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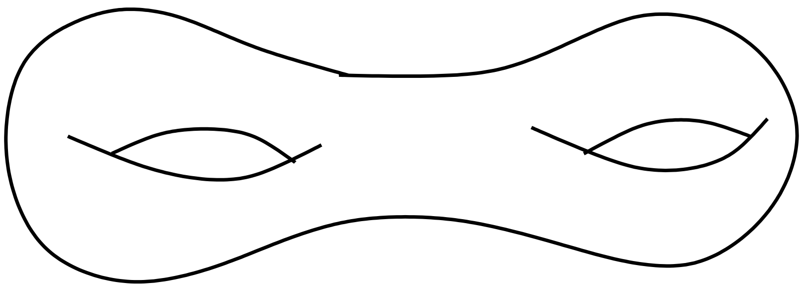
THE GEOMETRIZATION CONJECTURE AFTER R. HAMILTON AND G. PERELMAN

1. Introduction

This is the text of a Lagrange Lecture given in the mathematics department of the University of Torino. It aims at describing quite briefly the main milestones in the proof of the Geometrization conjecture due to G. Perelman using R. Hamilton's Ricci Flow. It is by no means exhaustive and intends to be a rough guide to the reading of the detailed literature on the subject. Extended notes have been published by H.-D. Cao and X.-P. Zhu ([5]), B. Kleiner and J. Lott ([16]) and J. Morgan and G. Tian ([17]). The reader may also look at the following survey papers [1, 3, 21] and the forthcoming monography [2].

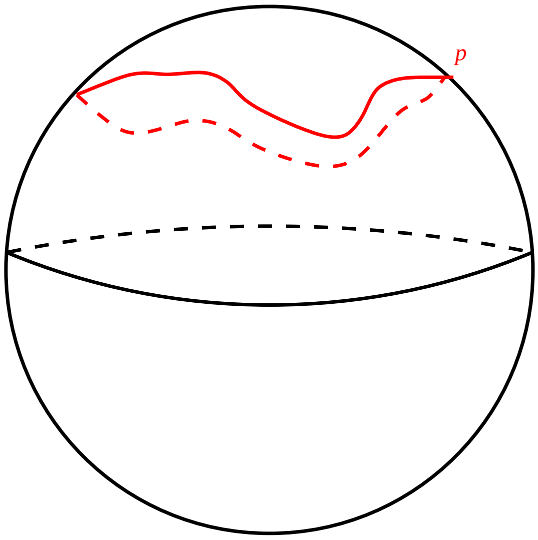
2. The classification of surfaces

It is known since the end of the 19th century that any closed orientable surface is the boundary of a ”bretzel”.



And to the question: how is the sphere characterized, among closed orientable surfaces? We can give a very simple answer. It is the only simply-connected surface.

*“LEZIONE LAGRANGIANA” given on January 30th, 2007.



This means that any continuous loop can be continuously deformed to a constant loop (to a point).

3. The 3-manifolds case

We may ask the same question, that is, let M be a closed, connected and orientable 3-manifold. How can we distinguish the sphere? The answer is the content of the famous Poincaré conjecture.

CONJECTURE 1 (Poincaré [22], 1904). If M^3 is simply connected then M is homeomorphic (diffeo) to the 3-sphere S^3.

The question was published in an issue of the Rendiconti del Circolo Matematica di Palermo ([22]). Let us recall that in dimension 3 the homeomorphism classes and the diffeomorphism classes are the same. The next conjecture played an important role in the understanding of the situation.

CONJECTURE 2 (Thurston [24], 1982). M^3 can be cut open into geometric pieces.

The precise meaning of this statement can be checked in [3]. It means that M can be cut open along a finite family of incompressible tori so that each piece left carries one of the eight geometries in dimension 3 (see [23]). These geometries are characterized by their group of isometries. Among them are the three constant curvature geometries: spherical, flat and hyperbolic. One finds also five others among which the one given by the Heisenberg group and the Sol group (check the details in [23]).

