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GLOBAL HYPOELLIPTICITY AND SOBOLEV ESTIMATES FOR GENERALIZED SG-PSEUDO-DIFFERENTIAL OPERATORS

Abstract. We prove global hypoellipticity and maximal Sobolev estimates for a class of generalized SG-pseudo-differential operators. Applications are given to partial differential operators with polynomial coefficients in $\mathbb{R}^n$.

1. Introduction

In this paper we consider linear partial differential operators or pseudo-differential operators $P = p(x,D)$ globally defined in $\mathbb{R}^n$, and we study for them the problem of the global hypoellipticity. Namely, writing $\mathcal{S}(\mathbb{R}^n)$ for the Schwartz space of the rapidly decaying functions and $\mathcal{S}'(\mathbb{R}^n)$ for its dual, we assume that $P : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ extends to a map $\mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$. We then recall that P is globally hypoelliptic if, for any given $f \in \mathcal{S}(\mathbb{R}^n)$, all the solutions $u \in \mathcal{S}'(\mathbb{R}^n)$ of the equation $Pu = f$ belong to $\mathcal{S}(\mathbb{R}^n)$. This definition does not imply the local regularity of the solutions of P, however it is natural to limit attention to operators P which are already known to be hypoelliptic in the Schwartz sense, i.e. $Pu \in C^\infty(\Omega)$ for $u \in \mathcal{D}'(\Omega)$ implies $u \in C^\infty(\Omega)$, for every $\Omega \subset \mathbb{R}^n$. Relevant classes of globally hypoelliptic operators were identified by Shubin [10], taking as basic example the harmonic oscillator of Quantum Mechanics $P = -\Delta + |x|^2$. A generalization of the classes of Shubin was then given by Boggiatto, Buzano, Rodino [3]. In a somewhat different direction, Parenti [9] and Cordes [5] introduced the so-called SG-classes. In the present paper we present $SG^m_{\rho,\delta}$ extensions of SG-classes, allowing new applications to global hypoellipticity.

Basic example, where to test the previous contributions and our results, is given by partial differential operators with polynomial coefficients in $\mathbb{R}^n$

$$P = \sum c_{\alpha,\beta} x^\alpha D^\beta \qquad (D_{x_j} = -i\partial_{x_j}) \tag{1}$$

where the coefficients $c_{\alpha,\beta}$ are complex numbers, and in the sum α, β run over a finite set of indices. Consider the symbol

$$p(x,\xi) = \sum c_{\alpha,\beta} x^\alpha \xi^\beta.$$

The Newton's polyhedron $\mathcal{N}_p$ of the polynomial $p(x,\xi)$ is defined as the convex hull in $\mathbb{R}^{2n}$ of all multi-indices $(\alpha,\beta) \in \mathbb{N}^{2n}$, with $c_{\alpha,\beta} \neq 0$, and the origin (see for example Gindikin, Volevich [7]). We then associate to $p(x,\xi)$ the function on $\mathbb{R}^{2n}$

$$\lambda(x,\xi) = \sum_{(\alpha,\beta)\in\mathcal{N}_p} \left| x^\alpha \xi^\beta \right|$$

and we say that P, or $p(x,\xi)$, is λ-elliptic if

$$c\lambda(x,\xi) \leq |p(x,\xi)| \leq C\lambda(x,\xi) \tag{2}$$

for positive constants c and C and large $|x|+|\xi|$. The λ-ellipticity grants the global ellipticity of P, if $\mathcal{N}_p$ satisfies suitable geometric properties. As basic example of polyhedron having a "good geometry" we have from Shubin [10] the simplex

$$\mathcal{N}_p = \{(\alpha,\beta),\, |\alpha|+|\beta| \leq m\} \tag{3}$$

where m is a fixed positive integer, and we let α,β run here in $\mathbb{R}^{2n}$. In Boggiatto, Buzano, Rodino [3], as generalization of (3), the Newton's polyhedron $\mathcal{N}_p$ was assumed complete, this means that it does not contain sides parallel to coordinate axes and the outer normals to the non-coordinate faces have strictly positive components. In the *SG*-case of Parenti [9] and Cordes [5] we have

$$\mathcal{N}_p = \{(\alpha,\beta),\, |\alpha| \leq m_2,\, |\beta| \leq m_1\} \tag{4}$$

where $m_1 \geq 0$, $m_2 \geq 0$ are fixed integers. Note that the polyhedron in (4) is not complete, since the outer normals to the non-coordinate faces contain zero components. Hence the results of Boggiatto, Buzano, Rodino [3] cannot be applied, nevertheless the corresponding λ-elliptic operators are globally hypoelliptic. In the present paper we shall go further with respect to the *SG*-case, considering for a fixed γ, $0<\gamma<1$:

$$P = \sum_{\substack{0\leq\beta\leq m_1\\ 0\leq\alpha\leq\gamma\beta}} c_{\alpha,\beta}x^{\alpha}D^{\beta} \tag{5}$$

where for simplicity we assume $\alpha\in\mathbb{N}$, $\beta\in\mathbb{N}$, i.e. P is an ordinary differential operator. Let us assume $m_2=\gamma m_1$ is an integer. The corresponding Newton's polyhedron in $\mathbb{R}^2$

$$\mathcal{N}_p = \{(\alpha,\beta),\, 0\leq\alpha\leq\gamma\beta,\, 0\leq\beta\leq m_1\} \tag{6}$$

has a face, namely the segment connecting $(0,0)$ and (m_2,m_1), with outer normal containing a negative component. The λ-ellipticity condition (2) can be written in a simplified form:

$$c\,(1+|\xi^{m_1}|+|x^{\gamma m_1}\xi^{m_1}|) \leq |p(x,\xi)| \leq C\,(1+|\xi^{m_1}|+|x^{\gamma m_1}\xi^{m_1}|)\,. \tag{7}$$

As an application of our pseudo-differential calculus, we shall obtain that P in (5) is globally hypoelliptic, under the assumption (7).

