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AN ELECTRO-VISCOELASTIC CONTACT PROBLEM WITH ADHESION AND DAMAGE

Abstract. We consider a quasistatic frictionless contact problem for an electro-viscoelastic body with damage. The contact is modelled with normal compliance. The adhesion of the contact surfaces is taken into account and modelled by a surface variable, the bonding field. We derive variational formulation for the model which is in the form of a system involving the displacement field, the electric potential field, the damage field and the adhesion field. We prove the existence of a unique weak solution to the problem. The proof is based on arguments of time-dependent variational inequalities, parabolic inequalities, differential equations and fixed point.

1. Introduction

The piezoelectric phenomenon represents the coupling between the mechanical and electrical behavior of a class of materials, called piezoelectric materials. In simplest tems, when a piezoelectric material is squeezed, an electric charge collects on its surface, conversely, when a piezoelectric material is subjected to a voltage drop, it mechanically deforms. Many crystalline materials exhibit piezoelectric behavior. A few materials exhibit the phenomenon strongly enough to be used in applications that take advantage of their properties. These include quartz, Rochelle salt, lead titanate zirconate ceramics, barium titanate and polyvinylidene flouride (a polymer film). Piezoelectric materials are used extensively as switches and actually in many engineering systems in radioelectronics, electroacoustics and measuring equipment. However, there are very few mathematical results concerning contact problems involving piezoelectric materials and therefore there is a need to extend the results on models for contact with deformable bodies which include coupling between mechanical and electrical properties. General models for elastic materials with piezoelectric effects can be found in [11,12,13,21,22] and more recently in [1,20]. The adhesive contact between deformable bodies, when a glue is added to prevent relative motion of the surfaces, has also received recently increased attention in the mathematical literature. Analysis of models for adhesive contact can be found in [3,4,6,7,15,16,17] and recently in the monographs [18,19]. The novelty in all these papers is the introduction of a surface internal variable, the bonding field, denoted in this paper by α, which describes the pointwise fractional density of adhesion of active bonds on the contact surface, and is sometimes referred to as the intensity of adhesion. Following [6,7], the bonding field satisfies the restriction $0 \leq \alpha \leq 1$, when $\alpha = 1$ at a point of the contact surface, the adhesion is complete and all the bonds are active, when $\alpha = 0$ all the bonds are inactive, severed, and there is no adhesion, when $0 < \alpha < 1$ the adhesion is partial and only a fraction α of the bonds is active.

The importance of this paper is to make the coupling of an electro-viscoelastic problem with damage and a frictionless contact problem with adhesion. We study a quasistatic problem of frictionless adhesive contact. We model the material behavior with an electro-viscoelastic constitutive law with damage and the contact with normal compliance with adhesion. We derive a variational formulation and prove the existence and uniqueness of the weak solution.

The paper is structured as follows. In section 2 we present notation and some preliminaries. The model is described in section 3 where the variational formulation is given. In section 4, we present our main result stated in Theorem 2 and its proof which is based on arguments of time-dependent variational inequalities, parabolic inequalities, differential equations and fixed point.

2. Notation and preliminaries

In this short section, we present the notation we shall use and some preliminary material. For more details, we refer the reader to $[2,5,14]$. We denote by S^d the space of second order symmetric tensors on $\mathbb{R}^d$ $(d=2,3)$, while "." and $|\ .\ |$ represent the inner product and the Euclidean norm on S^d and $\mathbb{R}^d$, respectively. We recall that the inner products and the corresponding norms on $\mathbb{R}^d$ and S^d are given by

$$\mathbf{u}\ .\ \mathbf{v} = u_i v_i,\ \ |\ \mathbf{v}\ | = (\mathbf{v},\mathbf{v})^{\frac{1}{2}} \quad \forall \mathbf{u},\mathbf{v} \in \mathbb{R}^d,$$

$$\sigma\ .\ \tau = \sigma_{ij}\tau_{ij},\ \ |\ \tau\ | = (\tau,\tau)^{\frac{1}{2}} \quad \forall \sigma,\tau \in S^d,$$

respectively. Here and below, the indices i and j run from 1 to d, the summation convention over repeated indices is used and the index that follows a comma indicates a partial derivative with respect to the corresponding component of the independent variable.

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with a regular boundary Γ and let ν denote the unit outer normal on Γ. We shall use the notation

$$H = L^2(\Omega)^d = \left\{\mathbf{u} = (u_i)\ /\ u_i \in L^2(\Omega)\ \right\},$$
$$H^1(\Omega)^d = \left\{\mathbf{u} = (u_i)\ /\ u_i \in H^1(\Omega)\ \right\},$$
$$\mathcal{H} = \left\{\sigma = (\sigma_{ij})\ /\ \sigma_{ij} = \sigma_{ji} \in L^2(\Omega)\ \right\},$$
$$\mathcal{H}_1 = \left\{\sigma \in \mathcal{H}\ /\ Div\ \sigma \in H\right\},$$

we consider that $\varepsilon : H^1(\Omega)^d \to \mathcal{H}$ and $Div : \mathcal{H}_1 \to H$ are the deformation and divergence operators, respectively, defined by

$$\varepsilon(\mathbf{u}) = (\varepsilon_{ij}(\mathbf{u})),\ \ \varepsilon_{ij}(\mathbf{u}) = \frac{1}{2}(u_{i,j} + u_{j,i}),\ Div\sigma = (\sigma_{ij,j}).$$

The spaces H, $H^1(\Omega)^d$, $\mathcal{H}$ and $\mathcal{H}_1$ are real Hilbert spaces endowed with the canonical inner products given by

$$(\mathbf{u},\mathbf{v})_H = \int_\Omega \mathbf{u}\ .\ \mathbf{v}\ dx \qquad \forall \mathbf{u},\mathbf{v} \in H,$$

$$(\mathbf{u},\mathbf{v})_{H^1(\Omega)^d} = \int_\Omega \mathbf{u} \,.\, \mathbf{v}\, dx + \int_\Omega \nabla\mathbf{u} \,.\, \nabla\mathbf{v}\, dx \qquad \forall \mathbf{u},\mathbf{v} \in H^1(\Omega)^d,$$

where

$$\nabla\mathbf{v} = (v_{i,j}) \quad \forall \mathbf{v} \in H^1(\Omega)^d,$$

$$(\sigma,\tau)_{\mathcal{H}} = \int_\Omega \sigma \,.\, \tau\, dx \qquad \forall \sigma,\tau \in \mathcal{H},$$

$$(\sigma,\tau)_{\mathcal{H}_1} = (\sigma,\tau)_{\mathcal{H}} + (Div\,\sigma, Div\,\tau)_H \quad \forall \sigma,\tau \in \mathcal{H}_1.$$

The associated norms on the spaces H, $H^1(\Omega)^d$, $\mathcal{H}$ and $\mathcal{H}_1$ are denoted by $|\,.\,|_H$, $|\,.\,|_{H^1(\Omega)^d}$, $|\,.\,|_{\mathcal{H}}$ and $|\,.\,|_{\mathcal{H}_1}$ respectively. Let $H_\Gamma = H^{\frac{1}{2}}(\Gamma)^d$ and let $\gamma : H^1(\Omega)^d \to H_\Gamma$ be the trace map. For every element $\mathbf{v} \in H^1(\Omega)^d$, we also use the notation $\mathbf{v}$ to denote the trace $\gamma\mathbf{v}$ of $\mathbf{v}$ on Γ and we denote by v_ν and $\mathbf{v}_\tau$ the normal and the tangential components of $\mathbf{v}$ on the boundary Γ given by

(1) $$v_\nu = \mathbf{v} \,.\, \nu,\ \mathbf{v}_\tau = \mathbf{v} - v_\nu \nu.$$

We note that v_ν is a scalar, whereas $\mathbf{v}_\tau$ is a tangent vector to Γ. In particular, in what follows, u_ν and $\mathbf{u}_\tau$ will represent the normal and tangential displacement. Similarly, for a regular (say C^1) tensor field $\sigma : \Omega \to S^d$ we define its normal and tangential components by

(2) $$\sigma_\nu = (\sigma\nu) \,.\, \nu,\ \sigma_\tau = \sigma\nu - \sigma_\nu \nu,$$

and we recall that the following Green's formula holds :

$$(\sigma, \varepsilon(\mathbf{v}))_{\mathcal{H}} + (Div\,\sigma, \mathbf{v})_H = \int_\Gamma \sigma\nu \,.\, \mathbf{v}\, da \qquad \forall \mathbf{v} \in H^1(\Omega)^d.$$

Finally, for any real Hilbert space X, we use the classical notation for the spaces $L^p(0,T;X)$ and $W^{k,p}(0,T;X)$, where $1 \le p \le +\infty$ and $k \ge 1$. We denote by $C(0,T;X)$ and $C^1(0,T;X)$ the space of continuous and continuously differentiable functions from $[0,T]$ to X, respectively, with the norms

$$|\,\mathbf{f}\,|_{C(0,T;X)} = \max_{t\in[0,T]} |\,\mathbf{f}(t)\,|_X,$$

$$|\,\mathbf{f}\,|_{C^1(0,T;X)} = \max_{t\in[0,T]} |\,\mathbf{f}(t)\,|_X + \max_{t\in[0,T]} |\dot{\mathbf{f}}\,(t)\,|_X,$$

respectively. Moreover, we use the dot above to indicate the derivative with respect to the time variable and, for a real number r, we use r_+ to represent its positive part, that is $r_+ = \max\{0,r\}$. For the convenience of the reader, we recall the following version of the classical theorem of Cauchy-Lipschitz (see, e.g., [19,p.48]).