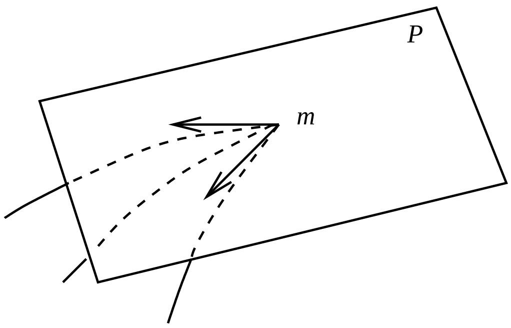
Thurston's conjecture has put the Poincaré conjecture in a geometric setting, namely the purely topological statement of Poincaré is understood in geometrical terms: a simply connected 3-manifold should carry a spherical geometry. Letting the geometry enter the picture opens the Pandora box; the analysis comes with the geometry.

4. Some basic differential geometry

The idea developed by R. Hamilton is to deform continuously the ”shape” of a manifold in order to let the various pieces, in Thurston's sense, appear. What do we mean by ”shape”?

4.1. Shape of a differentiable manifold: the sectional curvature

By ”shape” we mean a Riemannian metric denoted by g. Let us recall that it is a Euclidean scalar product on each tangent space, $T_m(M)$, for $m \in M$. Although the situation is infinitesimally Euclidean (*i.e.* on each tangent space) it is not locally. The defect to being locally Euclidean is given by the curvature. Giving the precise definitions, explanations and examples would be beyond the scope of this text. The reader is referred to the standard textbooks such as [6] or [9]. The paradigm is the 2-dimensional Gauß curvature. Let us take, for $m \in M$, a 2-plane $P \subset T_m(M)$.



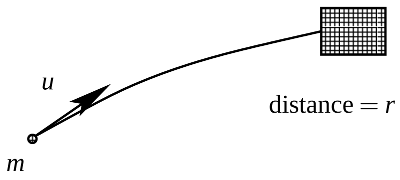
Let us consider the piece of surface obtained by the family of geodesics starting from m and tangent to a vector in P, then the sectional curvature associated to P, denoted by $K(P)$ is the Gauß curvature at m (and only at m) of this surface.

4.2. The Ricci curvature

The most important object is the Ricci curvature which is an average of the sectional curvature. Let $u \in T_m(M)$ be a unit vector and (u, e_2, e_3) and orthonormal basis of $T_m(M)$. Then,

$$\mathrm{Ricci}(u,u) = K(u,e_2) + K(u,e_3)\,.$$

This definition is however not enlightening. Let us look at the picture below,



the volume element at distance r from m in the direction of a unit vector u has an asymptotic expansion given by,

$$d\,\mathrm{vol} = (1 - \frac{r^2}{6}\mathrm{Ricci}_m(u,u) + o(r^2))d\,\mathrm{vol}_{\mathrm{eucl}}$$

where $d\,\mathrm{vol}_{\mathrm{eucl}}$ is the Euclidean volume element written in normal coordinates (see [9]). From this it is clear that Ricci is a bilinear form on $T_m(M)$ that is an object of the same nature than the Riemannian metric.

4.3. The scalar curvature

Finally, the simplest curvature although the weakest is the scalar curvature which is, at each point, the trace of the bilinear form Ricci with respect to the Euclidean structure g. It is a smooth real valued function on M.

5. Hamilton's Ricci flow

This is an evolution equation on the Riemannian metric g whose expected effect is to make the curvature look like one in the list of the geometries in 3 dimensions. This expectation is however far too optimistic and not yet proved to be achieved by this technique. Nevertheless, this ”flow” turns out to be sufficiently efficient to prove both Poincaré and Thurston's conjectures. The inspiration for this beautiful idea is explained in [10] and [4]. let (M, g_0) be a Riemannian manifold, we are looking for a family of Riemannian metrics depending on a parameter $t \in \mathbf{R}$, such that $g(0) = g_0$ and,

$$\frac{dg}{dt} = -2\mathrm{Ricci}_{g(t)}\,.$$

The coefficient 2 is completely irrelevant whereas the minus sign is crucial. This could be considered as a differential equation on the space of Riemannian metrics (see [4]), it is however difficult to use this point of view for practical purposes. It is more efficient to look at it in local coordinates in order to understand the structure of this equation ([10]). This turns out to be a non-linear heat equation, which is schematically like