Consider as an example

$$P = \left(1+x^2\right)D^4+1$$

with symbol

$$p(x,\xi) = \left(1+x^2\right)\xi^4+1$$

of the form (5) with $m_1 = 4$, $\gamma = 1/2$. The condition (7) is obviously satisfied. Our result cannot be extended to the case when $\gamma \geq 1$ in (5). As counter-example, consider the operator

$$P = \left(1+x^2\right)D^2 + 2. \tag{8}$$

The symbol $p(x,\xi) = \left(1+x^2\right)\xi^2 + 2$ is λ-elliptic with respect to

$$\mathcal{N}_p = \{(\alpha,\beta), 0 \leq \alpha \leq \beta, 0 \leq \beta \leq 2\},$$

that is we are in the case (6), but $\gamma = 1$. The corresponding homogeneous equation

$$Pu = -\left(1+x^2\right)u'' + 2u = 0$$

admits the solution

$$u(x) = 1 + x^2 \in \mathcal{S}'(\mathbb{R}),$$

which does not belong to $\mathcal{S}(\mathbb{R})$. Therefore P in (8) is not globally hypoelliptic. Let us conclude by observing that the characterization of the polyhedrons $\mathcal{N}_p$ for which λ-ellipticity grants global hypoellipticity, as well as the general characterization of the operators with polynomial coefficients (1) which are globally hypoelliptic, are widely open problems.

The contents of the paper are the following. In the next Section 2 we present our $SG^m_{\rho,\delta}$-pseudo-differential calculus. It is more general than the calculus of Cordes [5] and it does not enter the results of Beals [1]. However, it can be seen as a particular case of the so-called Weyl-Hörmander calculus, see Hörmander [6]. For this reason we omit the proofs, and emphasize only the peculiarities of the construction of the parametrices.

In the last part of this work, Section 4, we focus our attention to SG Sobolev spaces H^s of Cordes [5]: we prove that an operator with symbol in $SG^m_{\rho,\delta}$ is continuous from H^s into H^{s-m}. Finally, we define a Sobolev space H^* modelled on (5) and we show a maximal inequality that tie up H^*-norm with L^2-norm plus a Sobolev norm. To prove this assertion, proporties of Newton's polyhedron are fundamental. As a consequence of the maximal estimate, we obtain the Fredholm property of the map $P : H^* \to L^2$.

Recently Cappiello, Gramchev, Rodino [4] have proved that, in the case when the symbols of the pseudo-differential operators $SG^m_{\rho,\delta}$ satisfy additional analytic bounds, the regularity in $\mathcal{S}(\mathbb{R}^n)$ of the solution can be replaced by regularity in Gelfand-Shilov classes, expressing holomorphic extension and exponential decay. Via our results in Section 3, it is possible to apply the results of Cappiello, Gramchev, Rodino [4] to the operators with polynomial coefficients (5).

2. $SG^m_{\rho,\delta}$ operators

We introduce some notational conventions. Let $m = (m_1, m_2) \in \mathbb{R}^2$, and let $\rho_1, \rho_2, \delta_1, \delta_2$ be real numbers with $0 \leq \delta_j < \rho_j \leq 1$, $j = 1,2$, and denote $\rho = (\rho_1, \rho_2)$, $\delta = (\delta_1, \delta_2)$. Let $e_1 = (1,0)$, $e_2 = (0,1)$, $e = (1,1)$.

We define pseudo-differential operators of class $SG^m_{\rho,\delta}$ and investigate their basic properties.

DEFINITION 1. *We say that a function* $p(x,\xi)\in C^\infty(\mathbb{R}^{2n})$ *is a symbol of class* $SG^m_{\rho,\delta}$ *if there exists a positive constant* $C_{\alpha,\beta}$ *such that*

$$\left|D^\alpha_\xi D^\beta_x p(x,\xi)\right| \le C_{\alpha,\beta}\langle\xi\rangle^{m_1-\rho_1|\alpha|+\delta_1|\beta|}\langle x\rangle^{m_2-\rho_2|\beta|+\delta_2|\alpha|}$$

for every $(x,\xi)\in\mathbb{R}^{2n}$ *and* $\alpha,\beta\in\mathbb{N}^n$, *where* $\langle\xi\rangle=\left(1+|\xi|^2\right)^{1/2}$ *and* $\langle x\rangle=\left(1+|x|^2\right)^{1/2}$.

Some classes were treated by Cordes [5] under the assumption $\delta_2=0$.

We also define

$$SG^{-\infty}=\bigcap_{m\in\mathbb{R}^2}SG^m_{\rho,\delta}=\mathcal{S}(\mathbb{R}^{2n}),\quad SG^{\infty}=\bigcup_{m\in\mathbb{R}^2}SG^m_{\rho,\delta},$$

where $\mathcal{S}(\mathbb{R}^{2n})$ is the Schwartz space.

For $p(x,\xi)\in SG^m_{\rho,\delta}$ we define the semi-norms $N_{\alpha,\beta}(p)$, by

$$N_{\alpha,\beta}(p)=\sup_{(x,\xi)\in\mathbb{R}^{2n}}\langle\xi\rangle^{-m_1+\rho_1|\alpha|-\delta_1|\beta|}\langle x\rangle^{-m_2+\rho_2|\beta|-\delta_2|\alpha|}\left|D^\alpha_\xi D^\beta_x p(x,\xi)\right|.$$

Then $SG^m_{\rho,\delta}$ is a Fréchet space with respect to the topology induced by these semi-norms.