THEOREM 1. *Assume that $(X, |\,.\,|_X)$ is a real Banach space and $T > 0$. Let $F(t,.) : X \to X$ be an operator defined a.e. on $(0,T)$ satisfying the following conditions:*

1 - There exists a constant $L_F > 0$ such that

$$\mid F(t,x) - F(t,y) \mid_X \leq L_F \mid x - y \mid_X \ \forall x, y \in X, \text{ a.e. } t \in (0,T).$$

2 - There exists $p \geq 1$ such that $t \longmapsto F(t,x) \in L^p(0,T;X) \ \ \forall x \in X$.

Then for any $x_0 \in X$, there exists a unique function $x \in W^{1,p}(0,T;X)$ such that

$$\dot{x}\,(t) = F(t, x(t)) \qquad \text{a.e. } t \in (0,T),$$

$$x(0) = x_0.$$

Theorem 1 will be used in section 4 to prove the unique solvability of the intermediate problem involving the bonding field. Moreover, if X_1 and X_2 are real Hilbert spaces then $X_1 \times X_2$ denotes the product Hilbert space endowed with the canonical inner product $(.,.)_{X_1 \times X_2}$.

3. Mechanical and variational formulations

We describe the model for the process, we present its variational formulation. The physical setting is the following. An electro-viscoelastic body occupies a bounded domain $\Omega \subset \mathbb{R}^d$ $(d = 2,3)$ with outer Lipschitz surface Γ. The body undergoes the action of body forces of density $\mathbf{f}_0$ and volume electric charges of density q_0. It also undergoes the mechanical and electric constraint on the boundary. We consider a partition of Γ into three disjoint measurable parts Γ_1, Γ_2 and Γ_3, on one hand, and in two measurable parts Γ_a and Γ_b, on the other hand, such that *meas* $(\Gamma_1) > 0$, *meas* $(\Gamma_a) > 0$ and $\Gamma_3 \subset \Gamma_b$. Let $T > 0$ and let $[0,T]$ be the time interval of interest. The body is clamped on $\Gamma_1 \times (0,T)$, so the displacement field vanishes there. A surface tractions of density $\mathbf{f}_2$ act on $\Gamma_2 \times (0,T)$ and a body force of density $\mathbf{f}_0$ acts in $\Omega \times (0,T)$. We also assume that the electrical potential vanishes on $\Gamma_a \times (0,T)$ and a surface electric charge of density q_2 is prescribed on $\Gamma_b \times (0,T)$. The body is in adhesive contact with an obstacle, or foundation, over the contact surface Γ_3. We suppose that the body forces and tractions vary slowly in time, and therefore, the accelerations in the system may be neglected. Neglecting the inertial terms in the equation of motion leads to a quasistatic approach of the process. We denote by $\mathbf{u}$ the displacement field, by σ the stress tensor field and by $\varepsilon(\mathbf{u})$ the linearized strain tensor. We use an electro-viscoelastic constitutive law with damage given by

$$\sigma = \mathcal{A}\varepsilon(\dot{\mathbf{u}}) + \mathcal{G}(\varepsilon(\mathbf{u}),\beta) - \mathcal{E}^* E(\varphi),$$

$$\mathbf{D} = \mathcal{E}\varepsilon(\mathbf{u}) + BE(\varphi),$$

where $\mathcal{A}$ is a given nonlinear function, $E(\varphi) = -\nabla\varphi$ is the electric field, $\mathcal{E} = (e_{ijk})$ represents the third order piezoelectric tensor, $\mathcal{E}^*$ is its transpose and B denotes the electric permittivity tensor. $\mathcal{G}$ represents the elasticity operator where β is an internal variable describing the damage of the material caused by elastic deformations. The differential inclusion used for the evolution of the damage field is

$$\dot{\beta} - k \triangle \beta + \partial\varphi_K(\beta) \ni S(\varepsilon(\mathbf{u}),\beta),$$

where K denotes the set of admissible damage functions defined by

$$K = \{\xi \in H^1(\Omega) \ / \ 0 \le \xi \le 1 \quad \text{a.e. in } \Omega\},$$

k is a positive coefficient, $\partial\varphi_K$ denotes the subdifferential of the indicator function φ_K and S is a given constitutive function which describes the sources of the damage in the system. When $\beta = 1$ the material is undamaged, when $\beta = 0$ the material is completely damaged, and for $0 < \beta < 1$ there is partial damage. General models of mechanical damage, which were derived from thermodynamical considerations and the principle of virtual work, can be found in [8] and [9] and references therein. The models describe the evolution of the material damage which results from the excess tension or compression in the body as a result of applied forces and tractions. Mathematical analysis of one-dimensional damage models can be found in [10].

To simplify the notation, we do not indicate explicitly the dependence of various functions on the variables $\mathbf{x} \in \Omega \cup \Gamma$ and $t \in [0,T]$. Then, the classical formulation of the mechanical problem of electro-viscoelastic material, frictionless, adhesive contact may be stated as follows.

Problem P. Find a displacement field $\mathbf{u} : \Omega \times [0,T] \to \mathbb{R}^d$, *an electric potential field* $\varphi : \Omega \times [0,T] \to \mathbb{R}$, *a damage field* $\beta : \Omega \times [0,T] \to \mathbb{R}$ *and a bonding field* $\alpha : \Gamma_3 \times [0,T] \to \mathbb{R}$ *such that*

(3) $$\sigma = \mathcal{A}\varepsilon(\dot{\mathbf{u}}) + \mathcal{G}(\varepsilon(\mathbf{u}),\beta) + \mathcal{E}^*\nabla\varphi \text{ in } \Omega \times (0,T),$$

(4) $$\mathbf{D} = \mathcal{E}\varepsilon(\mathbf{u}) - B\nabla\varphi \text{ in } \Omega \times (0,T),$$

(5) $$\dot{\beta} - k \triangle \beta + \partial\varphi_K(\beta) \ni S(\varepsilon(\mathbf{u}),\beta) \text{ in } \Omega \times (0,T),$$

(6) $$Div\, \sigma + \mathbf{f}_0 = 0 \quad \text{in } \Omega \times (0,T),$$

(7) $$div\, \mathbf{D} = q_0 \text{ in } \Omega \times (0,T),$$

(8) $$\mathbf{u} = 0 \quad \text{on} \quad \Gamma_1 \times (0,T),$$

(9) $$\sigma\nu = \mathbf{f}_2 \quad \text{on} \quad \Gamma_2 \times (0,T),$$

(10) $$-\sigma_\nu = p_\nu(u_\nu) - \gamma_\nu\alpha^2 R_\nu(u_\nu) \quad \text{on} \quad \Gamma_3 \times (0,T),$$

(11) $$-\sigma_\tau = p_\tau(\alpha)\mathbf{R}_\tau(\mathbf{u}_\tau) \text{ on } \quad \Gamma_3 \times (0,T),$$

(12) $$\dot{\alpha} = -\left(\alpha(\gamma_\nu(R_\nu(u_\nu))^2 + \gamma_\tau \mid \mathbf{R}_\tau(\mathbf{u}_\tau) \mid^2 \ ) - \varepsilon_a\right)_+ \text{ on } \quad \Gamma_3 \times (0,T),$$

$$\frac{\partial\beta}{\partial\nu} = 0 \quad \text{on} \quad \Gamma \times (0,T)\,, \tag{13}$$

$$\varphi = 0 \quad \text{on} \quad \Gamma_a \times (0,T)\,, \tag{14}$$

$$\mathbf{D}\,.\,\nu = q_2 \quad \text{on} \quad \Gamma_b \times (0,T)\,, \tag{15}$$

$$\mathbf{u}(0) = \mathbf{u}_0, \beta(0) = \beta_0 \quad \text{in}\ \Omega, \tag{16}$$

$$\alpha(0) = \alpha_0 \quad \text{on} \quad \Gamma_3. \tag{17}$$

First, 3 and 4 represent the electro-viscoelastic constitutive law with damage, the evolution of the damage field is governed by the inclusion of parabolic type given by the relation 5, where S is the mechanical source of the damage growth, assumed to be rather general function of the strains and damage itself, $\partial\varphi_K$ is the subdifferential of the indicator function of the admissible damage functions set K. Equations 6 and 7 represent the equilibrium equations for the stress and electric-displacement fields while 8 and 9 are the displacement and traction boundary condition, respectively. Condition 10 represents the normal compliance condition with adhesion where γ_ν is a given adhesion coefficient and p_ν is a given positive function which will be described below. In this condition the interpenetrability between the body and the foundation is allowed, that is u_ν can be positive on Γ_3. The contribution of the adhesive to the normal traction is represented by the term $\gamma_\nu\alpha^2 R_\nu(u_\nu)$, the adhesive traction is tensile and is proportional, with proportionality coefficient γ_ν, to the square of the intensity of adhesion and to the normal displacement, but as long as it does not exceed the bond length L. The maximal tensile traction is $\gamma_\nu L$. R_ν is the truncation operator defined by

$$R_\nu(s) = \begin{cases} L & \text{if } s < -L, \\ -s & \text{if } -L \leq s \leq 0, \\ 0 & \text{if } s > 0. \end{cases}$$