$$\frac{\partial}{\partial t} = \Delta_{g(t)} + Q\,.$$

Here Δ is the Laplacian associated to the evoluting Riemannian metric $g(t)$. The minus sign in the definition of the Ricci flow ensures that this heat equation is not backward and thus have solutions, at least for small time. The expression encoded in Q is quadratic in the curvatures. Such equations are called **reaction-diffusion** equations. The diffusion term is Δ; indeed if Q is equal to zero then it is an honest (time dependent) heat equation whose effect is to spread the initial temperature density. The reaction term is Q; if Δ were not in this equation then the prototype would be the ordinary differential equation,

$$f' = f^2\,,$$

for a real valued function f. It is well-known that it blows up for a finite value of t. These two effects are opposite and the main question is: **who will win?**

5.1. Dimension 2

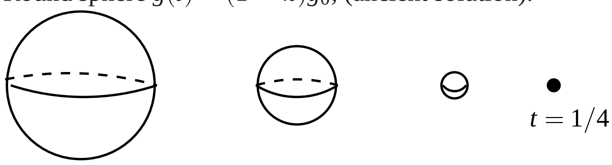
It is shown in [12] (see also[7]) that in 2-dimensions the diffusion wins, that is, starting from any (smooth) Riemannian metric on a compact and orientable surface, the flow converges, after rescaling, towards a constant curvature metric (even in the same conformal class). Full details are given in [7].

5.2. Examples in dimension 3

The following examples can be easily computed.

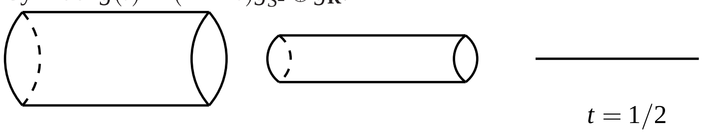
1. Flat tori, $g(t) \equiv g_0$; (it is said to be an eternal solution).

2. Round sphere $g(t) = (1-4t)g_0$; (ancient solution).



3. Hyperbolic space $g(t) = (1+4t)g_0$; (immortal solution).

4. Cylinder $g(t) = (1-2t)g_{S^2} \oplus g_{\mathbf{R}}$.



Two features deserve to be stressed. For the round sphere the flow stops in finite positive time but has an infinite past. For the hyperbolic manifolds, on the contrary, the flow has a finite past but an infinite future. We find these aspects in the core of the proofs of the two conjectures. Indeed, for the Poincaré conjecture one is led to (although it is not strictly necessary) show that starting from any Riemannian metric on a simply-connected 3-manifold the flow stops in finite time whereas for Thurston's conjecture one ought to study the long term behaviour of the evolution.

6. The seminal result

On a compact, connected and simply-connected 3 manifold M the idea is now to start with a ”shape” and deform it. We hope that, as for the example, the manifold M will contract to a point and that after rescaling the metric will become more and more round (*i.e.* constantly curved).

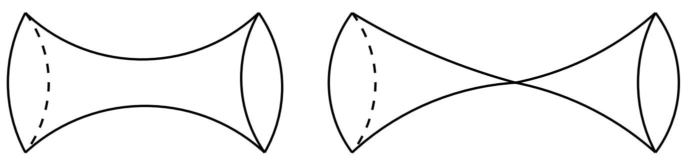
This is indeed the case when one adds some assumption on the metric.

THEOREM 3 ([10], R. Hamilton, 1982). *This scheme works if we start with a metric g_0 which has positive Ricci curvature.*

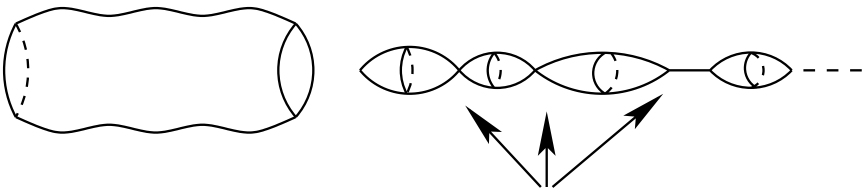
This is the seminal theorem of the theory. Clearly it is a step towards the proof of the Poincaré conjecture. The only restriction is important since it is not known whether a simply-connected manifold carries a metric of positive Ricci curvature. The proof is done along the lines mentioned above, the manifold becomes more and more round while contracting to a point.