It is easy to prove that the following relations hold:

1. If $m_j\le m'_j$, $\rho'_j\le\rho_j$, $\delta_j\le\delta'_j$, $j=1,2$, then $SG^m_{\rho,\delta}\subseteq SG^{m'}_{\rho',\delta'}$.

2. If $p\in SG^m_{\rho,\delta}$, then $D^\alpha_\xi D^\beta_x p(x,\xi)\in SG^{m''}_{\rho,\delta}$ with

$$m''=m-(\rho_1|\alpha|-\delta_1|\beta|)e_1-(\rho_2|\beta|-\delta_2|\alpha|)e_2.$$

Corresponding to $p(x,\xi)\in SG^m_{\rho,\delta}$, we define the pseudo-differential operator $P=p(x,D)$ by the standard formula

$$Pu(x)=p(x,D)u(x)=(2\pi)^{-n}\int_{\mathbb{R}^n}e^{i\langle x,\xi\rangle}p(x,\xi)\widehat{u}(\xi)d\xi,\quad u\in\mathcal{S}(\mathbb{R}^n),\tag{9}$$

where $\widehat{u}$ is the Fourier transform.

We shall denote by $OPSG^m_{\rho,\delta}$ the space of all operators of the form (9), with symbols in $SG^m_{\rho,\delta}$.

For a symbol $p\in SG^{m^0}_{\rho,\delta}$ and a sequence of symbols p_j, $j\in\mathbb{N}$, $p_j\in SG^{m^j}_{\rho,\delta}$, $m^j=(m^j_1,m^j_2)$, $m^0_k>m^1_k>\ldots>m^j_k>\ldots,m^j_k\to-\infty$, as $j\to\infty$, $k=1,2$, we will say that p

has the asymptotic expansion $\sum_{j=0}^{\infty} p_j$, written as $p \sim \sum_{j=0}^{\infty} p_j$, if for every $N \in \mathbb{N}$ we have $p - \sum_{j=0}^{N-1} p_j \in SG^{m^N}_{\rho,\delta}$.

The following theorems can be proved making simple changes to the proofs in Cordes [5], in Parenti [9] or else by applying the general Weyl-Hörmander calculus in [6], so we omit the proofs.

THEOREM 1. *Given $p \in SG^m_{\rho,\delta}$, the operator P defined by (9) is linear and continuous from the Schwartz space $\mathcal{S}$ into itself. Furthermore, P can be extended to a linear and continuous map from $\mathcal{S}'$ into itself.*

THEOREM 2. *Let $p_j \in SG^{m^j}_{\rho,\delta}$, $j \in \mathbb{N}$, where $m^j = (m^j_1, m^j_2,)$, with*

$$m^0_k > m^1_k > m^2_k > \dots, \quad m^j_k \to -\infty, \text{ as } j \to \infty, \quad k = 1,2.$$

There exists a symbol $p \in SG^{m^0}_{\rho,\delta}$ such that

$$p \sim \sum_{j=0}^{\infty} p_j.$$

THEOREM 3. *Let $P = p(x,D) \in OPSG^{m^1}_{\rho,\delta}$, $Q = q(x,D) \in OPSG^{m^2}_{\rho,\delta}$. Then, there exists a symbol $r \in SG^{m^1+m^2}_{\rho,\delta}$ such that $q(x,D)p(x,D) = r(x,D)$. Furthermore*

$$r(x,\xi) \sim \sum_{\alpha} \frac{1}{\alpha!} \partial^{\alpha}_{\xi} q(x,\xi) D^{\alpha}_x p(x,\xi).$$

DEFINITION 2. *A symbol $p \in SG^m_{\rho,\delta}$ is said to be hypoelliptic if there exist $R, C_0, C_{0,\alpha,\beta} > 0$ and $m' = (m'_1, m'_2) \in \mathbb{R}^2$ such that*

$$|p(x,\xi)| \geq C_0 \langle\xi\rangle^{m'_1} \langle x\rangle^{m'_2} \tag{10}$$

and

$$\left| D^{\alpha}_{\xi} D^{\beta}_x p(x,\xi) \right| \leq C_{0,\alpha,\beta} \, |p(x,\xi)| \, \langle\xi\rangle^{-\rho_1|\alpha|+\delta_1|\beta|} \langle x\rangle^{-\rho_2|\beta|+\delta_2|\alpha|} \tag{11}$$

for all $\alpha, \beta \in \mathbb{N}^n$ and $(x,\xi) \in \mathbb{R}^{2n}$ with $|x| + |\xi| > R$.

DEFINITION 3. *A symbol $p \in SG^m_{\rho,\delta}$ is called elliptic if it satisfies* (10) *with $m' = m$.*

Let us observe that (10) with $m' = m$ implies (11); therefore an elliptic symbol is also hypoelliptic.

We say that Q is a parametrix of P if we have $QP - I \in OPSG^{-\infty}$ and $PQ - I \in OPSG^{-\infty}$. We speak of a left (right) parametrix Q of P if only the second (first) condition holds.