Here $L > 0$ is the characteristic length of the bond, beyond which it does not offer any additional traction. The introduction of the operator R_ν, together with the operator $\mathbf{R}_\tau$ defined below, is motivated by mathematical arguments but it is not restrictive from the physical point of view, since no restriction on the size of the parameter L is made in what follows. Condition 11 represents the adhesive contact condition on the tangential plane, in which p_τ is a given function and $\mathbf{R}_\tau$ is the truncation operator given by

$$\mathbf{R}_\tau(\mathbf{v}) = \begin{cases} \mathbf{v} & \text{if} \quad |\,\mathbf{v}\,| \le L, \\ L\frac{\mathbf{v}}{|\mathbf{v}|} & \text{if} \quad |\,\mathbf{v}\,| > L. \end{cases}$$

This condition shows that the shear on the contact surface depends on the bonding field and on the tangential displacement, but as long as it does not exceed the bond length L. The frictional tangential traction is assumed to be much smaller than the adhesive one and, therefore, omitted.

Next, the equation 12 represents the ordinary differential equation which describes the evolution of the bonding field and it was already used in [3], see also [18, 19] for more details. Here, besides γ_ν, two new adhesion coefficients are involved, γ_τ and ε_a. Notice that in this model once debonding occurs bonding cannot be reestablished since, as it follows from 12, $\dot{\alpha} \le 0$. The relation 13 represents a homogeneous Neumann boundary condition where $\frac{\partial\beta}{\partial\nu}$ represents the normal derivative of β. 14 and 15 represent the electric boundary conditions. 16 represents the initial displacement field and the initial damage field. Finally 17 represents the initial condition in which α_0 is the given initial bonding field. To obtain the variational formulation of the problem 3-17, we introduce for the bonding field the set

$$Z = \left\{\theta \in C(0,T;L^2(\Gamma_3)) \;/\; 0 \le \theta(t) \le 1 \;\forall t \in [0,T], \text{ a.e. on } \Gamma_3\right\},$$

and for the displacement field we need the closed subspace of $H^1(\Omega)^d$ defined by

$$V = \left\{\mathbf{v} \in H^1(\Omega)^d \;/\; \mathbf{v} = 0 \text{ on } \Gamma_1\right\}.$$

Since *meas* $(\Gamma_1) > 0$, Korn's inequality holds and there exists a constant $C_k > 0$, that depends only on Ω and Γ_1, such that

$$|\,\varepsilon(\mathbf{v})\,|_{\mathcal{H}} \ge C_k \,|\,\mathbf{v}\,|_{H^1(\Omega)^d} \quad \forall \mathbf{v} \in V.$$

A proof of Korn's inequality may be found in [14, p.79]. On the space V we consider the inner product and the associated norm given by

$$(\mathbf{u},\mathbf{v})_V = (\varepsilon(\mathbf{u}),\varepsilon(\mathbf{v}))_{\mathcal{H}}, \quad |\,\mathbf{v}\,|_V = |\,\varepsilon(\mathbf{v})\,|_{\mathcal{H}} \quad \forall \mathbf{u},\mathbf{v} \in V. \tag{18}$$

It follows that $|\,.\,|_{H^1(\Omega)^d}$ and $|\,.\,|_V$ are equivalent norms on V and therefore $(V, |\,.\,|_V)$ is a real Hilbert space. Moreover, by the Sobolev trace Theorem and 18, there exists a constant $C_0 > 0$, depending only on Ω, Γ_1 and Γ_3 such that

$$|\,\mathbf{v}\,|_{L^2(\Gamma_3)^d} \le C_0 \,|\,\mathbf{v}\,|_V \qquad \forall \mathbf{v} \in V. \tag{19}$$

We also introduce the spaces

$$W = \left\{\phi \in H^1(\Omega) \;/\; \phi = 0 \text{ on } \Gamma_a\right\},$$

$$\mathcal{W} = \left\{ \mathbf{D} = (D_i) \ / \ D_i \in L^2(\Omega), \ div\, \mathbf{D} \in L^2(\Omega) \right\},$$

where $div\ \mathbf{D} = (D_{i,i})$. The spaces W and $\mathcal{W}$ are real Hilbert spaces with the inner products given by

$$(\varphi, \phi)_W = \int_\Omega \nabla\varphi \ . \ \nabla\phi \ dx,$$

$$(\mathbf{D}, \mathbf{E})_{\mathcal{W}} = \int_\Omega \mathbf{D} \ . \ \mathbf{E} \ dx + \int_\Omega div\, \mathbf{D} \ . \ div\, \mathbf{E} \ dx.$$

The associated norms will be denoted by $| \ . \ |_W$ and $| \ . \ |_{\mathcal{W}}$, respectively. Moreover, when $\mathbf{D} \in \mathcal{W}$ is a regular function, the following Green's type formula holds:

$$(\mathbf{D}, \nabla\phi)_H + (div\, \mathbf{D}, \phi)_{L^2(\Omega)} = \int_\Gamma \mathbf{D} \ . \ \nu \ \phi \ da \qquad \forall \phi \in H^1(\Omega).$$

Notice also that, since $meas(\Gamma_a) > 0$, the following Friedrichs-Poincaré inequality holds:

(20) $$| \nabla\phi |_H \geq C_F \ | \ \phi \ |_{H^1(\Omega)} \ \ \forall \phi \in W,$$

where $C_F > 0$ is a constant which depends only on Ω and Γ_a. In the study of the mechanical problem 3-17, we assume that the viscosity function $\mathcal{A} : \Omega \times S^d \to S^d$ satisfies

(21)
$$\begin{cases} (a) \text{ There exists a constant } L_{\mathcal{A}} > 0 \text{ Such that} \\ | \mathcal{A}(\mathbf{x}, \varepsilon_1) - \mathcal{A}(\mathbf{x}, \varepsilon_2) | \leq L_{\mathcal{A}} \ | \ \varepsilon_1 - \varepsilon_2 \ | \ \ \forall \varepsilon_1, \varepsilon_2 \in S^d, \text{ a.e. } \mathbf{x} \in \Omega. \\ (b) \text{ There exists a constant } m_{\mathcal{A}} > 0 \text{ Such that} \\ (\mathcal{A}(\mathbf{x}, \varepsilon_1) - \mathcal{A}(\mathbf{x}, \varepsilon_2)) \ . \ (\varepsilon_1 - \varepsilon_2) \geq m_{\mathcal{A}} \ | \ \varepsilon_1 - \varepsilon_2 \ |^2 \ \ \forall \varepsilon_1, \varepsilon_2 \in S^d, \text{ a.e. } \mathbf{x} \in \Omega. \\ (c) \text{ The mapping } \mathbf{x} \to \mathcal{A}(\mathbf{x}, \varepsilon) \text{ is Lebesgue measurable on } \Omega \text{ for any } \varepsilon \in S^d. \\ (d) \text{ The mapping } \mathbf{x} \to \mathcal{A}(\mathbf{x}, \mathbf{0}) \text{ belongs to } \mathcal{H}. \end{cases}$$

The elasticity Operator $\mathcal{G} : \Omega \times S^d \times \mathbb{R} \to S^d$ satisfies

(22)
$$\begin{cases} (a) \text{ There exists a constant } L_{\mathcal{G}} > 0 \text{ Such that} \\ | \mathcal{G}(\mathbf{x}, \varepsilon_1, \alpha_1) - \mathcal{G}(\mathbf{x}, \varepsilon_2, \alpha_2) | \leq L_{\mathcal{G}}(| \ \varepsilon_1 - \varepsilon_2 \ | + | \ \alpha_1 - \alpha_2 \ |) \\ \forall \varepsilon_1, \varepsilon_2 \in S^d, \ \forall \alpha_1, \alpha_2 \in \mathbb{R} \text{ a.e. } \mathbf{x} \in \Omega. \\ (b) \text{ The mapping } \mathbf{x} \to \mathcal{G}(\mathbf{x}, \varepsilon, \alpha) \text{ is Lebesgue measurable on } \Omega \\ \text{for any } \varepsilon \in S^d \text{ and } \alpha \in \mathbb{R}. \\ (c) \text{ The mapping } \mathbf{x} \to \mathcal{G}(\mathbf{x}, \mathbf{0}, 0) \text{ belongs to } \mathcal{H}. \end{cases}$$

The damage source function $S : \Omega \times S^d \times \mathbb{R} \to \mathbb{R}$ satisfies

