Rend. Sem. Mat. Univ. Pol. Torino - Vol. 66, 2 (2008)

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RANK 2 ARITHMETICALLY COHEN-MACAULAY VECTOR BUNDLES ON CERTAIN RULED SURFACES

Abstract. Here we study rank 2 arithmetically Cohen-Macaulay vector bundles on a a ruled surface over a smooth genus q curve, essentially proving their non-existence if $q \geq 2$ and the ruled surface is rather balanced.

1. Introduction

Let X be an integral n-dimensional projective variety, $n \geq 2$, defined over an algebraically closed field. Let η_+ denote the ample cone of $\mathrm{Pic}(X)$ and η_- its opposite. Let η_0 (resp. $\widetilde{\eta}_0$) denote the set of all line bundles on X algebraically equivalent to $\mathcal{O}_X$ (resp. numerically trivial). Set $\eta := \eta_+ \cup \eta_-$, $\gamma := \eta \cup \eta_0$ and $\widetilde{\gamma} := \eta \cup \widetilde{\eta_0}$. Let E be a vector bundle on X. We will say that E is ACM or *arithmetically Cohen-Macaulay* (resp. say that E is WACM or *weakly arithmetically Cohen-Macaulay*, resp. SACM or *strongly arithmetically Cohen-Macaulay*) if $H^i(X, E \otimes L) = 0$ for all $1 \leq i \leq n-1$ and all $L \in \gamma$ (resp. $L \in \eta$, resp. $L \in \widetilde{\gamma}$). Let C be a smooth and connected projective curve. Set $q := p_a(C)$. For any rank 2 vector bundle F on C set $s(F) = \deg(F) - 2 \cdot \deg(L)$, where L is a maximal degree rank 1 subsheaf of F. Hence F is stable (resp. semistable, resp. properly semistable) if and only if $s(F) > 0$ (resp. $s(F) \geq 0$, resp. $s(F) = 0$). A theorem of C. Segre and M. Nagata says that $s(F) \leq q$. If $s(F) \geq 0$, then set $e(F) := s(F)$. If $s(F) < 0$, then set $e(F) := 0$.

THEOREM 1. *Let C be a smooth curve of genus $q \geq 2$ and G a rank 2 vector bundle on C such that $2q - 3 \geq \max\{0, -s(G)\} + 3e(G)$. Set $X := \mathbf{P}(G)$. If $q \geq 2$, then there is no rank 2 WACM vector bundle on X.*

Of course, we will also check the rank 1 case (see Proposition 1). As obvious from that proof and the proof of Theorem 1 with no restriction on G there are very strong numerical restrictions for the WACM and ACM line bundles and rank 2 vector bundles on the ruled surface X. We stress the existence of rank 2 ACM vector bundles on X when $q = 1$ and $G = \mathcal{O}_C^{\oplus 2}$ ([1]) and of rank one ACM line bundles when $q = 0$, i.e. for Hirzebruch surfaces ([2]). For large e there are more (but always finitely many) isomorphism classes of line bundles on F_e ([2]).

*The author was partially supported by MIUR and GNSAGA of INdAM (Italy).

2. The proof and related results

Notice that on any scroll over a smooth curve numerical equivalence and algebraic equivalence are the same. Hence $\widetilde{\eta}_0 = \eta_0$ and $\widetilde{\gamma} = \gamma$ on any scroll over any smooth curve.

REMARK 1. Let C be a smooth curve of genus q and F a rank r vector bundle on C. If $h^0(C, F \otimes L) = 0$ for all $L \in \mathrm{Pic}^0(X)$, then $\deg(F) < (r-1)(q-1)$ ([4], Corollary at p. 252). Thus Riemann-Roch and Serre duality give that if $h^1(C, F \otimes L) = 0$ for all $L \in \mathrm{Pic}^0(C)$, then $\deg(F) > (r+1)(q-1)$.

