of [8], [9], taking into account the improvements of [5].

THEOREM 4. *Let $\Sigma \subset \mathbb{R}^n$ be a smooth closed surface of codimension* 1.

1. *Let $\gamma_0 = \gamma_0(\Sigma)$. Then it holds for all $f \in C^1(\Sigma)$*
$$\left| \int_\Sigma e^{ix\cdot\xi} f(\xi) d\xi \right| \le C\langle x\rangle^{-\frac{1}{\gamma_0}} \|f\|_{C^1}.$$
2. *Assume Σ is convex. Then with $\gamma = \gamma(\Sigma)$ and $r = \lceil (n-1)/\gamma \rceil + 1$ the estimate*
$$\left| \int_\Sigma e^{ix\cdot\xi} f(\xi) d\xi \right| \le C\langle x\rangle^{-\frac{n-1}{\gamma}} \|f\|_{C^r}$$
holds true for all $f \in C^r(\Sigma)$.

REMARK 4. It is enough to have $\Sigma \in C^{\gamma+1}$ in order to prove these statements. The original proof of Sugimoto for part 2, [8], uses real analyticity of the surface Σ, which was improved by [7], [5].

In order to derive dispersive estimates for the expressions in (15), we introduce the t-dependent family of level sets
$$\Sigma_t^{(j)} = \{\xi \in \mathbb{R}^n \mid \vartheta_j(t,\xi) = 1\}.$$
We restrict for the sake of simplicity to the case of convex surfaces. Then our estimates are based on the following assumption:

(B) The surfaces $\Sigma_t^{(j)}$ are strictly convex for all $t \ge t_0$ and converge in C^{γ_j+1} to a surface $\Sigma^{(j)}$ with $\gamma(\Sigma^{(j)}) = \gamma_j$.

THEOREM 5. *Assume (A1)$_{\ell,2k}$–(A4) in combination with (B) and let $\gamma_{\max} = \max_j \gamma(\Sigma^{(j)})$. If $\ell \ge k-1 \ge \frac{n-1}{\gamma_{\max}} + 1$, $\ell \ge \gamma_{\max} + 1$ then the dispersive estimate*
$$\|U(t,\cdot)\|_{L^\infty} \le C(1+t)^{-\frac{n-1}{\gamma_{\max}}(\frac{1}{p}-\frac{1}{q})} \|U_0\|_{H^{r,p}(\mathbb{R}^n)}$$
holds true for any solution $U = U(t,x)$ of (1) *where $p \in [1,2]$, $pq = p+q$ and $r > n(1/p - 1/q)$.*

REMARK 5. The stabilisation assumption (B) can be weakened to a uniformity assumption, in such a sense that for sufficiently big $t \ge t_0$ the constants appearing in the corresponding estimates of Theorem 4 are uniform in t.

REMARK 6. The corresponding result for non-convex surfaces holds true, but gives a much weaker decay rate.

5. Concluding remarks

1 If $A_0(t,\xi)$ is symmetric, the diagonaliser $M(t,\xi)$ can be chosen unitary and therefore $(D_t M^{-1})M$ is self-adjoint. If in addition $A(t,\xi) = A_0(t,\xi)$ is assumed to be homogeneous in ξ assumptions (A3) and (A4) are satisfied.

If we assume that $A(t,\mathrm{D})$ is a differential operator —which is a very restrictive assumption here—, we have a representation $A(t,\xi) = A_0(t,\xi) + A_1(t)$ and (A4) is equivalent to dissipativity, $\mathrm{Im} A_1(t) \geq 0$. If $A_0(t,\xi)$ is symmetric, (A3) reduces to the integrability of $\mathrm{Im}\,\mathrm{diag}\,(M^{-1}(t,\xi)A_1(t)M(t,\xi)) \geq 0$.

2 The results apply to hyperbolic equations of higher order. We consider a homogeneous equation of order m,

$$\mathrm{D}_t^m u + \sum_{k=0}^{m-1} \sum_{|\alpha|=m-k} a_{k,\alpha}(t)\mathrm{D}_t^k \mathrm{D}_x^\alpha u = 0, \qquad \mathrm{D}_t^k u(0,\cdot) = u_k, \tag{16}$$

with $a_{k,\alpha} \in \mathcal{S}^{*,\ell}\{0,0\}$ and assume uniform strict hyperbolicity. We rewrite it as a system in companion form, its eigenvalues $\lambda_j(t,\xi)$ are given by the (real) characteristic roots associated to (16). Assumption (A4) follows from homogeneity, assumption (A3) is equivalent to

$$\max_{j=1,\dots,m} \sup_{T>0,\omega\in\mathbb{S}^{n-1}} \left| \int_0^T \sum_{k\neq j} \frac{\partial_t \lambda_j(t,\omega)}{\lambda_j(t,\omega) - \lambda_k(t,\omega)} \mathrm{d}t \right| < \infty. \tag{17}$$

This assumption is necessary to have a generalised energy conservation for (16) under the symbol assumption $a_{k,\alpha} \in \mathcal{S}^{*,\infty}\{0,0\}$. In the treatment of [2] the condition $\partial_t a_{k,\alpha}(t) \in L^1(\mathbb{R}_+)$ implies (17).

Equations of higher order with arbitrary lower order terms but constant coefficients were considered in [6] and [7].

3 Most of the considerations transfer to problems bearing fast oscillations in the classification of Reissig-Yagdjian [3], [4]. The only major difference is that the corresponding statement of Theorem 2 is no longer valid.

It is an interesting question whether one can generalise the approach of [1] to higher order equations and larger systems. In this case the estimates for time-derivatives are weakened to an improvement of the form $(1+t)^{-p}$, $p < 1$ instead of $p = 1$ from Definition 1, but accompanied with a so-called stabilisation condition to treat an extended pseudo-differential zone.

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