(23)
$$\begin{cases} (a) \text{ There exists a constant } L_S > 0 \text{ such that} \\ \ | \ S(\mathbf{x}, \varepsilon_1, \alpha_1) - \ S(\mathbf{x}, \varepsilon_2, \alpha_2) \ | \leq L_S(| \ \varepsilon_1 - \varepsilon_2 \ | + | \ \alpha_1 - \alpha_2 \ |) \\ \ \forall \varepsilon_1, \varepsilon_2 \in S^d, \ \forall \alpha_1, \alpha_2 \in \mathbb{R} \text{ a.e. } \mathbf{x} \in \Omega. \\ (b) \text{ For any } \varepsilon \in S^d \text{ and } \alpha \in \mathbb{R}, \ \mathbf{x} \to S(\mathbf{x}, \varepsilon, \alpha) \text{ is Lebesgue measurable on } \Omega. \\ (c) \text{The mapping } \mathbf{x} \to S(\mathbf{x}, \mathbf{0}, 0) \text{ belongs to } L^2(\Omega). \end{cases}$$

The electric permittivity operator $B = (b_{ij})$: $\Omega \times \mathbb{R}^d \to \mathbb{R}^d$ satisfies

(24)
$$\begin{cases} (a)\ B(\mathbf{x},\mathbf{E}) = (b_{ij}(\mathbf{x})E_j)\ \forall \mathbf{E} = (E_i) \in \mathbb{R}^d,\ \text{a.e. } \mathbf{x} \in \Omega. \\ (b)\ b_{ij} = b_{ji},\ b_{ij} \in L^\infty(\Omega),\ \ 1 \le i, j \le d. \\ (c)\ \text{There exists a constant } m_B > 0 \ \text{ such that} \\ B\mathbf{E}.\mathbf{E} \ge m_B \mid \mathbf{E} \mid^2 \ \ \forall \mathbf{E} = (E_i) \in \mathbb{R}^d,\ \text{a.e. in } \Omega. \end{cases}$$

The piezoelectric operator $\mathcal{E} : \Omega \times S^d \to \mathbb{R}^d$ satisfies
(25)
$$\begin{cases} (a)\ \mathcal{E}(\mathbf{x},\tau) = (e_{ijk}(\mathbf{x})\tau_{jk}) \quad \forall \tau = (\tau_{ij}) \in S^d,\ \text{a.e. } \mathbf{x} \in \Omega. \\ (b)\ e_{ijk} = e_{ikj} \in L^\infty(\Omega),\ \ 1 \le i, j,k \le d. \end{cases}$$

The normal compliance function $p_\nu : \Gamma_3 \times \mathbb{R} \to \mathbb{R}_+$ satisfies

$$(26) \qquad \begin{cases} (a)\ \text{There exists a constant } L_\nu > 0 \text{ such that} \\ \mid p_\nu(\mathbf{x},r_1) - p_\nu(\mathbf{x},r_2) \mid \le L_\nu \mid r_1 - r_2 \mid \ \forall r_1, r_2 \in \mathbb{R},\ \text{a.e. } \mathbf{x} \in \Gamma_3. \\ (b)\ \text{The mapping } \mathbf{x} \to p_\nu(\mathbf{x},r) \text{ is measurable on } \Gamma_3,\ \text{for any } r \in \mathbb{R}. \\ (c)\ p_\nu(\mathbf{x},r) = 0 \text{ for all } r \le 0,\ \text{a.e. } \mathbf{x} \in \Gamma_3. \end{cases}$$

The tangential contact function $p_\tau : \Gamma_3 \times \mathbb{R} \to \mathbb{R}_+$ satisfies

$$(27) \qquad \begin{cases} (a)\ \text{There exists a constant } L_\tau > 0 \text{ such that} \\ \mid p_\tau(\mathbf{x},d_1) - p_\tau(\mathbf{x},d_2) \mid \le L_\tau \mid d_1 - d_2 \mid \ \forall d_1, d_2 \in \mathbb{R},\ \text{a.e. } \mathbf{x} \in \Gamma_3. \\ (b)\ \text{There exists } M_\tau > 0 \text{ such that } \mid p_\tau(\mathbf{x},d) \mid \le M_\tau\ \forall d \in \mathbb{R}, \text{a.e. } \mathbf{x} \in \Gamma_3. \\ (c)\ \text{The mapping } \mathbf{x} \to p_\tau(\mathbf{x},d) \text{ is measurable on } \Gamma_3,\ \text{for any } d \in \mathbb{R}. \\ (d)\ \text{The mapping } \mathbf{x} \to p_\tau(\mathbf{x},0) \in L^2(\Gamma_3). \end{cases}$$

We also suppose that the body forces and surface tractions have the regularity

$$(28) \qquad \mathbf{f}_0 \in C(0,T;H), \quad \mathbf{f}_2 \in C(0,T;L^2(\Gamma_2)^d),$$

$$(29) \qquad q_0 \in C(0,T;L^2(\Omega)), \quad q_2 \in C(0,T;L^2(\Gamma_b)).$$

$$(30) \qquad q_2(t) = 0 \text{ on } \Gamma_3 \quad \forall t \in [0,T].$$

Note that we need to impose assumption 30 for physical reasons, indeed the foundation is assumed to be insulator and therefore the electric charges (which are prescribed on $\Gamma_b \supset \Gamma_3$) have to vanish on the potential contact surface. The adhesion coefficients satisfy

$$(31) \qquad \gamma_\nu, \gamma_\tau \in L^\infty(\Gamma_3), \varepsilon_a \in L^2(\Gamma_3), \gamma_\nu, \gamma_\tau, \varepsilon_a \ge 0 \text{ a.e. on } \Gamma_3.$$

The initial displacement field satisfies

$$(32) \qquad \mathbf{u}_0 \in V,$$

the initial bonding field satisfies

$$\alpha_0 \in L^2(\Gamma_3),\ 0 \leq \alpha_0 \leq 1 \text{ a.e. on } \Gamma_3, \tag{33}$$

and the initial damage field satisfies

$$\beta_0 \in K. \tag{34}$$

We define the bilinear form $a : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ by

$$a(\xi, \varphi) = k \int_\Omega \nabla\xi \,.\, \nabla\varphi \, dx. \tag{35}$$

Next, we denote by $\mathbf{f} : [0,T] \to V$ the function defined by

$$(\mathbf{f}(t), \mathbf{v})_V = \int_\Omega \mathbf{f}_0(t) \,.\, \mathbf{v} \, dx + \int_{\Gamma_2} \mathbf{f}_2(t) \,.\, \mathbf{v} \, da \ \forall \mathbf{v} \in V,\ t \in [0,T], \tag{36}$$

and we denote by $q : [0,T] \to W$ the function defined by

$$(q(t), \phi)_W = \int_\Omega q_0(t) \,.\, \phi \, dx - \int_{\Gamma_b} q_2(t) \,.\, \phi \, da \ \forall \phi \in W,\ t \in [0,T]. \tag{37}$$

Next, we denote by $j : L^\infty(\Gamma_3) \times V \times V \to \mathbb{R}$ the adhesion functional defined by

$$j(\alpha, \mathbf{u}, \mathbf{v}) = \int_{\Gamma_3} p_\nu(u_\nu) v_\nu \, da + \int_{\Gamma_3} (-\gamma_\nu \alpha^2 R_\nu(u_\nu) v_\nu + p_\tau(\alpha) \mathbf{R}_\tau(\mathbf{u}_\tau) \,.\, \mathbf{v}_\tau) \, da. \tag{38}$$

Keeping in mind 26 and 27, we observe that the integrals 38 are well defined and we note that conditions 28 and 29 imply

$$\mathbf{f} \in C(0,T;V), \ \ q \in C(0,T;W). \tag{39}$$

Using standard arguments we obtain the variational formulation of the mechanical problem 3-17.

Problem PV. Find a displacement field $\mathbf{u} : [0,T] \to V$*, an electric potential field* $\varphi : [0,T] \to W$*, a damage field* $\beta : [0,T] \to H^1(\Omega)$ *and a bonding field* $\alpha : [0,T] \to L^\infty(\Gamma_3)$ *such that*

$$(\mathcal{A}\varepsilon(\dot{\mathbf{u}}(t)), \varepsilon(\mathbf{v}))_{\mathcal{H}} + (\mathcal{G}(\varepsilon(\mathbf{u}(t)), \beta(t)), \varepsilon(\mathbf{v}))_{\mathcal{H}} + (\mathcal{E}^*\nabla\varphi(t), \varepsilon(\mathbf{v}))_{\mathcal{H}}$$

$$+ j(\alpha(t), \mathbf{u}(t), \mathbf{v}) = (\mathbf{f}(t), \mathbf{v})_V \quad \forall \mathbf{v} \in V, t \in (0,T), \tag{40}$$

$$\beta(t) \in K \text{ for all } t \in [0,T], \quad (\dot{\beta}(t), \xi - \beta(t))_{L^2(\Omega)} + a(\beta(t), \xi - \beta(t))$$

$$\geq (S(\varepsilon(\mathbf{u}(t)), \beta(t)), \xi - \beta(t))_{L^2(\Omega)} \quad \forall \xi \in K, \tag{41}$$

(42) $\qquad (B\nabla\varphi(t),\nabla\phi)_H - (\mathcal{E}\,\varepsilon(\mathbf{u}(t)),\nabla\phi)_H = (q(t),\phi)_W \quad \forall\phi \in W, t \in (0,T),$

(43) $\qquad \dot{\alpha}\,(t) = -\big(\alpha(t)(\gamma_\nu(R_\nu(u_\nu(t)))^2 + \gamma_\tau \mid \mathbf{R}_\tau(\mathbf{u}_\tau(t)) \mid^2\ ) - \varepsilon_a\big)_+ \text{ a.e. } t \in (0,T),$

(44) $$\mathbf{u}(0) = \mathbf{u}_0, \beta(0) = \beta_0, \alpha(0) = \alpha_0.$$

We notice that the variational problem *PV* is formulated in terms of displacement field, an electrical potential field, damage field and bonding field. The existence of the unique solution of problem *PV* is stated and proved in the next section. To this end, we consider the following remark which is used in different places of the paper.