THEOREM 4. *Let P be an operator with a hypoelliptic symbol $p \in SG^m_{\rho,\delta}$. Then, it admits a parametrix Q with symbol $q \in SG^{-m'}_{\rho,\delta}$.*

Proof. As standard we define a sequence of symbols q_j ($j \geq 1$) inductively by

$$
\begin{cases}
q_0(x,\xi) = \dfrac{1}{p'(x,\xi)} \\
q_j(x,\xi) = -\left\{ \displaystyle\sum_{\substack{|\gamma| = j-k \\ k<j}} \frac{1}{\gamma!} \partial_\xi^\gamma q_k D_x^\gamma p' \right\} q_0, \quad j = 1, 2, \dots,
\end{cases}
\tag{12}
$$

for every $(x,\xi) \in \mathbb{R}^{2n}$, where $p'(x,\xi) = p(x,\xi)$ for $|x| + |\xi| \geq R$, $p'(x,\xi) \in C^\infty(\mathbb{R}^{2n})$ with $p'(x,\xi) \neq 0$. It is easy to prove that $q_j \in SG^{-m'-(\rho-\delta)je}_{\rho,\delta}$ by using (10), (11). Then, by Theorem 2 there exists an operator $Q = q(x,D) \in OPSG^{-m'}_{\rho,\delta}$ with symbol $q(x,\xi) \sim \sum_{j=0}^{\infty} q_j(x,\xi)$. Applying Theorem 3 we can easy verify that Q is a parametrix of P. □

THEOREM 5. *Let p be a hypoelliptic symbol in $SG^m_{\rho,\delta}$ and let $f \in \mathcal{S}$. If $u \in \mathcal{S}'$ is a solution of the corresponding equation $Pu = f$, then $u \in \mathcal{S}$. That is, P is globally hypoelliptic.*

Proof. By Theorem 4 there exist an operator Q with symbol in $SG^{-m'}_{\rho,\delta}$ and an operator R with symbol in $SG^{-\infty} = \mathcal{S}\left(\mathbb{R}^{2n}\right)$ such that

$$Qf = QPu = u + Ru.$$

We have

$$u = Qf - Ru.$$

Since a pseudo-differential operator with symbol in $\mathcal{S}\left(\mathbb{R}^{2n}\right)$ is regularizing, i.e. it maps $\mathcal{S}'\left(\mathbb{R}^{2n}\right)$ into $\mathcal{S}\left(\mathbb{R}^{2n}\right)$, then $u \in \mathcal{S}$. □

EXAMPLE 1. The symbol $p(x,\xi) = \langle x \rangle^\gamma \xi + i$, with $(x,\xi) \in \mathbb{R}^2$ is hypoelliptic as symbol in $SG^{(1,\gamma)}_{(1,1),(0,\gamma)}$ if $0 \leq \gamma < 1$. In fact, the condition (10) is fulfilled for $m' = (1,0)$. Moreover,

$$\left|D_\xi p(x,\xi)\right| = \langle x \rangle^\gamma = \frac{1+|\xi|}{1+|\xi|} \langle x \rangle^\gamma \leq \frac{|p(x,\xi)|}{1+|\xi|} \langle x \rangle^\gamma \sim |p(x,\xi)| \langle x \rangle^\gamma \langle \xi \rangle^{-1}$$

and

$$
\begin{aligned}
|D_x p(x,\xi)| &\sim |x| \langle x \rangle^{\gamma-2} |\xi| = \frac{|x|}{\langle x \rangle^2} \langle x \rangle^\gamma |\xi| \leq \frac{|x|}{\langle x \rangle^2} \left(\langle x \rangle^\gamma |\xi| + 1\right) \\
&\sim \frac{|x|}{\langle x \rangle^2} |p(x,\xi)| \leq \frac{|x|+1}{\langle x \rangle^2} |p(x,\xi)| \sim \langle x \rangle^{-1} |p(x,\xi)|
\end{aligned}
$$

when $|x|+|\xi|$ is large. It is easy to verify that also higher order derivatives satisfy (11).

3. Newton's polyhedron

Consider a polynomial
$$p(x,\xi)=\sum c_{\alpha,\beta}x^{\alpha}\xi^{\beta}$$
in the variable $(x,\xi)\in\mathbb{R}^2$, with constant complex coefficients $c_{\alpha,\beta}\in\mathbb{C}$.

Let $\mathbb{R}^2_+$ be the positive quadrant in the plane:
$$\mathbb{R}^2_+=\left\{(\alpha,\beta)\in\mathbb{R}^2,\alpha\geq 0,\beta\geq 0\right\}.$$

DEFINITION 4. *Let $p(x,\xi)$ be defined as before and consider the finite set of points of $\mathbb{R}^2_+$ $A=\left\{(\alpha,\beta)\in\mathbb{R}^2_+,c_{\alpha,\beta}\neq 0\right\}$. The Newton's polyhedron $\mathcal{N}_p$ of the polynomial $p(x,\xi)$ is the convex hull in $\mathbb{R}^2_+$ of $A\cup\{0\}$.*

We shall deal with polynomials in two variables of type:
$$p(x,\xi)=\sum_{\substack{0\leq\beta\leq m_1\\ 0\leq\alpha\leq\gamma\beta}}c_{\alpha,\beta}x^{\alpha}\xi^{\beta} \tag{13}$$
with $0<\gamma<1$. We define
$$\lambda(x,\xi)=1+|\xi^{m_1}|+|x^{\gamma m_1}\xi^{m_1}|\sim\sqrt{1+\xi^{2m_1}+x^{2\gamma m_1}\xi^{2m_1}}\sim\sum_{\substack{0\leq\beta\leq m_1\\ 0\leq\alpha\leq\gamma\beta}}\left|x^{\alpha}\xi^{\beta}\right|$$
and we suppose that there exists a constant $C>0$ such that
$$|p(x,\xi)|\geq C\lambda(x,\xi).$$
We then say that the polynomial (13) satisfies the condition of λ-ellipticity. Under these hypotheses it is obvious that $|p(x,\xi)|\sim\lambda(x,\xi)$ and $C_{0,m_1}\neq 0$, $C_{\gamma m_1,m_1}\neq 0$.

