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ON STRONG CONVERGENCE TO COMMON FIXED POINTS IN BANACH SPACES

Abstract. The purpose of this paper is to improve [9, Theorem 3.3] by removing the hypothesis of uniform convexity. We prove the following theorem: Let X be a reflexive Banach space which has the sequentially weakly continuous duality map with gauge J. Let C be a closed convex subset of X, let $\Gamma = \{T(t) : t \geq 0\}$ be a strongly continuous semigroup of nonexpansive mappings on C. For fixed point $u \in C$, define the sequence $\{x_n\}_{n\geq 1}$ by $x_n = \alpha_n u + (1-\alpha_n)T(t_n)x_n;$ for $n \geq 1$, where $\{\alpha_n\}$ and $\{t_n\}$ are real sequences satisfaying $\lim\limits_{n\to\infty} t_n = \lim\limits_{n\to\infty} \frac{t_n}{\alpha_n} = 0$. Then the sequence $\{x_n\}$ converges strongly to a point of $Fix\Gamma$.

1. Introduction

Let X be a Banach space, and let C be a nonempty closed convex subset of X. A mapping $T : C \to X$ is said to be nonexpansive if it satisfy $\| Tx - Ty \| \leq \| x - y \|$. It is already known that T has a fixed point while C is bounded closed convex subset of a Hilbert space, see for example [1]. Fix $u \in C$, then for each $\alpha \in (0,1)$ there exists a unique point x_α in C satisfying $x_\alpha = (1-\alpha)Tx_\alpha + \alpha u$ because of the nonexpansiveness of the mapping $x \to (1-\alpha)Tx + \alpha u$. In 1965 Browder proved the following theorem:

THEOREM 1 (Browder[3]). *Let C be a closed convex subset of a Hilbert space H and let T be a nonexpansive mapping on C with a fixed point. Let $\{\alpha_n\}$ be a sequence of $(0,1)$ converging to 0. Fix $u \in C$ and define a sequence $\{u_n\}$ by*

$$u_n = (1-\alpha_n)Tu_n + \alpha_n u, \; for\, n \in \mathbb{N}.$$

Then $\{u_n\}$ converges strongly to the element of $Fix(T)$ nearest to u.

In 1980 Reich proved in [5] the same result for uniformly smooth Banach space. This motivates a large number of authors to look for extending Browder's and Reich's results to the contraction semigroup case. However, only partial answers have been obtained.
In [6], Shioji and Takahashi introduce in Hilbert space the implicit iteration scheme

$$x_n = \alpha_n u + (1-\alpha_n)\sigma_{t_n}(x_n), \, n \geq 1,$$

Where $\{\alpha_n\}$ is a real sequence in $(0,1), \{t_n\}$ is a sequence of positive real numbers divergent to ∞, and for each $t > 0$ and $x \in C, \sigma_t(x)$ is the average given by

$$\sigma_t(x) = \frac{1}{t}\int_0^t T(s)xds.$$

Under certain appropriate assumptions on $\{\alpha_n\}$, the authors proved that the sequence $\{x_n\}$ converges strongly to an element of F.

Suzuki [7] is the first who introduce, again in Hilbert space, the implicit iteration scheme:

$$x_n = \alpha_n u + (1-\alpha_n)T(t_n)x_n,\, n \in \mathbb{N},$$

for the nonexpansive semigroup case. Under some assumptions on the parameter sequences $\{\alpha_n\}$ and $\{t_n\}$, Suzuki proved that his iteration converges strongly to a point of F. We should note here that in the iteration of Shioji and Takahashi x_n is constructed through the average of the semigroup on the segment $(0,t)$ while in the Suzuki's iteration is constructed directly from the semigroup, so the iteration scheme of Suzuki is stronger than the Shioji's and Takahashi's one.
Recently, Hong-Kun Xu [9] extends Suzuki's result to Uniform Banach spaces having weakly sequentially duality map, and proved that Suzuki's process converges strongly in such spaces. The purpose of this paper is to extend this result to a larger class of Banach spaces, more precisely we show that the Suzuki's result always holds in the reflexive Banach spaces which have a weakly sequentially duality map.

2. Preliminaries

Let X be a Banach space, suppose μ is a continuous strictly increasing real-valued function on $\mathbb{R}^+$ satisfying $\mu(0) = 0$ and $\lim\limits_{t\to\infty}\mu(t) = +\infty$.

DEFINITION 1. *A mapping $J : X \to X^*$ is called a duality mapping with gauge function μ if for every $x \in X$,*

$$\langle x, Jx\rangle = \| Jx \| \| x \| = \mu(\| x \|) \| x \| .$$

Such a mapping J is said to be weakly sequentially continuous if J is sequentially continuous relative to the weak topologies on both X and X^*. It is already known that the spaces $l^p, 1 < p < \infty$, possess duality mappings which are weakly sequentially continuous.