REMARK 1. We note that, in the problem *P* and in the problem *PV* we do not need to impose explicitly the restriction $0 \le \alpha \le 1$. Indeed, equation 43 guarantees that $\alpha(\mathbf{x},t) \le \alpha_0(\mathbf{x})$ and, therefore, assumption 33 shows that $\alpha(\mathbf{x},t) \le 1$ for $t \ge 0$, a.e. $\mathbf{x} \in \Gamma_3$. On the other hand, if $\alpha(\mathbf{x},t_0) = 0$ at time t_0, then it follows from 43 that $\dot{\alpha}\,(\mathbf{x},t) = 0$ for all $t \ge t_0$ and therefore, $\alpha(\mathbf{x},t) = 0$ for all $t \ge t_0$, a.e. $\mathbf{x} \in \Gamma_3$. We conclude that $0 \le \alpha(\mathbf{x},t) \le 1$ for all $t \in [0,T]$, a.e. $\mathbf{x} \in \Gamma_3$.

4. An existence and uniqueness result

Now, we propose our existence and uniqueness result.

THEOREM 2. *Assume that 21-34 hold. Then there exists a unique solution* $\{\mathbf{u},\varphi,\beta,\alpha\}$ *to problem PV. Moreover, the solution satisfies*

(45) $$\mathbf{u} \in C^1(0,T;V),$$

(46) $$\varphi \in C(0,T;W),$$

(47) $$\beta \in W^{1,2}(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)),$$

(48) $$\alpha \in W^{1,\infty}(0,T;L^2(\Gamma_3)) \cap Z.$$

The functions $\mathbf{u}$, φ, σ, $\mathbf{D}$, β and α which satisfy 3-4 and 40-44 are called a weak solution of the contact problem P. We conclude that, under the assumptions 21-34, the mechanical problem 3-17 has a unique weak solution satisfying 45-48. The regularity of the weak solution is given by 45-48 and, in term of stresses,

(49) $$\sigma \in C(0,T;\mathcal{H}_1),$$

(50) $$\mathbf{D} \in C(0,T;\mathcal{W}).$$

Indeed, it follows from 40 and 42 that $Div\ \sigma(t)+\mathbf{f}_0(t)=0$, $div\ \mathbf{D}=q_0(t)$ for all $t \in [0,T]$ and therefore the regularity 45 and 46 of $\mathbf{u}$ and φ, combined with 21-29 implies 49 and 50.

The proof of Theorem 2 is carried out in several steps that we prove in what follows, everywhere in this section we suppose that assumptions of Theorem 2 hold, and we consider that C is a generic positive constant which depends on $\Omega, \Gamma_1, \Gamma_3, p_\nu, p_\tau, \gamma_\nu, \gamma_\tau$ and L and may change from place to place. Let $\eta \in C(0,T;V)$ be given, in the first step we consider the following variational problem.

Problem PV_η. Find a displacement field $\mathbf{u}_\eta : [0,T] \to V$ such that

$$(\mathcal{A}\varepsilon(\dot{\mathbf{u}}_\eta(t)),\varepsilon(\mathbf{v}))_{\mathcal{H}}+(\eta(t),\mathbf{v})_V=(\mathbf{f}(t),\mathbf{v})_V \quad \forall \mathbf{v}\in V,\ t\in[0,T], \tag{51}$$

$$\mathbf{u}_\eta(0)=\mathbf{u}_0. \tag{52}$$

We have the following result for the problem.

LEMMA 1. *There exists a unique solution to problem PV_η which satisfies the regularity 45.*

Proof. We define the operator $A : V \to V$ such that

$$(A\mathbf{u},\mathbf{v})_V=(\mathcal{A}\varepsilon(\mathbf{u}),\varepsilon(\mathbf{v}))_{\mathcal{H}} \quad \forall \mathbf{u},\mathbf{v}\in V. \tag{53}$$

It follows from 53 and 21(a) that

$$|A\mathbf{u}-A\mathbf{v}|_V \le L_{\mathcal{A}}\ |\mathbf{u}-\mathbf{v}|_V \quad \forall \mathbf{u},\mathbf{v}\in V, \tag{54}$$

which shows that $A : V \to V$ is Lipschitz continuous. Now, by 53 and 21(b), we find

$$(A\mathbf{u}-A\mathbf{v},\mathbf{u}-\mathbf{v})_V \ge m_{\mathcal{A}}\ |\mathbf{u}-\mathbf{v}|_V^2 \quad \forall \mathbf{u},\mathbf{v}\in V, \tag{55}$$

i.e., that $A : V \to V$ is a strongly monotone operator on V. Therefore A is invertible and its inverse A^{-1} is also strongly monotone Lipschitz continuous on V. Moreover using Riesz Representation Theorem we may define an element $\mathbf{f}_\eta \in C(0,T;V)$ by

$$(\mathbf{f}_\eta(t),\mathbf{v})_V=(\mathbf{f}(t),\mathbf{v})_V-(\eta(t),\mathbf{v})_V.$$

It follows now from classical result (see for example [2]) that there exists a unique function $\mathbf{v}_\eta \in C(0,T;V)$ which satisfies

$$A\mathbf{v}_\eta(t)=\mathbf{f}_\eta(t). \tag{56}$$

Let $\mathbf{u}_\eta : [0,T] \to V$ be the function defined by

$$\mathbf{u}_\eta(t)=\int_0^t \mathbf{v}_\eta(s)\,ds+\mathbf{u}_0 \ \forall t\in[0,T]. \tag{57}$$

It follows from 53-57 that $\mathbf{u}_\eta$ is a solution of the variational problem PV_η and it satisfies the regularity expressed in 45. This concludes the existence part of lemma 1. The uniqueness of the solution follows from the uniqueness of the solution of the problem 56. $\square$

In the second step, let $\eta \in C(0,T;V)$, we use the displacement field $\mathbf{u}_\eta$ obtained in lemma 1 and we consider the following variational problem.

Problem QV_η. Find the electric potential field $\varphi_\eta : [0,T] \to W$ such that

$$(B\nabla\varphi_\eta(t), \nabla\phi)_H - (\mathcal{E}\varepsilon(\mathbf{u}_\eta(t)), \nabla\phi)_H$$

$$= (q(t), \phi)_W \quad \forall\phi \in W,\ t \in (0,T)\,, \tag{58}$$

we have the following result.

LEMMA 2. *QV_η has a unique solution φ_η which satisfies the regularity 46.*

Proof. We define a bilinear form: $b(.,.) : W \times W \to \mathbb{R}$ such that

$$b(\varphi,\phi) = (B\nabla\varphi, \nabla\phi)_H \quad \forall\varphi,\phi \in W. \tag{59}$$

We use 20 and 24 to show that the bilinear form b is continuous, symmetric and coercive on W, moreover using Riesz Representation Theorem we may define an element $q_\eta : [0,T] \to W$ such that

$$(q_\eta(t), \phi)_W = (q(t), \phi)_W + (\mathcal{E}\varepsilon(\mathbf{u}_\eta(t)), \nabla\phi)_H \quad \forall\phi \in W,\ t \in (0,T)\,.$$

We apply the Lax-Milgram Theorem to deduce that there exists a unique element $\varphi_\eta(t) \in W$ such that

$$b(\varphi_\eta(t), \phi) = (q_\eta(t), \phi)_W \quad \forall\phi \in W. \tag{60}$$

We conclude that $\varphi_\eta(t)$ is a solution of QV_η , let $t_1, t_2 \in [0,T]$, it follows from 20, 24 and 58 that

$$|\ \varphi_\eta(t_1) - \varphi_\eta(t_2)\ |_W \leq C(|\ \mathbf{u}_\eta(t_1) - \mathbf{u}_\eta(t_2)\ |_V + |\ q(t_1) - q(t_2)\ |_W),$$

the previous inequality and the regularity of $\mathbf{u}_\eta$ and q imply that $\varphi_\eta \in C(0,T;W)$. $\square$

In the third step, we let $\theta \in C(0,T;L^2(\Omega))$ be given and consider the following variational problem for the damage field.