Hence the Newton's polyhedron $\mathcal{N}_p$ of the polynomial (13), given by (6), is a right triangle with vertices $\mathcal{V}_p=\{(0,0),(0,m_1),(\gamma m_1,m_1)\}$.

We can prove the following result:

PROPOSITION 1. *If the symbol* (13) *is λ-elliptic and $\gamma<1$, then it is hypoelliptic in $SG^{(m_1,\gamma m_1)}_{(1,1),(0,\gamma)}$.*

To prove Proposition 1, we need a preliminary result.

LEMMA 1. *Given $x\in\left(\mathbb{R}^+_0\right)^k$, a finite subset $A\subset\left(\mathbb{R}^+_0\right)^k$ and a convex linear combination $\beta=\sum_{\alpha\in A}c_\alpha\alpha$, we have that*
$$x^{\beta}\leq\sum_{\alpha\in A}c_\alpha x^{\alpha}.$$

A proof of this lemma can be found in Boggiatto-Buzano-Rodino [3].

Proof of Proposition 1. It is obvius that the condition (10) is satisfied with $m' = (m_1, 0)$. Now we prove that the polynomial satysfies the condition (11) for the derivative of order one, the others inequalities follow immediately. We have

$$\begin{aligned}|D_x p(x,\xi)| &= \left| \sum_{\substack{0\le\beta\le m_1\\ 0<\alpha\le\gamma\beta}} \alpha c_{\alpha,\beta} x^{\alpha-1}\xi^\beta \right| \frac{1+|x|}{1+|x|} \\ &\sim \langle x\rangle^{-1} \left| \sum_{\substack{0\le\beta\le m_1\\ 0<\alpha\le\gamma\beta}} \alpha c_{\alpha,\beta} x^{\alpha-1}\xi^\beta (1+|x|) \right| \\ &\le \langle x\rangle^{-1} \left(\sum_{\substack{0\le\beta\le m_1\\ 0<\alpha\le\gamma\beta}} \alpha \left|c_{\alpha,\beta}\right| \left|x^{\alpha-1}\xi^\beta\right| + \sum_{\substack{0\le\beta\le m_1\\ 0<\alpha\le\gamma\beta}} \alpha \left|c_{\alpha,\beta}\right| \left|x^{\alpha}\xi^\beta\right| \right).\end{aligned}$$

Every term $\left|x^{\alpha-1}\xi^\beta\right|$ is such that $(\alpha-1,\beta) \in \mathcal{N}_p$. Using Lemma 1 and the λ-ellipticity of $p(x,\xi)$ we obtain

$$\left|x^{\alpha-1}\xi^\beta\right| \le \lambda(x,\xi) \le C\,|p(x,\xi)|\,.$$

Applying the same argument to the terms $\left|x^{\alpha}\xi^\beta\right|$, we have

$$\left|x^{\alpha}\xi^\beta\right| \le \lambda(x,\xi) \le |p(x,\xi)|\,.$$

Therefore we can conclude that:

$$|D_x p(x,\xi)| \le C\,|p(x,\xi)|\,\langle x\rangle^{-1}.$$

With analogous reasonings we have

$$\begin{aligned}\left|D_\xi p(x,\xi)\right| &= \left| |x|^\gamma \sum_{\substack{0<\beta\le m_1\\ 0\le\alpha\le\gamma\beta}} \beta c_{\alpha,\beta} x^\alpha |x|^{-\gamma} \xi^{\beta-1} \right| \frac{1+|\xi|}{1+|\xi|} \\ &\le \left(1+|x|^\gamma\right) \sum_{\substack{0<\beta\le m_1\\ 0\le\alpha\le\gamma\beta}} \beta \left|c_{\alpha,\beta}\right| |x|^{\alpha-\gamma} |\xi|^{\beta-1} \frac{1+|\xi|}{1+|\xi|} \\ &\sim \langle x\rangle^\gamma \langle\xi\rangle^{-1} \left(\sum_{\substack{0<\beta\le m_1\\ 0\le\alpha\le\gamma\beta}} \beta \left|c_{\alpha,\beta}\right| |x|^{\alpha-\gamma} |\xi|^{\beta-1} + \sum_{\substack{0<\beta\le m_1\\ 0\le\alpha\le\gamma\beta}} \beta \left|c_{\alpha,\beta}\right| |x|^{\alpha-\gamma} |\xi|^{\beta} \right).\end{aligned}$$

We have take out the term $|x|^\gamma$ from the sum, therefore all the terms contained in the two sums are associated to a point that belongs to the Newton's polyhedron $\mathcal{N}_p$ and we can apply the same argument used before. So we conclude that

$$\left|D_\xi p(x,\xi)\right| \le C\,|P(x,\xi)|\,\langle x\rangle^\gamma \langle\xi\rangle^{-1}.$$

$\square$

4. Sobolev estimates

DEFINITION 5. *The Sobolev space H^s, $s=(s_1,s_2)\in\mathbb{R}^2$ is defined by*

$$H^s = \left\{u \in \mathcal{S}'/\langle x\rangle^{s_2}\langle D\rangle^{s_1} u \in L^2(\mathbb{R}^n)\right\}.$$

The Sobolev space H^s is a Banach space with the norm

$$\|u\|_{H^s} = \|\langle x\rangle^{s_2}\langle D\rangle^{s_1} u\|_{L^2}, \quad u \in H^s. \tag{14}$$

In particular the space H^s is a Hilbert space with the inner product

$$(u,v)_{H^s} = (\langle x\rangle^{s_2}\langle D\rangle^{s_1} u, \langle x\rangle^{s_2}\langle D\rangle^{s_1} v)_{L^2}, \quad u,v \in H^s.$$

Note that the pseudo-differential operator $\langle x\rangle^{s_2}\langle D\rangle^{s_1} \in OPSG^s_{\rho,\delta}$ is invertible, as an operator from $\mathcal{S}$ into itself (or from $\mathcal{S}'$ into itself), with inverse $\langle D\rangle^{-s_1}\langle x\rangle^{-s_2} \in OPSG^{-s}_{\rho,\delta}$.
In particular $\langle x\rangle^{s_2}\langle D\rangle^{s_1}$ is elliptic of order s.
From (14) we conclude that $\langle x\rangle^{s_2}\langle D\rangle^{s_1}$ is an isometry from H^s into L^2.