REMARK 2. Fix $t \in \mathbb{Z}$. Fix a rank 2 vector bundle F on C. Set $d := \deg(F)$ and $s := s(F)$. Let L be a maximal degree rank one subsheaf of F. F/L is locally free, $\deg(L) = (d-2s)/2$ and $\deg(F/L) = (d+2s)/2$. Hence $s \equiv d \pmod 2$. $s(F \otimes R) = s(F)$ for all $R \in \mathrm{Pic}(C)$. $h^0(C, F \otimes M) = 0$ for all $M \in \mathrm{Pic}^t(C)$ if and only $\deg(L) + t \leq -1$, i.e. if and only if $(d-2s)/2 + t \leq -1$. Notice that $s(F^*) = s(F)$. Hence Serre duality shows that $h^1(C, F \otimes M) = 0$ for every $M \in \mathrm{Pic}^t(C)$ if and only $\deg(F/L) + t \geq 2q - 1$, i.e. if and only if $(d+2s)/2 + t \geq 2q - 1$.

NOTATION 1. Fix a smooth and connected curve C with genus q and the ruled surface $X = \mathbf{P}(G)$, where G is a rank 2 vector bundle on C. Let G_1 be a rank 1 subsheaf of G. Since G_1 has maximal degree, $G_2 := G/G_1$ is a line bundle. Set $a_i := \deg(G_i)$. Hence $\deg(G) = a_1 + a_2$ and $s(G) = a_2 - a_1$. Since $\mathbf{P}(G) \cong \mathbf{P}(G \otimes R)$ for any $R \in \mathrm{Pic}(C)$, we will always normalize G so that $G_2 \cong \mathcal{O}_C$. Hence $a_2 = 0$, $\deg(G) = a_1$ and $s(G) = -a_1$. Recall that $e(G) := 0$ if $a_1 \geq 0$ and $e(G) := -a_1$ if $a_1 < 0$. Notice thar $0 \leq e(G) \leq q$ for any X (Remark 2). Let $\pi : X \to C$ denote the ruling and $\mathcal{O}_\pi(1)$ the tautological π-ample line bundle on X. $\mathrm{Pic}(X) \cong \mathbb{Z}\mathcal{O}_\pi(1) \oplus \pi^*(\mathrm{Pic}(C))$. For every integer t and every $M \in \mathrm{Pic}(C)$ set $\mathcal{O}_\pi(t) := \mathcal{O}_\pi^{\otimes t}$ and $\mathcal{O}_X(t, M) := \mathcal{O}_\pi(t) \otimes \pi^*(M)$.

REMARK 3. Take $C, G, X, a_1, e(G)$ as in Notation 1. Fix $D \in \mathrm{Pic}(C)$. Notice that $X \cong \mathbf{P}(G \otimes D)$. Applying [3], Theorem III.1.1, to the vector bundle $G \otimes D$ we get that $\mathcal{O}_X(1, D)$ is ample if and only if $\deg(D) \geq 1 + e(G)$.

First Claim: For every integer $x > 0$ $S^x(G) \otimes D$ is an ample vector bundle if $\deg(D) \geq 1 + xe(G)$.

Proof of the First Claim: The vector bundle $S^x(G)$ has rank $x+1$ and it has an increasing filtration $\{F_i\}_{0 \leq i \leq x}$ such that $F_0 = 0$, $F_{x+1} = S^x(G)$, each F_i/F_{i-1}, $1 \leq i \leq x+1$, is a line bundle of degree ≥ 0 (case $e(G) = 0$) or degree $\geq -xe(G)$ (case $e(G) > 0$), and $\deg(F_1) = xa_1$. Just use that an extension of ample line bundles is ample and that a line bundle on C is ample if it has positive degree.

Second Claim: Fix an integer $x \geq 1$ and assume $\deg(D) \geq 1 + xe(G)$. Then $R := \mathcal{O}_X(x, D)$ is ample.