DEFINITION 2. *Let D be a subset of X. A mapping $T : D \to X$ is said to be accretive if for all $u, v \in D$, and some $j \in J(u-v)$,*

$$\langle Tu - Tv, j\rangle \geq 0.$$

Here J denotes the normalized duality mapping, i.e. for $x \in X$,

$$J(x) = \{ j \in X^* : \langle x, j\rangle = \| x \|^2 = \| j \|^2 \}.$$

One connection between accretive operators and nonexpansive mappings is immediate.

If $T : D \to X$ is nonexpansive, then for $U = I - T, x, y \in D$, and $j \in J(x-y)$,

$$\begin{aligned} \langle Ux - Uy, j \rangle &= \langle x - y - (Tx - Ty), j \rangle \\ &= \| x - y \|^2 - \langle Tx - Ty, j \rangle \\ &\geq \| x - y \|^2 - \| Tx - Ty \| \| x - y \| \\ &\geq 0. \end{aligned}$$

Thus U is accretive.
On the other hand not all accretive mappings are of the form $I - T$ with T nonexpansive.

Now if we set $\phi(t) = \int_0^t \mu(\tau) d\tau, t \geq 0,$ then we obtain $J(x) = \partial\phi(\|x\|), \forall x \in X$, where ∂ denotes the subdifferential in the sense of convex analysis.
The first part of the following lemma is an immediate consequence of the subdifferential inequality and the proof of the second part can be found in [4]; see also [9].

LEMMA 1. *Let X be a Banach space, and assume that it has weakly sequentially continuous duality mapping J with gauge μ, then we have:*

(i) For all $x, y \in X$, there holds the inequality

$$\phi(\| x + y \|) \leq \phi(\| x \|) + \langle y, J(x+y) \rangle.$$

(ii) Assume that a sequence $\{x_n\} \in X$ is weakly convergent to a point x. Then there holds the identity

$$\limsup_{n\to\infty} \phi(\| x_n - y \|) = \limsup_{n\to\infty} \phi(\| x_n - x \|) + \phi(\| y - x \|)$$

for all $x, y \in X$. In particular, X satisfies Opial's property; that is, if $\{x_n\}$ is a sequence weakly convergent to x, then holds the inequality

$$\limsup_{n\to\infty} \| x_n - x \| < \limsup_{n\to\infty} \| x_n - y \|, y \in X, y \neq x.$$

3. Main result

Recall that a one parameter strongly continuous semigroup of nonexpansive mappings on C is a family $\Gamma = \{T(t) : t \geq 0\}$ which satisfy:

(i) $T(0)x = x$ for all $x \in C$;

(ii) $T(t+s)x = T(t)T(s)x$ for $t, s \geq 0$ and $x \in C$;

(iii) $\lim_{t\to 0^+} T(t)x = x$ for $x \in C$;

(iv) for each $t>0, T(t)$ is nonexpansive.

We denote by $\mathfrak{F}$ the set of common fixed points of Γ; that is,

$$\mathfrak{F}=Fix(\Gamma)=\{x\in C: T(t)x=x, \forall t>0\}=\bigcap_{t>0}Fix(T(t)).$$

We know that $\mathfrak{F}$ is nonempty while C is a bounded closed convex of a Hilbert space; see [2].
Now we are able to give our main result.

THEOREM 2. *Let $\mathfrak{X}$ be a reflexive Banach space having a weakly sequentially continuous duality mapping J, C a closed convex subset of X and $\Gamma=\{T(t): t\geq 0\}$ a strongly continuous semigroup of nonexpansive mappings on C such that $\mathfrak{F}\neq\emptyset$. Define a sequence $\{x_n\}$ in C implicitly by the fixed point iteration process*

$$x_n=\alpha_n u+(1-\alpha_n)T(t_n)x_n,$$

where $u\in C$ is an arbitrarily fixed element in C and $\{\alpha_n\}$ and $\{t_n\}$ are sequences of real numbers such that $\alpha_n\in(0,1)$ and $t_n>0$ for all $n\geq 1$, and $\lim_{n\to\infty}t_n=\lim_{n\to\infty}(\frac{\alpha_n}{t_n})=0$. Then $\{x_n\}$ converges strongly to a point of $\mathfrak{F}$.