We now prove the boundedness of the $OPSG^m_{\rho,\delta}$ operators on the spaces H^s, following the lines of Parenti [9], Cordes [5].

PROPOSITION 2. *Given $p(x,\xi)\in SG^m_{\rho,\delta}$, the operator P defined by* (9) *is continuous from H^s into H^{s-m} if and only if the operator*

$$\langle x\rangle^{s_2-m_2}\langle D\rangle^{s_1-m_1} P \langle D\rangle^{-s_1}\langle x\rangle^{-s_2}$$

is continuous from L^2 into L^2.

Proof. Let $u\in H^s$. There exists $v\in L^2$ such that

$$u = \langle D\rangle^{-s_1}\langle x\rangle^{-s_2} v \tag{15}$$

and we have

$$\|u\|_{H^s} = \left\|\langle D\rangle^{-s_1}\langle x\rangle^{-s_2} v\right\|_{H^s} = \|v\|_{L^2}.$$

Using (15) and (14), we obtain

$$\begin{aligned}\|Pu\|_{H^{s-m}} &= \left\|\langle x\rangle^{s_2-m_2}\langle D\rangle^{s_1-m_1} Pu\right\|_{L^2}\\ &= \left\|\langle x\rangle^{s_2-m_2}\langle D\rangle^{s_1-m_1} P\langle D\rangle^{-s_1}\langle x\rangle^{-s_2} v\right\|_{L^2}.\end{aligned}$$

We can conclude that

$$\|Pu\|_{H^{s-m}} \leq C\|u\|_{H^s}, \quad u \in H^s$$

if and only if

$$\left\|\langle x\rangle^{s_2-m_2}\langle D\rangle^{s_1-m_1}P\langle D\rangle^{-s_1}\langle x\rangle^{-s_2}v\right\|_{L^2} \leq C\|v\|_{L^2}, \quad v \in L^2.$$

□

PROPOSITION 3. *Let $P = p(x,D)$ be an operator in $OPSG^m_{\rho,\delta}$. Then $Q = q(x,D) = \langle x\rangle^{s_2-m_2}\langle D\rangle^{s_1-m_1}P\langle D\rangle^{-s_1}\langle x\rangle^{-s_2}$ is in $OPSG^{(0,0)}_{\rho,\delta}$.*

This is an obvious consequence of Theorem 3.

PROPOSITION 4. *Let $p(x,\xi) \in SG^{(0,0)}_{\rho,\delta}$, then the associated pseudo-differential operator P is continuous from L^2 into itself.*

The L^2-boundedness of operators with symbol in $SG^{(0,0)}_{\rho,\delta}$ was proved by Cordes [5] when $\delta_2 = 0$, see also Beals [1], Beals-Fefferman [2]. In the present paper the most relevant case is represented by the case $\delta_2 > 0$, which remained unexplored in the above mentioned articles. However, Proposition 4 can be seen now as a consequence of the results in Hörmander [6]. In fact $SG^m_{\rho,\delta}$ classes are included in the Weyl-Hörmander calculus and actually we get L^2-boundedness also for $0 \leq \delta_j \leq \rho_j \leq 1$, $\delta_j \neq 1$, $j = 1,2$, $m = 0$.

THEOREM 6. *Given a symbol $p(x,\xi) \in SG^m_{\rho,\delta}$, the operator P defined by* (9) *is continuous from H^s into H^{s-m}.*

Proof. The proof is an obvious conseguence of Proposition 2, Proposition 3, Proposition 4. □

Applying Theorem 6 to the parametrix $Q \in OPSG^{-m'}_{\rho,\delta}$ in Theorem 4 and writing $u = QPu - Ru$ as in the proof of Theorem 5 we get for any $t \in \mathbb{R}^2$ the estimate

$$\|u\|_{m'} \leq C\left(\|Pu\|_{L^2} + \|u\|_{H^t}\right), \quad u \in L^2.$$

So for example for the operator with polynomial coefficients P in Section 3, in the proof of Proposition 1 we had $m' = (m_1, 0)$, so we obtain

$$\sum_{0\leq\beta\leq m_1}\left\|D^\beta u\right\|_{L^2} \leq C\left(\|Pu\|_{L^2} + \|u\|_{H^t}\right).$$

In the following we shall improve this estimates by adding in the sum in the left-hand side all the other terms $\left\|x^\alpha D^\beta u\right\|_{L^2}$ with $0 < \alpha \leq \gamma\beta$.