Problem PV_θ. Find a damage field $\beta_\theta : [0,T] \to H^1(\Omega)$ such that

$$\beta_\theta(t) \in K,\ (\ \dot{\beta}_\theta\ (t), \xi - \beta_\theta(t))_{L^2(\Omega)} + a(\beta_\theta(t), \xi - \beta_\theta(t))$$

$$\geq (\theta(t), \xi - \beta_\theta(t))_{L^2(\Omega)} \quad \forall\xi \in K \ \text{ a.e. } t \in (0,T)\,, \tag{61}$$

$$\beta_\theta(0) = \beta_0. \tag{62}$$

To solve PV_θ, we recall the following standard result for parabolic variational inequalities (see, e.g., [19, p.47]).

THEOREM 3. *Let $V \subset H \subset V'$ be a Gelfand triple. Let K be a nonempty closed, and convex set of V. Assume that $a(.,.) : V \times V \to \mathbb{R}$ is a continuous and symmetric bilinear form such that for some constants $\zeta > 0$ and c_0,*

$$a(v,v) + c_0 \mid v \mid_H^2 \geq \zeta \mid v \mid_V^2 \quad \forall v \in V.$$

Then, for every $u_0 \in K$ and $f \in L^2(0,T;H)$, there exists a unique function $u \in H^1(0,T;H) \cap L^2(0,T;V)$ such that $u(0) = u_0$, $u(t) \in K$ for all $t \in [0,T]$, and for almost all $t \in (0,T)$,

$$(\dot{u}\,(t), v - u(t))_{V' \times V} + a(u(t), v - u(t)) \geq (f(t), v - u(t))_H \quad \forall v \in K.$$

We apply this theorem to problem PV_θ.

LEMMA 3. *Problem PV_θ has a unique solution β_θ such that*

$$\beta_\theta \in W^{1,2}(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)). \tag{63}$$

Proof. The inclusion mapping of $(H^1(\Omega), \mid . \mid_{H^1(\Omega)})$ into $(L^2(\Omega), \mid . \mid_{L^2(\Omega)})$ is continuous and its range is dense. We denote by $(H^1(\Omega))'$ the dual space of $H^1(\Omega)$ and, identifying the dual of $L^2(\Omega)$ with itself, we can write the Gelfand triple

$$H^1(\Omega) \subset L^2(\Omega) \subset (H^1(\Omega))'.$$

We use the notation $(.,.)_{(H^1(\Omega))' \times H^1(\Omega)}$ to represent the duality pairing between $(H^1(\Omega))'$ and $H^1(\Omega)$. We have

$$(\beta, \xi)_{(H^1(\Omega))' \times H^1(\Omega)} = (\beta, \xi)_{L^2(\Omega)} \quad \forall \alpha \in L^2(\Omega), \xi \in H^1(\Omega),$$

and we not that K is a closed convex set in $H^1(\Omega)$. Then, using the definition 35 of the bilinear form a, and the fact that $\beta_0 \in K$ in 34, it is easy to see that lemma 3 is a straight consequence of Theorem 3. □

In the fourth step, we use the displacement field $\mathbf{u}_\eta$ obtained in lemma 1 and we consider the following initial-value problem.

Problem PV_α. Find the adhesion field $\alpha_\eta : [0,T] \to L^2(\Gamma_3)$ such that for a.e. $t \in (0,T)$

$$\dot{\alpha}_\eta\,(t) = -\left(\alpha_\eta(t)(\gamma_\nu(R_\nu(u_{\eta\nu}(t)))^2 + \gamma_\tau \mid \mathbf{R}_\tau(\mathbf{u}_{\eta\tau}(t)) \mid^2\ ) - \varepsilon_a\right)_+, \tag{64}$$

$$\alpha_\eta(0) = \alpha_0. \tag{65}$$

We have the following result.

LEMMA 4. *There exists a unique solution $\alpha_\eta \in W^{1,\infty}(0,T;L^2(\Gamma_3)) \cap Z$ to problem PV_α.*

Proof. For the sake of simplicity we suppress the dependence of various functions on Γ_3, and note that the equalities and inequalities below are valid a.e. on Γ_3. Consider the mapping $F_\eta : [0,T] \times L^2(\Gamma_3) \to L^2(\Gamma_3)$ defined by

(66) $$F_\eta(t,\alpha) = -\left(\alpha(\gamma_\nu(R_\nu(u_{\eta\nu}(t)))^2 + \gamma_\tau \mid \mathbf{R}_\tau(\mathbf{u}_{\eta\tau}(t)) \mid^2\ ) - \varepsilon_a\right)_+,$$

for all $t \in [0,T]$ and $\alpha \in L^2(\Gamma_3)$. It follows from the properties of the truncation operator R_ν and $\mathbf{R}_\tau$ that F_η is Lipschitz continuous with respect to the second variable, uniformly in time. Moreover, for all $\alpha \in L^2(\Gamma_3)$, the mapping $t \to F_\eta(t,\alpha)$ belongs to $L^\infty(0,T;L^2(\Gamma_3))$. Thus using a version of Cauchy-Lipschitz Theorem given in Theorem 1 we deduce that there exists a unique function $\alpha_\eta \in W^{1,\infty}(0,T;L^2(\Gamma_3))$ solution to the problem PV_α. Also, the arguments used in Remark 1 show that $0 \le \alpha_\eta(t) \le 1$ for all $t \in [0,T]$, a.e. on Γ_3. Therefore, from the definition of the set Z, we find that $\alpha_\eta \in Z$, which concludes the proof of the lemma. □

Finally as a consequence of these results and using the properties of the operator $\mathcal{G}$, the operator $\mathcal{E}$, the functional j and the function S, for $t \in [0,T]$, we consider the element

(67) $$\Lambda(\eta,\theta)(t) = (\Lambda_1(\eta,\theta)(t), \Lambda_2(\eta,\theta)(t)) \in V \times L^2(\Omega),$$

defined by the equalities

$$(\Lambda_1(\eta,\theta)(t),\mathbf{v})_V = (\mathcal{G}(\varepsilon(\mathbf{u}_\eta(t)),\beta_\theta(t)),\varepsilon(\mathbf{v}))_{\mathcal{H}} + (\mathcal{E}^*\nabla\varphi_\eta(t),\varepsilon(\mathbf{v}))_{\mathcal{H}}$$

(68) $$+ j(\alpha_\eta(t),\mathbf{u}_\eta(t),\mathbf{v}) \quad \forall \mathbf{v} \in V,\ t \in [0,T],$$

(69) $$\Lambda_2(\eta,\theta)(t) = S(\varepsilon(\mathbf{u}_\eta(t)),\beta_\theta(t)),\ t \in [0,T].$$

We have the following result.

LEMMA 5. *For* $(\eta,\theta) \in C(0,T;V \times L^2(\Omega))$, *the function* $\Lambda(\eta,\theta) : [0,T] \to V \times L^2(\Omega)$ *is continuous, and there is a unique element* $(\eta^*,\theta^*) \in C(0,T;V \times L^2(\Omega))$ *such that* $\Lambda(\eta^*,\theta^*) = (\eta^*,\theta^*)$.

Proof. Let $(\eta,\theta) \in C(0,T;V \times L^2(\Omega))$, and $t_1,t_2 \in [0,T]$. Using 19, 22, 25, 26 and 27, the definition of R_ν, $\mathbf{R}_\tau$ and the remark 1, we have

$$\mid \Lambda_1(\eta,\theta)(t_1) - \Lambda_1(\ \eta,\theta)(t_2) \mid_V$$

$$\le \mid \mathcal{G}(\varepsilon(\mathbf{u}_\eta(t_1)),\beta_\theta(t_1)) - \mathcal{G}(\varepsilon(\mathbf{u}_\eta(t_2)),\beta_\theta(t_2)) \mid_{\mathcal{H}}$$
$$+ \mid \mathcal{E}^*\nabla\varphi_\eta(t_1) - (\mathcal{E}^*\nabla\varphi_\eta(t_2) \mid_{\mathcal{H}}$$
$$+ C \mid p_\nu(u_{\eta\nu}(t_1)) - p_\nu(u_{\eta\nu}(t_2)) \mid_{L^2(\Gamma_3)}$$

$$+C\mid \alpha_\eta^2(t_1)R_\nu(u_{\eta\nu}(t_1))-\alpha_\eta^2(t_2)R_\nu(u_{\eta\nu}(t_2))\mid_{L^2(\Gamma_3)}$$
$$+C\mid p_\tau(\alpha_\eta(t_1))\mathbf{R}_\tau(\mathbf{u}_{\eta\tau}(t_1))-p_\tau(\alpha_\eta(t_2))\mathbf{R}_\tau(\mathbf{u}_{\eta\tau}(t_2))\mid_{L^2(\Gamma_3)}.$$
$$\leq C(\mid \mathbf{u}_\eta(t_1)-\mathbf{u}_\eta(t_2)\mid_V+\mid \varphi_\eta(t_1)-\varphi_\eta(t_2)\mid_W$$