Proof. Remark that the sequence $\{x_n\}$ is bounded, indeed, let $z\in\mathfrak{F}$, then we have:

$$\begin{aligned}\| x_n-z\| &\leq \alpha_n\| u-z\|+(1-\alpha_n)\| T(t_n)x_n-z\|\\ &\leq \alpha_n\| u-z\|+(1-\alpha_n)\| x_n-z\|\end{aligned}$$

hence $\| x_n-z\|\leq\| u-z\|$.
Since $\mathfrak{X}$ is reflexive, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which converges weakly to some $x\in C$.
Let $t>0$, we can assume that $t>t_n$, since $\{t_n\}$ converges to 0.
If we set

$$\tilde{t}_k=t_{n_k}, m=[\frac{t}{\tilde{t}_k}], \tilde{x}_k=x_{n_k} \text{ and } \tilde{\alpha}_k=\alpha_{n_k},$$

then

$$T(\tilde{t}_k)\tilde{x}_k-\tilde{x}_k=\tilde{\alpha}_k(T(\tilde{t}_k)-u),$$

so we have

$$\begin{aligned}\| x_{n_k}-T(t)x\| &\leq \sum_{i=0}^{m-1}\| T((i+1)\tilde{t}_k)\tilde{x}_k-T(i\tilde{t}_k)\tilde{x}_k\|+\| T(m\tilde{t}_k)\tilde{x}_k-T(m\tilde{t}_k)x\|\\ &\quad+\| T(m\tilde{t}_k)x-T(t)x\|\\ &\leq m\| T(\tilde{t}_k)\tilde{x}_k-\tilde{x}_k\|+\|\tilde{x}_k-x\|+\| T(t-m\tilde{t}_k)x-x\|\\ &\leq t\frac{\tilde{\alpha}_k}{\tilde{t}_k}\| T(\tilde{t}_k)\tilde{x}_k-\tilde{x}_k\|+\|\tilde{x}_k-x\|\\ &\quad+\max_{0\leq s\leq\tilde{t}_k}\| T(s)x-x\| .\end{aligned}$$

As $k \to \infty$, we get:

$$\limsup_{k\to\infty} \| x_{n_k} - T(t)x \| \leq \limsup_{k\to\infty} \| x_{n_k} - x \| .$$

Hence, by Opial's property, we conclude that $T(t)x = x,\;$ for all $t \in \mathbb{R}^+$, and then $x \in \mathfrak{F}$.
Now if we assume that there exist $\{x_{n_p}\}$, and $\{x_{n_q}\}$ two subsequences of $\{x_n\}$ such that $\{x_{n_p}\}$ and $\{x_{n_q}\}$ converge weakly to x and y respectively.
Then as it was seen above x and y are in $\mathfrak{F}$.
In one hand we have:

$$\begin{aligned}
\langle u - x_{n_p}, J(y - x_{n_p})\rangle &= \left(\frac{1}{\alpha_{n_p}} - 1\right)\langle (I - T(t_{n_p}))x_{n_p}, J(y - x_{n_p})\rangle \\
&= -\left(\frac{1}{\alpha_{n_p}} - 1\right)\langle (I - T(t_{n_p}))y - (I - T(t_{n_p}))x_{n_p}, J(y - x_{n_p})\rangle \\
&\leq 0
\end{aligned}$$

Because $I - T(t_n)$ is accretive as it be seen earlier.
In the other hand, and by the same argument as above, also we get that

$$\langle u - x_{n_q}, J(x - x_{n_q})\rangle \leq 0.$$

As we tend p and q to ∞ we get $\langle u - x, J(y - x)\rangle \leq 0$ and $\langle u - y, J(x - y)\rangle \leq 0$.
Thus we get by adding this two inequalities:

$$\langle x - y, J(x - y)\rangle \leq 0$$

Then $\| x - y \| \phi(\| x - y \|) \leq 0$. So we conclude that $x = y$, and the sequence $\{x_n\}$ converges weakly to x.
We claim that $\{x_n\}$ converges strongly to x.
In fact, we know that there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty} \langle u - x, J(x_n - x)\rangle = \lim_{k\to\infty} \langle u - x, J(x_{n_k} - x)\rangle$$

But the last term of the above identity converges to 0 since J is weakly sequentially continuous.
In the other hand, by **(i)** of Lemma 1, we know that:

$$\begin{aligned}
\phi(\| x_n - x \|) &= \phi(\| (1 - \alpha_n)(T(t_n)x_n - x) + \alpha_n(u - x) \|) \\
&\leq \phi((1 - \alpha_n) \| T(t_n)x_n - x \|) + \alpha_n \langle u - x, J(x_n - x)\rangle \\
&\leq (1 - \alpha_n)\phi(\| x_n - x \|) + \alpha_n \langle u - x, J(x_n - x)\rangle
\end{aligned}$$

Thus $\phi(\| x_n - x \|) \leq \langle u - x, J(x_n - x)\rangle$
Hence $\limsup_{n\to\infty} \phi(\| x_n - x \|) \leq \limsup_{n\to\infty} \langle u - x, J(x_n - x)\rangle = 0$.
Then $\| x_n - x \|$ converges to 0, and $\{x_n\}$ converges strongly to x and the proof is complete. □

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