7. The idea of surgery

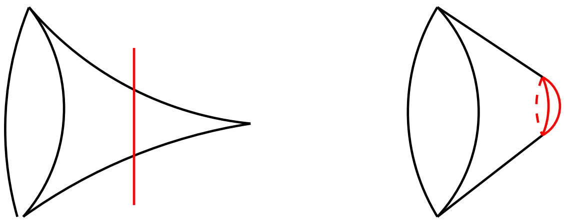
The question is now what happens if we start with a random metric g_0? It turns out that there are examples showing that the manifold may become singular, *i.e.* that the scalar curvature may become infinite on a subset of M. This is the case for the neckpinch:



which is a metric on a cylinder which develops a singularity in finite time (see [7]). But it could also be worse,



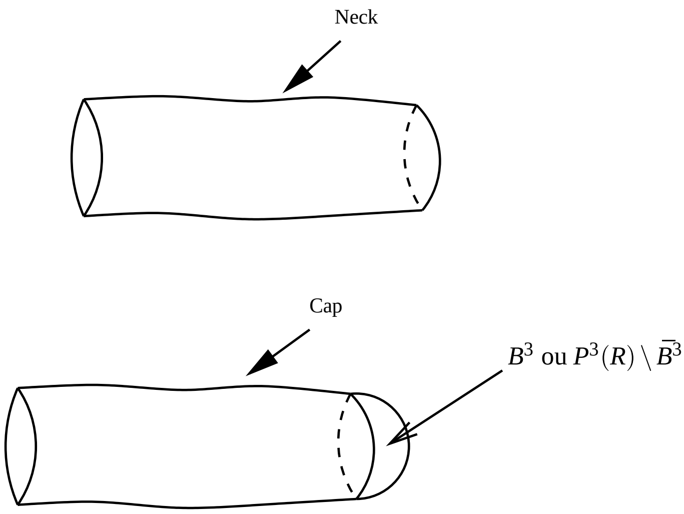
it could be that the singularities appear in a cylinder spread on a cantor subset of transversal spheres. Notice however that there are, at the moment, no explicit examples of such a behaviour. The idea introduced by R. Hamilton in [11] is to do surgery in the necks and restart the flow with a new metric on a (possibly) new manifold. It is schematically summarized by the picture below,



The picture on the left represents a so-called horn. The precise definitions are quite involved and the reader is referred to the original papers by G. Perelman ([18, 20]) or the monographies written on this work ([16, 17, 5]).

8. An important breakthrough

One of the technical achievements obtained by G. Perelman is the so-called canonical neighbourhood theorem (see [18], 12.1). Roughly, it shows that there exists a universal number r_0 such that if we start with a suitably normalised metric g_0 then the points of scalar curvature larger than r_0^{-2} have a neighbourhood in which the geometry is close to a model. There is a finite list of such model geometries and hence a very restricted list of topologies. The neighbourhood is either a cylinder, called a neck, with a metric close to a standard round cylinder, a so-called cap which is a metric on a ball or on the complement of a ball in the projective space which looks like a cylinder out of a small set, or the manifold M is a quotient of the 3-sphere by a subgroup Γ of $O(3)$.



9. The scheme of the proof

Let us now start with a Riemannian manifold (M, g_0), where g_0 is an arbitrary metric. We start the flow and let it go up to the first singular time, that is up to the first time when the scalar curvature reaches $+\infty$. Two cases may occur:

i) the curvature becomes big everywhere

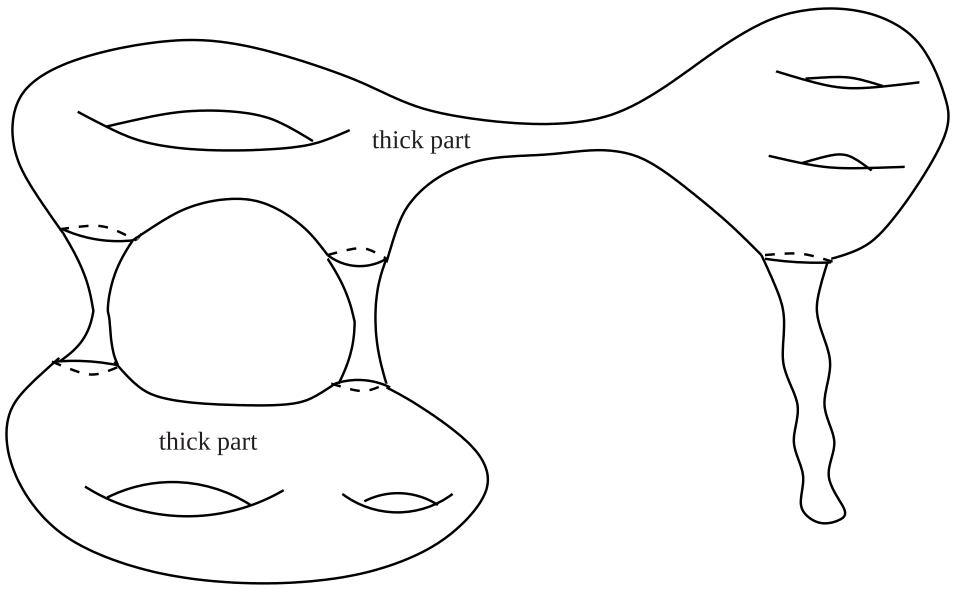
Just before the singular time the curvature may be big everywhere and the manifold may be entirely covered by canonical neighbourhoods. In that case we say that the manifold becomes extinct. Pasting together these neighbourhoods whose topology is known, leads to the following result

THEOREM 4 (Perelman, [20]). *If the manifold becomes extinct then it is,*
i) S^3/Γ, $(\Gamma \subset SO(4))$,
ii) $S^1 \times S^2$ *or* $(S^1 \times S^2)/\mathbf{Z}^2 = \mathbf{P}^3(\mathbf{R}) \# \mathbf{P}^3(\mathbf{R})$.

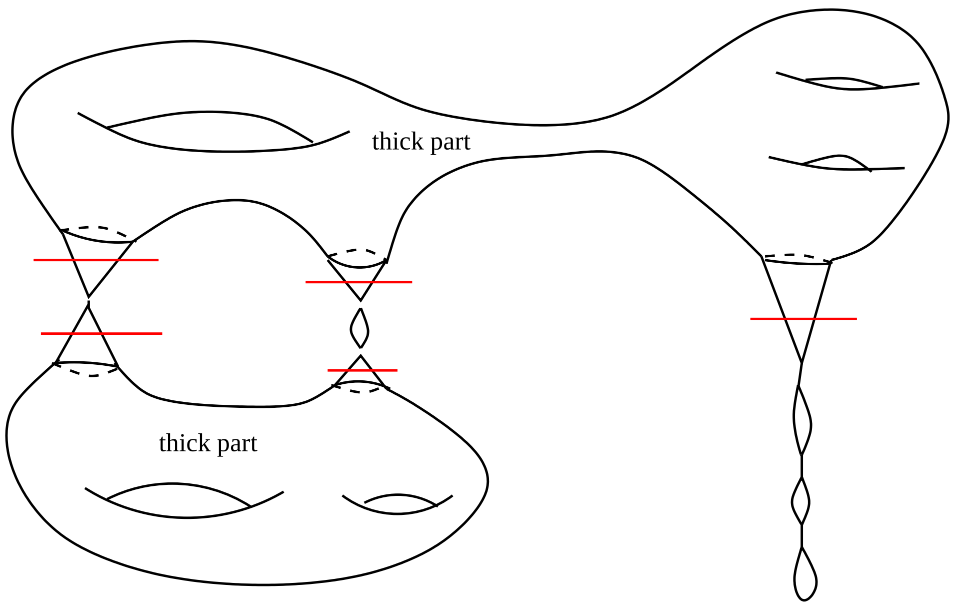
In that case we can stop the process since we have understood the topology of M. This is why it is said that the manifold becomes extinct (the curvature is high hence the manifold is small!).