Proof of the Second Claim: By Nakai criterion ([3], Theorem I.5.1) it is sufficient to prove that $R^2 > 0$ and that $\mathcal{O}_X(T) \cdot R > 0$ for every integral curve $T \subset X$. $R^2 = 2x \cdot \deg(D) + x^2 a_1 > 0$. Take an integral curve $T \subset X$ and set $\mathcal{O}_X(y, M) := \mathcal{O}_X(T)$. Notice that $y \geq 0$ and that $y = 0$ if and only if T is a fiber of π. $\mathcal{O}_X(T) \cdot R = xya_1 + x \cdot$

$\deg(M) + y \cdot \deg(D)$. If $y = 0$, then $\mathcal{O}_X(T) \cdot R = x > 0$. From now on we assume $y > 0$. First assume $a_1 \geq 0$. Hence $e(G) = 0$ and $\mathcal{O}_X(T) \cdot R \geq xya_1 + x \cdot \deg(M) + y > x(ya_1 + \deg(M))$. Hence it is sufficient to prove that $\deg(M) \geq -ya_1$. Assume $\deg(M) \leq -ya_1 - 1$. To get a contradiction it is sufficient to show that $h^0(X, \mathcal{O}_X(y,M)) = 0$. Since $y > 0$, $h^0(X, \mathcal{O}_X(y,M)) = h^0(C, S^y(G) \otimes M)$. The vector bundle $S^y(G)$ has rank $y+1$ and it has an increasing filtration $\{F_i\}_{0 \leq i \leq x}$ such that $F_0 = 0$, $F_{y+1} = S^y(G)$, each F_i/F_{i-1}, $1 \leq i \leq y+1$, is a line bundle of degree $(y+1-i)a_1$. Hence $h^0(X, \mathcal{O}_X(y,M)) = 0$. Now assume $a_1 < 0$. Hence $e(G) = -a_1$ and $\mathcal{O}_X(T) \cdot R \geq y + x \cdot \deg(M)$. Hence it is sufficient to observe that the same filtration of $S^y(G)$ used in the previous case gives $h^0(C, S^y(G) \otimes M) = 0$ if $\deg(M) < 0$.

REMARK 4. Take $C, G, X, a_1, e(G)$ as in Notation 1. Let F be a rank 2 vector bundle on C.

(a) Set $E := \pi^*(F)$. We want to check that E is not WACM if $3e(G) \leq 2q-3$. Assume that E is WACM. $h^1(X, E(1,D)) = h^1(C, G \otimes F \otimes D)$. If $h^1(C, G \otimes F \otimes D) = 0$, then $h^1(C, G_2 \otimes F \otimes D) = 0$. Recall that $G_2 \cong \mathcal{O}_C$ and that $\mathcal{O}_X(1,D)$ is ample if $\deg(D) \geq 1 + e(G)$. Varying $D \in \mathrm{Pic}^{1+e(D)}(C)$ and applying Remark 1 we get $\deg(F) \geq 3q - 3 - e(G)$. Set $J := \mathcal{O}_X(2,M)$ with $M \in \mathrm{Pic}^{1+2e(G)}(C)$. J is ample. Serre duality gives $h^1(X, E \otimes J^*) = h^1(X, E^*(0, F^* \otimes \omega_C \otimes \det(G) \otimes M \otimes A^*)) = h^1(C, F^* \otimes \omega_C \otimes \det(G) \otimes M) = h^0(C, F \otimes M^*)$. Remark 1 shows that if $h^0(C, F \otimes M^*) = 0$ for all M, then $\deg(F) \leq q - 2 + 1 + 2e(G)$.