(70) $$+\mid \beta_\theta(t_1)-\beta_\theta(t_2)\mid_{L^2(\Omega)}+\mid \alpha_\eta(t_1)-\alpha_\eta(t_2)\mid_{L^2(\Gamma_3)}).$$

Recall that above $u_{\eta\nu}$ and $\mathbf{u}_{\eta_\tau}$denote the normal and the tangential component of the function $\mathbf{u}_\eta$ respectively. Next, due to the regularities of $\mathbf{u}_\eta,\varphi_\eta,\beta_\theta$ and α_η expressed in 45, 46, 47 and 48, respectively, we deduce from 70 that $\Lambda_1(\eta,\theta)\in C(0,T;V)$. By a similar argument, from 69 and 23 it follows that

$$\mid \Lambda_2(\eta,\theta)(t_1)-\Lambda_2(\eta,\theta)(t_2)\mid_{L^2(\Omega)}$$

(71) $$\leq C(\mid \mathbf{u}_\eta(t_1)-\mathbf{u}_\eta(t_2)\mid_V+\mid \beta_\theta(t_1)-\beta_\theta(t_2)\mid_{L^2(\Omega)}).$$

Therefore, $\Lambda_2(\eta,\theta)\in C(0,T;L^2(\Omega))$ and $\Lambda(\eta,\theta)\in C(0,T;V\times L^2(\Omega))$. Let now $(\eta_1,\theta_1),(\eta_2,\theta_2)\in C(0,T;V\times L^2(\Omega))$. We use the notation $\mathbf{u}_{\eta_i}=\mathbf{u}_i,\dot{\mathbf{u}}_{\eta_i}=\mathbf{v}_{\eta_i}=\mathbf{v}_i,\varphi_{\eta_i}=\varphi_i,\beta_{\theta_i}=\beta_i$ and $\alpha_{\eta_i}=\alpha_i$ for $i=1,2$. Arguments similar to those used in the proof of 70 and 71 yield

$$\mid \Lambda(\eta_1,\theta_1)(t)-\Lambda(\eta_2,\theta_2)(t)\mid^2_{V\times L^2(\Omega)}$$
$$\leq C(\mid \mathbf{u}_1(t)-\mathbf{u}_2(t)\mid_V^2+\mid \varphi_1(t)-\varphi_2(t)\mid_W^2$$

(72) $$+\mid \beta_1(t)-\beta_2(t)\mid^2_{L^2(\Omega)}+\mid \alpha_1(t)-\alpha_2(t)\mid^2_{L^2(\Gamma_3)}).$$

Since

$$\mathbf{u}_i(t)=\int_0^t \mathbf{v}_i(s)ds+\mathbf{u}_0,\quad t\in[0,T],$$

we have

(73) $$\mid \mathbf{u}_1(t)-\mathbf{u}_2(t)\mid_V^2\leq \int_0^t \mid \mathbf{v}_1(s)-\mathbf{v}_2(s)\mid_V^2 ds\quad \forall t\in[0,T].$$

Moreover, from 51 we obtain that

$$(\mathcal{A}\varepsilon(\mathbf{v}_1)-\mathcal{A}\varepsilon(\mathbf{v}_2),\varepsilon(\mathbf{v}_1-\mathbf{v}_2))_{\mathcal{H}}+(\eta_1-\eta_2,\mathbf{v}_1-\mathbf{v}_2)_V=0.$$

We use the assumption 21 and condition 18 to find that

(74) $$\mid \mathbf{v}_1(t)-\mathbf{v}_2(t)\mid_V^2\leq C\mid \eta_1(t)-\eta_2(t)\mid_V^2.$$

We use 58, 24, 25 and 20 to obtain

(75) $$\mid \varphi_1(t)-\varphi_2(t)\mid_W^2\leq C\mid \mathbf{u}_1(t)-\mathbf{u}_2(t)\mid_V^2.$$

On the other hand, from the Cauchy problem 64-65 we can write

$$\alpha_i(t) = \alpha_0 - \int_0^t (\alpha_i(s)(\gamma_\nu(R_\nu(u_{i\nu}(s)))^2 + \gamma_\tau \mid \mathbf{R}_\tau(\mathbf{u}_{i\tau}(s)) \mid^2) - \varepsilon_a)_+ \, ds,$$

and then

$$\begin{aligned} &\mid \alpha_1(t) - \alpha_2(t) \mid_{L^2(\Gamma_3)} \\ &\leq C \int_0^t \mid \alpha_1(s)(R_\nu(u_{1\nu}(s)))^2 - \alpha_2(s)(R_\nu(u_{2\nu}(s)))^2 \mid_{L^2(\Gamma_3)} ds \\ &+ C \int_0^t \mid \alpha_1(s) \mid \mathbf{R}_\tau(\mathbf{u}_{1\tau}(s)) \mid^2 - \alpha_2(s) \mid \mathbf{R}_\tau(\mathbf{u}_{2\tau}(s)) \mid^2 \mid_{L^2(\Gamma_3)} ds. \end{aligned}$$

Using the definition of R_ν and $\mathbf{R}_\tau$ and writing $\alpha_1 = \alpha_1 - \alpha_2 + \alpha_2$, we get

$$\begin{aligned} &\mid \alpha_1(t) - \alpha_2(t) \mid_{L^2(\Gamma_3)} \\ &\leq C\left(\int_0^t \mid \alpha_1(s) - \alpha_2(s) \mid_{L^2(\Gamma_3)} ds + \int_0^t \mid \mathbf{u}_1(s) - \mathbf{u}_2(s) \mid_{L^2(\Gamma_3)^d} ds\right). \end{aligned}$$

Next, we apply Gronwall's inequality to deduce

$$\mid \alpha_1(t) - \alpha_2(t) \mid_{L^2(\Gamma_3)} \leq C \int_0^t \mid \mathbf{u}_1(s) - \mathbf{u}_2(s) \mid_{L^2(\Gamma_3)^d} ds,$$

and from the relation 19 we obtain

$$\mid \alpha_1(t) - \alpha_2(t) \mid^2_{L^2(\Gamma_3)} \leq C \int_0^t \mid \mathbf{u}_1(s) - \mathbf{u}_2(s) \mid^2_V ds. \tag{76}$$

From 61 we deduce that

$$\begin{aligned} &(\dot{\beta}_1 - \dot{\beta}_2, \beta_1 - \beta_2)_{L^2(\Omega)} + a(\beta_1 - \beta_2, \beta_1 - \beta_2) \\ &\leq (\theta_1 - \theta_2, \beta_1 - \beta_2)_{L^2(\Omega)} \quad \text{a.e. } t \in (0,T). \end{aligned}$$

Integrating the previous inequality with respect to time, using the initial conditions $\beta_1(0) = \beta_2(0) = \beta_0$ and inequality $a(\beta_1 - \beta_2, \beta_1 - \beta_2) \geq 0$ to find

$$\frac{1}{2} \mid \beta_1(t) - \beta_2(t) \mid^2_{L^2(\Omega)} \leq \int_0^t (\theta_1(s) - \theta_2(s), \beta_1(s) - \beta_2(s))_{L^2(\Omega)} ds,$$

which implies that

$$\begin{aligned} &\mid \beta_1(t) - \beta_2(t) \mid^2_{L^2(\Omega)} \\ &\leq \int_0^t \mid \theta_1(s) - \theta_2(s) \mid^2_{L^2(\Omega)} ds + \int_0^t \mid \beta_1(s) - \beta_2(s) \mid^2_{L^2(\Omega)} ds. \end{aligned}$$

This inequality combined with Gronwall's inequality lead to

$$\mid \beta_1(t) - \beta_2(t) \mid^2_{L^2(\Omega)} \leq C \int_0^t \mid \theta_1(s) - \theta_2(s) \mid^2_{L^2(\Omega)} ds \quad \forall t \in [0,T]. \tag{77}$$

We substitute 75 and 76 in 72 and use 73 to obtain

$$| \Lambda(\eta_1,\theta_1)(t) - \Lambda(\eta_2,\theta_2)(t) |^2_{V\times L^2(\Omega)}$$

$$\leq C(| \mathbf{u}_1(t) - \mathbf{u}_2(t) |^2_V + \int_0^t | \mathbf{u}_1(s) - \mathbf{u}_2(s) |^2_V \, ds + | \beta_1(t) - \beta_2(t) |^2_{L^2(\Omega)})$$

$$\leq C(\int_0^t | \mathbf{v}_1(s) - \mathbf{v}_2(s) |^2_V \, ds + | \beta_1(t) - \beta_2(t) |^2_{L^2(\Omega)}).$$

It follows now from the previous inequality, the estimates 74 and 77 that

$$| \Lambda(\eta_1,\theta_1)(t) - \Lambda(\eta_2,\theta_2)(t) |^2_{V\times L^2(\Omega)}$$

$$\leq C\int_0^t | (\eta_1,\theta_1)(s) - (\eta_2,\theta_2)(s) |^2_{V\times L^2(\Omega)} \, ds, \tag{78}$$

thus

$$| \Lambda(\eta_1,\theta_1)(t) - \Lambda(\eta_2,\theta_2)(t) |^2_{V\times L^2(\Omega)}$$

$$\leq Ct \, | (\eta_1,\theta_1) - (\eta_2,\theta_2) |^2_{C(0,T;V\times L^2(\Omega))} \, .$$