ii) The manifold does not completely disappear

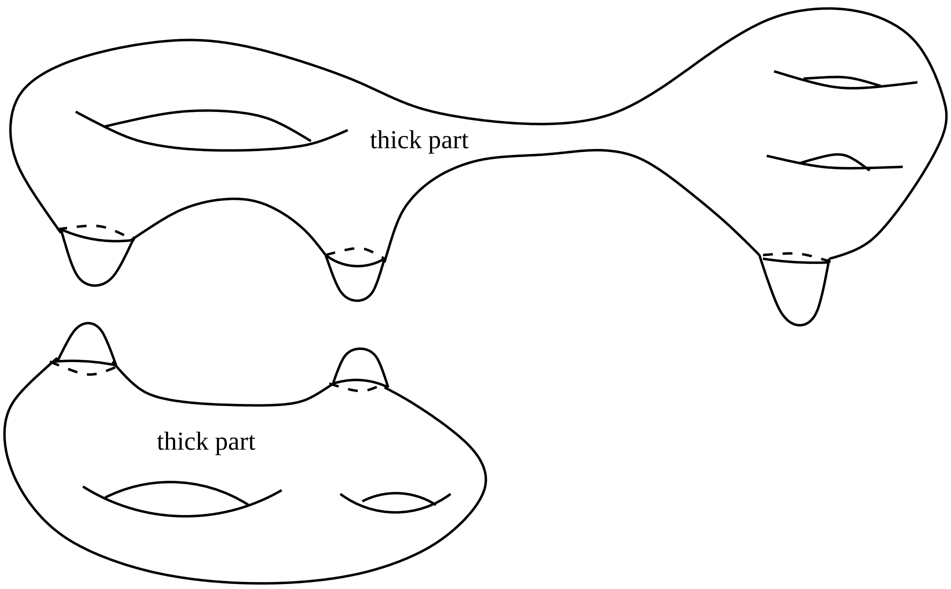
An open subset Ω is left. This is the subset of M where the scalar curvature is finite at the singular time. Just before the singular time, the manifold splits in a thick part where the scalar curvature is smaller than the scale of the canonical neighbourhoods and a thin part. So it looks like the drawing below.



At the singular time, applying surgery in the horns as shown below,



leads to a new manifold M_1, possibly not connected.



From this new Riemannian manifold one starts the Ricci flow up to the next singular time. To each connected component we apply the same dichotomy. The question is now to know whether this can be done for all time.

10. The main results

In [20] section 5 it is shown that this procedure leads to the Ricci flow with surgery, which is a non continuous version of the smooth Ricci flow (the manifold is not even fixed) defined for all time. Stating precisely the result would be too technical and beyond the scope of this note; the reader is referred to the text mentioned above. The key step is to show that the surgeries do not accumulate, that is, on a given finite interval of time there are only finitely many of them. Globally, there may be infinitely many surgeries to perform. The question whether on an infinite interval of time we reach some special geometry will be discussed later.

The proof of this result is quite involved. The point is that if we start with a normalised metric (see the references for a precise definition) then after the first surgery it is not any more. Thus the surgery parameter r_0 has changed. It changes in fact after each surgery and it may be that it goes to zero in finite time which will stop the procedure and corresponds to an accumulation of surgeries. Showing that it is not the case is a *tour de force* which is a masterpiece of Riemannian geometry.

Let us say that the solution becomes extinct if all connected component are covered by canonical neighbourhoods at some time. Then one has the

THEOREM 5 ([19], [8]). *If M is simply connected, the solution extinct in finite time.*

The proof given in [19] relies on an idea that was developed by R. Hamilton in [13] and the details are in [5]. The proof given in [8] relies on the use of harmonic maps. In both cases the idea is that a certain Riemannian invariant decreases along the flow at a certain speed whereas it is never zero, this leads to a contradiction if the flow with surgery can be done for all time without extinction.