(b) Set $E := \pi^*(F)(-1, \mathcal{O}_C)$. $h^1(X, E \otimes L) = 0$ for all $L \in \eta_0$. Here we check that E is not WACM if $3e(G) \leq 2q-2$. Assume that E is WACM. $h^1(X, E(1,D)) = 0$ if and only if $h^1(X, F \otimes D) = 0$. Hence we get $h^1(X, F \otimes D) = 0$ for all $D \in \mathrm{Pic}^{1+e(G)}(C)$. Remark 1 gives $\deg(F) + 2 + 2e(G) > 3(q-1)$, i.e. $\deg(F) \geq 3q - 2 - 2e(G)$. Serre duality shows that $h^1(X, E(-1,M)) = 0$ if and only if $h^1(X, \pi^*(F^*)(0, M^* \otimes \omega_C)) = 0$, i.e. if and only $h^1(C, F^* \otimes M^* \otimes \omega_C) = 0$, i.e. if and only if $h^0(C, F \otimes M) = 0$. Varying M in $\mathrm{Pic}^{-1-e(G)}(C)$ we get $\deg(F) \leq q - 1 + e(G)$.

(c) Set $E := \pi^*(F)(-2, \mathcal{O}_C)$. Serre duality and part (a) shows that E is not WACM if $3e(G) \leq 2q-3$.

PROPOSITION 1. *Take $C, G, X, a_1, e(G)$ as in Notation 1. If $q \geq 2$ and $2q - 3 \geq \max\{0, a_1\} + 3e(G)$, then there is no WACM line bundle on X.*

Proof. Fix any $R := \mathcal{O}_X(x,A) \in \mathrm{Pic}(X)$ and assume that R is WACM.

(a) Here we assume $x \geq -1$. Take any $L := \mathcal{O}_X(1,D)$ such that $\deg(D) = 1 + e(G)$. L is ample (Remark 3). Since $x+1 \geq 0$, $h^1(X, R \otimes L) = 0$ if and only if $h^1(C, S^{x+1}(G) \otimes A \otimes D) = 0$. Since $\mathcal{O}_C = G_2$ is a quotient of G, $\mathcal{O}_C$ is a quotient of $S^t(G)$ for any $t > 0$. Hence if $t > 0$, $M \in \mathrm{Pic}(C)$ and $h^1(C, S^t(G) \otimes M) = 0$, then $h^1(C, M) = 0$. Varying D in $\mathrm{Pic}^{1+e(G)}(C)$ we see that if R is WACM, then $\deg(A) + 1 + e(G) \geq 2q - 1$, i.e. $\deg(A) \geq 2q - 2 - e(G)$.

(b) Here we assume $x > 0$. Set $L := \mathcal{O}_X(x,D)$ with $\deg(D) \gg 0$. Since L is ample and $h^1(X, R \otimes L^*) = h^1(C, A \otimes D^*) > 0$ if $\deg(D) \gg 0$, R is not WACM.

(c) Here we assume $x = 0$. Take $L := \mathcal{O}_X(2,D)$ with $\deg(D) = 2 \cdot e(G) + 1$.

Hence L is ample (Remark 4). Serre duality gives $h^1(X, R\otimes L^*) = h^1(X, \mathcal{O}_X(0, \omega_C \otimes \det(G)\otimes D\otimes A^*)) = h^1(C, \omega_C\otimes \det(G)\otimes D\otimes \otimes A^*)$. Varying D in $\mathrm{Pic}^{1+2e(G)}(C)$ we see that R is not WACM if $\det(G)+1+2\cdot e(G)-\deg(A)\le 0$, i.e. if $\deg(A)\ge a_1+1+2e(G)$. If $2q-2-e(G)\ge a_1+1+2e(G)$, then part (a) shows that R is not WACM.

(d) Here we assume $x=-1$. Take $L := \mathcal{O}_X(1,D)$ with $\deg(D) = 1+e(G)$. L is ample. $h^1(X,R\otimes L) = h^0(C, A\otimes D)$. Hence varying D in $\mathrm{Pic}^{1+e(G)}(C)$ we see that if R is WACM, then $\deg(A)+1+e(G)\ge 2q-1$, i.e. $\deg(A)\ge 2q-2-e(G)$. Serre duality gives $h^1(X,R\otimes L^*) = h^1(X,\mathcal{O}_X(0, D\otimes A^*\otimes \omega_C\otimes \det(G)))$. Hence If R is WACM, then $1+e(G)-\deg(A)+2q-2+a_1\ge 2q-1$, i.e. $\deg(A)\le e(G)+a_1$. Thus if R is WACM, then $2q-2-e(G)\le \deg(A)\le e(G)+a_1$. First assume $a_1\le 0$. Hence $e(G) = -a_1$. Since $q\ge 2$, we get a contradiction. Now assume $a_1>0$. Hence $e(G)=0$. In this case the contradiction comes from the assumption $2q-1\ge a_1$.