LEMMA 2. *Let $p(x,\xi)$ be the symbol* (13)*. If $p(x,\xi)$ is hypoelliptic and $q(x,\xi)$ is the symbol of its parametrix with asymptotic expansion $\sum_{l=0}^{\infty} q_l(x,\xi)$, cf. the proof of Theorem 4, then the following inequality holds:*

$$\left|D_\xi^\alpha D_x^\beta q_l\right| \leq C_{\alpha,\beta}\,|q_0(x,\xi)|\,\langle\xi\rangle^{-(\alpha+l)}\langle x\rangle^{-(\beta+l)+\gamma(\alpha+l)},\ l=0,1,2,\ldots. \tag{16}$$

Proof. First we prove that the inequality holds for q_1. Using (12) and Leibniz' formula we obtain

$$\begin{aligned}
\left|D_\xi^\alpha D_x^\beta q_1(x,\xi)\right| &= \left|D_\xi^\alpha D_x^\beta\left(\partial_\xi q_0(x,\xi) D_x p(x,\xi) q_0(x,\xi)\right)\right| \\
&\leq \sum_{\substack{\alpha_1+\alpha_2=\alpha \\ \alpha_3+\alpha_4=\alpha_2}} \sum_{\substack{\beta_1+\beta_2=\beta \\ \beta_3+\beta_4=\beta_2}} \binom{\alpha}{\alpha_1}\binom{\alpha_2}{\alpha_3}\binom{\beta}{\beta_1}\binom{\beta_2}{\beta_3}\left|D_\xi^{\alpha_1}\partial_\xi D_x^{\beta_1} q_0(x,\xi)\right| \\
&\qquad \cdot\left|D_\xi^{\alpha_3} D_x^{\beta_3+1} p(x,\xi)\right|\left|D_\xi^{\alpha_4} D_x^{\beta_4} q_0(x,\xi)\right|.
\end{aligned}$$

Using hypoellipticity of the symbols $q_0(x,\xi)$ and $p(x,\xi)$, we can deduce (16). Now we suppose that inequality (16) is true until l and we prove that it holds for $l+1$. Using the same arguments, we obtain

$$\begin{aligned}
&\left|D_\xi^\alpha D_x^\beta q_{l+1}(x,\xi)\right| \\
&\quad = \left|D_\xi^\alpha D_x^\beta\left(\sum_{j=0}^{l}\frac{1}{(l+1-j)!}\partial_\xi^{l+1-j} q_j(x,\xi) D_x^{l+1-j} p(x,\xi) q_0(x,\xi)\right)\right| \\
&\quad \leq \sum_{j=0}^{l}\frac{1}{(l+1-j)!}\sum_{\substack{\alpha_1+\alpha_2=\alpha \\ \alpha_3+\alpha_4=\alpha_2}} \sum_{\substack{\beta_1+\beta_2=\beta \\ \beta_3+\beta_4=\beta_2}} \binom{\alpha}{\alpha_1}\binom{\alpha_2}{\alpha_3}\binom{\beta}{\beta_1}\binom{\beta_2}{\beta_3} \\
&\qquad \cdot\left|D_\xi^{\alpha_1}\partial_\xi^{l+1-j} D_x^{\beta_1} q_j(x,\xi)\right|\left|D_\xi^{\alpha_3} D_x^{\beta_3+l+1-j} p(x,\xi)\right|\left|D_\xi^{\alpha_4} D_x^{\beta_4} q_0(x,\xi)\right|.
\end{aligned}$$

Using inductive hypothesis on the first term and hypoellipticity of $p(x,\xi)$ and $q(x,\xi)$ respectively to the others two terms, we easily obtain the conclusion. □

PROPOSITION 5. *Let P be the operator with symbol* (13) *and Q its parametrix. Then $x^\alpha D^\beta Q$ is continuous from L^2 into itself, for every (α,β) in the Newton's polyhedron $\mathcal{N}_p$ of $p(x,\xi)$.*

By applying the composition Theorem 3, since $Q \in SG_{\rho,\delta}^{(-m_1,0)}$, $x^\alpha \in SG_{\rho,\delta}^{(0,\gamma m_1)}$, $\xi^\beta \in SG_{\rho,\delta}^{(m_1,0)}$ for $(\alpha,\beta)\in\mathcal{N}_p$, we obtain $x^\alpha\xi^\beta Q \in OPSG_{\rho,\delta}^{(0,\gamma m_1)}$, that does not grant L^2-boundedness. A more refined argument is therefore necessary.

Proof. First we prove that $x^\alpha\xi^\beta q_l(x,\xi) \in SG_{(1,1),(0,\gamma)}^{(-l,-l+\gamma l)}$, $l=0,1,2,\ldots.$

By Leibniz' formula we have

$$(17)\quad \left|D_\xi^n D_x^m \left(x^\alpha \xi^\beta q_l(x,\xi)\right)\right| \le \sum_{j=0}^{m}\sum_{k=0}^{n}\binom{m}{j}\binom{n}{k} C_{\alpha,j}C_{\beta,k} \left|x^{\alpha-j}\xi^{\beta-k}D_\xi^{n-k}D_x^{m-j}q_l(x,\xi)\right|$$

Using (16), we obtain

$$\begin{aligned}(18)\quad \left|x^{\alpha-j}\xi^{\beta-k}D_\xi^{n-k}D_x^{m-j}q_l(x,\xi)\right| &= \left|x^{k\gamma}\right|\left|x^{\alpha-j-k\gamma}\xi^{\beta-k}\right|\frac{1+|\xi|^k}{1+|\xi|^k}\cdot\frac{1+|x|^j}{1+|x|^j}\\ &\quad\cdot\left|D_\xi^{n-k}D_x^{m-j}q_l(x,\xi)\right|\\ &\le \left|x^{\alpha-j-k\gamma}\xi^{\beta-k}\right|\left(1+|\xi|^k\right)\left(1+|x|^j\right)\langle x\rangle^{k\gamma}\langle\xi\rangle^{-k}\langle x\rangle^{-j}\\ &\quad\cdot|q_0(x,\xi)|\,\langle\xi\rangle^{-(n-k+l)}\langle x\rangle^{-(m-j+l)+\gamma(n-k+l)}.\end{aligned}$$

Now, it can easily be seen that

$$(19)\quad \left|x^{\alpha-j-k\gamma}\xi^{\beta-k}\right|\left(1+|\xi|^k\right)\left(1+|x|^j\right)\le C\,|p(x,\xi)|,$$

in fact every term on the first member of (19) has exponent that belongs to the Newton's polyhedron $\mathcal{N}_p$ of the polynomial $p(x,\xi)$.