The conclusion is that if the manifold is simply connected the Ricci flow with surgery behaves like in the previous example. The list of such manifolds is then known and this proves the Poincaré conjecture. It also proves the so-called space form conjecture since the above theorem also applies to manifolds with finite fundamental group. The list of such manifolds is

$$M = \underset{\text{finite}}{\#} S^3/\Gamma \underset{\text{finite}}{\#} S^1 \times S^2.$$

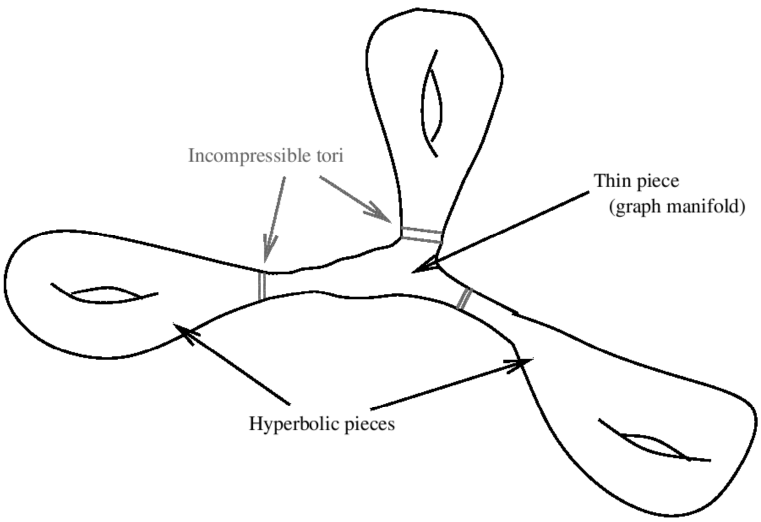
which shows that if M is simply connected then $M \simeq S^3$.

11. Manifolds with infinite fundamental group

In this case, as is shown in the example of the hyperbolic manifolds, we have to allow the flow with surgery to live for an infinite time. The main result is summarized in the following rough claim.

CLAIM 6 (Perelman [20], sections 6-8). For large t, $(M, g(t))$ decomposes into thick and thin pieces (possibly empty).

The following picture gives a hint of what may happen,



hyperbolic pieces emerge from the thick part and are bounded by incompressible tori, that is tori whose fundamental group injects in the one of the manifold. The thin pieces are collapsing, that is the volume of balls goes to zero while the sectional curvature is bounded below. The reader should check the precise definition of collapsing since it differs from the familiar one.

It is in fact the rescaled metric $\frac{1}{4t}g(t)$ which becomes thick or thin as in the case of the hyperbolic metric which behaves like $(1+4t)g_0(t)$. The assertion is that the thin part is a graph manifold.

What is a graph manifold?

Roughly speaking it is a bunch of Seifert bundles glued along their boundaries (which are tori) and a Seifert bundle is a circle bundle over a 2-orbifold with some exceptional fibers. For more precise definitions the reader is referred to [15].

The proof of the fact that the thin part is graphed is not completely clarified at the moment. An alternative proof can be found in [2].

12. Some technical aspects

We shall give below some hints about few technical issues.

12.1. Singularities

As for ordinary differential equations a smooth solution exists on a maximal interval $[0,T[$, for some $T > 0$. If T is finite one can show that

$$\lim_{t\to T^-} \sup_M |\mathrm{Riem}(g(t))| = +\infty,$$

where $|\mathrm{Riem}| =$ largest sectional curvature at a point (in absolute value).

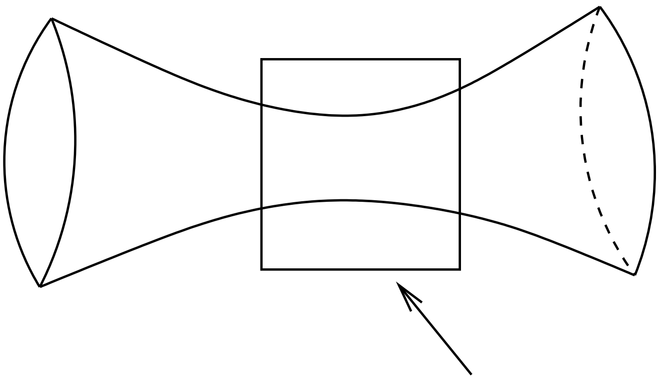
In order to describe what happens at a point where the curvature blows up, one uses the zoom technique which is familiar to all analysts. Let us assume that (x_i, t_i) is a sequence of space-time points such that

$$Q_i = |\mathrm{Riem}(x_i, t_i)| \underset{i\to\infty}{\longrightarrow} +\infty.$$