(e) Here we assume $x\le -2$. Serre duality shows that R is not WACM under the same assumptions we used in the case $x\ge 0$. Notice that if $x<-2$, then no assumption at all is needed. □

Proof of Theorem 1. Let E be a rank 2 WACM vector bundle on X. Since $\mathrm{Pic}(X)\cong \mathbb{Z}\mathcal{O}_\pi(1)\oplus \pi^*(\mathrm{Pic}(C))$, there are an integer x and $A\in \mathrm{Pic}(C)$ such that $\det(E)\cong \mathcal{O}_X(x,A)$. By [1], proof of Theorem 2, and [2], Theorem 1, $-4\le x\le 0$ and there are an integer $z\in\{-2,-1,0\}$, $N\in \mathrm{Pic}(C)$, and an exact sequence

$$0\to \mathcal{O}_X(z,N)\to E\to \mathcal{O}_X(x-z, A\otimes N^*)\to 0 \qquad (1)$$

Moreover, $x\le 2z$.

(a) Here we assume $x=2z$. A base-change theorem ([5], p. 11) says that $F := \pi_*(E(-z,\mathcal{O}_C))$ is a rank 2 vector bundle on C and that the natural map $\pi^*(F)\to E(-z,\mathcal{O}_C)$ is an isomorphism. Apply Proposition 1. Hence from now on in the proof we will assume $x<2z$ and in particular $z\in\{-1,0\}$.

(b) Here we assume $z=-1$. Hence $x\in\{-4,3\}$. Fix any $D\in \mathrm{Pic}^{1+e(G)}(G)$ and set $L := \mathcal{O}_X(1,D)$. L is ample (Remark 3). Since $x-z+1<0$, $h^0(X,\mathcal{O}_X(x-z+1, A\otimes N^*\otimes D)) = 0$. Since E is WACM, the exact sequence (1) gives $h^1(X,\mathcal{O}_X(-1,N)\otimes L) = 0$. Since $x-z-1<0$, $h^0(X,\mathcal{O}_X(x-z-1, A\otimes N^*\otimes D^*)) = 0$. Since E is WACM, we get $h^1(X,\mathcal{O}_X(-1,N)\otimes L^*) = 0$. Part (d) of the proof of Proposition 1 gives a contradiction, because $q\ge 2$ and $2q-1\ge a_1$.

(c) Here we assume $z=0$ and $x\le -2$. Take L as in part (b). Since $h^0(X,\mathcal{O}_X(x-z+1, A\otimes N^*\otimes D)) = h^0(X,\mathcal{O}_X(x-z-1, A\otimes N^*\otimes D^*)) = 0$, we conclude as in part (b).

(d) Here we consider the case $(z,x) = (0,-1)$, i.e. the unique remaining case. Fix any $D\in \mathrm{Pic}^{1+e(G)}(G)$ and set $L := \mathcal{O}_X(1,D)$. L is ample (Remark 3). Set $R := \mathcal{O}_X(-1, A\otimes N^*)$. Since $h^2(X,\mathcal{O}_X(1,N\otimes D)) = h^2(X,\mathcal{O}_X(-1,N\otimes D^*)) = 0$ and E is WACM, the exact sequence (1) gives $h^1(X,R\otimes L) = h^1(X,R\otimes L^*) = 0$. Part (d) of the proof of Proposition 1 gives a contradiction, because $q\ge 2$ and $2q-1\ge a_1$. □

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AMS Subject Classification: 14J60; 14H60.

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Lavoro pervenuto in redazione il 13.04.2008 e, in forma definitiva, il 14.05.2008.