Using (17), (18), (19), we obtain

$$\left|x^{\alpha-j}\xi^{\beta-k}D_\xi^{n-k}D_x^{m-j}q_l(x,\xi)\right|\le C_{n,k,m,j}\langle\xi\rangle^{-(n+l)}\langle x\rangle^{-(m+l)+\gamma(n+l)},$$

so $x^\alpha\xi^\beta q_l(x,\xi)\in SG^{(-l,-l+\gamma l)}_{(1,1),(0,\gamma)}$, $l=0,1,2,\dots$.

We denote $r(x,D)=x^\alpha D^\beta Q$. By Theorem 3, the symbol $r(x,\xi)$ has the following asymptotic expansion:

$$(20)\quad r(x,\xi)\sim x^\alpha\xi^\beta q(x,\xi)+\binom{\beta}{1}x^\alpha\xi^{\beta-1}D_x q(x,\xi) +\dots+\binom{\beta}{\beta-1}x^\alpha\xi D_x^{\beta-1}q(x,\xi)+x^\alpha D_x^\beta q(x,\xi).$$

We note that the sum in (20) is a finite sum. By construction, $q(x,\xi)$ has asymptotic expansion $q(x,\xi)\sim\sum_{l=0}^{\infty}q_l(x,\xi)$ where the $q_l(x,\xi)$ are defined in (12), $l=0,1,2,\dots$.

It is obvious that

$$x^\alpha\xi^\beta q(x,\xi)\sim\sum_{l=0}^{\infty}x^\alpha\xi^\beta q_l(x,\xi).$$

For every $x^\alpha\xi^\beta q_l(x,\xi)$, $l = 0,1,2,\ldots$, the hypothesis of the Theorem 2 are satisfied with $m^0 = (0,0)$, therefore it exists a symbol $s(x,\xi) \in SG^{0,0}_{\rho,\delta}$ that has the asymptotic expansion $\sum_{l=0}^{\infty} x^\alpha\xi^\beta q_l(x,\xi)$. We conclude that symbols $s(x,\xi)$ and $x^\alpha\xi^\beta q(x,\xi)$ have the same asymptotic expansion, then the symbol $x^\alpha\xi^\beta q(x,\xi)$ belongs to $SG^{0,0}_{\rho,\delta}$.

Using the properties of the Newton's polyhedron associated to the symbol (13) and arguing similarly, we prove that

$$x^\alpha\xi^{\beta-j}D_x^j q(x,\xi) \in SG^{(-j,-j+\gamma j)}_{(1,1),(0,\gamma)}, \quad j = 1,\ldots,\beta.$$

So we have proved that $r(x,\xi) \in SG^{(0,0)}_{(1,1),(0,\gamma)}$. By Proposition 4, we have that $x^\alpha D^\beta Q$ is continuous from L^2 into itself. $\square$

Let $\mathcal{N}_p$ be the Newton's polyhedron of the symbol (13).

DEFINITION 6. *We set*

$$H^* = \left\{ u \in \mathcal{S}'/x^\alpha D^\beta u \in L^2,\, (\alpha,\beta) \in \mathcal{N}_p \right\},$$

with norm

$$\|u\|_{H^*} = \sum_{\substack{0\le\beta\le m_1 \\ 0\le\alpha\le\gamma\beta}} \left\| x^\alpha D^\beta u \right\|_{L^2}.$$

THEOREM 7. *Let P be the operator with symbol* (13). *The following inequality holds for any $t \in \mathbb{R}^2$, for a suitable $C > 0$:*

$$\|u\|_{H^*} \le C\left(\|Pu\|_{L^2} + \|u\|_{H^t}\right), \quad u \in L^2.$$

Proof. The operator P is hypoelliptic, therefore, if we call Q its parametrix, we have

$$QPu = u + Ku, \tag{21}$$

where $K \in OPSG^{-\infty}$. Obviously, we can write (21) as $u = QPu - Ku$. Using the norm H^* we have

$$\|u\|_{H^*} \le \|QPu\|_{H^*} + \|Ku\|_{H^*}.$$

We would like to prove that $\|QPu\|_{H^*} \le C\|Pu\|_{L^2}$. This is true if the operator $Q : L^2 \to H^*$ is continuous. By definition of H^* norm, this is equivalent to $\sum_{\substack{0\le\beta\le m_1 \\ 0\le\alpha\le\gamma\beta}} \left\| x^\alpha D^\beta Qv \right\|_{L^2} \le C\|v\|_{L^2}$. In other words, we have to prove that $x^\alpha D^\beta Q : L^2 \to L^2$ is continuous. Using Proposition 5 we know that this assertion is valid.

It is easy to prove that $\|Ku\|_{H^*} \le C\|u\|_{H^t}$ for any $t \in \mathbb{R}^2$. $\square$

Observe finally that the boundedness of the parametrix $Q : L^2 \to H^*$ implies the Fredholm property of P. Namely, we have $P : H^* \to L^2$, as evident from Definition 6. Moreover $PQ = I + K_1$, $QP = I + K_2$, where K_1, K_2 are regularizing, hence compact in H^* and L^2. It follows that the map $P : H^* \to L^2$ is Fredholm.

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