We can zoom around each x_i slowing down the time, which is translated in mathematical terms by changing the metric to

$$g_i(t) = Q_i g(t_i + t/Q_i).$$

This is called a parabolic dilation. The metrics $g_i(t)$ are also solutions of the Ricci flow equation.



The solutions $(M, g_i(t))$ are defined on a backward interval getting larger with i. A compactness theorem for flows proved by R. Hamilton (see [14]) then shows that a sub-sequence converges towards a Ricci flow $(N_\infty, g_\infty(t))$, defined on $]-\infty, 0]$. Such a solution is called an ancient solution. Now, the classification of singularities rely on the classification of ancient solutions (with extra properties). They are infinitesimal models for singularities.

12.2. Ancient solutions

An important tool is a by-product of the maximal principle for parabolic systems (see [11]), which is called the Hamilton-Ivey pinching property. It could be summarized by the inequality below. If g_0 is normalised then, for any $x \in M$ and any 2-plane tangent to M at x, one has

$$R(x,t) + 2\phi(R(x,t)) \geq K(P,t) \geq -\phi(R(x,t)),$$

where $R(x,t)$ is the scalar curvature of the metric $g(t)$ at x and $K(P,t)$ is the sectional curvature of the 2-plane P tangent at x for the metric $g(t)$. Here ϕ is the inverse function of $x \mapsto x \ln x - x$. This inequality says that the scalar curvature controls all curvatures which is not surprising in dimension 3 but makes this fact quantitative.

One particular feature is that $\phi(y)/y \underset{y\to\infty}{\longrightarrow} 0$. In particular one can apply this to the metrics g_i, for which, in rough terms one obtains,

$$K_i(P,t_i) = \frac{K(P,t_i)}{Q_i} \geq -\frac{\phi(R_i(x_i,t_i))}{Q_i} \underset{i\to\infty}{\longrightarrow} 0\,.$$

for any 2-plane P tangent at x. This shows that the curvature operator of the limit space (N_∞, g_∞) is non-negative. This is one property of the ancient solutions which appears as limit of blow-ups. It is also possible to show that $(N_\infty, g_\infty(t))$ has bounded sectional curvatures for all t (the bound may depend on t) and that they are not collapsed. In dimension 3 this leads to a complete classification, which yields the list below:

- S^3/Γ where S^3 is the round sphere

- diffeomorphic to S^3 or $\mathbf{P}^3\mathbf{(R)}$
- $\mathbf{R} \times S^2$ or $(\mathbf{R} \times S^2)/\mathbf{Z}_2 = \mathbf{P}^3\mathbf{(R)} \setminus B^3$ with canonical metric,
- B^3 with positive curvature.

This classification is given in section 11 of [18] and is finished in [20].

13. Conclusion

This short note is a very brief account of the main ideas underlying the proofs of the Poincaré and Geometrization conjectures. Although there are still some issues to clarify, at this moment it is accepted that both are proved. We skipped the technical details and definitions since they are really involved and the reader is referred to the references mentioned. This masterpiece of Riemannian geometry is now at the stage where people are trying to simplify some of the arguments and there are several direction in which this could be improved. For example one could try to prove that the metric does evolve towards some kind of canonical one on each piece. At the moment the only situation in which we know that is when the limit is (after rescaling) hyperbolic. A probable by-product of that is that there could always be finitely many surgeries which would be a great simplification. Beyond a proof without surgeries at all is a dream that could be achieved if one finds a way to construct a new and more efficient flow.

Study of the Ricci flow for higher dimensions and in particular for Kähler manifolds is the most promising direction of research at the moment as well as improvements of the Hamilton-Ivey pinching inequality which lead to rigidity theorems such as the sphere theorem. There is no doubt that more is to come.

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