Using the inequality 78 with (η_1,θ_1) and (η_2,θ_2) replaced by $\Lambda(\eta_1,\theta_1)$ and $\Lambda(\eta_2,\theta_2)$, respectively, we obtain

$$| \Lambda^2(\eta_1,\theta_1)(t) - \Lambda^2(\eta_2,\theta_2)(t) |^2_{V\times L^2(\Omega)}$$

$$\leq C\int_0^t | \Lambda(\eta_1,\theta_1)(s) - \Lambda(\eta_2,\theta_2)(s) |^2_{V\times L^2(\Omega)} \, ds$$

$$\leq C^2\int_0^t s \, | (\eta_1,\theta_1) - \Lambda(\eta_2,\theta_2) |^2_{C(0,T;V\times L^2(\Omega))} \, ds$$

$$= \frac{C^2}{2!}t^2 \, | (\eta_1,\theta_1) - \Lambda(\eta_2,\theta_2) |^2_{C(0,T;V\times L^2(\Omega))} \, ds.$$

Reiterating this inequality m times leads to

$$| \Lambda^m(\eta_1,\theta_1) - \Lambda^m(\eta_2,\theta_2) |^2_{C(0,T;V\times L^2(\Omega))}$$

$$\leq \frac{C^m T^m}{m!} \, | (\eta_1,\theta_1) - (\eta_2,\theta_2) |^2_{C(0,T;V\times L^2(\Omega))} \, .$$

Thus, for m sufficiently large, Λ^m is a contraction on the Banach space $C(0,T;V\times L^2(\Omega))$, and so Λ has a unique fixed point. □

Now, we have all the ingredients to prove Theorem 2.

Proof. Existence. Let $(\eta^*,\theta^*) \in C(0,T;V\times L^2(\Omega))$ be the fixed point of Λ defined by 67-69 and denote

(79) $$\mathbf{u}_* = \mathbf{u}_{\eta^*}, \varphi_* = \varphi_{\eta^*}, \beta_* = \beta_{\theta^*}, \alpha_* = \alpha_{\eta^*}.$$

Let $\sigma_*\colon [0,T] \to \mathcal{H}$ be defined by

(80) $$\sigma_*(t) = \mathcal{A}\varepsilon(\dot{\mathbf{u}}_*(t)) + \mathcal{G}(\varepsilon(\mathbf{u}_*(t)),\beta_*(t)) + \mathcal{E}^*\nabla\varphi_*(t) \quad \forall t\in[0,T],$$

and let $\mathbf{D}_*\colon [0,T]\to H$ the function be defined by

(81) $$\mathbf{D}_*(t) = -B\nabla\varphi_*(t) + \mathcal{E}\varepsilon(\mathbf{u}_*(t)) \quad \forall t\in[0,T].$$

We prove that the quadruplet $(\mathbf{u}_*,\varphi_*,\beta_*,\alpha_*)$ satisfies 40-44 and the regularity 45-48. Indeed, we write 51 for $\eta=\eta^*$ and use 79 to find

(82) $$(\mathcal{A}\varepsilon(\dot{\mathbf{u}}_*(t)),\varepsilon(\mathbf{v}))_{\mathcal{H}} + (\eta^*(t),\mathbf{v})_V = (\mathbf{f}(t),\mathbf{v})_V \quad \forall \mathbf{v}\in V,\ t\in[0,T],$$

and we write 61 for $\theta=\theta^*$ and use 79 to obtain

$$\beta_*(t)\in K,\ (\dot{\beta}_*(t),\xi-\beta_*(t))_{L^2(\Omega)} + a(\beta_*(t),\xi-\beta_*(t))$$

(83) $$\geq (\theta^*(t),\xi-\beta_*(t))_{L^2(\Omega)} \quad \forall\xi\in K \text{ a.e. } t\in(0,T).$$

Equalities $\Lambda_1(\eta^*,\theta^*)=\eta^*$ and $\Lambda_2(\eta^*,\theta^*)=\theta^*$ combined with 68-69 show that

$$(\eta^*(t),\mathbf{v})_V = (\mathcal{G}(\varepsilon(\mathbf{u}_*(t)),\beta_*(t)),\varepsilon(\mathbf{v}))_{\mathcal{H}} + (\mathcal{E}^*\nabla\varphi_*(t),\varepsilon(\mathbf{v}))_{\mathcal{H}}$$

(84) $$+\,j(\alpha_*(t),\mathbf{u}_*(t),\mathbf{v}) \quad \forall\mathbf{v}\in V,$$

(85) $$\theta^*(t) = S(\varepsilon(\mathbf{u}_*(t)),\beta_*(t)).$$

We now substitute 84 in 82 to obtain

$$(\mathcal{A}\varepsilon(\dot{\mathbf{u}}_*(t)),\varepsilon(\mathbf{v}))_{\mathcal{H}} + (\mathcal{G}(\varepsilon(\mathbf{u}_*(t)),\beta_*(t)),\varepsilon(\mathbf{v}))_{\mathcal{H}} + (\mathcal{E}^*\nabla\varphi_*(t),\varepsilon(\mathbf{v}))_{\mathcal{H}}$$

(86) $$+\,j(\alpha_*(t),\mathbf{u}_*(t),\mathbf{v}) = (\mathbf{f}(t),\mathbf{v})_V \quad \forall\mathbf{v}\in V, t\in(0,T),$$

and we substitute 85 in 83 to have

$$\beta_*(t)\in K \text{ for all } t\in[0,T], \quad (\dot{\beta}_*(t),\xi-\beta_*(t))_{L^2(\Omega)} + a(\beta_*(t),\xi-\beta_*(t))$$

(87) $$\geq (S(\varepsilon(\mathbf{u}_*(t)),\beta_*(t)),\xi-\beta_*(t))_{L^2(\Omega)} \quad \forall\xi\in K,$$

we write now 58 for $\eta = \eta^*$ and use 79 to see that

$$(88) \qquad (B\nabla\varphi_*(t), \nabla\phi)_H - (\mathcal{E}\varepsilon(\mathbf{u}_*(t)), \nabla\phi)_H = (q(t), \phi)_W \quad \forall\phi \in W, t \in (0,T),$$

and we write 64 for $\eta = \eta^*$ and use 79 to find

$$(89) \quad \dot{\alpha}_*(t) = -\left(\alpha_*(t)(\gamma_\nu(R_\nu(u_{*\nu}(t)))^2 + \gamma_\tau \mid \mathbf{R}_\tau(\mathbf{u}_{*\tau}(t)) \mid^2) - \varepsilon_a\right)_+ \text{ a.e. } t \in (0,T).$$

The relations 86, 87, 88 and 89 allow us to conclude that $(\mathbf{u}_*, \varphi_*, \beta_*, \alpha_*)$ satisfies 40-43. Next, 44 and the regularity 45-48 follow from Lemmas 1, 2, 3 and 4, since $(\mathbf{u}_*, \varphi_*)$ satisfies 45-46, it follows from 80 that

$$(90) \qquad \sigma_* \in C(0,T;\mathcal{H}).$$

We choose $\mathbf{v} = \omega \in D(\Omega)^d$ in 86, we use 80 and 36 to obtain

$$Div\sigma_*(t) = -\mathbf{f}_0(t) \quad \forall t \in [0,T],$$

where $D(\Omega)^d = \{\mathbf{u} = (u_i) \ / \ u_i \in D(\Omega)\}$ and $D(\Omega)$ is the space of infinitely differentiable real functions with a compact support in Ω, we use 28 and 90 to find

$$\sigma_* \in C(0,T;\mathcal{H}_1).$$

Let $t_1, t_2 \in [0,T]$, by using 24, 25, 18 and 81 we deduce that

$$\mid \mathbf{D}_*(t_1) - \mathbf{D}_*(t_2) \mid_H \leq C(\mid \varphi_*(t_1) - \varphi_*(t_2) \mid_W + \mid \mathbf{u}_*(t_1) - \mathbf{u}_*(t_2) \mid_V),$$

the previous inequality and the regularity of $\mathbf{u}_*$ and φ_* given by 45-46 imply

$$(91) \qquad \mathbf{D}_* \in C(0,T;H).$$

We choose $\phi \in D(\Omega)$ in 88 and use 37 we find

$$div\mathbf{D}_*(t) = q_0(t) \quad \forall t \in [0,T],$$

by 29 and 91 we obtain

$$\mathbf{D}_* \in C(0,T;\mathcal{W}).$$

Finally we conclude that the weak solution $(\mathbf{u}_*, \sigma_*, \varphi_*, \mathbf{D}_*, \beta_*, \alpha_*)$ of the piezoelectric contact problem P has the regularity 45-50, which concludes the existence part of Theorem 2.

Uniqueness. The uniqueness of the solution is a consequence of the uniqueness of the fixed point of the operator Λ defined by 67-69. □

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